1.3 The ultraviolet divergence due to the electrons (Version 160421)

In this section we will follow closely Freeman Dyson's computation of the vacuum polarization of the Dirac sea:

Advanced Quantum Mechanics,
F. Dyson, http://arxiv.org/abs/quant-ph/0608140

Due to the ill-defined equation of motion, also here we will come across several infinities in the computation, some of which can be argued "away for physical reasons, and others which remain without introducing a renormalization scheme. Heuristically, we may interpret the occurrence of these infinities as reminiscent of the infinitely many particles making up the electrodynamic vacuum.

The necessary renormalization scheme will allow to arrive at a finite quantity up to second order but will lead to the infamous "Landau pole" problem when carried out self-consistently.

This section will serve as the motivation of a rigorous construction of the time evolution that will be the content of the last mathematical part of this lecture.

Let us reconsider the QED Hamiltonian

$$i\partial_t \Psi_t = H \Psi_t$$

$$H = \int d^3x \; \overline{\psi}(0,\underline{x})(-i\underline{\gamma}\cdot\nabla_{\underline{x}} + m)\psi(0,\underline{x}) + \sum_\lambda \int d^3k \frac{|k|}{2} a^*_{\underline{k},\lambda} a_{\underline{k},\lambda}$$

$$+ e\int d^3x \; \overline{\psi}(0,\underline{x}) \not{A}(0,\underline{x}) \psi(0,\underline{x})$$

But this time let us neglect the photons and all their discussed problems to see what we will have to deal with concerning the electrons. We will therefore replace A^μ by a smooth external four-vector potential. The resulting so-called external field Hamiltonian reads:

$$H^A = \int d^3x \left[\overline{\psi}(0,\underline{x})(-i\underline{\gamma}\cdot\nabla_{\underline{x}} + m)\psi(0,\underline{x}) + e\,\overline{\psi}(0,\underline{x}) \not{A}(0,\underline{x}) \psi(0,\underline{x})\right]$$

Already without further interaction, the generated dynamics bares an ultraviolet problem. To understand its origin it is useful to consider for example the vacuum expectation value of the current:

$$j^\mu_A(x) = e\langle \Omega, U^A(x^0,0)^* \overline{\psi}(0,\underline{x})\gamma^\mu \psi(0,\underline{x}) U^A(x^0,0)\Omega\rangle \qquad \boxed{1.1}$$

$$= e\langle \Omega, \overline{\psi}^A(x)\gamma^\mu \psi^A(x)\,\Omega\rangle$$

where the field operators $\psi^A(x) = \psi^A(x^0,\underline{x})$ and $\overline{\psi}^A(x)$ that fulfill

and $\psi(x) = \psi^{A=0}(x)$ in our old notation

$$(i\not{\partial} - m)\psi^A(x) = e\not{A}(x)\psi^A(x)$$
$$\overline{\psi}^A(x)(-i\overleftarrow{\not{\partial}} - m) = e\,\overline{\psi}^A(x)\not{A}(x) \qquad \boxed{1.2}$$

Using the adv./ret. Green's functions

$$(i\not{\partial} - m)S_\pm(x) = \delta^4(x), \qquad S_\pm(x) = 0 \text{ for } \pm x^0 > 0$$

we may write $\boxed{1.2}$ as integral equations

$$\psi^A(x) = \psi(x) + e\int d^4y\, S_-(x-y)\, \not{A}(y)\, \psi^A(y)$$
$$\bar\psi^A(x) = \bar\psi(x) + e\int d^4y\, \bar\psi^A(y)\, \not{A}(y)\, S_+(y-x)$$

An informal expansion of $\boxed{1.1}$ up to e^2 gives

$$j^\mu_A(x) = e\langle\Omega, \bar\psi(x)\,\gamma^\mu\,\psi(x)\,\Omega\rangle \qquad \boxed{2.1}$$
$$+ e^2\int d^4y\,\langle\Omega, \bar\psi(y)\,\not{A}(y)\,S_+(y-x)\,\gamma^\mu\,\psi(x) + \bar\psi(x)\,\gamma^\mu\,S_-(x-y)\,\not{A}(y)\,\psi(y)\,\Omega\rangle \qquad \boxed{2.2}$$
$$+ \text{Rest}\,(e^4)$$

Let's start with

$$\boxed{2.1} = e\,\gamma^\mu_{ab}\,\langle\Omega, \bar\psi_a(x)\,\psi_b(x)\,\Omega\rangle$$

using $\langle\Omega, \bar\psi_a(x)\,\psi_b(y)\,\Omega\rangle = -i\,S^{(-)}_{ba}(x-y)$ HW

for $S^{(-)}(x) = -\frac{i}{(2\pi)^3}\int d^4k\, e^{-ik_\mu x^\mu}\,(\not{k}+m)\,\delta(k^2-m^2)\,\Theta(-k_0)$

$$= -i\,e\,\mathrm{tr}\left[\gamma^\mu S^{(-)}(0)\right]$$

using $\mathrm{tr}\,\gamma^\mu = 0$, $\mathrm{tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$

$$= -\frac{e}{(2\pi)^3}\int d^4k\,\delta(k^2-m^2)\,\Theta(-k_0)\,\mathrm{tr}\left[\gamma^\mu(\not{k}+m)\right]$$

$$= -\frac{e}{(2\pi)^3}\int d^4k\,\delta(k^2-m^2)\,\Theta(-k_0)\,k^\mu$$

This integral clearly diverges.

- It is the expectation value of the current of the sea of infinite negative energy electrons which fill the Dirac sea.
- Note that the sea particles are so homogeneously distributed that the expectation does not depend on the space-time point x.
- Since we are interested in the relative current that is generated due to the potential A one argues that we may neglect this "free" current. A procedure that is encoded into the so-called "normal ordering" of the current.

Let us continue with the relative current

$$\Delta j^\mu_A = j^\mu_A(x) - j^\mu_{A=0}(x) = \boxed{2.2} + \mathrm{Rest}(e^3)$$

$$\Rightarrow = e^2 \int d^4y \langle \Omega, \bar\psi_a(y) \not{A}_{ab}(y) S_{+\,bc}(x-y) \gamma^\mu_{cd} \psi_d(x) + \bar\psi(x)\gamma^\mu S_-(x-y) \not{A}(y) \psi(y) \Omega\rangle$$

using $\langle \Omega, \bar\psi_a(x) \psi_b(y) \Omega\rangle = -i\, S^{(-)}_{ba}(x-y)$

for $S^{(-)}(x) = -\frac{i}{(2\pi)^3} \int d^4k\, e^{-ik_\mu x^\mu} (\not{k}+m)\, \delta(k^2-m^2)\, \Theta(-k_0)$

$$= -i\, e^2 \int d^4y\, \mathrm{tr}\left[\not{A}(y) S_+(y-x)\gamma^\mu S^{(-)}(x-y) + \not{A}(y) S^{(-)}(y-x)\gamma^\mu S_-(x-y)\right]$$

$$S_\pm(x) = \lim_{\delta^0 \searrow 0} \frac{1}{(2\pi)^4} \int d^4k\, e^{-i(k\pm i\delta)x} \frac{\not{k}+m}{m^2-k^2} \qquad \delta = (\overset{\overset{\mathbb{R}^+}{\cup}}{\delta^0}, 0, 0, 0)$$

$$= -e^2 \frac{1}{(2\pi)^7} \int d^4y \int d^4k \int d^4\ell \Big(A_\nu(y)\, e^{-i(k+i\delta)(y-x)}\, e^{-i\ell\cdot(x-y)} \frac{\delta(\ell^2-m^2)\Theta(-\ell_0)}{m^2-k^2}\, \mathrm{tr}\left[\gamma^\nu (\not{k}+m)\gamma^\mu(\not{\ell}+m)\right]$$
$$+ A_\nu(y)\, e^{-i\ell(y-x)}\, e^{-i(k-i\delta)(x-y)} \frac{\delta(\ell^2-m^2)\Theta(-\ell_0)}{m^2-k^2}\, \mathrm{tr}\left\{\gamma^\nu(\not{\ell}+m)\gamma^\mu(\not{k}+m)\right\}\Big)$$

Since everything is in Fourier space except A^μ we transform it and furthermore introduce a substitution to make the phase factors δ independent.

$$= -e^2 \frac{1}{(2\pi)^7} \int d^4y \int d^4k \int d^4\ell \int d^4q \Big[e^{-ik\cdot(y-x)}\, e^{-i\ell\cdot(x-y)} \frac{\delta(\ell^2-m^2)\Theta(-\ell_0)}{m^2-(k-i\delta)^2}\, \mathrm{tr}\left[\gamma^\nu(\not{k}-i\not{\delta}+m)\gamma^\mu(\not{\ell}+m)\right]$$
$$+ e^{-i\ell\cdot(y-x)}\, e^{-ik\cdot(x-y)} \frac{\delta(\ell^2-m^2)\Theta(-\ell_0)}{m^2-(k+i\delta)^2}\, \mathrm{tr}\left\{\gamma^\nu(\not{\ell}+m)\gamma^\mu(\not{k}+i\not{\delta}+m)\right\}\Big] \cdot e^{-iq\cdot y}\, \hat{A}_\nu(q)$$

$$= -e^2 \frac{1}{(2\pi)^7} \int d^4y \int d^4k \int d^4\ell \int d^4q \; \hat{A}_\nu(q)$$

$$\Big[e^{-ik\cdot(y-x)} \, e^{-i\ell\cdot(x-y)} \cdot e^{-iq\cdot y} \frac{\delta(\ell^2-m^2)\Theta(-\ell_0)}{m^2-(k-i\delta)^2} \, \mathrm{tr}\left[\gamma^\nu (\not{k} - i\not{\delta} + m)\gamma^\mu(\not{\ell}+m)\right]$$

$$+ e^{-i\ell\cdot(y-x)} \, e^{-ik\cdot(x-y)} \cdot e^{-iq\cdot y} \frac{\delta(\ell^2-m^2)\Theta(-\ell_0)}{m^2-(k+i\delta)^2} \, \mathrm{tr}\left\{\gamma^\nu (\not{\ell}+m)\gamma^\mu(\not{k}+i\not{\delta}+m)\right\} \Big]$$

This expression can be simplified changing the labels of the integration variables.

$$= -e^2 \frac{1}{(2\pi)^7} \int d^4y \int d^4k \int d^4\ell \int d^4q \; \hat{A}_\nu(q) \; e^{-ix\cdot(\ell-k)} \; e^{-iy\cdot(k-\ell+q)}$$

$$\Big[\frac{\delta(\ell^2-m^2)\Theta(-\ell_0)}{m^2-(k-i\delta)^2} \, \mathrm{tr}\left[\gamma^\nu (\not{k} - i\not{\delta} + m)\gamma^\mu(\not{\ell}+m)\right]$$

$$+ \frac{\delta(k^2-m^2)\Theta(-k_0)}{m^2-(\ell+i\delta)^2} \, \mathrm{tr}\left\{\gamma^\nu (\not{k}+m)\gamma^\mu(\not{\ell}+i\not{\delta}+m)\right\} \Big]$$

Now we can carry out the $d^3\ell$ and d^3y integration which effectively results in setting: $k-\ell+q=0$

$\ell = k+q$

$$= -e^2 \frac{1}{(2\pi)^3} \int d^4k \int d^4q \; \hat{A}_\nu(q) \; e^{-ix\cdot q}$$

$$\Big[\frac{\delta((k+q)^2-m^2)\Theta(-k_0-q_0)}{m^2-(k-i\delta)^2} \, \mathrm{tr}\left[\gamma^\nu (\not{k} - i\not{\delta} + m)\gamma^\mu(\not{k}+\not{q}+m)\right]$$

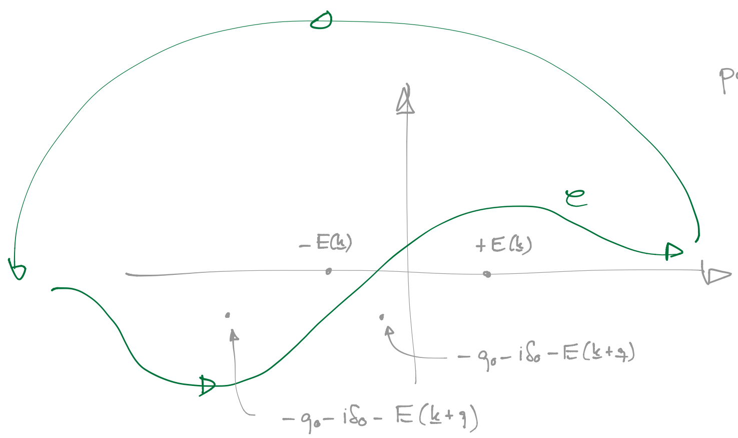
$$+ \frac{\delta(k^2-m^2)\Theta(-k_0)}{m^2-(k+q+i\delta)^2} \, \mathrm{tr}\left\{\gamma^\nu (\not{k}+m)\gamma^\mu(\not{k}+\not{q}+i\not{\delta}+m)\right\} \Big]$$

$$= -e^2 \frac{1}{(2\pi)^3} \int d^4q\, \hat{A}_\nu(q)\, e^{-ix\cdot q} \int d^3k$$

$$\Bigg[-\frac{1}{2E(k+q)} \frac{1}{m^2-(k-i\delta)^2} \operatorname{tr}\left[\gamma^\nu (\not{k} - i\not{\delta} + m)\gamma^\mu(\not{k}+\not{q}+m)\right]\Bigg|_{k_0=-q_0-E(k+q)}$$

$$-\frac{1}{2E(k)} \frac{1}{m^2-(k+q+i\delta)^2} \operatorname{tr}\left[\gamma^\nu(\not{k}+m)\gamma^\mu(\not{k}+\not{q}+i\not{\delta}+m)\right]\Bigg|_{k_0=-E(k)} \Bigg]$$

$$F^{\mu\nu}(k,q) = \operatorname{tr}\left[\gamma^\nu(\not{k}+m)\gamma^\mu(\not{k}+q+m)\right] \frac{1}{m^2-k^2} \frac{1}{m^2-(k+q+i\delta)^2}$$



poles:

$k_0 = \pm E(\underline{k})\,, \qquad E(\underline{k}) = \sqrt{\underline{k}^2+m^2}$

$k_0 = -q_0 - i\delta_0 \pm E(\underline{k}+q)$

$$= +e^2 \frac{i}{(2\pi)^4} \int d^4q\, \hat{A}_\nu(q)\, e^{-iq\cdot x} \underbrace{\int_{\mathcal{C}} d^4k\, \operatorname{tr}\left[\gamma^\nu(\not{k}+m)\gamma^\mu(\not{k}+\not{q}+i\not{\delta}+m)\right] \frac{1}{m^2-k^2}\frac{1}{m^2-(k+q+i\delta)^2}}_{\Pi^{\mu\nu}(q)}$$

Contour in k_0
d^3k integral in $\underline{k}$

To compute $\Pi^{\mu\nu}(q)$, first, we consider the case of m sufficiently large s.t. the poles included in contour $\mathcal{C}$ lie to the left of $i\mathbb{R}$ and the other two to the right and take $\delta = 0$.

Using Feynman's parametrization

$$\frac{1}{ab} = \int_0^1 dz \frac{1}{[az+b(1-z)]^2}$$

$$\Pi^{\mu\nu}(q) = \int_0^1 dz \int_e d^4k \; tr\left[\gamma^\nu(\not k + m)\gamma^\mu(\not k + \not q + m)\right] \frac{1}{\left[\underbrace{m^2 - k^2 - (2k_\mu q^\mu + q^2)z}_{= m^2 - (k+qz) + q^2(z^2 - z)}\right]^2}$$

$$= \int_0^1 dz \int_{-i\infty}^{+i\infty} dk^0 \int d^3k \; tr\left[\gamma^\nu(\not k - \not q z + m)\gamma^\mu(\not k + \not q(1-z) + m)\right] \frac{1}{\left[m^2 - k^2 + q^2(z^2 - z)\right]^2}$$

Exploiting the trace formulas

$$tr\, \gamma^\nu \gamma^\mu = 4 g^{\nu\mu}$$
$$tr\, \gamma^\mu \gamma^\nu \gamma^\zeta = 0$$
$$tr\, \gamma^\nu \gamma^\zeta \gamma^\mu \gamma^\tau = 4 g^{\nu\zeta} g^{\mu\tau} - 4 g^{\nu\mu} g^{\zeta\tau} + 4 g^{\nu\tau} g^{\zeta\mu}$$

$$tr\left[\gamma^\nu(\not k - \not q z + m)\gamma^\mu(\not k + \not q(1-z) + m)\right] = tr\left[\gamma^\nu\gamma^\zeta\gamma^\mu\gamma^\tau\right](k - qz)_\zeta (k + q(1-z))_\tau + tr\left[\gamma^\nu\gamma^\mu\right] m^2$$

$$= 4\left(g^{\nu\zeta}g^{\mu\tau} - g^{\nu\mu}g^{\zeta\tau} + g^{\nu\tau}g^{\zeta\mu}\right)(k - qz)_\zeta (k + q(1-z))_\tau + 4 g^{\nu\mu} m^2$$

$$= 4(k - qz)^\nu (k + q(1-z))^\mu - 4 g^{\nu\mu}\left[(k - qz)(k + q(1-z)) - m^2\right] + 4(k - qz)^\mu (k + q(1-z))^\nu$$

This can be simplified by observing that all terms odd in k vanish under the integral

$$\Pi^{\mu\nu}(q) = \int_0^1 dz \int_{-i\infty}^{+i\infty} dk^0 \int d^3k \; \frac{4\left[2k^\mu k^\nu - 2q^\mu q^\nu(z - z^2)\right] - 4g^{\nu\mu}\left[k^2 - q^2(z - z^2) - m^2\right]}{\left[m^2 - k^2 + q^2(z^2 - z)\right]^2}$$

Again using the symmetry under the integral we find

$$\int_{-i\infty}^{+i\infty} dk^0 \int d^3k \; k^\mu k^\nu A_\nu \overbrace{(\dots)}^{\text{even in } k} = \int_{-i\infty}^{+i\infty} dk^0 \int d^3k \; \frac{k^2}{4} A^\mu (\dots)$$

$\boxed{2.2}$ $$= + e^2 \frac{i}{(2\pi)^4} \int d^4q \, \hat{A}_\nu(q) e^{-iq\cdot x} \, \Pi^{\mu\nu}(q)$$

$$= e^2 \frac{i}{(2\pi)^4} \int d^4q \, \hat{A}_\nu(q) e^{-iqx} \int_0^1 dz \int_{-i\infty}^{+i\infty} dk^0 \int d^3k \; \frac{4\left[-2q^\mu q^\nu(z - z^2) - g^{\nu\mu}\left[\frac{1}{2}k^2 - q^2(z - z^2) - m^2\right]\right]}{\left[m^2 - k^2 + q^2(z^2 - z)\right]^2}$$

At least here, it should be clear that these were only symbolic manipulations because the integral is divergent.

We argue further that $\partial_\mu j^\mu_A(x) \overset{!}{=} 0$ should hold. Hence, instead of being ill-defined it would be nice if the following expression would be zero:

$$\partial_\mu \boxed{2.2}^\mu = e^2 \frac{i}{(2\pi)^4} \int d^4q\, \hat{A}_\nu(q)\, e^{-iqx} \int_0^1 dz \int_{-i\infty}^{+i\infty} dk^0 \int d^3k\, \frac{4\left[-2 q^2 q^\nu (z-z^2) - q^\nu \left[\frac{1}{2} k^2 - q^2(z-z^2) - m^2\right]\right]}{\left[m^2 - k^2 + q^2(z^2-z)\right]^2}$$

$$= e^2 \frac{i}{(2\pi)^4} \int d^4q\, q\cdot\hat{A}(q)\, e^{-iqx} \int_0^1 dz \int_{-i\infty}^{+i\infty} dk^0 \int d^3k\, \frac{(-4)\left(\frac{1}{2}k^2 + q^2(z-z^2) - m^2\right)}{\left[m^2 - k^2 + q^2(z^2-z)\right]^2} \overset{!}{=} 0$$

Rewriting $\boxed{2.2}$, thus, may be interpreted as

$$\boxed{2.2} = e^2 \frac{i}{(2\pi)^4} \int d^4q\, \hat{A}_\nu(q)\, e^{-iqx} \int_0^1 dz \int_{-i\infty}^{+i\infty} dk^0 \int d^3k\, \frac{4\left[-2 q^\mu q^\nu (z-z^2) - g^{\nu\mu}\left(\overbrace{\left[\frac{1}{2} k^2 + q^2(z-z^2) - m^2\right] + 2q^2(z-z^2)}^{\text{"defined" to vanishes under integration}}\right)\right]}{\left[m^2 - k^2 + q^2(z^2-z)\right]^2}$$

$$= e^2 \frac{i}{(2\pi)^4} \int d^4q\, \hat{A}_\nu(q)\, e^{-iqx}\, 8\left(q^2 g^{\nu\mu} - q^\mu q^\nu\right) \int_0^1 dz\, (z-z^2) \int_{-i\infty}^{+i\infty} dk^0 \int d^3k\, \frac{1}{\left[m^2 - k^2 + q^2(z^2-z)\right]^2} \qquad \boxed{7.1}$$

$$+ \frac{e^2}{4\pi} \int d^4q\, \hat{A}_\nu(q)\, e^{-iqx}\, G(q) \Big\} \text{ the term which we set to zero by requiring } \partial_\mu j^\mu_A(x) \overset{!}{=} 0$$

This integral is still divergent. We substract from it the contribution from the unperturbed sea:

$$\boxed{7.1} = e^2 \frac{i}{(2\pi)^4} \int d^4q\, \hat{A}_\nu(q)\, e^{-iqx}\, 8\left(q^2 g^{\nu\mu} - q^\mu q^\nu\right) \int_0^1 dz\, (z-z^2)$$

$$\int_{-i\infty}^{+i\infty} dk^0 \int d^3k \left(\frac{1}{(m^2-k^2)^2} + \frac{1}{\left[m^2 - k^2 + q^2(z^2-z)\right]^2} - \frac{1}{(m^2-k^2)^2} \right)$$

$$\int_{-i\infty}^{+i\infty} dk^0 \int d^3k \left[\frac{1}{(\Lambda - k^2)^2} - \frac{1}{(m^2-k^2)^2} \right] = i \int dk^0 \cdots \int dk^3 \left(\frac{1}{[k_0^2+k_1^2+k_2^2+k_3^2+\Lambda]^2} - \frac{1}{[k_0^2+k_1^2+k_2^2+k_3^2+m^2]^2} \right)$$

$$= 2\pi^2 i \int_0^\infty \left(\frac{1}{(k^2+\Lambda)^2} - \frac{1}{(k^2+m^2)^2} \right) k^3 dk = 2\pi^2 i \int_0^\infty dx \times \left(\frac{1}{(x+\Lambda)^2} - \frac{1}{(x+m^2)^2} \right)$$

$$= \pi^2 i \log \frac{\mu^2}{\Lambda}$$

$$\| \quad \int_{-i\infty}^{+i\infty} dk^0 \int d^3k \frac{1}{(m^2-k^2)^2} = 2i\pi^2 \left(\frac{3\pi}{2} R \right) \|$$

logarithmically divergent in $|k|$

$$\boxed{2.2} = e^2 \frac{i}{(2\pi)^4} \int d^4q \, \hat{A}_\nu(q) e^{-iqx} \left(q^2 g^{\nu\mu} - q^\mu q^\nu \right) 8\pi^2 i \left[\frac{\pi}{2} R - \int_0^1 dz \, (z-z^2) \log\left(1 - \frac{(z-z^2) q^2}{m^2} \right) \right]$$

Recall that this formula only holds for large enough μ.
The case of small m can however be recovered by analytic continuation noting the branch cut of log:

$$\text{For } q \to q + i\delta : \quad \log\left(1 - \frac{(z-z^2) q^2}{\mu^2}\right) = \log\left| 1 - \frac{(z-z^2)q^2}{m^2} \right| + \begin{cases} -i\pi \operatorname{sign}(q_0) & \text{for } \frac{(z-z^2)q^2}{m^2} > 1 \\ 0 & \text{else} \end{cases}$$

$$\boxed{2.2} = -\frac{e^2}{2\pi^2} \int d^4q \, \hat{A}_\nu(q) e^{-iqx} \left(q^2 g^{\nu\mu} - q^\mu q^\nu \right) \Bigg[\frac{\pi}{2} R - \int_0^1 dz \, (z-z^2) \left[\log\left| 1 - \frac{(z-z^2)q^2}{m^2} \right| - i\pi \operatorname{sign}(q^0) \, \mathbb{1}\left(\frac{(z-z^2)q^2}{m^2} > 1 \right) \right] \Bigg]$$

Substitution $x \mapsto 4(z-z^2)$ gives

$$\boxed{2.2} = -\frac{e^2}{2\pi^2} \int d^4q \, \hat{A}_\nu(q) e^{-iqx} \left(q^2 g^{\nu\mu} - q^\mu q^\nu \right) \Bigg[\frac{\pi}{2} R - \frac{1}{8} \int_0^1 dx \, \frac{x}{\sqrt{1-x}} \log\left| 1 - \frac{x q^2}{4m^2} \right| + \frac{i\pi}{8} \operatorname{sign}(q^0) \int_0^1 dx \, \frac{x}{\sqrt{1-x}} \, \mathbb{1}\left(x > \frac{4m^2}{q^2} \right) \Bigg]$$

Or in units of the fine structure constant $\alpha = \frac{e^2}{4\pi}$

We find the following situation:

$$j^\mu_A(x) = e\,\langle \Omega, U^A(x^0)^* \,\overline{\psi}(0,\underline{x})\,\gamma^\mu\, \psi(0,\underline{x})\, U^A(x^0)\Omega\rangle$$

$$= \overbrace{j^\mu_{A=0}(x)}^{\infty \text{ contribution from entire sea indep. of } A^\mu}$$

$$- \alpha \int d^4q\, \hat{A}_\nu(q)\, e^{-iqx}\, \overbrace{G(q)}^{\infty \text{ contribution which however breaks current conservation}}$$

external current generating $\hat{A}_\nu$: $\hat{j}^{ext}_\nu(q) := \hat{A}_\nu(q)\,(q^2 g^{\nu\mu} - q^\mu q^\nu)$

$$- \alpha \int d^4q\, \hat{j}^{ext}_\nu(q)\, \Big[\overbrace{R}^{\infty \text{ contribution from entire sea prop. to } A^\mu} - \Delta\Big] \quad + \underbrace{\mathrm{Rest}(\alpha^2)}_{\text{all other orders are in fact finite}}$$

$$\Delta := + \overbrace{\frac{1}{4\pi}\int_0^1 dx\, \frac{x}{\sqrt{1-x}} \log\left|1 - \frac{x q^2}{4m^2}\right|}^{\text{Vacuum polarization}} \underbrace{- \frac{1}{4}\,\mathrm{sgn}(q^0)\int_0^1 dx\, \frac{x}{\sqrt{1-x}}\, \mathbb{1}\left(x > \frac{4m^2}{q^2}\right)}_{\text{only kicks in if energy } q \text{ of } A^\mu \text{ mode is large enough to produce a pair}}$$

Dropping the first two infinities we arrive at

$$\hat{j}^A = -\alpha\, \hat{j}^{ext}(R - \Delta) + O(\alpha^2) \qquad \boxed{9.1}$$

still containing the logarithmically divergent $R = O(\log|\overbrace{k_{max}}^{\text{UV cut-off}}|)$.

$\boxed{9.1}$ can be interpreted as "answer" to a purely external perturbation $\hat{j}^{ext}$.
However, $\hat{j}^A$ and $\hat{j}^{ext}$ are not experimentally separable.
Measurable is $\hat{j}^{tot} = \hat{j}^{ext} + \hat{j}^A$ which leads the following argument:

polarization due to total current and not just the external

$$\alpha\,\hat{j}^{tot} = \alpha(\hat{j}^A + \hat{j}^{ext}) = \alpha\,\hat{j}^{ext} - \alpha^2\,\hat{j}^{tot}(R - \Delta) + O(\alpha^4)$$

$$\Longleftrightarrow \quad (1 + \alpha R)\,\alpha\,\hat{j}^{tot} = \alpha\,\hat{j}^{ext} + \alpha^2\,\hat{j}^{tot}\Delta + O(\alpha^4)$$

$$\Longleftrightarrow \quad \alpha\,\hat{j}^{tot} = \frac{\alpha}{1+\alpha R}\,\hat{j}^{ext} + \frac{\alpha}{1+\alpha R}\,\alpha\,\hat{j}^{tot}\Delta + O(\alpha^4) \qquad \boxed{9.2}$$

for $q=0$ one can show that $\Delta(q)$ and $\mathcal{O}(\alpha^4)$ vanish:

$$\alpha \, \hat{j}^{tot}(q=0) = \frac{\alpha}{1+\alpha R} \, \hat{j}^{ext}(q=0)$$

This suggest the scaling

$$\frac{\alpha}{1+\alpha R} \longrightarrow \alpha_{exp} \qquad \text{and} \qquad \alpha \hat{j}^{tot} \longrightarrow \alpha_{exp} \hat{j}^{tot}_{exp}$$

which turns $\boxed{9.2}$ into

$$\alpha_{exp} \hat{j}^{tot} = \alpha_{exp} \hat{j}^{ext} + \alpha_{exp}^2 \hat{j}^{tot}_{exp} \Delta + \mathcal{O}(\alpha^4_{exp})$$

a relation that now contains only finite terms.

However, $\alpha_{exp} = \dfrac{\alpha}{1+\alpha R} \iff \alpha = \dfrac{\alpha_{exp}}{1-\alpha_{exp} R}$ where $R = \mathcal{O}(\log |k_{max}|)$ and $\alpha = \dfrac{1}{137}$

and therefor α increases with the UV cut-off $|k_{max}|$ and reaches ∞ already for a finite cut-off !

This behavior is called "Landau pole problem."

Hence, even granted the dropping of unphysical terms, the perturbative treatment of the self-consistent vacuum polarization is very unsatisfactory. Recall that the hope for a perturbative treatment was based on the smallness of α. Now we find that the renormalization method requires to treat large α !

Dirac '75: Most physicists are very satisfied with the situation. They say: 'Quantum electrodynamics is a good theory and we do not have to worry about it any more.'

I must say that I am very dissatisfied with the situation, because this so-called 'good theory' does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way.

This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small – not neglecting it just because it is infinitely great and you do not want it!

Further reading:

Recently more and more attempts are being made to treat the external QED model in full mathematical rigor. Here are some examples that may serve as an introduction to the topic:

1) Hartree-Fock approximation of the ground state, e.g:

On the Vacuum Polarization Density Caused by an External Field,
C. Hainzl, AHP, 2004

Two Hartree-Fock models for the vacuum polarization
P. Gravejat, C. Hainzl, M. Lewin and É. Séré, JEDP, 2012

2) Construction of the time evolution

Time Evolution of the External Field Problem in QED
D.-A.D., D. Dürr, F. Merkl, M. Schottenloher, JMP, 2010

A Perspective on External Field QED,
D.-A.D., F. Merkl, Quantum Mathematical Physics, 2015

3) Construction of the scattering operator

Scattering matrix in external field problems,
E. Langmann, J. Mickelsson, JMP, 1996

Finite Quantum Electrodyanmcis,
G. Scharf, Springer, 1989

Appendix A: Computation of the Green's functions for Dirac's equation

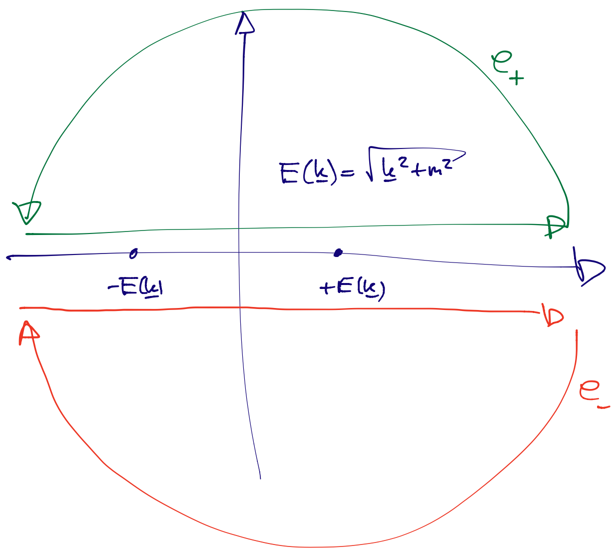
$$(\partial_\mu \partial^\mu - m^2) D_\pm(x) = \delta^4(x)\,, \quad D_\pm(x) = 0 \text{ for } \pm x^0 > 0$$

$$\mathbb{R} \ni \quad \delta = (\delta^0, 0, 0, 0)$$

$$\Rightarrow \quad \int d^4k \; e^{-ik\cdot x} (-k_\mu k^\mu + m^2) \hat{D}_\pm(k) = \int d^4k \; e^{-ik\cdot x} \frac{1}{(2\pi)^{3/2}}$$

$$\Rightarrow \quad \hat{D}(k) = \frac{1}{(2\pi)^{3/2}} \frac{1}{m^2 - k_\mu k^\mu} \quad \text{and} \quad D(x + i\delta) = \frac{1}{(2\pi)^4} \int d^4k \; e^{-ik\cdot(x+i\delta)} \frac{1}{m^2 - k_\mu k^\mu}$$

Say $x^0 < 0 \Rightarrow \operatorname{Re} ik^0 x^0 < 0$ for $k^0 \in \mathbb{R} - i\mathbb{R}$



$\mathcal{C}_+$ contour for D_+

$\mathcal{C}_-$ contour for D_-

$$D_\pm(x) = \frac{1}{(2\pi)^4} \int_{\mathcal{C}_\pm} dk_0 \int d^3k \; e^{-ik\cdot x} \frac{1}{(E(k) - k^0)(E(k) + k^0)} = \lim_{\delta \searrow 0} \frac{1}{(2\pi)^4} \int d^4k \; e^{-ik\cdot x} \frac{1}{m^2 - (k \mp i\delta)^2}$$

$$= \lim_{\delta \searrow 0} \frac{1}{(2\pi)^4} \int d^4k \; e^{-i(k \pm i\delta)x} \frac{1}{m^2 - k^2} = \lim_{\delta \searrow 0} D(x \pm i\delta)$$

$$S_\pm(x) = (i\partial\!\!\!/ + m) D_\pm(x)$$

$$\Rightarrow \quad (i\partial\!\!\!/ - m) S_\pm(x) = \delta^4(x)$$

because $(i\partial\!\!\!/ - m)(i\partial\!\!\!/ + m) = -\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + \cancel{i\partial\!\!\!/ m} - \cancel{i\partial\!\!\!/ m} - m^2 = -\Box - m^2$

$$S_\pm(x) = \frac{1}{(2\pi)^4} \int d^4k \; e^{-i(k \pm i\delta)x} \frac{k\!\!\!/ + m}{m^2 - k^2}$$

Appendix B:

Unfortunately Dyson uses his own notation in his great Advanced Quantum Mechanics lecture notes. Above we have use the standard notation and below are a few hints how to translate Dyson's notation into ours.

Relativistic notation

- no distinction between co- and contravariant vectors
  – instead:
  $$(x^\mu)_{\mu=1,2,3,4} = (x^1, x^2, x^3, i x^0) \quad \text{and} \quad ct = x^0$$
  $$x^2 = \sum_{\mu=1}^{4} x^\mu x^\mu = \underline{x}^2 - t^2 \quad \text{for } c=1$$

- $\gamma^\mu = (-i\beta \underline{\alpha}, \beta)$

  $\Rightarrow$ Dirac equation takes the form:

  $$0 = \left(\sum_{\mu=1}^{4} \gamma^\mu \frac{\partial}{\partial x^\mu} + \mu \right) \psi = \left(-i\beta \underline{\alpha} \cdot \nabla + \beta \frac{\partial}{\partial (it)} + \mu \right) \psi \qquad \text{again } c=1$$
  $$= \left(-i\beta \underline{\alpha} \cdot \nabla - i\beta \partial_t + \mu \right) \psi$$

- Feynman slash $\not{p} = \sum_{\mu=1}^{4} \gamma^\mu p^\mu = -i\beta \underline{\alpha} \cdot \underline{p} + \beta i p^0$

  $\Rightarrow$ Dirac equation takes the form

  $$0 = (\not{p} - im)\hat{\psi} = \left(\sum_{\mu=1}^{4} \gamma^\mu p^\mu - im \right) \hat{\psi} = \left(-i\beta \underline{\alpha} \cdot \underline{p} + i\beta p^0 - im \right) \hat{\psi}$$
  $$\Leftrightarrow \left(\beta p^0 - \beta \underline{\alpha} \cdot \underline{p} - m \right) \hat{\psi} = 0$$