1.2 Infrared divergences due to the photon field (Version 160815)

In the simple toy model of Section 1.1 we will now observe another well-known problem that occurs with massless fields called the "infrared catastrophe". Contrary to the conceptual problem of the ultraviolet problem discussed in Section 1.1, it is only a representational problem and could loosely be described as: "Two charges dressed with their appropriate fields for different asymptotic velocities do not live in the same Fock space."

It was first observed in the form that the ground state of the system cannot be represented in standard Fock space. Later it was again discovered in scattering theory when considering asymptotic states as mentioned above.

It is important to note that this problem does not occur during finite times but only in idealizations, such as those employed in scattering theory, when sending time to plus or minus infinity. When evolving a well-defined state over a finite time period, the new field that may build up during the time evolution may only have finite spatial support due to the relativistic dispersion relation. Only in the limit, when sending time to plus or minus infinity, the spatial support may grow to infinity. As consequence, infinitely many infrared photon modes may be necessary to represent the change of the fields at spatial infinity between the initial and final state and that change may not be representable anymore in one and the same Fock space. Furthermore, it should be noted that a non-zero field mass provides an effective cut-off on the infrared modes of the bosons.

It can readily be observed in our simple toy model:

For a fixed source at position $q \in \mathbb{R}^3$ we found that the ground state was given by

$$\mathrm{Dress}_\Lambda \Omega \quad \text{for} \quad \mathrm{Dress}_\Lambda = \exp\left(-g \int d^3k \, \frac{\gamma_k^\Lambda}{\omega_k} \left(a_k^+ e^{-ikq} - a_k e^{+ikq}\right)\right)$$

The factor $\frac{\gamma_k^\Lambda}{\omega_k}$ arose from the time integration of the interaction Hamiltonian. Recall:

For test states $\Phi \in \mathcal{C}_c^\infty$:

$$\langle \Phi, U_\Lambda(0,t_0)\Omega \rangle = \langle \Phi, \Omega \rangle - i \underbrace{\langle \Phi, \int_{t_0}^{0} ds \, g \int d^3k \, \gamma_k^\Lambda \, a_k^+ e^{-ikq + i\omega_k s} \, \Omega \rangle} + \mathrm{Rest}(g^2)$$

$$= -ig \int d^3k \, \phi^*(k) \int_{t_0}^{0} ds \, \gamma_k^\Lambda e^{-ikq + i\omega_k s} = -ig \int d^3k \, \phi^*(k) \frac{\gamma_k^\Lambda}{i\omega_k} e^{-ikq} \left(1 - e^{+i\omega_k t_0}\right) = \boxed{1.1}$$

Now suppose the charge is not fixed but moves on a straight line $q(t) = q + vt$

this time we find

$$\boxed{1.1} = -ig \int d^3k \, \phi^*(k) \int_{t_0}^{0} ds \, \gamma_k^\Lambda e^{-ik(q+vs) + i\omega_k s}$$

$$= -ig \int d^3k \, \phi^*(k) \frac{\gamma_k^\Lambda}{\omega_k - k \cdot v} e^{-ikq} \left(1 - e^{-kvs + i\omega_k s}\right)$$

Carrying out the computation to all orders and employing a similar stationary phase argument, we find:

$$\langle \Phi, U_\Lambda(0,t_0)\Omega \rangle \xrightarrow[t_0 \to -\infty]{} \langle \Phi, \mathrm{Dress}_\Lambda^v \Omega \rangle$$

$$\text{for} \quad \mathrm{Dress}_\Lambda^v = \exp\left(-g \int d^3k \, \frac{\gamma_k^\Lambda}{\omega_k - k \cdot v} \left(a_k^+ e^{-ikq} - a_k e^{+ikq}\right)\right)$$

Hence, the state $\mathrm{Dress}_\Lambda^v\,\Omega$ can be interpreted as a charge dressed with its Yukawa field moving at asymptotic velocity v.

For a fixed ultraviolet cut-off $\Lambda < \infty$ and a non-zero mass $\mu > 0$ this sate is well-defined since:

$$\frac{\gamma_k^\Lambda}{\omega_k - k\cdot v} = \mathcal{O}_{|k|\to 0}\left(\frac{1}{\mu}\right) \quad \text{and due to } \quad \rho_\Lambda \text{ decays fast enough to be in } L^2$$

For zero mass this is not the case anymore. In fact, already for $\Lambda < \infty$

$$\int d^3k \left|\frac{\gamma_k^\Lambda}{\omega_k - k\cdot v}\right|^2 \sim \int d^3k \left|\frac{1}{\sqrt{2\omega_k}}\frac{\rho_\Lambda(k)}{\omega_k - k\cdot v}\right|^2 \sim \int_{|k|<\Lambda} d^3k\, \frac{1}{|k|}\left|\frac{1}{|k| - k\cdot v}\right|^2 \quad \text{because } \omega_k = \sqrt{k^2+\mu^2} \text{ but now } \mu = 0$$

which diverges logarithmically for $|k| \to 0$

It is natural to employ the same trick as we did when dealing with the ultraviolet problem, namely changing the representation by defining a new Fock space $\widetilde{\mathcal{F}}_\Lambda^v$ according to

$$b_k^v := \mathrm{Dress}_\Lambda^v\, a_k\, \mathrm{Dress}_\Lambda^{v\,*} = a_k - g\,\frac{\gamma_k^\Lambda}{\omega_k - k\cdot v}\, e^{-ikq}$$
$$b_k^{v\,*} := \mathrm{Dress}_\Lambda^v\, a_k^*\, \mathrm{Dress}_\Lambda^{v\,*} = a_k^* - g\,\frac{\gamma_k^\Lambda}{\omega_k - k\cdot v}\, e^{+ikq}$$

note again $[b_k^v, b_\ell^{v*}] = \delta^3(k-\ell)$ and $b_k^v\,\widetilde{\Omega}_\Lambda^v = 0$ for $\widetilde{\Omega}_\Lambda^v = \mathrm{Dress}_\Lambda^v\,\Omega$

The trouble is, however, that all these representations are not unitary equivalent as $\mu \to 0$ because

$$b_\Lambda^{v'} = b_\Lambda^v + \left[f_{v'}(k) - f_v(k)\right] e^{-ikq} \quad \text{and} \quad f_{v'} - f_v \in L^2 \iff v = v' \;!$$

In a more complicated model allowing for a typical scattering situation that changes the aszmptotic velocity, this implies:

$$\widetilde{\Omega}_\Lambda^{v_{out}} \in \widetilde{\mathcal{F}}_\Lambda^{v_{out}} \not\ni \widetilde{\Omega}_\Lambda^{v_{in}}$$

scattering zone

$$\mathcal{F}_\Lambda^{v_{in}} \ni \widetilde{\Omega}_\Lambda^{v_{in}}$$

⚠ So generically the S matrix must be a map between varying Fock spaces!

It is important to note that the proof of self-adjointness of the interaction Hamiltonian goes through even for $\mu = 0$. Recall that we only needed the boundedness of $\left\| \frac{\hat{\gamma}_k}{\sqrt{\omega_k}} \right\|_2$ which is given for all choices of the photon mass.

However, e.g., the infinite time limit employed computing the ground state wleads out of Fock space:

the time evolved expectation value of the field $\varphi(t,x)$

$$\overline{\varphi}_\Omega(t,x) = \langle \Omega, U_{I,\Lambda}(t)^* \varphi(t,x) U_{I,\Lambda}(t) \Omega \rangle$$

for a charge fixed at $q \in \mathbb{R}^3$ fulfills

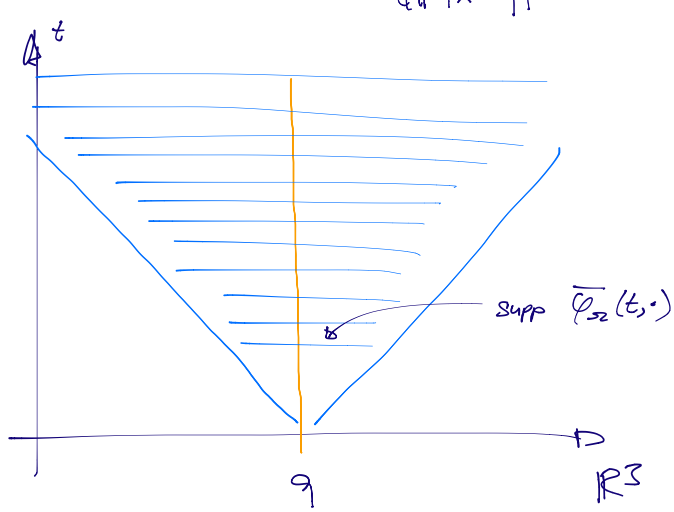
$$(\partial_t^2 - \Delta_x)\, \overline{\varphi}_\Omega(t,x) = -g\, \delta^3(x-q) \quad \text{and} \quad \overline{\varphi}_\Omega(0,x) = 0$$

using Kirchhoff's formulas we can compute the unique solution

$$\overline{\varphi}_\Omega(t,x) = \int_0^t ds\, K_{t-s} * \left[-g\, \delta^3(\cdot - q)\right](x)$$

$$= -g \int_0^t ds \int d^3y\, \frac{1}{4\pi |y|}\, \delta(|y| - (t-s))\, \delta^3(x-y-q)$$

$$= -g\, \frac{1_{B_t(q)}(x)}{4\pi |x-q|}$$



As an ultraviolet cut-off implies extended charges, an infrared cut-off implies finite spatial extend or sufficient decay. The typical Coulomb potentials do not have enough decay, i.e., "too many" photons of low energy are needed to represent them. However, for finite time there is a natural infrared cut-off due to the speed of light.

Further reading:

Single particles interacting with their massless fields can nowadays be treated in full rigor. Here some articles that may also serve as an introduction to the field:

Dynamics of charged particles and their radiation fields,
H. Spohn, Cambridge, 2004

One particle (improper) eigenstates in Nelson's massless model,
A. Pizzo, AHP, 2004

Infrared-Finite Algorithms in QED: The Groundstate of an Atom Interacting with the Quantized Radiation Field,
V. Bach, J. Fröhlich, A. Pizzo, CMP, 2006