Introduction to Semigroup Theory

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The Way Up: Opening

The prototype of a parabolic PDE is given by the heat equation, this is

$$\begin{aligned} (\partial_t - \Delta)u &= 0 && \text{in } \mathbb{R}^d \times (0,\infty) \\ u &= f && \text{on } \mathbb{R}^d \times \{t=0\} \end{aligned}$$

Solutions are given by

$$S(t)f(x) \equiv \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp\left(\frac{-|x-\zeta|^2}{4t}\right) f(\zeta)\, d\zeta \qquad (1)$$

This is strongly dependent on $f \in X$ - what is X?

$\longrightarrow \mathcal{S}(\mathbb{R}^d) \subset \bigcap_{1\leq p\leq\infty} L^p(\mathbb{R}^d)$ and thus is well-defined for any $f \in L^q(\mathbb{R}^d)$, $1 \leq q \leq \infty$.

$\longrightarrow$ Let us investigate $S(t)$!

Properties of $S(t)$

Recall the Gauss-Weierstraß kernel

$$W_t(x) \equiv \frac{1}{(4\pi t)^{d/2}} \exp\left(\frac{-|x|^2}{4t}\right)$$

for $t > 0$.

Write $S(t)f = W_t * f$ for $t > 0$ and additionally $S(0) = I$.

The following is obvious:

- $S(t)$ is a well-defined, bounded linear operator on $L^p(\mathbb{R}^d)$ for all $p < \infty$
- $||S(t)||_{L^p \longrightarrow L^p} \leq 1$

Moreover: $S(s+t) = S(s)S(t)$ for all $s, t \geq 0$

Key topic: Semigroups

Definition

Let X be Banach and $\mathfrak{A} \equiv (S(t))_{t\geq 0} \subset \mathcal{L}(X)$ a family of linear, bounded operators $S(t)\colon X \longrightarrow X$ for all $t \geq 0$. $\mathfrak{A}$ is called a *semigroup* if and only if

$$S(0) = I \quad \text{and} \quad S(s+t) = S(s)S(t)\ \forall t, s \geq 0$$

Example: Exponential Series

Let $A \in \mathcal{L}(X)$, X Banach. Then set $S(t) = \exp(tA)$, i.e.

$$\exp(tA) \equiv I + \sum_{n \in \mathbb{N}} \frac{t^k A^k}{k!}$$

Check by

$$||\exp(tA)|| \leq \exp(t||A||) < \infty$$

and $A^k A^l = A^l A^k$ that this indeed defines a semigroup. Even more is valid: For any $x \in X$,

$$||S(t)x - x|| \leq \sum_{k \in \mathbb{N}} \frac{t^k}{k!} ||A||^k ||x|| = ||x||(\exp(t||A||) - 1) \longrightarrow 0, \; t \longrightarrow 0+$$

UC and C_0-semigroups

Due to our last example the following notion makes sense:

Definition

A semigroup $\mathfrak{A} \equiv \{S(t)\}_{t \geq 0} \subset \mathcal{L}(X)$ on X Banach is called a C_0- or *strongly continuous semigroup* if and only if

$$||S(t)x - x|| \longrightarrow 0,\ t \longrightarrow 0^+ \quad \forall x \in X$$

Moreover, it is called a *uniformly continuous or UC* semigroup if and only if

$$||S(t) - I|| \longrightarrow 0, \quad t \longrightarrow 0^+$$

Clearly, UC $\Longrightarrow C_0$. The converse is not true!

Infinitesimal Generators

Definition

Let $\mathfrak{A} \equiv (S(t))_{t\geq 0} \subset \mathcal{L}(X)$ be a semigroup on X Banach. Set

$$D(A) \equiv \left\{ x \in X : \; \exists \lim_{t \longrightarrow 0^+} \frac{S(t)x - x}{t} \right\}$$

$$Ax \equiv \lim_{t \longrightarrow 0^+} \frac{S(t)x - x}{t} \quad \text{for } x \in D(A)$$

A is called the *(infinitesimal) generator* of $\mathfrak{A}$. We write

$$\langle A \rangle_{gen} = \mathfrak{A}$$

Intermezzo: Example - The Heat Semigroup

Recall our basic example

$$S(t)f(x) \equiv \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp\left(\frac{-|x-\zeta|^2}{4t}\right) f(\zeta)\, d\zeta, t > 0$$

$$S(0) = I$$

We refer to this semigroup as the heat semigroup and write $\mathfrak{A}_{heat}$

Our goal:

We want to show $\langle \Delta \rangle_{gen} = \mathfrak{A}_{gen}$.

Recall:

- $*\colon \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$
- $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$ bijectively
- $\mathcal{F}\colon L^2(\mathbb{R}^d) \xrightarrow{\cong} L^2(\mathbb{R}^d)$
- Alternative characterization of $W^{m,2}$-Sobolev functions:

$$W^m(\mathbb{R}^d)(= W^{m,2}(\mathbb{R}^d)) = \left\{ f \in L^2(\mathbb{R}^d)\colon\ (1 + |\zeta|^2)^{m/2}\mathcal{F}f \in L^2(\mathbb{R}^d) \right\}$$

- $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$

Density $\longrightarrow$ it suffices to show the claim for the Schwartz class!

Claim:

$$\lim_{t \longrightarrow 0+} \frac{W_t * f - f}{t} = \Delta f \text{ for } f \in \mathcal{S}(\mathbb{R}^d)$$

(L^2 convergence).

$$\lim_{t \longrightarrow 0} \frac{(2\pi)^{d/2}(\mathcal{F}W_t) \cdot (\mathcal{F}f) - (\mathcal{F}f)}{t} = \mathcal{F}(\Delta f)$$

By Cauchy's integral formula we easily check that $\zeta \mapsto \exp(-\zeta^2)$ is a fixed point of $\mathcal{F}$, whereby a change of variables implies

$$\mathcal{F}W_t(\zeta) = \frac{1}{(2\pi)^{d/2}} \exp(-t\zeta^2), \;\; \mathcal{F}(\Delta f)(\zeta) = -\zeta^2 \mathcal{F}f(\zeta)$$

$$\lim_{t \longrightarrow 0} \frac{\exp(-t\zeta^2)g - g}{t} = -\zeta^2 g \qquad \text{for all } g \in \mathcal{S}(\mathbb{R}^d), \zeta \in \mathbb{R}^d$$

or

$$\lim_{t \longrightarrow 0+} \frac{\exp(tv)g - g}{t} = vg \qquad \text{for all } g \in \mathcal{S}(\mathbb{R}^d), \zeta \in \mathbb{R}^d$$

with $v(\zeta) = -\zeta^2$.

Set

$$\Phi(z) = \frac{\exp(z) - 1}{z} \equiv \sum_{n\geq 2} \frac{z^{n-1}}{n!}, \quad -1 \leq \Phi(z) \leq 0 \text{ for } z \leq 0$$

Thus

$$\left\| \frac{\exp(tv)g - g}{t} - vg \right\|^2 = \int_{\mathbb{R}^d} |\Phi(-t\zeta^2)| \, |\zeta^2 g(\zeta)|^2 \, d\zeta \longrightarrow 0$$

This estimate is valid for **any** v such that $v\mathcal{F}f \in L^2 \longrightarrow W^{2,2}(\mathbb{R}^d)$.
Conclusion: $\langle \Delta \rangle_{\text{gen}} = \mathfrak{A}$ and $D(A) = W^{2,2}(\mathbb{R}^d)$

Differential properties of Generators

Theorem

Assume $u \in D(A)$ and $\sup_{t\geq 0} ||S(t)|| < \infty$. Then

(i) $S(t)u \in D(A)$ *for all* $t \geq 0$

(ii) $AS(t)u = S(t)Au$ *for all* $t \geq 0$

(iii) $t \mapsto S(t)u$ *is differentiable for each* $t > 0$ *and* $\frac{d}{dt}S(t)u = AS(t)u$ *for* $t > 0$

Proof of (iii) & (iv): For $u \in D(A)$, $h > 0$ and $t > 0$. Consider

$$\lim_{h\longrightarrow 0+}\left(\frac{S(t)u - S(t-h)u}{h} - S(t)Au\right)$$

Topological Properties of Generators

Note: Under the assumptions of the last theorem, continuity of $t \mapsto AS(t)u = S(t)Au$ implies that $t \mapsto S(t)u$ is of class $C^1((0,\infty),X)$ for $u \in D(A)$

Recall: An operator $T\colon X \longrightarrow X$ is called closed iff its graph is closed with respect to the product topology on $X \times X$.

Theorem

The generator of a C_0-semigroup is densely defined and closed.

Exponential bounds for C_0-semigroups

Theorem

Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X Banach. Then there exist $\omega \in \mathbb{R}$, $M \geq 1$ such that

$$||S(t)|| \leq Me^{\omega t} \quad \forall t > 0$$

Proof. At first, there exists $\tau > 0$ such that

$$M \equiv \sup_{0\leq t\leq\tau} ||S(t)|| < \infty$$

Now, rescale: For $t \geq 0$ write $t = n\tau + \theta$ with $n \in \mathbb{N}$ and $0 \leq \theta < \tau$.

The goal of the game: Cauchy Problems

Theorem

Let A be the infinitesimal generator of some C_0-semigroup and let $u \in D(A) \subset X$. Then the mapping $u\colon [0,\infty) \ni t \mapsto S(t)u \in X$ is C^1, $D(A)$-valued and a solution to

$$y' = Ay \quad \& \quad y(0) = u$$

Proof. Differentiability: Compute, compute and estimate. Then: $\frac{d}{dt}S(s-t)v(t) = 0$ for $s \leq t$. $\Longrightarrow \Phi\colon [0,s] \ni t \mapsto S(s-t)v(t) \in X$ satisfies $\frac{d}{dt}(\ell \circ \Phi) = \ell \circ \frac{d}{dt}\Phi(t) = 0$ + Hahn-Banach $\Longrightarrow$ Uniqueness.

Peripety: C_0-contraction semigroups

We have: A generates a UC-semigroup $\Longleftrightarrow A \in \mathcal{L}(X)$

Question: A generates a C_0-semigroup $\Longleftrightarrow$???

We know: If A is a generator of a C_0-semigroup, then A is densely defined and closed

We need: Criterion to decide whether a densely defined, closed operator generates a C_0-semigroup $\longrightarrow$ Available for 'Contraction Semigroups'

Recall: Any C_0-semigroup satisfies $||S(t)|| \leq Me^{\omega t}$ for all $t \geq 0$.

Definition

If one can choose $\omega = 0$ and $M = 1$ in the last theorem, then the C_0-semigroup is called contractive or contraction semigroup.

On the Road to Hille-Yosida - Spectral Theory

Spectral Theory in Linear Algebra: $\sigma(A) =$ Eigenvalues

Spectral Theory in Functional Analysis: For $A\colon D(A) \longrightarrow X$ (un)bounded and closed operator, define the resolvent set

$$\lambda \in \rho(A) \subset \mathbb{C} \Longleftrightarrow \lambda - A\colon D(A) \longrightarrow X \text{ bijective}$$

and the resolvent operator

$$R_\lambda\colon X \ni u \mapsto (\lambda - A)^{-1} u$$

Then the spectrum is the complement of the resolvent set:

$$\sigma(A) = \rho(A)^c = \mathbb{C} \setminus \rho(A)$$

Standard Assumption from now on: A is a closed linear operator on X Banach.

The Hilla - Yosida Theorem

Theorem

An operator A is the infinitesimal generator of a C_0-contraction semigroup if and only if A is densely defined and closed, $(0, \infty) \subset \rho(A)$ and $||R_\lambda|| \leq \lambda^{-1} \forall \lambda > 0$.

Proof. IDEA: For $\lambda > 0$ define the bounded Yosida approximations

$$A_\lambda = \lambda A(\lambda - A)^{-1} = \lambda^2(\lambda - A)^{-1} - \lambda \in \mathcal{L}(X)$$

and define semigroups $(\exp(tA_\lambda))_{t \geq 0}$. Show: $\exists S(t)x \equiv \lim_{\lambda \to \infty} e^{tA_\lambda} x$ and this defines semigroup with operator A.

The Homogeneous Heat Equation I

$$\begin{aligned} (\partial_t - \Delta)u &= 0 && \text{in } \Omega \times (0,\infty) \\ u &= f && \text{on } \partial\Omega \times (0,\infty) \\ u(\cdot,0) &= u_0 && \text{in } \Omega \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ bounded, open, $\partial\Omega \in C^{0,1}$.

CLAIM:

Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then there exists exactly one solution

$$u \in C^1\left([0,\infty)\,;L^2(\Omega)\right) \cap C^0([0,\infty)\,;H^2(\Omega) \cap H_0^1(\Omega))$$

of the above problem such that $||u(t)||_{L^2} \leq ||u_0||_{L^2} \quad \forall t \geq 0$

The Homogeneous Heat Equation II

Proof. Set

$$X = L^2(\Omega), \quad A = \Delta, \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega)$$

Densely defined: Obvious.

$(0, \infty) \subset \rho(A)$: Show $\forall \lambda \in \mathbb{C}, \Re(\lambda) > 0$: $\mathcal{R}(\lambda - A) = X$.

Equivalently: For $\lambda \in \mathbb{C}, \Re(\lambda) > 0$ the problem

$$-\Delta u + \lambda u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has a unique solution for every $f \in L^2(\Omega)$.

Apply Lax-Milgram!

The Homogeneous Heat Equation III

Set

$$B[u,v] \equiv \int_\Omega \nabla u \cdot \overline{\nabla v} + \lambda u \overline{v}\, dx,\;\; u,v \in H_0^1(\Omega)$$
$$F(v) \equiv \int_\Omega f \overline{v}\, dx\;\, u \in H_0^1(\Omega)$$

$\Longrightarrow$ B bounded and coercive BLF, F bounded linear functional

$\Longrightarrow \exists! u \in H_0^{1,2}(\Omega) \forall v \in H_0^1(\Omega)$

USE: $u \in H^2(\Omega) \longrightarrow$ Regularity Theory.

$\Longrightarrow u \in D(A)$.

The Homogeneous Heat Equation IV

$\forall\lambda > 0$: $||R_\lambda|| \leq \lambda^{-1}$: Let $\lambda > 0$. Then

$$\int_\Omega |\nabla u|^2 + \lambda |u|^2 dx = \int_\Omega f\overline{u} dx \leq \frac{1}{2\lambda}\int_\Omega |f|^2 dx + \frac{\lambda}{2}\int_\Omega |u|^2 dx$$

$$\Longrightarrow \frac{\lambda}{2}\int_\Omega |u|^2 dx \leq \frac{1}{2\lambda}\int_\Omega |f|^2 dx \Longrightarrow ||u||_{L^2} \leq \lambda^{-1}||f||_{L^2}$$

For $u_1, u_2 \in D(A)$ solutions to the PDE

$$-\Delta u + \lambda u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

we have $u_1 = u_2$.

$\Longrightarrow R_\lambda$ exists, is continuous and $||R_\lambda f|| \leq \lambda^{-1}||f||$

The Homogeneous Heat Equation V

A is closed: Let $x_n \in D(A)$, $x_n \longrightarrow x$ in $D(A)$ and $Ax_n \longrightarrow y$ in X, $n \longrightarrow \infty$.

Show: $x \in D(A)$ and $Ax = y$.

We know: For $x + \lambda y$, there exists $z \in D(A)$ such that $(\lambda I - A)z = \lambda x - y$

$$||x_n - z|| \leq \lambda^{-1}||(\lambda - A)(x_n - z)||$$
$$= \lambda^{-1}||\lambda x_n - Ax_n - (\lambda x - y)||$$
$$\leq ||x_n - x|| + \lambda^{-1}||Ax_n - y|| \longrightarrow 0 \Longrightarrow x = z,\ Ax = y$$

Hille-Yosida $\longrightarrow$ A generates contraction semigroup $(S(t))_{t \geq 0}$.

$t \mapsto S(t)u_0 \in D(A)$ continuous

$t \mapsto u'(t) = AS(t)u_0 = S(t)Au_0 \in X$ continuous $\longrightarrow$ claim.

Second-order parabolic PDE

Recall that an elliptic differential operator of 2nd order in divergence form is given by

$$Lu \equiv -\sum_{1\leq i,j\leq d} \partial_{x_j}(a^{ij}(x)\partial_{x_i}u) + \sum_{1\leq i\leq d} b^i(x)\partial_{x_i}u + c(x)u$$

where $\zeta \cdot A(x)\zeta \geq \theta|\zeta|^2$.

Assume: $a^{ij}, b^i, c \in L^\infty(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ for all $i, j \leq d$ and they are not time-dependent. Moreover: $\partial U \in C^\infty$. Consider the general parabolic equation

$$\begin{aligned} (\partial_t + L)u &= 0 && \text{in } U_T \\ u &= 0 && \text{on } \partial U \times [0, T] \\ u &= g && \text{on } U \times \{t = 0\} \end{aligned}$$

Final Aria

Then: A defined by $Au \equiv -Lu$ for $u \in D(A) = H_0^{1,2}(U) \cap H^2(U)$ generates a γ-contraction semigroup! The problem is SOLVED.