

I QED: Its state and its problems (Version 160815)

The aim of this introductory part is to gain an overview on the conceptual and mathematical problems in the current formulation of QED.

We start by recalling some key ideas that led to the method of second quantization and discuss the resulting difficulties in finding an equation of motion. The absence of a well-defined equation of motion led people to abandon it in favor of an informal expansion of a scattering matrix. This approach gave rise to the so-called "perturbative formulation" of quantum field theory.

A straight-forward expansion of the scattering matrix, however, inherits all the difficulties from the ill-defined equation of motion it is based on. These difficulties express themselves in terms of infinities when computing scattering matrix coefficients, which then have to be "renormalized". This renormalization is implemented by means of symbolic manipulation employing an algorithm that can be cast conveniently in terms of Feynman diagrams and the appropriate rules to assign complex number values to them. With this algorithm, each order in the expansion of scattering matrix coefficients in QED can be assigned a finite value -- a fact that is referred to as "renormalizability of QED".

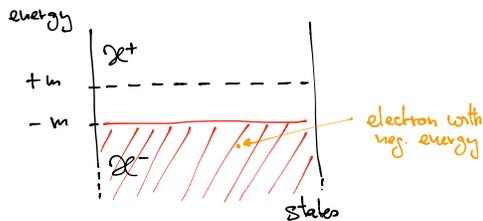
However, even after renormalization of each order in the expansion, the question arises whether the sum of renormalized orders converges. We briefly discuss the mathematical obstacle involved by means of a simple example. To date, the convergence behavior of the expansion in QED in $3+1$ dimension is unknown, though toy models suggest that it is doubtful. This means that currently we may only trust lower order terms up to an unknown order to give good approximations to physical entities we want to compute. Furthermore, informal approximations of this kind also come without good hints as to in which regime they are valid -- not to mention providing an estimate of the error.

Nevertheless, it is a fact that these lower order terms served us very well in scattering experiments such as conducted, e.g., at CERN. Next-generation experiments might, however, be able to probe QED beyond this perturbative regime and call for a "non-perturbative" formulation of QED, i.e., a formulation where at least the expansion can be controlled.

Key ideas that led to QED

HW: Spectrum of free Dirac Hamiltonian

- vacuum consists of "sea of electrons" electrons [Dirac 1934]

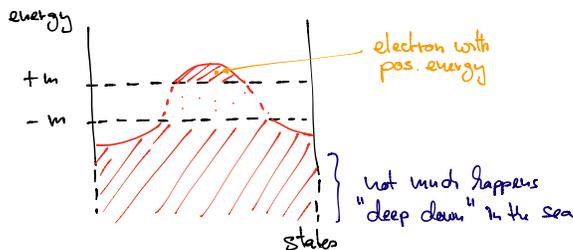


non-interacting case:

$$\Omega = \psi_1^- \wedge \psi_2^- \wedge \psi_3^- \wedge \dots$$

for ONB $(\psi_n^\pm)_{n \in \mathbb{N}}$ of \mathcal{H}^\pm

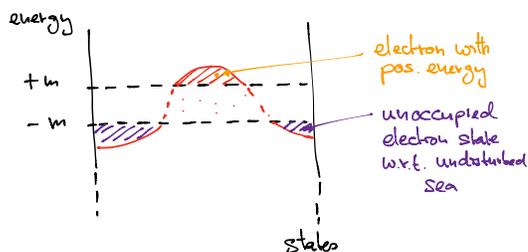
- with available energies, the vacuum is only disturbed at "sea level"



for example:

$$\Phi = \underbrace{\psi_1^+ \wedge \psi_2^+}_{\text{"above" sea level}} \wedge \underbrace{\psi_3^- \wedge \psi_4^-}_{\text{"below sea level"}} \wedge \dots$$

- a more economic description, taking account only of the perturbations of the undisturbed vacuum



the "net" excitations:

$$\Phi' = \underbrace{\psi_1^+ \wedge \psi_2^+}_{\text{additional electron states}} \otimes \underbrace{\psi_1^- \wedge \psi_2^-}_{\text{missing electron states or "holes"}}$$

- need to allow for a varying number of excitations!
- naturally leads to Fock space description employing creation/annihilation operators



Consequently, the economic description is w.r.t. one special state such as Ω .

- e.g., ground state of the free theory like above;
- ground state of the interaction theory -- more natural but extremely complicated;
- choice has to be adapted to construct the time evolution -- discussed later.
- best viewed as choice of coordinates

Short-hand notation [Schweber Ch. 8]

• $a^*(f) \varphi_1 \wedge \varphi_2 \wedge \dots = f \wedge \varphi_1 \wedge \varphi_2 \wedge \dots$

$$a^*(f) = b^*(f) + c^*(f), \quad b^*(f) = a^*(P_+ f) \quad \text{and} \quad c^*(f) = a^*(P_- f)$$

$$\mathbb{F} = b^*(\varphi_1^+) b^*(\varphi_2^+) \underbrace{c(\varphi_1^-) c(\varphi_2^-)}_{= \Omega} \varphi_1^- \wedge \varphi_2^- \wedge \varphi_3^- \wedge \dots$$

in the economic picture
these should be interpreted
as creation operators

therefore one sets $d^*(f) = c(\bar{f})$ to ensure that d^* is linear

so that

$$\mathbb{F}' = \underbrace{b^*(\varphi_1^+) b^*(\varphi_2^+)}_{\text{creation of electrons}} \underbrace{d^*(\varphi_1^-) d^*(\varphi_2^-)}_{\text{creation of holes}} \Omega$$

note $b(f)\Omega = 0 = d(g)\Omega$

$$\{b(f), b^*(g)\} = \langle f, g \rangle$$

for all $f, g_1, g_2 \in \mathcal{D}^+$

$$\{d(g_1), d^*(g_2)\} = \langle g_1, g_2 \rangle$$

$g, g_1, g_2 \in \mathcal{D}^-$

all other anticommutators zero

• A similarly economic description is useful for the electrodynamic fields:

- in terms of excitations w.r.t. to a reference field, e.g., the Coulomb field of a charge
- consequently, the number of excitations varies
- which leads to a bosonic Fock space description [Schweber Ch. 9]

Since one usually works in the economic picture the names a, a^* are free again and the correspondingly bosonic creation and annihilation operators fulfill

$$[a(f), a^*(g)] = \langle f, g \rangle, \quad a(f)1 = 0 \quad \text{for all } f, g \in \mathcal{D}_{\text{boson}}$$

all other commutators zero

These insights led to what is called second-quantization of the coupled non-linear field equations of quantum electrodynamics:

$$(i\not{\partial} - m)\psi(x) = e A(x)$$

$$\square A^\mu(x) = 4\pi j^\mu(x) \quad \text{for} \quad j^\mu(x) = e \bar{\psi}(x) \gamma^\mu \psi(x)$$

The corresponding classical action reads

$$S[\psi, \bar{\psi}, A^\mu] = \int d^4x \left[\bar{\psi}(x) (i\not{\partial} - m) \psi(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}(x) \right] \quad [3.1]$$

$$- e \int d^4x \bar{\psi}(x) A(x) \psi(x)$$

Since every state can be described in terms of annihilation and creation operators of excitations it is convenient to define the time evolution w.r.t. to these operators:

Informally, this can be achieved by finding a representation of the above CCRs, e.g.,

$$A^\mu(x) = \sum_{\lambda} \int d^3k \frac{1}{\sqrt{2k_0}} \left[\underbrace{e_{\lambda,1}^\mu e^{-ik_\nu x^\nu}}_{\text{generalized eigenvectors of } \square = \partial_\mu \partial^\mu} a_{k,1\lambda} + e_{\lambda,2}^\mu e^{ik_\nu x^\nu} a_{k,2\lambda}^* \right]_{k^0 = |k|}$$

$$\psi(x) = \sum_{\uparrow} \int d^3p \frac{m}{\sqrt{2p_0}} \left[\underbrace{\eta_{\uparrow,1}^+ e^{-ip_\nu x^\nu}}_{\text{generalized eigenvectors of } (i\not{\partial} - m)} b_{\uparrow,1} + \eta_{\uparrow,1}^- e^{ip_\nu x^\nu} d_{\uparrow,1}^* \right]_{p^0 = \sqrt{m^2 + p^2}}$$

and writing down a Schrödinger equations w.r.t. a time zero Hamiltonian, derived from the action in [3.1], in terms of the field operators above:

$$i\partial_t \Psi_t = H \Psi_t$$

$$H = \int d^3x \bar{\Psi}(0, \underline{x}) (-i \underline{x} \cdot \nabla_{\underline{x}} + m) \Psi(0, \underline{x}) + \sum_{\lambda} \int d^3k |k| a_{k, \lambda}^* a_{k, \lambda} \\ + e \int d^3x \bar{\Psi}(0, \underline{x}) A(0, \underline{x}) \Psi(0, \underline{x})$$

or in the interaction picture

$$i\partial_t \bar{\Phi}_t = H_{\text{I}}(t) \bar{\Phi}_t \quad \text{with} \quad H_{\text{I}}(t) = e \bar{\Psi}(t, \underline{x}) A(t, \underline{x}) \Psi(t, \underline{x})$$

Immediate questions

- In which sense are the field operators defined?
- On which space do they act?
- Can we define an initial value problem?

This program has never been successfully carried out.

In fact, finding an equation of motion was abandoned completely.

Instead, one adopts the informal S matrix expansion from quantum mechanics:

$$\bullet \quad S = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n H_{\text{I}}(t_1) \dots H_{\text{I}}(t_n)$$

Unfortunately, not even order by order, not to mention the limits, this can be interpreted as a well-defined expression.

Many symbolic manipulations have to be conducted on the integrand such as:

- normal ordering due to the infinitely many charges in the Dirac sea;
- mass renormalization due to the ultraviolet/infrared divergence of the photon field;
- charge renormalization, again due to the infinitely many charges in the Dirac sea.

Only then, the n-th sum and of the above expression makes sense.

Even after all these manipulations, convergence of the sequence is unknown.

- Not every Taylor series has a sufficiently large convergence radius
- A simple example that caricatures the problem is the following:

$$\bullet \quad f(\alpha) = \pi^{-1/2} \int_{-\infty}^{+\infty} dx \underbrace{e^{-x^2} e^{-\alpha x^4}}_{g(\alpha, x)}$$

$$g(\alpha, x) = e^{-x^2} \sum_{m=0}^{\infty} (-x^4)^m \frac{\alpha^m}{m!} \text{ converges for all } x, \alpha$$

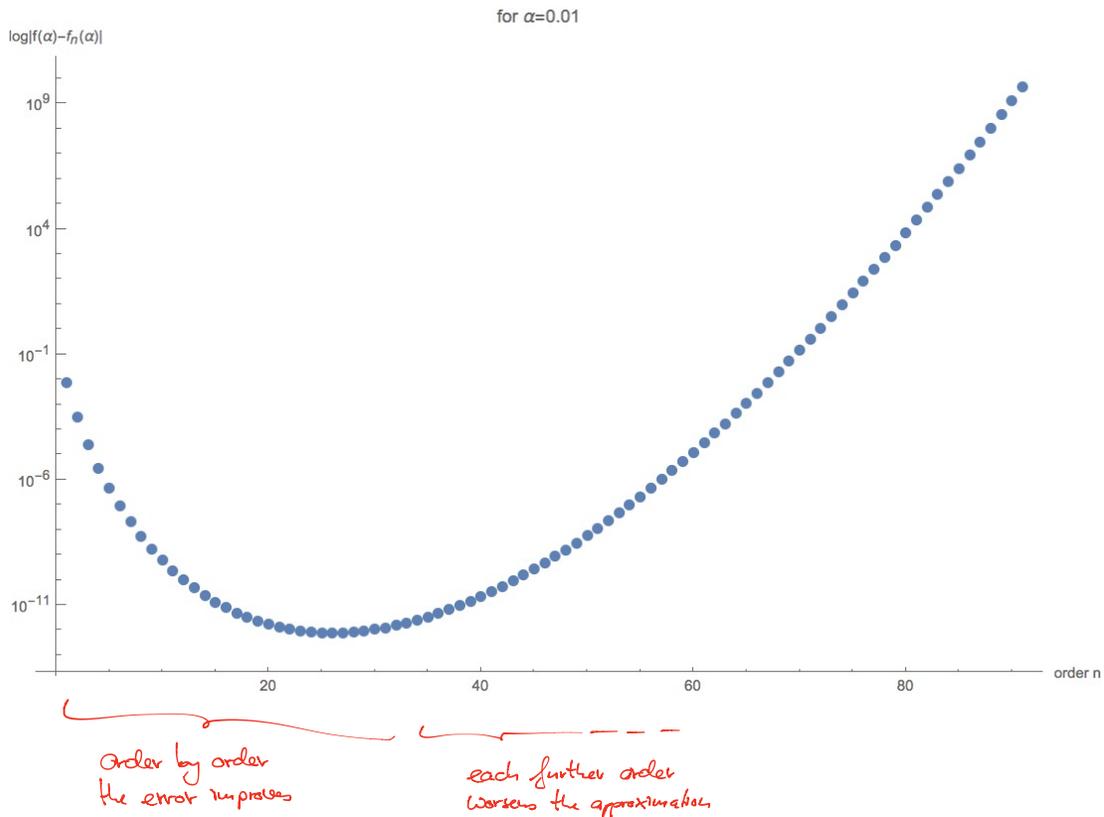
Can we commute the two limits?

$$f(\alpha) = \pi^{-1/2} \int_{-\infty}^{+\infty} dx \sum_{m=0}^{\infty} (-x^4)^m e^{-x^2} \frac{\alpha^m}{m!}$$

$$f_n(\alpha) = \sum_{m=0}^n \frac{\alpha^m}{m!} \pi^{-1/2} \int_{-\infty}^{+\infty} dx (-x^4)^m e^{-x^2} = \sum_{m=0}^n \frac{\alpha^m}{m!} (-1)^m \left(\frac{d^{2m}}{dx^{2m}} \lambda^{1/2} \Big|_{\lambda=1} \right) \\ = \frac{(4m)!}{4^{2m} (2m)!}$$

$$\text{Therefore: } f_n(\alpha) \xrightarrow{n \rightarrow \infty} \begin{cases} 0 & \text{for } \alpha = 0 \\ \infty & \text{for } \alpha > 0 \end{cases}$$

Nevertheless, the first couple of orders give a good approximation even though the series diverges:



State-of-the-art:

- QED lacks an equation of motion;
- The informal S matrix expansion can only be trusted to some unknown finite order;
- The exact regime is unknown in which the first order give a good approximation;
- In ϕ^4 theory, the informal S matrix expansion in 3+1 dimensions diverges.
- In fact, the theory becomes trivial, which raises question about the perturbation results;
- Yet, there is not a single non-trivial QFT in 3+1 dimensions that can be analyzed non-perturbatively;

For an overview of the non-perturbative state see

[Strocchi, 2013: An introduction to non-perturbative foundations of QFT]

Beside these mathematical problems, it would be silly to neglect the tremendous success of perturbative QFT.

But, we should be prepared to find out that some of the perturbative recipes are may be too naive and do not apply anymore in extreme regimes in which electrons are exposed to strong fields for long times.

Next generation experiments such as the ELI may already call for a non-perturbative understanding of QED.

In the mathematical part of this lecture, we will discuss the major problems in finding an equation of motion or an S matrix as well as some recent advances.

