Distributions - Part I

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Motivation

We would like $\delta(x) : \delta(x) = 0$ for $x \neq 0 \wedge \int_{\mathbb{R}^d} \delta(x)\,dx = 1$.

This allows $\int_{\mathbb{R}^d} \delta(x)\varphi(x)\,dx = \varphi(0)$, for $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Consider now:

$$\int_{-1}^{1} |x|\varphi'(x)\,dx = -\int_{-1}^{1} \text{sign}(x)\varphi(x)\,dx,$$

$$\int_{-1}^{1} \text{sign}(x)\varphi'(x)dx = \int_{-1}^{0} -\varphi'(x)\,dx + \int_{0}^{1} \varphi'(x)\,dx = -2\varphi(0).$$

Test functions - $\mathcal{D}(\Omega)$

Let $\Omega \subset \mathbb{R}^{\mathrm{d}}$ and $C_0^\infty(\Omega) = \{\varphi \in C^\infty(\Omega) | \operatorname{supp}(\varphi) \subset\subset \Omega\}$.

Whereby $\operatorname{supp}(\varphi) = \overline{\{x \in \Omega | \varphi(x) \neq 0\}}$.

With the usual pointwise addition and skalar multiplication this is a vector space (over $\mathbb{C}$).

We further define the following convergence:

$$\lim_{j\to\infty} \varphi_j = \varphi \text{ in } C_0^\infty(\Omega), \text{ if}$$

- $\exists K \subset\subset \Omega \ \forall j : \ \operatorname{supp}(\varphi_j) \subset K$,
- $\forall \alpha \in \mathbb{N}^{\mathrm{d}} : D^\alpha \varphi_j \to D^\alpha \varphi$ uniformly, i. e. $\lim_{j\to\infty} \sup_{x\in K} |D^\alpha \varphi_j(x) - D^\alpha \varphi(x)| = 0$.

Distributions - $\mathcal{D}'(\Omega)$

The map $T : \mathcal{D}(\Omega) \to \mathbb{C}$ is a distribution, if and only if it is a sequentially continuous linear form, i.e.

$$(\varphi_j \to \varphi \Rightarrow T(\varphi_j) \to T(\varphi)) \;\wedge\; T(\varphi + \lambda\psi) = T(\varphi) + \lambda T(\psi).$$

Equivalently:

$\forall K \subset\subset \Omega \, \exists c, N$ constant, such that:

$\forall \varphi \in \mathcal{D}(K) : \; |T(\varphi)| \leq c||\varphi||_{N,\infty;K} = c \sum_{|\alpha| \leq N} \sup_{x \in K} |\partial^\alpha \varphi(x)|.$

If in the above a certain N suffices for all compact K, then the smallest of those is called the order of T.

Regular Distributions

Let $\emptyset \neq \Omega \subset \mathbb{R}^d$ open, $(\mathbb{R}^d, \lambda^d)$ the Lebesgue measure space and $f \in L^1_{\text{loc}}(\Omega)$, then following is a Distribution of order 0:

$$T_f : C_0^\infty(\Omega) \to \mathbb{C},\ \varphi \mapsto \int_\Omega f \cdot \varphi \, \mathrm{d}\lambda^d. \quad \Big(= (f, \varphi) \Big)$$

If we additionally let $\alpha \in \mathbb{N}^d$, then

$$(D^\alpha T_f)(\varphi) := (-1)^{|\alpha|} T_f(D^\alpha \varphi) = (-1)^{|\alpha|} \int_\Omega f \cdot D^\alpha \varphi \, \mathrm{d}\lambda^d$$

is a Distribution, since for $\varphi_j \to \varphi$:

$$|(D^\alpha T_f)(\varphi_j - \varphi)| = |\int_K f \cdot D^\alpha(\varphi_j - \varphi)\, \mathrm{d}\lambda^d| \leq \|D^\alpha(\varphi_j - \varphi)\|_{\infty;K} \|f\|_{1;K} \to 0.$$

Properties $\mathcal{D}'(\Omega)$

By definition, $\mathcal{D}'$ permits a $\mathbb{C}$ vectorspace structure. Denoting $T(\varphi) = (T, \varphi)$, we get a bilinear form on $\mathcal{D}'$.

Let now $a \in C^\infty(\Omega)$ and $\alpha, \beta \in \mathbb{N}^d$.
With the following definitions we get:

$$(aT, \varphi) := (T, a\varphi) \Rightarrow aT \in \mathcal{D}',$$
$$(D^\alpha T, \varphi) := (-1)^{|\alpha|}(T, D^\alpha \varphi) \Rightarrow D^\alpha T \in \mathcal{D}'.$$

And further:

$$D_i(aT) = D_i(a)T + aD_i(T),$$
$$D^{\alpha+\beta} T = D^\alpha(D^\beta T) = D^\beta(D^\alpha T).$$

Localization

For $T \in \mathcal{D}'(\Omega)$ we define its restriction on $\Omega_0 \subset \Omega$, by

$$T\Big|_{\Omega_0}(\varphi) = T(\varphi) \;\; \forall \varphi \in \mathcal{D}(\Omega_0).$$

Using this we define the support of $T \in \mathcal{D}'(\Omega)$:

$$\text{supp}(T) = \{x \in \overline{\Omega} | \; \forall \delta > 0 : T\Big|_{\Omega \cap B_\delta(x)} \neq 0\}.$$

For regular distributions, we have: $\text{supp}(T_f) = \text{supp}(f)$.

We also have the general implication for $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$:

$$\text{supp}(T) \cap \text{supp}(\varphi) = \emptyset \Rightarrow (T, \varphi) \equiv 0.$$

Convolution with functions

For $T \in \mathcal{D}'(\Omega)$ and $\psi \in \mathcal{D}(\Omega)$ we define their convolution at $x \in \Omega$, by

$$(T * \psi)(x) := T(\psi(x - \cdot)) = (T, \psi(x - \cdot)).$$

For a regular distribution T_f we have: $(T_f * \psi)(x) = \int_\Omega f(y)\psi(x - y)\,\mathrm{d}y$.

This convolution has the following properties:

- $T * \varphi \in C^\infty(\mathbb{R}^\mathrm{d})$,
- $\mathrm{supp}(T * \varphi) \subset \mathrm{supp}(T) + \mathrm{supp}(\varphi)$,
- $D^\alpha(T * \varphi) = (D^\alpha T) * \varphi = T * (D^\alpha \varphi)$.

For $\eta \in \mathcal{D}$ as well we get further: $(T * \eta) * \varphi = T * (\eta * \varphi)$.

Convergence in $\mathcal{D}'$

We define the following convergence on $\mathcal{D}'$:

$$\lim_{k\to\infty} T_k = T \;:\Leftrightarrow\; \forall\varphi\in\mathcal{D}: \lim_{k\to\infty}(T_k,\varphi) = (T,\varphi).$$

This convergence makes $\mathcal{D}'$ a complete space.

With the mollifier $J_\varepsilon(x) = \varepsilon^{-\mathrm{d}} J(\varepsilon^{-1}x) = \varepsilon^{-\mathrm{d}} c\exp(-\frac{1}{1-|\varepsilon^{-1}x|^2})1_{\{|\varepsilon^{-1}x|<1\}}$ we get the convergence:

$$T * J_\varepsilon \to T \quad \text{in } \mathcal{D}'.$$

So the space C^∞ is dense in $\mathcal{D}'$ and analogously for the space with compact support.

Delta Distribution

Let $a \in \Omega$ and define,

$$\delta_0 \in \mathcal{D}' : \quad \delta_0(\varphi) = \varphi(0) \ \forall \varphi \in \mathcal{D}. \qquad \left[\int_\Omega f(x)\varphi(x)\,\mathrm{d}x \right]$$

Approximation via Dirac sequences $(t_k)_k \subset L^1(\mathbb{R}^\mathrm{d})$:

1. $\forall x \in \mathbb{R}^\mathrm{d} \ \forall k \in \mathbb{N} : t_k(x) \geq 0,$
2. $\forall k \in \mathbb{N} : \int_{\mathbb{R}^\mathrm{d}} t_k(x)\mathrm{d}x = 1,$
3. $\forall \varepsilon > 0 : \lim_{k\to\infty} \int_{\mathbb{R}^\mathrm{d} \setminus B_\varepsilon(0)} t_k(x)\mathrm{d}x = 0.$

For example:

$$t_k(x) = \frac{1}{\sqrt{2\pi k^{-1}}} \exp\left(-\frac{x^2}{2k^{-1}}\right) \qquad \text{or} \qquad t_k(x) = J_{\varepsilon=\frac{1}{k}}(x)$$

Radon Measures

Let $(X, \mathcal{B}, \mu)$ with X Hausdorff, such that

- μ is locally finite, i.e. $\forall x \in X\ \exists U$ open $: \mu(U) < \infty$,
- μ is inner regular, i.e. $\forall A \in \mathcal{B} : \mu(A) = sup\{\mu(K) | K \subset A : K$ compact$\}$.

Let $M(\Omega)$ be the set of Radon Measures on Ω and $\mu \in M(\Omega)$, then

$$(\mu, \varphi) := \int_\Omega \varphi(x)\, \mathrm{d}\mu(x), \text{ for } \varphi \in \mathcal{D}, \quad \text{defines a Distribution.}$$

In particular, the Dirac measure $\delta(A) = \begin{cases} 1, & 0 \in A, \\ 0, & 0 \notin A \end{cases}$ $(A \subset \mathbb{R}^{\mathrm{d}})$ gives the

$$\text{Delta Distribution:} \quad (\delta_0, \varphi) = \int_\Omega \varphi(x)\, \mathrm{d}\delta_0(x) = \varphi(0) = \delta_0(\varphi) \quad \text{for } \varphi \in \mathcal{D}.$$

Distributions - Part II

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Distributional Solution

Let $T \in \mathcal{D}'$, $f \in \mathcal{C}(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ and $P(D) = \sum_{|\alpha| \leq k} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$.

T is a distributional solution to $P(D)u = f$, if and only if

$$P(D)T = T_f \Leftrightarrow (P(D)T - T_f, \varphi) = 0 \quad \forall \varphi \in \mathcal{D}.$$

For the Laplacian Operator Δ on T we have:

$$\Delta T = \sum_{i=1}^{d} D_i D_i T \Rightarrow (\Delta T, \varphi) = (T, \Delta \varphi), \quad \text{for } \varphi \in \mathcal{D}.$$

If $g \in C^2(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ and $T = T_g$ we have the classical Green fromula

$$\int (\varphi \Delta g - g \Delta \varphi) \, \mathrm{d}x = 0.$$

Fundamental Solution

For $P(D) = \sum_{|\alpha| \leq k} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}$, the solution γ of

$$P(D)u(x) = 0, \text{ i.e. } \qquad \boxed{P(D)\gamma = \delta_0}$$

is called a fundamental solution.

The inhomogeneous solution to

$$P(D)u(x) = f(x) \quad \text{is} \quad u(x) = (\gamma * f)(x) = \int \gamma(x-y) f(y)\, dy.$$

Reminder

We have seen:

$$\int_{-1}^{1} |x|\varphi'(x)\,\mathrm{d}x = -\int_{-1}^{1} \mathrm{sign}(x)\varphi(x)\,\mathrm{d}x$$

$$\int_{-1}^{1} \mathrm{sign}(x)\varphi'(x)\,\mathrm{d}x = -2\varphi(0) = -\int_{-1}^{1} 2\delta_0(x)\varphi(x)\,\mathrm{d}x.$$

So:

$$(\mathrm{sign}', \varphi) = -(\mathrm{sign}, \varphi') = -(-2\delta_0, \varphi) \;\Rightarrow\; \mathrm{sign}' = 2\delta_0 \quad \left[= \int 2\delta_0 \cdot \,\mathrm{d}x \right]$$

Example - I

For the Heaviside function $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ we have:

$$(H', \varphi) = -(H, \varphi') = -\int_0^\infty 1\varphi'(x)\,\mathrm{d}x = \varphi(0) = (\delta_0, \varphi) \;\Rightarrow\; \frac{\partial}{\partial x}H(x) = \delta_0(x)$$

So for the Laplacian Δ in $\mathbb{R}$:

$$\Delta\gamma = \frac{\partial^2}{\partial x^2}\gamma = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\gamma\right) = \delta_0 \Rightarrow \frac{\partial}{\partial x}\gamma = H(x) + c$$

$$\Rightarrow \gamma = xH(x) + cx + c' \quad \overset{c=\frac{1}{2}, c'=0}{\Rightarrow} \quad \gamma(x) = \frac{1}{2}|x|$$

Example - II

From classical theory in $\mathbb{R}^d$:

$$\gamma(x) = \begin{cases} \frac{1}{2\pi}\ln(|x|) \\ \frac{1}{(d-2)\omega_d}|x|^{2-d} \end{cases} \quad \text{and} \quad \int_{|x|\leq c} |\gamma(x)| = \begin{cases} \int_0^c |\ln(r)| r\,\mathrm{d}r, & d=2, \\ \frac{1}{(d-2)}\int_0^c r^{2-d} r^{d-1}, & d \geq 2 \end{cases}$$

Then with R sufficiently large,

$$(\Delta\gamma, \varphi) = (\gamma, \Delta\varphi) = \int \gamma(x)\Delta\varphi(x)\,\mathrm{d}x = \lim_{\varepsilon\searrow 0} \int_{\varepsilon<|x|<R} \gamma(x)\Delta\varphi(x)\,\mathrm{d}x = \lim_{x\searrow 0} J_\varepsilon.$$

Using Green's second identity:

$$J_\varepsilon = \int_{\varepsilon<|x|<R} \varphi\Delta\gamma\,\mathrm{d}x + \int_{|x|=\varepsilon} \Big(\gamma\frac{\partial\varphi}{\partial\nu} - \varphi\frac{\partial\gamma}{\partial\nu}\Big)\,\mathrm{d}\sigma.$$

Example - II cont.

Since $\Delta\gamma = 0$ for $|x| > \varepsilon$ and for $|x| = \varepsilon$ we have $\frac{\partial}{\partial\nu} = \frac{\partial}{\partial r}$:

$$J_\varepsilon = \int_{|x|=\varepsilon} \Big(\gamma \frac{\partial\varphi}{\partial\nu} - \varphi \frac{\partial\gamma}{\partial\nu}\Big)\, \mathrm{d}\sigma = \int_{|x|=\varepsilon} \Big(\gamma(x)\frac{\partial\varphi}{\partial r} - \varphi(x)\frac{1}{(2-d)\omega_d}(2-d)|x|^{1-d}\Big)\, \mathrm{d}\sigma$$

$$= \frac{1}{\omega_d}\int_{|x|=\varepsilon} \Big(\frac{\varepsilon^{2-d}}{d-2}\frac{\partial\varphi(x)}{\partial r} - \varphi(x)\varepsilon^{1-d}\Big)\, \mathrm{d}\sigma$$

Using the mean value theorem with $|x'| = |x''| = \varepsilon$:

$$J_\varepsilon = \frac{1}{\omega_d}\Big(\frac{\varepsilon^{2-d}}{d-2}\frac{\partial(x')}{\partial r} - \varepsilon^{1-d}\varphi(x'')\Big)\omega_d\varepsilon^{d-1} = \frac{\varepsilon}{d-2}\frac{\partial\varphi(x')}{\partial r} + \varphi(x'').$$

As the derivative is finite for $\varepsilon \searrow 0$ we get:

$$(\Delta\gamma, \varphi) = \lim_{\varepsilon\searrow 0} J_\varepsilon = \varphi(0) = (\delta_0, \varphi).$$

The Schwartz Space $\mathcal{S}$

The vector space of all rapidly decreasing functions

$$\mathcal{S}(\mathbb{R}^d) := \{\phi \in C^\infty(\mathbb{R}^d) \mid \forall \alpha, \beta \in \mathbb{N}_0^d : \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta \phi(x)| < \infty\}$$

is called Schwartz space. We have $\mathcal{D} \subsetneq \mathcal{S}$. $\left[\text{e.g. } \exp(-|x|^2)\right]$

Topology and convergence are induced by the seminorms

$$||\phi||_N = \sup_{x \in \mathbb{R}^d} \max_{|\alpha|,|\beta|<N} |x^\alpha D^\beta \varphi(x)| \quad \Leftrightarrow \quad p_{k,l}(\phi) = \sup_{x \in \mathbb{R}^d} (|x|^k + 1) \sum_{|\beta| \leq l} |D^\beta \phi(x)|$$

$$\text{and} \quad \phi_k \to \phi \text{ in } \mathcal{S} \quad :\Leftrightarrow \quad \forall N \in \mathbb{N}_0 : ||\phi_k - \phi||_N \to 0.$$

Fourier Transform $\mathcal{F}$

For some $\phi \in \mathcal{S}$ its Fourier transform is defined as

$$\mathcal{F}\phi(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^{\mathrm{d}}} e^{-ix\cdot\xi}\phi(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}^{\mathrm{d}}.$$

which maps as $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ and is bijective and bicontinuous with inverse

$$(\mathcal{F}^{-1}\phi)(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^{\mathrm{d}}} e^{ix\cdot\xi}\phi(\xi)\,\mathrm{d}\xi, \quad \varphi \in \mathcal{S}.$$

For $\phi \in \mathcal{S}$ we also get $x^\alpha\phi, D^\alpha\phi, \mathcal{F}\phi, D^\alpha\mathcal{F}\phi, \mathcal{F}(D^\alpha\phi) \in \mathcal{S}$ and

$$D^\alpha\mathcal{F}\phi = (-i)^{|\alpha|}\mathcal{F}(x^\alpha\phi), \qquad \xi^\alpha\mathcal{F}\phi = (-i)^{|\alpha|}\mathcal{F}(D^\alpha\phi)$$

So, in particular $\quad \mathcal{F}(D^\alpha\phi) = (-i)^{-|\alpha|}\xi^\alpha\mathcal{F}\phi$

Tempered Distributions $\mathcal{S}'$

The space $\mathcal{S}'$ contains all linear forms $\mathcal{S} \to \mathbb{C}$, which are sequentially continous, i. e.

$$\phi_k \to \phi \text{ in } \mathcal{S} \Rightarrow T(\phi_k) \to T(\phi).$$

Since $$\phi_k \to \phi \text{ in } \mathcal{D} \Rightarrow \phi_k \to \phi \text{ in } \mathcal{S} \Rightarrow T(\phi_k) \to T(\phi),$$

we have $T \in \mathcal{S}' \Rightarrow T \in \mathcal{D}'$. And by

$$\mathcal{F}T(\varphi) := T(\mathcal{F}\varphi)$$

we define the Fourier transform of $T \in \mathcal{S}'$.

Example - III

The regular distribution T_e is not tempered. Let $\varphi \in \mathcal{D}(\mathbb{R})$, then

$$\psi_k(x) := e^{-k}\varphi(x-k) \to 0 \quad \text{in } \mathcal{S}(\mathbb{R}),$$

with $\psi_k \in \mathcal{D}(\mathbb{R})$, but

$$T_e(\psi_k) = \int_{\mathbb{R}} e^x \psi_k(x)\,\mathrm{d}x = \int_{\mathbb{R}} e^x e^{-k}\varphi(x-k)\,\mathrm{d}x$$

$$= \int_{\mathbb{R}} e^y \varphi(y)\,\mathrm{d}y \neq 0 = T_e(0).$$

Example - IV.1

Consider the heat equation for distributions:

$$\begin{array}{lcll} \frac{\partial u}{\partial t} u(t,x) = \Delta_x u(t,x) & \text{i.e.} & (\frac{\partial}{\partial t} T, \varphi) = (\Delta_x T, \varphi) & (0,\infty) \times \mathbb{R}^d \\ u(t,x) = f(t,x) & & (T,\varphi) = (T_f, \varphi) & \{t = 0\} \times \mathbb{R}^d \end{array}$$

Determine its Fourier transform for x:

$$\mathcal{F}_x(\Delta_x T, \varphi) = \mathcal{F}_x(T, \Delta_x \varphi) = (T, \mathcal{F}_x(\Delta_x \varphi)) = (T, (-i)^{-2} \xi^{(0,2)} \mathcal{F}_x \varphi)$$

Thus we get:

$$\mathcal{F}_x(\frac{\partial}{\partial t} T, \varphi) = \mathcal{F}_x(-T, \frac{\partial}{\partial t}\varphi) = (-T, \frac{\partial}{\partial t}\mathcal{F}_x \varphi) =$$

$$(\frac{\partial}{\partial t} T, \mathcal{F}_x \varphi) = \frac{\partial}{\partial t}(T, \mathcal{F}_x \varphi) \overset{!}{=} (-|\xi|^2 T, \mathcal{F}_x \varphi)$$

Example - IV.2

We have now the ordinary differential equation in $\mathcal{F}_x(T)$:

$$\begin{array}{l} \frac{\partial}{\partial t}\mathcal{F}_x(T,\varphi) = -|\xi|^2\mathcal{F}_x(T,\varphi) \\ \mathcal{F}_x(T,\varphi) = \mathcal{F}_x(T_f,\varphi) \end{array} \quad \text{i.e.} \quad \begin{array}{ll} \frac{\partial}{\partial t}\mathcal{F}_x(T) = -|\xi|^2\mathcal{F}_x(T) & (0,\infty)\times\mathbb{R}^{\mathrm{d}} \\ \mathcal{F}_x(T) = \mathcal{F}(T_f) & \{t=0\}\times\mathbb{R}^{\mathrm{d}} \end{array}$$

Thus we get:

$$\mathcal{F}_x(T) = e^{-t|\xi|^2}\mathcal{F}(T_f) \Rightarrow (T,\mathcal{F}_x\varphi) = e^{-t|\xi|^2}(T_f,\mathcal{F}\varphi)$$

$$\Rightarrow \mathcal{F}_\xi^{-1}(T,\mathcal{F}\varphi) = \mathcal{F}_\xi^{-1}(T_f, e^{-t|\xi|^2}\mathcal{F}\varphi) \Rightarrow (T,\varphi) = \left(T_f, \mathcal{F}_\xi^{-1}(e^{-t|\xi|^2}\mathcal{F}\varphi)\right)$$

Example - IV.3

The convolution theorem for $T_1 \in \mathcal{S}'$ and $T_2 \in \mathcal{E}'$:

$$\mathcal{F}(T_1 * T_2) = (2\pi)^{\frac{d}{2}} \mathcal{F}(T_1)\mathcal{F}(T_2)$$

With this we get:

$$\mathcal{F}_\xi^{-1}(e^{-t|\xi|^2}\mathcal{F}_x\varphi) = (2\pi)^{-\frac{d}{2}} \mathcal{F}_\xi^{-1}(e^{-t|\xi|^2}) * \varphi$$

Further

$$\mathcal{F}_\xi^{-1}(e^{-t|\xi|^2}) = (2\pi)^{-\frac{d}{2}} \int e^{ix\cdot\xi} e^{-t|\xi|^2}\, \mathrm{d}\xi = \int_{\mathbb{R}^\mathrm{d}} e^{ix\cdot\xi - t|\xi|^2}\, \mathrm{d}\xi$$

$$= (2t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}} = g(t,x)$$

Example - IV.4

So finally:

$$(T_f, g * \varphi) = \int f(t,x)(2\pi)^{-\frac{d}{2}} \int g(t-t', x-x')\varphi(t',x')\, \mathrm{d}(t',x')\, \mathrm{d}(t,x) =$$

$$\int \varphi(t',x')(2\pi)^{-\frac{d}{2}} \int f(t,x) g(t-t', x-x')\, \mathrm{d}(t,x)\, \mathrm{d}(t',x') =$$

$$\int \Big((2\pi)^{-\frac{d}{2}} \int f(t,x) g(t-t', x-x')\, \mathrm{d}(t,x) \Big) \varphi(t',x')\, \mathrm{d}(t',x') =$$

$$\int \Big((4\pi(t-t'))^{-\frac{d}{2}} f(t,x) e^{-\frac{|x-x'|^2}{4(t-t')}}\, \mathrm{d}(t,x) \Big) \varphi(t',x')\, \mathrm{d}(t',x') =$$

$$\int \Big((4\pi(-t'))^{-\frac{d}{2}} \int f(x) e^{-\frac{|x-x'|^2}{4(-t')}}\, \mathrm{d}(x) \Big) \varphi(t',x')\, \mathrm{d}(t',x').$$

Example - IV.5

The solution is the regular distribution T_h with:

$$h(t,x) = (4\pi t)^{-\frac{d}{2}} \int f(x) e^{-\frac{|x-x'|^2}{4t}} \, \mathrm{d}(x)$$

$$\ddot{\smile}$$