# On motions of Prandtl-Eyring fluids in 2D

Lars Diening

LMU Munich, Germany

Bielefeld 2011, November 1st

# Stationary Navier Stokes equations (1/2)

Find velocity $\mathbf{v}$ and pressure $q$ such that

$$\begin{aligned} -\nu\Delta\mathbf{v} + [\nabla\mathbf{v}]\mathbf{v} + \nabla q &= \mathbf{f} \qquad \text{on } \Omega, \\ \operatorname{div}\mathbf{v} &= 0 \qquad \text{on } \Omega, \\ \mathbf{v} &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $\nu > 0$, $\mathbf{f}$ is external force and $\Omega \subset \mathbb{R}^2$.

Goal: Find a weak solution $\mathbf{v} \in W^{1,2}_{0,\text{div}}$ and $q \in L^2_0$.

# Stationary Navier Stokes equations (2/2)

Strategy for construction of weak solution:

- Hide pressure in weak formulation: Find $\mathbf{v} \in W^{1,2}_{0,\mathrm{div}}$ with

$$\nu\langle \nabla \mathbf{v}, \nabla \boldsymbol{\xi}\rangle + \langle [\nabla \mathbf{v}]\mathbf{v}, \boldsymbol{\xi}\rangle = \langle \mathbf{f}, \boldsymbol{\xi}\rangle \qquad \text{for } \boldsymbol{\xi} \in W^{1,2}_{0,\mathrm{div}}.$$

- $-\Delta \mathbf{v}$ is monotone operator on $W^{1,2}_{0,\mathrm{div}}$
- $[\nabla \mathbf{v}]\mathbf{v}$ is compact perturbation (for $\mathbb{R}^2$)
- $\langle [\nabla \mathbf{v}]\mathbf{v}, \mathbf{v}\rangle = 0$, since $\operatorname{div} \mathbf{v} = 0 \quad \Rightarrow \quad$ coerciveness.
- Recover pressure by De Rahm (negative norm theorem).

$\Rightarrow$ Existence!

# Power law fluids (1/4)

Find velocity $\mathbf{v}$ and pressure $q$ such that

$$\begin{aligned} -\nu \operatorname{div}\big((1+|\varepsilon(\mathbf{v})|)^{p-2}\varepsilon(\mathbf{v})\big) + [\nabla \mathbf{v}]\mathbf{v} + \nabla q &= \mathbf{f} && \text{on } \Omega,\\ \operatorname{div} \mathbf{v} &= 0 && \text{on } \Omega,\\ \mathbf{v} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $1 < p < \infty$ and $\varepsilon(\mathbf{v}) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is symmetric gradient.

shear thinning: $1 < p < 2$ (ketchup, blood)

# Power law fluids (2/4)

Weak formulation without pressure

$$\nu\langle \mathbf{S}(\varepsilon(\mathbf{v})), \varepsilon(\boldsymbol{\xi})\rangle + \langle [\nabla \mathbf{v}]\mathbf{v}, \boldsymbol{\xi}\rangle = \langle \mathbf{f}, \boldsymbol{\xi}\rangle \qquad \text{for } \boldsymbol{\xi} \in W^{1,p}_{0,\operatorname{div}}.$$

with $\mathbf{S}(\varepsilon(\mathbf{v})) := (1 + |\varepsilon(\mathbf{v})|)^{p-2}\varepsilon(\mathbf{v})$.

Goal: Find weak solution $\mathbf{v} \in W^{1,p}_{0,\operatorname{div}}$

Problem: $[\nabla \mathbf{v}]\mathbf{v}$ compact perturbation for $p > \frac{3}{2}$ (in $\mathbb{R}^2$)

Idea: Rewrite $\langle [\nabla \mathbf{v}]\mathbf{v}, \boldsymbol{\xi}\rangle$ as $\langle \mathbf{v} \otimes \mathbf{v}, \varepsilon(\boldsymbol{\xi})\rangle$

Just need $\mathbf{v} \otimes \mathbf{v} \in L^1$ for distributional solutions, i.e. $p > 1$ (in $\mathbb{R}^2$)

# Power law fluids fluids (3/4)

Weak formulation

$$\nu\langle \mathbf{S}(\varepsilon(\mathbf{v})), \varepsilon(\boldsymbol{\xi})\rangle + \langle \mathbf{v} \otimes \mathbf{v}, \varepsilon(\boldsymbol{\xi})\rangle = \langle \mathbf{f}, \boldsymbol{\xi}\rangle \qquad \text{for } \boldsymbol{\xi} \in W^{1,\infty}_{0,\mathrm{div}}.$$

**Approach for** $1 < p \leq \frac{3}{2}$

- Stabilize system such that $\langle \mathbf{v} \otimes \mathbf{v}, \varepsilon(\boldsymbol{\xi})\rangle$ is again compact perturbation
  $\Rightarrow$ Approximate solutions $\mathbf{v}_n \in W^{1,p}_{0,\mathrm{div}}$
- Weak convergent subsequence $\mathbf{v}_n \rightharpoonup \mathbf{v}$
- Problem: Identify limit $\mathbf{S}(\varepsilon(\mathbf{v}_n)) \to \mathbf{S}(\varepsilon(\mathbf{v}))$

# Power law fluids (4/4)

**Need:** $\langle \mathbf{S}(\varepsilon(\mathbf{v}_n)), \varepsilon(\xi)\rangle \to \langle \mathbf{S}(\varepsilon(\mathbf{v})), \varepsilon(\xi)\rangle$ for smooth $\xi$

**Rough idea:** Test function $\mathbf{v}^n - \mathbf{v} \in L^p(W_0^{1,p})$

$$0 \leq \big\langle \mathbf{S}(\varepsilon(\mathbf{v}^n)) - \mathbf{S}(\varepsilon(\mathbf{v})), \varepsilon(\mathbf{v}^n) - \varepsilon(\mathbf{v})\big\rangle \xrightarrow{\text{equation}} 0. \qquad (1)$$

Strict monotonicity implies $\mathbf{S}(\varepsilon(\mathbf{v}^n)) \to \mathbf{S}(\varepsilon(\mathbf{v}))$ a.e.

**Problem:** $\operatorname{div}(\mathbf{v}^n \otimes \mathbf{v}^n) \notin (W_0^{1,p})^*$ Only: $(W_0^{1,\infty})^*$

**Idea:** Approximate $\mathbf{w}^n := \mathbf{v}^n - \mathbf{v}$ by $\mathbf{w}_\lambda^n \in W_0^{1,\infty}$

which allows to proceed as in (1).

# Power law fluids (4/4)

**Need:** $\langle \mathbf{S}(\varepsilon(\mathbf{v}_n)), \varepsilon(\xi)\rangle \to \langle \mathbf{S}(\varepsilon(\mathbf{v})), \varepsilon(\xi)\rangle$ for smooth $\xi$

**Rough idea:** Test function $\mathbf{v}^n - \mathbf{v} \in L^p(W_0^{1,p})$

$$0 \leq \big\langle \mathbf{S}(\varepsilon(\mathbf{v}^n)) - \mathbf{S}(\varepsilon(\mathbf{v})), \varepsilon(\mathbf{v}^n) - \varepsilon(\mathbf{v})\big\rangle \xrightarrow{\text{equation}} 0. \tag{1}$$

Strict monotonicity implies $\mathbf{S}(\varepsilon(\mathbf{v}^n)) \to \mathbf{S}(\varepsilon(\mathbf{v}))$ a.e.

**Problem:** $\operatorname{div}(\mathbf{v}^n \otimes \mathbf{v}^n) \notin (W_0^{1,p})^*$ Only: $(W_0^{1,\infty})^*$

**Idea:** Approximate $\mathbf{w}^n := \mathbf{v}^n - \mathbf{v}$ by $\mathbf{w}^n_\lambda \in W_0^{1,\infty}$

which allows to proceed as in (1).

# Lipschitz truncation – cutting the gradients

- For $\mathbf{w} \in W_0^{1,1}(\Omega)$ we have

$$|\mathbf{w}(x) - \mathbf{w}(y)| \leq c\,|x - y|\big(M(\nabla\mathbf{w})(x) + M(\nabla\mathbf{w})(y)\big),$$

where $M(\nabla\mathbf{w})(x) = \sup_{B \ni x} \fint_B |\nabla\mathbf{w}|\,dy$.

- $\mathbf{w}$ is Lipschitz outside the small bad set $\{M(\nabla\mathbf{w}) > \lambda\}$.
- Cut out the bad set and extend to $\mathbf{w}_\lambda \in W_0^{1,\infty}(\Omega)$ with $|\nabla\mathbf{w}_\lambda| \leq c\lambda$
- By choosing *good* $\lambda$
$\|\nabla\mathbf{w}_\lambda\,\chi_{\{\mathbf{w}\neq\mathbf{w}_\lambda\}}\|_p \leq \|\lambda\chi_{\{M(\nabla\mathbf{w})>\lambda\}}\|_p \leq \delta(\lambda)\,\|\nabla\mathbf{w}\|_p$.

# Lipschitz truncation – cutting the gradients

- For $\mathbf{w} \in W_0^{1,1}(\Omega)$ we have

$$|\mathbf{w}(x) - \mathbf{w}(y)| \leq c\,|x-y|\big(M(\nabla \mathbf{w})(x) + M(\nabla \mathbf{w})(y)\big),$$

where $M(\nabla \mathbf{w})(x) = \sup_{B \ni x} \fint_B |\nabla \mathbf{w}|\,dy$.

- $\mathbf{w}$ is Lipschitz outside the small bad set $\{M(\nabla \mathbf{w}) > \lambda\}$.
- Cut out the bad set and extend to $\mathbf{w}_\lambda \in W_0^{1,\infty}(\Omega)$ with $|\nabla \mathbf{w}_\lambda| \leq c\lambda$
- By choosing *good* $\lambda$
$\|\nabla \mathbf{w}_\lambda\, \chi_{\{\mathbf{w} \neq \mathbf{w}_\lambda\}}\|_p \leq \|\lambda \chi_{\{M(\nabla \mathbf{w}) > \lambda\}}\|_p \leq \delta(\lambda)\, \|\nabla \mathbf{w}\|_p$.

# Lipschitz truncation – Conclusion

**Theorem (Lipschitz truncation; Diening, Málek, Steinhauer '07)**

*For* $\mathbf{w}_n = \mathbf{v}^n - \mathbf{v} \rightharpoonup 0 \in W_0^{1,p}$ *exists* $\mathbf{w}^{n,j} \in W_0^{1,\infty}$ *such that*

- $\varepsilon(\mathbf{w}^{n,j}) \overset{n}{\to} 0$ **-weakly in* $L^\infty$,
- $\limsup_{n\to\infty} \|\nabla \mathbf{w}^{n,j} \, \chi_{\{\mathbf{w}^n \neq \mathbf{w}^{n,j}\}}\|_p \leq 2^{-j}$,

So we get $\mathbf{S}(\varepsilon(\mathbf{v}^n)) \to \mathbf{S}(\varepsilon(\mathbf{v}))$ almost everywhere by

$$
\begin{aligned}
0 &\leq \int_{\{\mathbf{w}^n = \mathbf{w}^{n,j}\}} (\mathbf{S}(\varepsilon(\mathbf{v}^n)) - \mathbf{S}(\varepsilon(\mathbf{v}))) : (\varepsilon(\mathbf{v}_n) - \varepsilon(\mathbf{v}))\, dx \\
&= \int (\mathbf{S}(\varepsilon(\mathbf{v}^n)) - \mathbf{S}(\varepsilon(\mathbf{v}))) : \varepsilon(\mathbf{w}^{n,j})\, dx && \overset{n}{\to} 0 \text{ by equation} \\
&\quad - \int_{\{\mathbf{w}^n \neq \mathbf{w}^{n,j}\}} (\mathbf{S}(\varepsilon(\mathbf{v}^n)) - \mathbf{S}(\varepsilon(\mathbf{v}))) : \varepsilon(\mathbf{w}^{n,j})\, dx && \leq 2^{-j} \text{ after } n \to \infty
\end{aligned}
$$

# Lipschitz truncation – Conclusion

**Theorem (Lipschitz truncation; Diening, Málek, Steinhauer '07)**

*For* $\mathbf{w}_n = \mathbf{v}^n - \mathbf{v} \rightharpoonup 0 \in W^{1,p}_0$ *exists* $\mathbf{w}^{n,j} \in W^{1,\infty}_0$ *such that*

- $\varepsilon(\mathbf{w}^{n,j}) \overset{n}{\to} 0$ **-weakly in* $L^\infty$,
- $\limsup_{n\to\infty} \|\nabla \mathbf{w}^{n,j}\, \chi_{\{\mathbf{w}^n \neq \mathbf{w}^{n,j}\}}\|_p \leq 2^{-j}$,

So we get $\mathbf{S}(\varepsilon(\mathbf{v}^n)) \to \mathbf{S}(\varepsilon(\mathbf{v}))$ almost everywhere by

$$
\begin{aligned}
0 &\leq \int_{\{\mathbf{w}^n = \mathbf{w}^{n,j}\}} (\mathbf{S}(\varepsilon(\mathbf{v}^n)) - \mathbf{S}(\varepsilon(\mathbf{v}))) : (\varepsilon(\mathbf{v}_n) - \varepsilon(\mathbf{v}))\, dx \\
&= \int (\mathbf{S}(\varepsilon(\mathbf{v}^n)) - \mathbf{S}(\varepsilon(\mathbf{v}))) : \varepsilon(\mathbf{w}^{n,j})\, dx && \overset{n}{\to} 0 \text{ by equation} \\
&\quad - \int_{\{\mathbf{w}^n \neq \mathbf{w}^{n,j}\}} (\mathbf{S}(\varepsilon(\mathbf{v}^n)) - \mathbf{S}(\varepsilon(\mathbf{v}))) : \varepsilon(\mathbf{w}^{n,j})\, dx && \leq 2^{-j} \text{ after } n \to \infty
\end{aligned}
$$

# Things omitted

- **Convective term:**

$$\langle \mathbf{v}^n \otimes \mathbf{v}^n, \varepsilon(\mathbf{w}^{n,j}) \rangle \overset{n}{\to} 0$$

by $\varepsilon(\mathbf{w}^{n,j}) \overset{n}{\to} 0$ *-weakly in $L^\infty$ and $W_0^{1,p} \hookrightarrow\hookrightarrow L^2$ for $p > 1$

- **Pressure:** Lipschitz truncation is not solenoidal, i.e. $\operatorname{div} \mathbf{w}^{n,j} \neq 0$

Correct $\mathbf{w}^{n,j}$ by solution $\psi^{n,j} \in W_0^{1,p}$ of $\operatorname{div} \psi^{n,j} = \chi_{\{\mathbf{w}^n \neq \mathbf{w}^{n,j}\}} \operatorname{div} \mathbf{w}^{n,j}$.

**Theorem (Frehse, Málek, Steinhauer '03; +Diening '07)**

*There exists a weak solution of power-law fluids in $\mathbb{R}^2$ for all $p > 1$.*

# Prandtl-Eyring Fluids

Find velocity $\mathbf{v}$ and pressure $q$ such that

$$\begin{aligned} -\nu \operatorname{div}(\mathbf{S}(\varepsilon(\mathbf{v}))) + [\nabla \mathbf{v}]\mathbf{v} + \nabla q &= \mathbf{f} && \text{on } \Omega, \\ \operatorname{div} \mathbf{v} &= 0 && \text{on } \Omega, \\ \mathbf{v} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with constitutive law

$$\mathbf{S}(\varepsilon(\mathbf{v})) = \frac{\log(1 + |\varepsilon(\mathbf{v})|)}{|\varepsilon(\mathbf{v})|}\varepsilon(\mathbf{v})$$

- Prandtl-Eyring model is an approximation of perfectly plastic fluids
- features well the behaviour of lubricants

# Weak formulation

Weak formulation without pressure

$$\nu\langle \mathbf{S}(\varepsilon(\mathbf{v})), \varepsilon(\boldsymbol{\xi})\rangle + \langle \mathbf{v}\otimes\mathbf{v}, \varepsilon(\boldsymbol{\xi})\rangle = \langle \mathbf{f}, \boldsymbol{\xi}\rangle \qquad \text{for } \boldsymbol{\xi} \in W^{1,\infty}_{0,\mathrm{div}}.$$

with $\mathbf{S}(\varepsilon(\mathbf{v})) = \frac{\log(1+|\varepsilon(v)|)}{|\varepsilon(\mathbf{v})|}\varepsilon(\mathbf{v})$.

Natural function space

$$V := \left\{\mathbf{v} \in W^{1,1}_{0,\mathrm{div}} \,:\, \varepsilon(\mathbf{v}) \in L^{t\ln(1+t)}\right\}$$

Corresponds almost to the bad case $p = 1$.

Need Lipschitz truncation technique, since $\mathbf{v}\otimes\mathbf{v} \in L^{t\ln^2(1+t)}$.

# Problems (1/2)

- Korn's inequality:

$$\text{Failure:} \qquad \|\nabla \mathbf{v}\|_{t\ln(1+t)} \not\leq c\, \|\varepsilon(\mathbf{v})\|_{t\ln(1+t)},$$
$$\text{Only:} \qquad \|\nabla \mathbf{v}\|_{1} \leq c\, \|\varepsilon(\mathbf{v})\|_{t\ln(1+t)}$$

Solution: Work directly with $\varepsilon(\mathbf{v})$ in space definition.

- Maximal function:

$$\text{Failure:} \qquad \|Mg\|_{t\ln(1+t)} \not\leq c\, \|g\|_{t\ln(1+t)},$$
$$\text{Only:} \qquad \|Mg\|_{1} \leq c\, \|g\|_{t\ln(1+t)}$$

Solution: Delicate use of weak type estimates.

# Problems (1/2)

- Korn's inequality:

$$\text{Failure:} \qquad \|\nabla \mathbf{v}\|_{t\ln(1+t)} \not\leq c\, \|\varepsilon(\mathbf{v})\|_{t\ln(1+t)},$$
$$\text{Only:} \qquad \|\nabla \mathbf{v}\|_{1} \leq c\, \|\varepsilon(\mathbf{v})\|_{t\ln(1+t)}$$

  Solution: Work directly with $\varepsilon(\mathbf{v})$ in space definition.

- Maximal function:

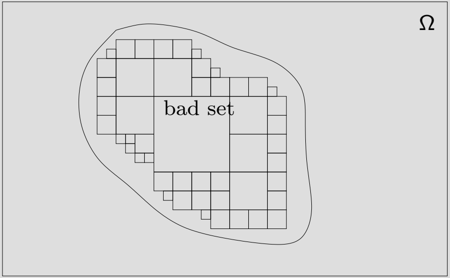
$$\text{Failure:} \qquad \|Mg\|_{t\ln(1+t)} \not\leq c\, \|g\|_{t\ln(1+t)},$$
$$\text{Only:} \qquad \|Mg\|_{1} \leq c\, \|g\|_{t\ln(1+t)}$$

  Solution: Delicate use of weak type estimates.

# Problems (2/2)

- Solenoidal correction: fails on $L^{t\ln(1+t)}$.
  Solution: Use Whitney type extension

$$\mathbf{w}_\lambda := \begin{cases} \mathbf{w} & \text{on good set} \\ \sum_j \varphi_j \mathbf{w}_j & \text{on good set} \end{cases}$$




Correct divergence of $\varphi_j \mathbf{w}_j$ to get divergence free Lipschitz truncation

# Prandtl Eyring fluids – Existence of weak solutions

**Theorem (Breit, Diening, Fuchs '11)**

*There exists a weak solution* $\mathbf{v}$ *in* $\Omega \subset \mathbb{R}^2$ *of*

$$\begin{aligned} -\nu \operatorname{div}\left(\frac{\ln(1+|\varepsilon(\mathbf{v})|)}{|\varepsilon(\mathbf{v})|}\varepsilon(\mathbf{v})\right) + [\nabla \mathbf{v}]\mathbf{v} + \nabla q &= \mathbf{f} && \text{on } \Omega, \\ \operatorname{div} \mathbf{v} &= 0 && \text{on } \Omega, \\ \mathbf{v} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

Summary of proof:

- Use solenoidal Lipschitz truncation in pressure free formulation
- Recover pressure in $L^1$

# More applications of the Lipschitz truncation

$\mathcal{A}$-harmonic approximation: [in version of Diening, Stroffolini, Verde]

For every almost harmonic function, i.e. (for small $\delta > 0$)

$$\fint_Q \nabla \mathbf{u} \cdot \nabla \boldsymbol{\xi} \, dx \leq \delta \fint_Q |\nabla \mathbf{u}| \, dx \, \|\nabla \boldsymbol{\xi}\|_\infty \qquad \text{for all } \boldsymbol{\xi} \in C_0^\infty(Q)$$

exists a harmonic $\mathbf{h}$ on $Q$ with $\mathbf{h} = \mathbf{u}$ on $\partial Q$ and

$$\fint_Q |\nabla \mathbf{u} - \nabla \mathbf{h}|^2 \, dx \leq \varepsilon \left( \fint_Q |\nabla \mathbf{u}|^{2s} \, dx \right)^{\frac{1}{s}}$$

for small $\varepsilon > 0$ and $s > 1$.

Constructive proof! Also: coefficients, Orlicz spaces, quasi-convex.