A thesis submitted for the degree of

Master of Science

Ludwig-Maximilians Universität München
Fakultät der Physik

# Second Quantization of Liénard-Wiechert Fields

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October 26, 2015

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**Abstract**

This thesis is about the question whether there is a quantized analogue of the Liénard-Wiechert fields.
Classically, the computation of the Liénard-Wiechert fields is well-understood: The solution consists of two different types of fields. There is a initial field as well as a retarded field. In the classical computation, starting with some initial data at $t_0 \to -\infty$, we can show that for certain conditions only the retarded field survives. It has two contributions, a (boosted) Coulomb term and a radiation term which is proportional to the acceleration of the charge.
In this thesis, we consider the Nelson model with just one nucleon since this model shares many features with quantum electrodynamics. It describes the interaction of a field of spinless nucleons with a scalar meson field. We will calculate the corresponding fields in this model in various conditions and compare them to the classical situation.
We start with a semi-classical situation, where we consider the trajectory of the nucleon to be given. At first, we look at a nucleon at a fixed point and then we consider a nucleon with a given classical trajectory. In both situations, we obtain for $t_0 \to -\infty$ qualitatively the same result, namely, the initial field (i.e. the free field) vanishing for similar conditions as in the classical case and a (boosted) Yukawa or Coulomb potential solving the inhomogeneous wave equation having a source term at the position of the charge. Nevertheless, there is no radiation term in this setting.
Finally, we look at a particle whose dynamics are determined by the free Schrödinger equation. We will see that in this case it is not as easy to obtain a general result. Therefore, we consider a nucleon whose dynamics are determined by a quantum mechanical harmonic oscillator. Then, we find that the field consists again of the free field part and a second part, which is the convolution of the eigenstates of the harmonic oscillator with a potential. In case of the ground state, this potential is just the Coulomb potential again. In addition, there appear terms solving the homogeneous wave equation and hence we may interpret them as radiation.
It is well-known that in quantum electrodynamics there are ultraviolet as well as infrared divergences. Nevertheless, this model offers a good chance to understand the origin of them. This is an additional goal of this thesis.

### Style of writing

Although this master thesis is written by only one author, the chosen form of writing employs the use of first person plural throughout the work for two reasons: First, research is never done by a single person alone. In this sense phrases like "we conclude" are used to recall all people who contributed to a "conclusion" in one way or another. Second, for an interested reader phrases like "we prove" are also meant in the sense that the author and the reader go through a "proof" together to check if it is correct.

### Revision logbook

- correction of some typos
- correction of a minus sign and its consequences in section 5

# Contents

# 1 Roadmap

This first chapter should give an overview over the rest of the thesis and help the reader to find the main results in it.

This work deals with the second quantization of fields, in particular, we would like to find a second quantized version of the Liénard-Wiechert fields (LWFs).
In order to do so, we work with certain scalar field interaction models. Of course, we would actually like to find the second quantized fields in quantum electrodynamics (QED). Nevertheless, for simplicity we restrict ourself to Nelson's model (in [6]) with one nucleon. It describes the interaction of a field of spinless nucleons with a scalar meson field, where the nucleons are treated non-relativistically. Note that in this model the scalar field is usually massive, i.e. the bosons have field mass $\mu$. However, in the limit $\mu \to 0$, the scalar field can be interpreted as a photon field without spin. Hence the model is already pretty close to QED, when neglecting pair-creation and spin. So this should give us a good intuition for what happens if we consider QED.

This work is divided into four main sections. 2, 3, 4, 5. Section 2 gives a short introduction on the classical LWFs. Section 3 is about the second quantized fields in the Nelson model for one nucleon with a given trajectory. Further, we will explaining divergences appearing in this particular toy model. Section 4 deals with the same model, but the dynamics of a free nucleon determined by the quantum mechanical equation. Finally, section 5 describes again the same interaction, but dynamics of the nucleon determined by a quantum mechanical harmonic oscillator.

In the following, we give a brief summary over the main results in the sections:

**<u>Section 2 - Introduction:</u>**

This section gives some introduction on the (well-known) classical computation of the LWFs. We will show that, given some initial data at $t_0$, the field consists of two terms, the initial field and the retarded field. Actually, in addition to that, there could also be a advanced field, but usually physicists find arguments not to consider this solution. If the initial fields decay just a little bit (spatially), we can show that only the retarded (and the advanced) field survives in the limit $t_0 \to -\infty$. Then, the retarded field has two contributions, namely a radiation field and a (boosted) Coulomb field. Note that the radiation term is proportional to the acceleration of the particle, but there is no notion of acceleration in quantum mechanics. Hence, in will be difficult to identify those terms in a second quantized version.

**Section 3 - Interaction between a scalar field and a spinless fermion:**

This section deals with the interaction of a spinless fermion field and a (massive) scalar field. We assume the fermion momenta to be very small, i.e. the dispersion of the fermions is just $\sqrt{m_0^2 + p^2} \approx m_0$. Soon, we restrict the model to only one fermion as this yields the same results, but the calculations get easier. The Hamiltonian for this situation looks like the following:

$$H = m_0 + \int \mathrm{d}^3\mathrm{k}\omega_k a_k^* a_k + \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \left(a_k e^{ik\cdot x} + a_k^* e^{-ik\cdot x}\right) \tag{1.1}$$

where $a_k$, $a_k^*$ are the annihilation, creation operators of the meson field, $\gamma_k = \frac{f(k^2)}{\sqrt{2\omega_k}}$, $\omega_k = \sqrt{k^2 + \mu^2}$ and $f(k^2)$ is an appropriate cutoff function, most of the time we will choose $f \in C_0^\infty(\mathbb{R}^3)$ and such that it removes the ultraviolet as well as the infrared divergences.
Then, the first important result is that the time evolution in the interaction picture weakly converges to a dressing operator $D_f = \exp\left\{\frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\frac{\gamma_k}{\omega_k} \left(a_k e^{ik\cdot x} - a_k^* e^{-ik\cdot x}\right)\right\}$.
Note that by weak convergence, we mean in the sense of matrix elements and by integrals over operators, we actually mean those operators acting on a state in the Hilbert space and performing the integration afterwards. We call $D_f$ the dressing operator, since it dresses the nucleon with the right field.
With this at hand we calculate the expectation of the field operator for some initial data at $t_0 \to -\infty$. The result is the free (meson) field and a Yukawa potential. Note that only the free field depends on the initial data and again we show that for certain conditions (e.g. $f \in C_0^\infty(\mathbb{R}^3)$) it vanishes. So in the limit $\mu \to 0$ only a Coulomb potential survives, which is independent of the initial data. This agrees with our classical results.
In a next step, we consider a nucleon on a given trajectory. For simplicity we choose just a straight line, i.e. a fermion with constant velocity $v$. In this case, we get analogous results. The only difference in the field is we get the a Coulomb potential boosted with this velocity $v$.
A second goal of this section is to understand the origin of the divergences in this toy model. There appear two different types of divergences: infrared and ultraviolet divergence. In any model we consider, we start with a cutoff function, which is smooth and which has compact support in between the two balls $\mathcal{B}_\Lambda(0)$ and $\mathcal{B}_\kappa(0)$, $(\Lambda > \kappa > 0)$. Having such a cutoff function, all integral are finite and we do not have any problems with divergences. However, in the end we are interested in the situation without a cut-off, i.e. in the limit $f \to 1$, $(\Lambda \to \infty, \kappa \to 0)$. Now, if we look at the different types of divergences separately, then we notice that their origins are completely different. The appearance of the ultraviolet divergence has basically just the same reason as it has in classical calculations. The Maxwell equations are divergent at the origin. Hence, if we calculate the self-interaction, we need the evaluate these equations at its divergent point. However, the infrared divergence is a new problem, which occurs in the limit of

a massless scalar field and it is due to the fact that there is no notion of probability implemented in the Maxwell equations. They simpy allow more solutions than just the ones, which are square integrable. On the other hand, the quantum description restricts us to just consider these ones. Hence, it sounds reasonable that for example if we evolve the vacuum state in time over an infinitely long period, we may end up with a state which is not in the same Fock space anymore. The main result in this thesis is that if the vacuum state gets "dressed" with its appropriate field, i.e. if it gets evolved in time from $-\infty$ to 0, then it is not in the original Fock space anymore. Luckily, we can construct a new Fock space with the "dressed vacuum state" being the new vacuum state of this Fock space and some new annihilation and creation operators given by an algebraic relation to the old ones. Hence, the infrared problem is essentially just a self-made problem and it can be resolved by just changing the Fock space appropriately. Similarly, we obtain that the dressed vacuum state belonging to a Coulomb field boosted with velocity $v$ and the dressed vacuum state belonging to a Coulomb field boosted with a different velocity $v'$ can not be in the same Fock space.

**Section 4 - Interaction between a scalar field and a spinless fermion field with a quantum mechanical motion:**

This section is about the same interaction between a scalar field and one spinless fermion as before. However, so far, we have just considered a semi-classical model, where the trajectory of the fermion is given. In this chapter, we examine the case of a quantum mechanical motion. At first, we consider the supposed easiest case, namely a free fermion. Then, the Hamiltonian looks like

$$H = \frac{\hat{p}^2}{2m_0} + \int \mathrm{d}^3\mathrm{k} \omega_k a_k^* a_k + \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \gamma_k \left( a_k e^{ik\cdot\hat{x}} + a_k^* e^{-ik\cdot\hat{x}} \right) \tag{1.2}$$

which is pretty much the same as before, except that $\hat{x}$ and $\hat{p}$ are operators now. Basically, we can do the same calculations as in the previous chapter and obtain similar results. Albeit, $\hat{x}$ and $\hat{p}$ do not commute anymore and therefore these computations get way more complicated. Hence, in this thesis, we will state the results just up to first order in the coupling constant $\lambda$. Further, the results depend on the initial state in the Fock space. We can not just give a trajectory of the fermion and calculate the field, since the whole dynamics are given by the quantum mechanical equations. In order to calculate an explicit field, we would have to consider a particular initial wave function. We will see that for most choices, the expectation value of the field is just zero. The reason for this is that we consider a free particle and the dynamics generated by the corresponding part in the Hamiltonian, namely $\frac{\hat{p}^2}{2m_0}$, pushes the wave function to zero in the limit $t_0 \to -\infty$. In order to get a useful result, we would have to choose very special initial data by considering scattering theory.
However, we can still show that the field solves the inhomogeneous wave equation with a source term at the position of the charge at time $t$, namely at $\hat{x}(t)$.
In order to avoid the problem described above, we continue with the nucleon being in a quantum mechanical harmonic oscillator, since we know that the eigenstates of the

harmonic oscillator are bound states and therefore we can hope that this problem does not occur in this setting.

## Section 5 - Charge in a harmonic oscillator

In this section, we examine of a more specific situation, namely an oscillating fermion, i.e. a fermion in a quantum mechanical harmonic oscillator. Calculating the field explicitly for the ground state of the harmonic oscillator and its first excitation basically just yields the convolution of the wave function with a potential. This potential consists of different types of terms. It always contains the Yukawa potential, which solves the inhomogeneous wave equation with a source term at the position of the nucleon and thus, similarly as the first term in the classical LWFs, it can be interpreted as the field attached to the particle. Further, there might appear additional terms, solving the homogeneous wave equation and. These terms may be interpreted as radiation term or the far field.
This result is nice, since it is the very similar to the result in the static case, just the field gets smeared out a little bit due to the oscillation of the fermion.

# 2 Introduction

As this thesis is about relating the classical Liénard-Wiechert fields (LWFs) to their second quantized relatives, it is useful to get familiar with the classical computations at first. The LWFs describes the dynamics of a well-localized charge coupled to its own field. Nonetheless, they do not involve any quantum effects. This will be the aim of this thesis. At first, we will have a short look at the derivation of the classical LWFs as this gives us a good feeling for the general situation.
We start with the Maxwell equations in the following form:

$$\partial_t B(x,t) = -\nabla \times E(x,t) \tag{2.1}$$
$$\partial_t E(x,t) = \nabla \times B(x,t) - j(x,t) \tag{2.2}$$

$$\nabla \cdot B(x,t) = 0 \tag{2.3}$$
$$\nabla \cdot E(x,t) = \rho(x,t) \tag{2.4}$$

Note that $x, E, B, j \in \mathbb{R}^3$ are three dimensional spatial vectors and $t, \rho \in \mathbb{R}$ are just numbers.
Combining these equations, we obtain the continuity equation (charge conservation):

$$\partial_t \rho(x,t) + \nabla \cdot j(x,t) = 0 \tag{2.5}$$

Now we will find a solution of this equations. First, we notice that if we choose the right initial data at $t = t_0$, namely

$$\nabla \cdot B(x,t_0) = 0 \tag{2.6}$$
$$\nabla \cdot E(x,t_0) = \rho(x,t_0) \tag{2.7}$$

Then, charge conservation (2.5) ensures that the constraints (2.3) and (2.4) are fulfilled not only for $t = t_0$ but for all times $t \in \mathbb{R}$.
Switching to the momentum space by performing a Fourier transformation, the system of partial differential equations can be solved in the usual way (first solve the homogeneous problem and then find a solution for the full problem). The well-known calculation done in many works (e.g. by H. Spohn in [3] or also in standard literature like [10]) yields to

the following result:

$$\begin{aligned} E(k,t) &= \cos\Big(|k|t\Big)E(k,t_0) + \frac{\sin\Big(|k|t\Big)}{|k|} ik \times B(k,t_0) \\ &+ \int_{t_0}^{t} \mathrm{d}s \left\{ -\frac{\sin\Big(|k|(t-s)\Big)}{|k|} ik\rho(k,s) - \cos\Big(|k|(t-s)\Big) j(k,s) \right\} \\ &= E_{initial}(k,t) + E_{retarded}(k,t) \end{aligned} \tag{2.8}$$

$$\begin{aligned} B(k,t) &= \cos\Big(|k|t\Big)B(k,t_0) - \frac{\sin\Big(|k|t\Big)}{|k|} ik \times E(k,t_0) \\ &+ \int_{t_0}^{t} \mathrm{d}s \frac{\sin\Big(|k|(t-s)\Big)}{|k|} ik \times j(k,s) \\ &= B_{initial}(k,t) + B_{retarded}(k,t) \end{aligned} \tag{2.9}$$

Here $k \in \mathbb{R}^3$ is the three dimensional momentum vector and $s \in \mathbb{R}$.
In order to get back in the physical space, we use the propagator of the wave equation $G_t(x)$, which is the Fourier transformation of $\frac{1}{(2\pi)^{\frac{3}{2}}|k|} \sin\Big(|k|t\Big)$ and satisfies the wave equation:

$$\Big(\partial_t^2 - \Delta\Big) G = \delta^{(3)}(x)\delta(t) \tag{2.10}$$

i.e.

$$G_t(x) = \frac{1}{2\pi}\delta\Big(|x|^2 - t^2\Big) \tag{2.11}$$

This gives

$$G_t(\boldsymbol{x}) = \frac{1}{4\pi|t|}\left\{\delta(|\boldsymbol{x}| - t) + \delta(|\boldsymbol{x}| + t)\right\} \tag{2.12}$$

Putting everything together, the solution in the physical space reads like the following:

$$\begin{aligned} E(t,x) &= \partial_t G_{t-t_0} \star E(t_0,x) + \nabla_x \times G_{t-t_0} \star B(t_0,x) \\ &- \int_{t_0}^{t} \mathrm{d}s \left\{ \nabla_x G_{t-s} \star \rho(s,x) + \partial_t G_{t-s} \star j(s,x) \right\} \\ &= E_{initial}(t,x) + E_{retarded}(t,x) \end{aligned} \tag{2.13}$$

$$\begin{aligned} B(t,x) &= \partial_t G_{t-t_0} \star B(t_0,x) - \nabla_x \times G_{t-t_0} \star E(t_0,x) - \int_{t_0}^{t} \mathrm{d}s \nabla_x \times G_{t-s} \star j(s,x) \\ &= B_{initial}(t,x) + B_{retarded}(t,x) \end{aligned} \tag{2.14}$$

where $\star$ denotes the convolution. The convolution of two functions $f, g$ is defined by $(f \star g)(x) = \int \mathrm{d}^3 y f(x-y) g(y)$.

Now, we look at the special case where the fields are generated by a single, well-localized point charge, i.e. we calculate the Liénard-Wiechert fields. This calculation is well-known and one can find it in almost every standard literature on electrodynamics, but still it is a good exercise to get a feeling for what happens to the initial data in the classical case before we try to find a second quantized version.

Let us assume the trajectory of our point charge is given by its position $q(t) \in \mathbb{R}^3$ and its velocity $v(t) = \dot{q}(t) \in \mathbb{R}^3$. Then the charge distribution and the current look like the following:

$$\rho(x,t) = e\delta^{(3)}(x - q(t)) \tag{2.15}$$
$$j(x,t) = e\dot{q}(t)\delta^{(3)}(x - q(t)) \tag{2.16}$$

Note that at every spacetime point $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ the contribution from the initial fields vanishes for $t_0 \to -\infty$, of the initial fields decay fast enough. Heuristically, this makes sense, since for $t_0 \to -\infty$ our charge is "infinitely far away". It could only give a contribution to the field if it is non-zero on the backwards light-cone. Since we send $t_0 \to -\infty$, we could only get contributions from points, which are spatially infinitely far out. Hence, if the initial fields are spatially decaying fast enough, we do not get any contribution at all. Or in other words, the initial fields only give a local contribution. In the following, we show this behavior mathematically.

The initial part of the $E$-field consists of two terms and the same is true for the initial part of the $B$-field. Actually, we have to show that each of this terms vanishes in the limit $t_0 \to -\infty$. Nevertheless, the calculation is very similar for each of the terms, hence we will do it only for the $\partial_t G_{t-t_0} \star E(t_0, x)$ term.

$$\begin{aligned}
&\partial_t G_{t-t_0} \star E(t_0, x) \\
&= \partial_t \int \mathrm{d}^3 y G_{t-t_0}(y) E(t_0, x - y) \\
&= \partial_t \int \mathrm{d}^3 y \frac{1}{4\pi|t - t_0|} \left[\delta(|y| - t + t_0) + \delta(|y| + t - t_0)\right] E(t_0, x - y) \\
&= \partial_t \left\{ \frac{1}{4\pi|t - t_0|} \int \mathrm{d}^3 y \left[\delta(|y| - t + t_0) + \delta(|y| + t - t_0)\right] E(t_0, x - y) \right\} \\
&= \partial_t \left\{ \frac{1}{4\pi|t - t_0|} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^{\infty} \mathrm{d}|y| |y|^2 \left[\delta(|y| - t + t_0) + \delta(|y| + t - t_0)\right] E(t_0, x - y) \right\} \\
&= \partial_t \left\{ \frac{1}{4\pi|t - t_0|} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \sin\theta |t - t_0|^2 \left[E(t_0, x - |t - t_0| e_r) + E(t_0, x + |t - t_0| e_r)\right] \right\}
\end{aligned} \tag{2.17}$$

$$
\begin{aligned}
&= \partial_t \left\{ \frac{1}{4\pi} |t - t_0| \int \mathrm{d}\Omega \, 2\, E(t_0, x + |t - t_0| e_r) \right\} \\
&= \partial_t \left\{ \frac{1}{4\pi} |t - t_0| \int_{\partial B_1(x)} \mathrm{d}\Omega \, 2\, E(t_0, |t - t_0| e_r) \right\} \\
&= \frac{\pm 1}{2\pi} \int_{\partial \mathcal{B}_1(x)} \mathrm{d}\Omega \, E(t_0, |t - t_0| e_r) + \frac{1}{2\pi} |t - t_0| \int_{\partial \mathcal{B}_1(x)} \mathrm{d}\Omega \, \partial_t E(t_0, |t - t_0| e_r) \\
&= \frac{1}{2\pi} \int_{\partial \mathcal{B}_1(x)} \mathrm{d}\Omega \, \left[ E(t_0, |t - t_0| e_r) \pm |t - t_0| \partial_t E(t_0, |t - t_0| e_r) \right]
\end{aligned} \tag{2.18}
$$

where $e_r$ is the unit vector in $r$ direction and $\mathrm{d}\Omega$ is the angular measure in spherical coordinates and $\partial \mathcal{B}_1(x)$ is the unit sphere around $x \in \mathbb{R}$, i.e. in the end, we integrate over the unit sphere around $x$.
Therefore, the integral together with its measure $\int \mathrm{d}\Omega$ is of order $\mathcal{O}(1)$. In order to get our desired statement, we need

$$
\left| \partial_t G_{t-t_0} \star E(t_0, x) \right| \to 0 \quad \text{as} \;\; t_0 \to -\infty \tag{2.19}
$$

Together with the calculation above this gives us the following conditions the fields:

$$
\begin{aligned}
&|E(t_0, |t - t_0| e_r)| + |t - t_0| |\partial_t E(t_0, |t - t_0| e_r)| \to 0 \;\; \text{as} \; t_0 \to -\infty \\
\Leftrightarrow &|E(t_0, x)| + |x| |\nabla_x \cdot E(t_0, x)| \to 0 \;\; \text{as} \; |x| \to \infty
\end{aligned} \tag{2.20}
$$

This is fulfilled if

$$
\mathcal{O}(|E(t_0, x)|) = |x|^{-\epsilon} \tag{2.21}
$$

for any $\epsilon > 0$ arbitrary small.

Performing a similar calculation and putting everything together, we end up with the desired result: For any initial data fulfilling the following condition

$$
|E(t_0, x)| + |x| |\nabla_x \cdot E(t_0, x)| + |B(t_0, x)| + |x| |\nabla_x \cdot B(t_0, x)| \to 0 \tag{2.22}
$$

as $|x| \to \infty$, the initial fields are vanishing in the limit $t_0 \to -\infty$.

Hence, if they are decaying just a little bit (spatially), i.e.

$$
\mathcal{O}(|E(t, x)|) = \mathcal{O}(|B(t, x)|) = |x|^{-\epsilon} \tag{2.23}
$$

for any $\epsilon > 0$ arbitrary small, then the initial data vanishes in the limit $t_0 \to -\infty$. So in the classical case, we obtain conditions on our initial data at time $t_0$, if we want that the initial fields will be forgotten in the limit $t_0 \to -\infty$. A similar thing will happen in the second quantized case. This is not really surprising, since the Maxwell fields are already very closely related to Quantum mechanics.

This has been the interesting part of the calculation, since the same thing will happen in the second quantized version again. If we would do the whole calculation for the retarded fields, we would end up with the usual result, namely the Liénard-Wiechert fields:

$$E(t,x) = \frac{e}{4\pi}\left[\frac{(1-v^2)(n-v)}{(1-v\cdot n)^3|x-q|^2} + \frac{n\times\Big((n-v)\times\dot{v}\Big)}{(1-v\cdot n)^3|x-q|}\right]\Bigg|_{t=t_{ret}} \tag{2.24}$$

$$B(x,t) = n \times E(t,x) \tag{2.25}$$

where

$$n = \frac{x-q(t_{ret})}{|x-q(t_{ret})|} \tag{2.26}$$

and the retarded time $t_{ret}$ is the solution of $t_{ret} = t - |x - q(t_{ret})|$.

This fields are smooth everywhere except on the world line of the charge $x = q(t)$. The first summand describes the field attached to the particle, whereas the second one describes the far field, i.e. the radiation field that comes from infinitely far away.

# 3 Interaction between a scalar field and a spinless fermion

## 3.1 Setting and Definition of the model

In this chapter we examine the interaction between a meson, described by a scalar field and a nucleon, described by a spinless fermion field. One could ask, why we consider a scalar field even though we know that actually the electromagnetic field is a vector field. The reason is that in this case, we do not have to worry about spin indices. This would just complicate the calculations. Of course, the model we are examining in the following is just a toy model then, but qualitatively, the result we do get in this toy model, can be related well to QED.
As a first step, we restrict the problem to the case where the nucleon is at a fixed point, i.e. it has no momentum. Hence, the energy of the nucleon is given by its mass $m_0$. The dispersion relation for the meson field is given by $\omega_k = \sqrt{k^2 + \mu^2}$, where $k \in \mathbb{R}^3$ is the momentum of the meson. In the end, we want to allow dispersion relations as for the photon, i.e. we will consider the limit $\mu \to 0$. Unfortunately, in addition to the usual ultraviolet divergences, there appear infrared divergences in this limit. Therefore, we start with a meson field mass $\mu \neq 0$ and consider the limit $\mu \to 0$ afterwards. In this case, we either need an infrared cutoff in addition to the ultraviolet cutoff or the initial fields have to fulfill certain extra conditions.
The system can be described by the following Hamiltonian:

$$H = H_0 + V \tag{3.1}$$

$$H_0 = m_0 \int \mathrm{d}^3\mathrm{p} \Psi_p^* \Psi_p + \int \mathrm{d}^3\mathrm{k} \omega_k a_k^* a_k \tag{3.2}$$

$$V = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{p} \int \mathrm{d}^3\mathrm{k} \gamma_k \Psi^*(p+k)\Psi(p) \left(a_{-k}^* + a_k\right) \tag{3.3}$$

with $\gamma_k = \frac{f(k^2)}{\sqrt{2\omega_k}}$.

The operators $\Psi_p^*, \Psi_p$ are the creation, annihilation operators for the fermion field. They fulfill the usual anti-commutation relations:

$$\left\{\Psi_p, \Psi_{p'}^*\right\} = \delta^{(3)}(p - p') \tag{3.4}$$

Similarly, $a_k, a_k^*$ are the creation and annihilation operators for the meson field, i.e. they fulfill the usual commutation relations:

$$[a_k, a_{k'}^*] = \delta^{(3)}(k - k') \tag{3.5}$$

(all the other commutators give zero).

After perfoming a Fourier transformation in p, i.e.

$$\Psi_p = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x}\Psi(x)e^{-ip\cdot x} \tag{3.6}$$

$$\Psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{p}\Psi_p e^{ip\cdot x} \tag{3.7}$$

we obtain

$$H = H_0 + V$$

$$H_0 = m_0 \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x) + \int \mathrm{d}^3\mathrm{k}\omega_k a_k^* a_k \tag{3.8}$$

$$V = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k}\gamma_k \left(a_k e^{ik\cdot x} + a_k^* e^{-ik\cdot x}\right) \tag{3.9}$$

and also the anti-commutation relations translate in a nice way:

$$\{\Psi(x), \Psi^*(x')\} = \delta^{(3)}(x - x') \tag{3.10}$$

Note that all these integral have to be understood in the weak sense, i.e. acting on a state in the Hilbert space and performing the integration afterwards.
Up to this point, this is only formal, in the following we define the model mathematically

**Definition:** We call

$$\mathcal{H} := \mathcal{F}_{(fermion)} \otimes \mathcal{F}_{(meson)} \tag{3.11}$$

the Hilbert space of our system, where

$$\mathcal{F}_{(fermion)} := \bigoplus_{j=0}^{\infty} \mathcal{F}^j_{(fermion)}, \quad \mathcal{F}^0_{(fermion)} := \mathbb{C}, \quad \mathcal{F}^{j\geq 1}_{(fermion)} := \bigodot_{l=1}^{j} L^2(\mathbb{R}^3, \mathbb{C}, d^3x) \tag{3.12}$$

and

$$\mathcal{F}_{(meson)} := \bigoplus_{j=0}^{\infty} \mathcal{F}^j_{(meson)}, \quad \mathcal{F}^0_{(meson)} := \mathbb{C}, \quad \mathcal{F}^{j\geq 1}_{(meson)} := \bigodot_{l=1}^{j} L^2(\mathbb{R}^3, \mathbb{C}, d^3k) \tag{3.13}$$

Hence an element $\psi \in \mathcal{H}$ is a sequence of functions $\left\{\psi^{(N)}_{(fermion)} \otimes \psi^{(n)}_{(meson)}\right\}$ on $\mathbb{R}^{3N+3n}$ with $\|\Psi\| < \infty$, where $\|\cdot\| := \sqrt{\langle\cdot|\cdot\rangle}$ is the norm induced by the scalar product of $\mathcal{H}$ defined in the following:
Let $\psi = \psi_{(fermion)} \otimes \psi_{(meson)}, \xi = \xi_{(fermion)} \otimes \xi_{(meson)} \in \mathcal{H} = \mathcal{F}_{(fermion)} \otimes \mathcal{F}_{(meson)}$, then the scalar product of these elements is given by

$$\begin{aligned}\langle\psi|\xi\rangle := \sum_{n,N=0}^{\infty} \int \mathrm{d}^3x_1...\mathrm{d}^3x_N \overline{\psi^{(N)}_{(fermion)}(x_1, ..., x_N)} \xi^{(N)}_{(fermion)}(x_1, ..., x_N) \\ \times \int \mathrm{d}^3k_1...\mathrm{d}^3k_n \overline{\psi^{(n)}_{(meson)}(k_1, ..., k_n)} \xi^{(n)}_{(meson)}(k_1, ..., k_n)\end{aligned} \tag{3.14}$$

where $\psi^{(n)}_{(meson)}, \xi^{(n)}_{(meson)}$ are symmetric in their arguments (bosons) and $\psi^{(N)}_{(fermion)}, \xi^{(N)}_{(fermion)}$ are antisymmetric in their arguments (fermions).
Further, we define the meson annihilation and creation operators $a_k, a^*_k$

$$(a_k\psi) := \psi_{(fermion)} \otimes \left(a_k\psi_{(meson)}\right), \quad (a^*_k\psi) := \psi_{(fermion)} \otimes \left(a^*_k\psi_{(meson)}\right) \tag{3.15}$$

with

$$\begin{aligned}\left(a_k\psi_{(meson)}\right)^{(n)}(k_1,...,k_n) &:= \sqrt{n+1}\,\psi^{(n+1)}_{(meson)}(k,k_1,...,k_n)\\ \left(a^*_k\psi_{(meson)}\right)^{(n)}(k_1,...,k_n) &:= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\delta^{(3)}(k-k_i)\psi^{(n-1)}_{(meson)}(k_1,...,k_{i-1},k_{i+1},...,k_n)\end{aligned} \tag{3.16}$$

and similar we define the fermion annihilation and creation operators $\Psi(x), \Psi^*(x)$

$$(\Psi(x)\psi) := \left(\Psi(x)\psi_{(fermion)}\right) \otimes \psi_{(meson)}, \quad (\Psi^*(x)\psi) := \left(\Psi^*(x)\psi_{(fermion)}\right) \otimes \psi_{(meson)} \tag{3.17}$$

with

$$\begin{aligned}\left(\Psi(x)\psi_{(fermion)}\right)^{(N)}(x_1,...,x_N) &:= \sqrt{N+1}\,\psi^{(N+1)}_{(fermion)}(x,x_1,...,x_N)\\ \left(\Psi^*(x)\psi_{(fermion)}\right)^{(N)}(x_1,...,x_N) &:= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(-1)^{1+i}\delta^{(3)}(x-x_i)\psi^{(N-1)}_{(fermion)}(x_1,...,x_{i-1},x_{i+1},...,x_N)\end{aligned} \tag{3.18}$$

Hence for any element $|\psi\rangle \in \mathcal{H}$, we can write

$$\begin{aligned}|\psi\rangle = &\sum_{N=0}^{\infty}\frac{1}{\sqrt{N!}}\int \mathrm{d^3x_1...d^3x_N}\psi^{(N)}_{fermion}(x_1,...,x_N)\Psi^*(x_1)...\Psi^*(x_n)\,|0\rangle\\ &\otimes \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\int \mathrm{d^3k_1...d^3k_n}\psi^{(n)}(k_1,...,k_n)a^*_{k_1}...a^*_{k_n}\,|0\rangle\end{aligned} \tag{3.19}$$

where $|0\rangle \otimes |0\rangle$ is the vacuum state of $\mathcal{H}$ and $\langle 0|0\rangle = 1$. Formally, this is implemented by the commutation relations mentioned before.
Further, let n be the meson particle number operator defined by

$$(n\psi) := \psi_{(fermion)} \otimes \left(n\psi_{(meson)}\right), \quad \left(n\psi_{(meson)}\right)^{(n)} := n\psi^{(n)}_{(meson)} \tag{3.20}$$

on the domain $\mathcal{D}(n)$ of all $\psi \in \mathcal{H}$, such that $\left\{n\psi^{(n)}\right\}$ are again in $\mathcal{H}$.
Similarly, let N be the fermion particle number operator defined by

$$(N\psi) := \left(N\psi_{(fermion)}\right) \otimes \psi_{(meson)}, \quad \left(N\psi_{(fermion)}\right)^{(N)} := N\psi^{(N)}_{(fermion)} \tag{3.21}$$

on the domain $\mathcal{D}(N)$ of all $\psi \in \mathcal{H}$, such that $\left\{N\psi^{(N)}\right\}$ are again in $\mathcal{H}$.
In the following, if we talk about states in the Hilbert space, we always mean the normalized ones.

**Remark:** With the definition above we know that for any function $h \in L^2(\mathbb{R}^3)$ the operators $\int d^3k h(k) a_k$, $\int d^3k h(k) a_k^*$, $\int d^3x h(x)\Psi(x)$, $\int d^3x h(x)\Psi^*(x)$ defined by

$$\left(\int d^3k h(k) a_k \psi_{meson}\right)^{(n)}(k_1, ..., k_n) := \sqrt{n+1}\int d^3k h(k)\, \psi^{(n+1)}_{(meson)}(k, k_1, ..., k_n)$$
$$\left(\int d^3k h(k) a_k^* \psi_{(meson)}\right)^{(n)}(k_1, ..., k_n) := \frac{1}{\sqrt{n}}\sum_{i=1}^{n} h(k_i)\psi^{(n-1)}_{(meson)}(k_1, ..., k_{i-1}, k_{i+1}, ..., k_n) \tag{3.22}$$

$$\left(\int d^3x h(x)\Psi(x)\psi_{(fermion)}\right)^{(N)}(x_1, ..., x_N) := \sqrt{N+1}\int d^3x h(x)\, \psi^{(N+1)}_{(fermion)}(x, x_1, ..., x_N)$$
$$\left(\int d^3x h(x)\Psi^*(x)\psi_{(fermion)}\right)^{(N)}(x_1, ..., x_N) := \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(-1)^{1+i} h(x_i)\psi^{(N-1)}_{(fermion)}(x_1, ..., x_{i-1}, x_{i+1}, ..., \tag{3.23}$$

are well-defined operators on the domain $\mathcal{D}(n^{\frac{1}{2}})$ of all $\psi \in \mathcal{H}$, such that $\left\{n^{\frac{1}{2}}\psi^{(n)}\right\}$ are again in $\mathcal{H}$, respectively on the domain $\mathcal{D}(N^{\frac{1}{2}})$of all $\psi \in \mathcal{H}$, such that $\left\{N^{\frac{1}{2}}\psi^{(N)}\right\}$ are again in $\mathcal{H}$. This can be seen by Schwarz inequality, for example

$$\left\|\int d^3k h(k) a_k \psi\right\| \leq \|h\|_2 \left\|n^{\frac{1}{2}}\psi\right\| \tag{3.24}$$

and analogously for the others.
Further, for any $f \in C_0^\infty$, we know $\|\omega_k\|_2 < \infty$. Hence, the integral operator $\int d^3\mathrm{k}\,\omega_k a_k^* a_k$ is well-defined on $\mathcal{D}(n)$. This can be shown again by Schwarz inequality

$$\left\|\int d^3\mathrm{k}\,\omega_k a_k^* a_k \psi\right\| \leq \|\omega_k\|_2 \|n\psi\| \tag{3.25}$$

Unfortunately, $\frac{1}{\sqrt{2\omega_k}} \notin L^2(\mathbb{R}^3)$, hence we need to introduce a cutoff function in order to make the interacting part of the Hamiltonian well-defined.

Having this on hand we write down the following well-defined Hamiltonian

**Definition:** Let $f \in C_0^\infty$ be a cutoff function, $\omega_k = \sqrt{k^2+\mu^2}$ and $\gamma_k = \frac{f(k^2)}{\sqrt{2\omega_k}}$, then we define the Hamiltonian by

$$H = H_0 + V$$
$$H_0 = m_0 \int d^3\mathrm{x}\,\Psi^*(x)\Psi(x) + \int d^3\mathrm{k}\,\omega_k a_k^* a_k =: H_0^{(fermion)} + H_0^{(meson)} \tag{3.26}$$
$$V = \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int d^3\mathrm{x}\,\Psi^*(x)\Psi(x)\int d^3\mathrm{k}\,\gamma_k\left(a_k e^{ik\cdot x} + a_k^* e^{-ik\cdot x}\right) \tag{3.27}$$

with $\mathcal{D}(H_0) = \mathcal{D}(N) \cap \mathcal{D}(n)$ and $\mathcal{D}(V) = \mathcal{D}(N) \cap \mathcal{D}(n^{\frac{1}{2}})$.

It is well-known that $\mathcal{D}(n) \subset \mathcal{D}(n^{\frac{1}{2}})$, hence

$$\mathcal{D}(H) = \mathcal{D}(N) \cap \mathcal{D}(n) \tag{3.28}$$

This Hamiltonian is self-adjoint. This can be shown by noticing that $H_0$ is self-adjoint and $V$ is relatively bounded with respect to $H_0$, then one concludes that by the Kato-Rellich theorem that also $H$ is self-adjoint on the domain $\mathcal{D}(H) = \mathcal{D}(H_0)$. This was shown by Nelson in his original paper [6].

In the following, if we write down an integral containing operators, we always understand this integral as acting on a state $|\psi\rangle \in \mathcal{H}$, in particular an element of the domain of this integral operator. In most of the upcoming cases, we are just interested in what happens to the meson part of this domain, which is a subspace of $\mathcal{F}_{(meson)}$. Sometimes, we will call this meson Fock space also $\mathcal{F}_{|0\rangle}$, where $|0\rangle$ indicates the vacuum state.

## 3.2 Evolution operator

First of all, it needs to be clarified that (as mentioned before) in the following if we write down integral operator like $U_I(t_0, t)$, those integrals are always meant in the weak sense, i.e. they always have to be understood acting on a state in our Fock space and then performing the integration (not the other way round). This is no limitation in our context, since we only talk about weak convergences is this chapter anyways.

We start with the calculation of the evolution operator $U_I(t, t_0)$ in the interaction picture. It is defined by

$$i\partial_t U_I(t, t_0) = V(t)U_I(t, t_0) \tag{3.29}$$

where $V(t) = e^{iH_0 t} V e^{-iH_0 t}$.
A formal solution to this partial differential equation is given by the Born series

$$U_I(t, t_0) = 1 - i \int_{t_0}^{t} \mathrm{dt_1} V(t_1) U_I(t_1) \tag{3.30}$$

Iterating this yields

$$U_I(t, t_0) = 1 + ... + (-i)^n \int_{t_0}^{t} \mathrm{dt_1} ... \int_{t_0}^{t_{n-1}} \mathrm{dt_n} V(t_1)...V(t_n) + ... \tag{3.31}$$

Note that $t_1 > t_2 > ... > t_n$, therefore we can introduce the time ordering operator $T[\cdot]$ and write

$$\begin{aligned} U_I(t, t_0) &= 1 + ... + (-i)^n \int_{t_0}^{t} \mathrm{dt_1} ... \int_{t_0}^{t_{n-1}} \mathrm{dt_n} T\Big[V(t_1)...V(t_n)\Big] \\ &= 1 + .. + \frac{(-i)^n}{n!} \int_{t_0}^{t} \mathrm{dt_1} ... \mathrm{dt_n} T\Big[V(t_1)...V(t_n)\Big] \\ &=: \sum_{n=0}^{\infty} \frac{1}{n!} U_I^{(n)}(t, t_0) \end{aligned} \tag{3.32}$$

where

$$U_I^{(n)}(t,t_0) := (-i)^n \int_{t_0}^{t} \mathrm{dt_1...dt_n} T\Big[V(t_1)...V(t_n)\Big] \tag{3.33}$$

Here we used that $T\Big[V(t_1)...V(t_n)\Big]$ is symmetric in its arguments.
In order to compute $U_I^{(n)}(t,t_0)$, we need to know $V(t)$ at first. Hence we compute $a_k(t)$ in the interaction picture by the standard operator identity, i.e.

$$\begin{aligned} a_k(t) &= e^{iH_0t} a_k e^{-iH_0t} \\ &= a_k + [iH_0t, a_k] + \frac{1}{2!}[iH_0t,[iH_0t,a_k]] + ... \end{aligned} \tag{3.34}$$

It is easy to see that

$$\begin{aligned} [iH_0t, a_k] &= it \int \mathrm{d^3k'} \omega_k' [a_{k'}^* a_{k'}, a_k] \\ &= it \int \mathrm{d^3k'} \omega_k' [a_{k'}^*, a_k] a_{k'} \\ &= -it \int \mathrm{d^3k'} \omega_k' \delta^{(3)}(k-k') a_{k'} \\ &= -it\omega_k a_k \end{aligned} \tag{3.35}$$

hence

$$\begin{aligned} a_k(t) &= a_k\Big(1 + (-it\omega_k) + ... + \frac{1}{n!}(-it\omega_k)^n + ...\Big) \\ &= a_k e^{-i\omega_k t} \end{aligned} \tag{3.36}$$

Together with

$$[H_0, \Psi^*(x)\Psi(x)] = 0 \tag{3.37}$$

(which obviously holds by $[H_0, H_0] = 0$) this gives

$$\begin{aligned} V(t) &= e^{iH_0t} V e^{-iH_0t} \\ &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d^3x} \Psi^*(x)\Psi(x) \int \mathrm{d^3k} \gamma_k \left(e^{iH_0t} a_k e^{-iH_0t} e^{ik\cdot x} + e^{iH_0t} a_k^* e^{-iH_0t} e^{-ik\cdot x}\right) \\ &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d^3x} \Psi^*(x)\Psi(x) \int \mathrm{d^3k} \gamma_k \left(a_k e^{-i\omega_k t} e^{ik\cdot x} + a_k^* e^{i\omega_k t} e^{-ik\cdot x}\right) \end{aligned} \tag{3.38}$$

Our goal is to calculate $U_I(t,-\infty) = \lim_{t_0\to-\infty} U_I(t,t_0)$ and we want to show that it converges to the operator $T_f$, defined by

$$T_f := \exp\left\{\frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d^3x} \Psi^*(x)\Psi(x) \int \mathrm{d^3k} \frac{\gamma_k}{\omega_k} \left(a_k e^{ik\cdot x} - a_k^* e^{-ik\cdot x}\right)\right\} \tag{3.39}$$

Later, we will consider just one nucleon at a fixed point $y \in \mathbb{R}^3$, i.e.
$\Psi^*(x)\Psi(x) = \delta^{(3)}(x-y)$, then $T_f$ reduces to the dressing operator $D_f$ found in [2]:

$$D_f = \exp\left\{ \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{\omega_k} \left( a_k e^{ik\cdot x} - a_k^* e^{-ik\cdot x} \right) \right\} \tag{3.40}$$

This operator transforms the non-interacting vacuum state $|0\rangle$ into a new vacuum state $|\psi^{gs}\rangle$. This is the dressed vacuum state of the system, appropriately dressed with its field, i.e. $D_f$ is the operator transforming a state in the Fock space with to the non-interacting vacuum $\mathcal{F}_{|0\rangle}$ into the corresponding state in a Fock space with a new, dressed vacuum state $|\psi^{gs}\rangle := D_f\,|0\rangle$. Note that there might be situations, where these two "vacuum" states cannot be described in the same Fock space. This special situations will be discussed later. Therefore, we have

$$D_f : \mathcal{F}_{|0\rangle} \to \mathcal{F}_{D_f|0\rangle} \tag{3.41}$$

It is easy to check that this new vacuum is stable under the production of new mesons, since one can easily calculate that

$$a_k D_f\,|0\rangle = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} D_f\,|0\rangle \tag{3.42}$$

Hence, the new Fock space with the dressed vacuum state $|\psi^{gs}\rangle = D_f\,|0\rangle$ can be constructed by the following annihilation operator

$$b_k = D_f^* a_k D_f = a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} \tag{3.43}$$

and its adjoint, the creation operator. It is easy to see that $b_k$ annihilates the new vacuum state

$$b_k\,|\psi^{gs}\rangle = 0 \tag{3.44}$$

All these properties are just algebraic properties, no operator equalities. Nevertheless, we may use them in they way they are written down. We will verify them later in this thesis.
We start with the proof of the weak convergence of $U_I(0,t_0)$ to $T_f$ for $t_0 \to -\infty$, i.e. we show that for any $|\xi\rangle, |\phi\rangle \in \mathcal{H}$

$$\langle\phi|\, U_I(0,t_0)\,|\xi\rangle \xrightarrow{t_0\to-\infty} \langle\phi|\, T_f\,|\xi\rangle \tag{3.45}$$

We will do this proof in the following step by step, at first we will look at the situation where there is a cutoff $f(k^2)$. Later on, we will discuss the problem without a cutoff, i.e. in the limit $f(k^2) \to 1$. This will lead to restrictions on $|\xi\rangle, |\phi\rangle$ and further, we will have to redefine our Fock space. We may also get additional problems for the massless case, i.e. if we let the mass of the mesonic field go to zero $(\mu \to 0)$.

Note that the dressing operator $D_f$ is basically the same as the operator $T_f$, $D_f$ is the part of $T_f$, which is acting on the $\mathcal{F}_{(meson)}$ part of the Hilbert space $\mathcal{H}$. Later, we will restrict the situation to the case with only one nucleon. In this situation, $T_f$ almost reduces to $D_f$ anyways. This means, the dressing operator $D_f$ is the relevant part of the time evolution in the interaction picture "$U_I(t, -\infty)$".

### 3.2.1 Proof of the weak convergence with cutoff

First of all, one could ask why we are only aiming to prove weak convergence and not a stronger form of convergence? This is due to the fact that, in this toy model, there is always radiation running out of the system, hence the evolution operator can not converge in any stronger sense.

In the following, we will show the weak convergence step by step. We start with consider the first order term of $U_I(t, t_0)$ ($U_I^{(0)}(t, t_0) = 1$):

$$\begin{aligned} U_I^{(1)}(t, t_0) &= -i \int_{t_0}^{t} \mathrm{dt}_1 V(t_1) \\ &= \frac{-i\lambda}{(2\pi)^{\frac{3}{2}}} \int_{t_0}^{t} \mathrm{dt}_1 \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k} \gamma_k (a_k e^{-i\omega_k t_1} e^{ik\cdot x} + a_k^* e^{i\omega_k t_1} e^{-ik\cdot x}) \\ &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{\omega_k} \Big[ (a_k e^{ik\cdot x - i\omega_k t} - a_k^* e^{-ik\cdot x + i\omega_k t}) \\ &- (a_k e^{ik\cdot x - i\omega_k t_0} - a_k^* e^{-ik\cdot x + i\omega_k t_0}) \Big] \end{aligned} \tag{3.46}$$

We immediately see that the first part is exactly the same as the first order term in the expansion of $T_f$. Hence, up to first order, the only thing we have to show, is that the second term in the equation above (the part containing $t_0$) vanishes as $t_0 \to -\infty$ (in the weak sense). This will give us a recipe how we should proceed with higher orders.
In order to make sense out of the limit $t_0 \to -\infty$, we perform a little trick (as $e^{i\omega_k t_0}$ does not converge in this limit:

$$\partial_{|k|} e^{-i\omega_k t_0} = -it_0 e^{-i\omega_k t_0} \partial_{|k|}\omega_k = -it_0 e^{-i\omega_k t_0} \frac{|k|}{\omega_k} \tag{3.47}$$

which yields

$$e^{-i\omega_k t_0} = \frac{-i}{t_0} \frac{\omega_k}{|k|} \partial_{|k|} e^{-i\omega_k t_0} \tag{3.48}$$

Hence, the second part of $U_I^{(1)}(t, t_0)$ (the part with $t_0$) can be written as

$$\frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{\omega_k} \left( a_k e^{ik\cdot x - i\omega_k t_0} - a_k^* e^{-ik\cdot x + i\omega_k t_0} \right) \tag{3.49}$$

$$= \frac{-i\lambda}{(2\pi)^{\frac{3}{2}}} \frac{1}{t_0} \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{|k|} \left( a_k e^{ik\cdot x} \partial_{|k|} e^{-i\omega_k t_0} + a_k^* e^{-ik\cdot x} \partial_{|k|} e^{i\omega_k t_0} \right) \tag{3.50}$$

Thus, it is sufficient to show that for any $|\xi\rangle, |\psi\rangle \in \mathcal{F}_{(meson)}$

$$\langle\phi| \frac{1}{t_0} \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{|k|} \Big( a_k e^{ik\cdot x} \partial_{|k|} e^{-i\omega_k t_0} + a_k^* e^{-ik\cdot x} \partial_{|k|} e^{i\omega_k t_0} \Big) |\xi\rangle \xrightarrow{t_0 \to -\infty} 0 \tag{3.51}$$

which is true if

$$\frac{1}{t_0} \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{|k|} e^{-ik\cdot x} \partial_{|k|} e^{i\omega_k t_0} \langle\phi| a_k^* |\xi\rangle \xrightarrow{t_0 \to -\infty} 0 \tag{3.52}$$

since the other term is just the complex conjugated of this. Define the following

**Definition:**

$$I(\phi, \xi) := \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{|k|} e^{-ik\cdot x} (\partial_{|k|} e^{i\omega_k t_0}) \langle\phi| a_k^* |\xi\rangle \tag{3.53}$$

Then, the condition above is equivalent to the condition that the integral exists, i.e. $|I(\phi, \xi)| < \infty$.

**Definition:** For later purposes, let us also define the integral $I_t(\phi, \xi)$, which is the same as $I(\phi, \xi)$ but $t_0$ is replaced by $t$.

$|\phi\rangle$ and $|\xi\rangle$ live in the (meson) Fock space $\mathcal{F}$, hence they can be written as

$$|\phi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int \mathrm{d}^3\mathrm{k}_1 ... \mathrm{d}^3\mathrm{k}_\mathrm{n} \phi^{(n)}(k_1, ..., k_n) a_{k_1}^* ... a_{k_n}^* |0\rangle \tag{3.54}$$

$$|\xi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int \mathrm{d}^3\mathrm{k}_1 ... \mathrm{d}^3\mathrm{k}_\mathrm{n} \xi^{(n)}(k_1, ..., k_n) a_{k_1}^* ... a_{k_n}^* |0\rangle \tag{3.55}$$

Note that talking about states in the Fock space, we mean the normalized ones:

$$1 = \langle\phi|\phi\rangle = \sum_{n=0}^{\infty} \|\phi^{(n)}\|_2^2 \tag{3.56}$$

This implies that $\|\phi^{(n)}\|_2 \xrightarrow{n \to \infty} 0$.

In the following, we will examine the condition we need for the weak convergence (up to first order). We start with a simple case, where we set $|\xi\rangle = |0\rangle$. Then

$$\langle\phi| a_k^* |0\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int \mathrm{d}^3\mathrm{k}_1 ... \mathrm{d}^3\mathrm{k}_\mathrm{n} \overline{\phi^{(n)}(k_1, ..., k_n)} \langle 0| a_{k_1} ... a_{k_n} a_k^* |0\rangle \tag{3.57}$$

Exploiting the commutation relation and the fact $a_k |0\rangle = 0$, one can easily see that the only summand giving a non-zero contribution is the one for $n = 1$. Hence

$$\langle\phi| a_k^* |0\rangle = \overline{\phi^{(1)}(k)} \tag{3.58}$$

So

$$I(\phi, 0) = \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{|k|} e^{-ik\cdot x} \overline{\phi^{(1)}(k)} (\partial_{|k|} e^{i\omega_k t_0}) \tag{3.59}$$

Note that setting $|\phi\rangle = |0\rangle$ will yield the same result except for a complex conjugation.

Now, we estimate this integral by transforming to spherical coordinates and integrating by parts:

$$\begin{aligned} I(\phi, 0) &= \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{|k|} e^{-ik\cdot x} \overline{\phi^{(1)}(k)} (\partial_{|k|} e^{i\omega_k t_0}) \\ &= \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{1} \mathrm{d}(\cos\theta) \int_0^{\infty} \mathrm{d|k|} \frac{\gamma_k |k|^2}{|k|} e^{-i|k||x|cos\theta} \overline{\phi^{(1)}(|k|, \theta, \varphi)} (\partial_{|k|} e^{i\omega_k t_0}) \\ &= -\int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{1} \mathrm{d}(\cos\theta) \int_0^{\infty} \mathrm{d|k|} e^{i\omega_k t_0} \partial_{|k|} \Big( \gamma_k |k| e^{-i|k||x|cos\theta} \overline{\phi^{(1)}(|k|, \theta, \varphi)} \Big) \end{aligned} \tag{3.60}$$

Here we have used that the boundary terms vanish. Out of this integrating by parts comes the first condition on $\phi$. The formula is only valid if not only $\phi \in L^2$ but also $\partial_{|k|}\phi \in L^2$. This condition will appear again later.

Now, we apply the the triangle inequality and remember $\gamma_k = \frac{f(k^2)}{\sqrt{2\omega_k}}$:

$$\begin{aligned} |I(\phi, 0)| &\leq \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{1} \mathrm{d}(\cos\theta) \int_0^{\infty} \mathrm{d|k|} \Big| \partial_{|k|} \Big( \gamma_k |k| e^{-i|k||x|cos\theta} \overline{\phi^{(1)}(|k|, \theta, \varphi)} \Big) \Big| \\ &= \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{1} \mathrm{d}(\cos\theta) \int_0^{\infty} \mathrm{d|k|} |\phi^{(1)}(|k|, \theta, \varphi)| |f(k^2)| \Big| \partial_{|k|} \Big( \frac{|k|}{\sqrt{2}(k^2+\mu^2)^{\frac{1}{4}}} \Big) \Big| \\ &+ \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{1} \mathrm{d}(\cos\theta) \int_0^{\infty} \mathrm{d|k|} |\phi^{(1)}(|k|, \theta, \varphi)| \frac{|k|}{\sqrt{2}(k^2+\mu^2)^{\frac{1}{4}}} |\partial_{|k|} f(k^2)| \\ &+ \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{1} \mathrm{d}(\cos\theta) \int_0^{\infty} \mathrm{d|k|} |\phi^{(1)}(|k|, \theta, \varphi)| \frac{|f(k^2)||k|}{\sqrt{2}(k^2+\mu^2)^{\frac{1}{4}}} |\partial_{|k|} e^{-i|k||x|cos\theta}| \\ &+ \int \mathrm{d}^3\mathrm{k} \frac{|f(k^2)||k|}{\sqrt{2}(k^2+\mu^2)^{\frac{1}{4}}} |\partial_{|k|} \phi^{(1)}(|k|, \theta, \varphi)| \end{aligned} \tag{3.61}$$

After some easy computations we obtain

$$
\begin{aligned}
|I(\phi,0)| \leq & \underbrace{\int \mathrm{d^3k} |\phi^{(1)}(k)| \frac{(3k^2+2\mu^2)|f(k^2)|}{2\sqrt{2}k^2(k^2+\mu^2)^{\frac{5}{4}}}}_{=:I_1(\phi,0)} \\
& + \underbrace{\int \mathrm{d^3k} |\phi^{(1)}(k)| \frac{|\partial_k f(k^2)|}{\sqrt{2}|k|(k^2+\mu^2)^{\frac{1}{4}}}}_{=:I_2(\phi,0)} \\
& + |x| \underbrace{\int \mathrm{d^3k} |\phi^{(1)}(k)| \frac{|f(k^2)|}{\sqrt{2}|k|(k^2+\mu^2)^{\frac{1}{4}}}}_{=:I_3(\phi,0)} \\
& + \underbrace{\int \mathrm{d^3k} |\partial_{|k|}\phi^{(1)}(k)| \frac{|f(k^2)|}{\sqrt{2}|k|(k^2+\mu^2)^{\frac{1}{4}}}}_{=:I_4(\phi,0)}
\end{aligned}
\tag{3.62}
$$

All these integrals $I_1(\phi,0), ..., I_4(\phi,0)$ are positive and thus $I(\phi,0)$ exists if $I_1(\phi,0), ..., I_4(\phi,0)$ have an upper bound each.
In this section, we consider to be a real cutoff function that makes all integrals finite, for example a smooth function with compact support ($f \in C_0^\infty(\mathbb{R}^3)$). In this case, we only need

$$\phi^{(1)} \in \mathrm{L}^2(\mathbb{R}^3) \tag{3.63}$$

$$(\partial_{|k|}\phi^{(1)}) \in \mathrm{L}^2(\mathbb{R}^3) \tag{3.64}$$

for the integral being finite.
Nevertheless, we can integrate by parts again and note that $f(k^2)$ is a smooth function with compact support. Then, we have obtain that (3.63) is implied by (3.64), which is true for every element in the Fock space.

We started with trying to estimate $I(\phi,0)$ just to get a feeling for what is happening. However, this does not complete the proof, we need to show that $I(\phi,\xi)$ is finite. In order to do this, we have to calculate

$$
\begin{aligned}
\langle\phi|\, a_k^* \,|\xi\rangle = & \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{n!}\sqrt{m!}} \int \mathrm{d^3k_1 ... d^3k_n} \int \mathrm{d^3k'_1 ... d^3k'_m} \overline{\phi^{(n)}(k_1,...,k_n)} \xi^{(m)}(k'_1,...,k'_m) \\
& \cdot \langle 0|\, a_{k_1} ... a_{k_n} a_k^* a_{k'_1}^* ... a_{k'_m}^* \,|0\rangle
\end{aligned}
\tag{3.65}
$$

This can be calculated step by step, analogously to the easy case before by using the commutation relations in order to get a normal ordering. We start with bringing $a_k^*$ to the left. This gives us (n-1) terms, but relabeling and using $[a_k, a_{k'}] = 0$ shows that all of them are the same. Also note that the integral only gives a contribution if $n = m+1$.

Hence

$$\langle\phi| a_k^* |\xi\rangle = \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{n!}\sqrt{m!}}(n-1)\delta_{n,m+1} \int \mathrm{d^3k_2...d^3k_n} \int \mathrm{d^3k'_1...d^3k'_m} \cdot \overline{\phi^{(n)}(k,k_2,...,k_n)}\xi^{(m)}(k'_1,...,k'_m) \langle 0| a_{k_2}...a_{k_n} a^*_{k'_1}...a^*_{k'_m} |0\rangle \tag{3.66}$$

In the next step, we bring $a^*_{k'_1}$ to the left, this gives us similar as above (n-2) identical terms and so on. Proceeding like this yields:

$$\langle\phi| a_k^* |\xi\rangle = \sum_{m=0}^{\infty} \int \mathrm{d^3k_2...d^3k_{m+1}} \overline{\phi^{(m+1)}(k,k_2,...,k_{m+1})}\xi^{(m)}(k_2,...,k_{m+1}) \tag{3.67}$$

In the end, by using the triangle inequality we get

$$\begin{aligned}
&|I(\phi,\xi)| \\
&= \int \mathrm{d^3k} \left| \frac{\gamma_k}{|k|} e^{-ik\cdot x} (\partial_{|k|} e^{i\omega_k t_0}) \langle\phi| a_k^* |\xi\rangle \right| \\
&= \sum_{m=0}^{\infty} \int \mathrm{d^3kd^3k_2...d^3k_{m+1}} \left| (\partial_{|k|} e^{i\omega_k t_0}) \frac{\gamma_k}{|k|} e^{-ik\cdot x} \overline{\phi^{(m+1)}(k,k_2,...,k_{m+1})}\xi^{(m)}(k_2,...,k_{m+1}) \right| \\
&=: \sum_{m=0}^{\infty} I^{(m)}(\phi,\xi)
\end{aligned} \tag{3.68}$$

Again, we want to show that $|I(\phi,\xi)| < \infty$. This is true if $I^{(m)}(\phi,\xi)$ in decaying fast enough (for $m \to \infty$). Therefore, in this general case it not sufficient to show that all the integrals are finite, we also need to show that they sum over them is finite.
Therefore, we examine $I^{(m)}(\phi,\xi)$ a bit further. Basically, this is the same calculation as in the easy part and it will give us four terms:

$$\begin{aligned}
&I^{(m)}(\phi,\xi) \\
&\leq \underbrace{\int \mathrm{d^3k_1d^3k_2...d^3k_{m+1}} \left|\phi^{(m+1)}(k_1,...,k_{m+1})\right|\left|\xi^{(m)}(k_2,...,k_{m+1})\right| \frac{(3k_1^2+2\mu^2)|f(k_1^2)|}{2\sqrt{2}k_1^2(k_1^2+\mu^2)^{\frac{5}{4}}}}_{=:I_1^{(m)}(\phi,\xi)} \\
&+ \underbrace{\int \mathrm{d^3k_1d^3k_2...d^3k_{m+1}} \left|\phi^{(m+1)}(k_1,...,k_{m+1})\right|\left|\xi^{(m)}(k_2,...,k_{m+1})\right| \frac{|\partial_{|k_1|}f(k_1^2)|}{\sqrt{2}|k_1|(k_1^2+\mu^2)^{\frac{1}{4}}}}_{=:I_2^{(m)}(\phi,\xi)} \\
&+ |x| \underbrace{\int \mathrm{d^3k_1d^3k_2...d^3k_{m+1}} \left|\phi^{(m+1)}(k_1,...,k_{m+1})\right|\left|\xi^{(m)}(k_2,...,k_{m+1})\right| \frac{|f(k_1^2)|}{\sqrt{2}|k_1|(k_1^2+\mu^2)^{\frac{1}{4}}}}_{=:I_3^{(m)}(\phi,\xi)} \\
&+ \underbrace{\int \mathrm{d^3k_1d^3k_2...d^3k_{m+1}} \left|\partial_{|k_1|}\phi^{(m+1)}(k_1,...,k_{m+1})\right|\left|\xi^{(m)}(k_2,...,k_{m+1})\right| \frac{|f(k_1^2)|}{\sqrt{2}|k_1|(k_1^2+\mu^2)^{\frac{1}{4}}}}_{=:I_4^{(m)}(\phi,\xi)}
\end{aligned} \tag{3.69}$$

Now, we have to estimate each of those integrals in a way, such that the sum of them is still finite. We will do this in a similar way as in the easy case.
Again, let $f$ be a smooth function with compact support. This guarantuees the existence of the integrals $I_l^{(m)}(\phi,\xi),\ l=0,1,2,3,4$.

Hence, the only problem we have to deal with, is the question whether the sum over m is still finite. First, observe that the integrals $I_1^{(m)}(\phi,\xi), I_2^{(m)}(\phi,\xi), I_3^{(m)}(\phi,\xi)$ are of the same form

$$I_l^{(m)}(\phi,\xi) = \int \mathrm{d^3k_1 d^3k_2 ... d^3k_{m+1}} \left|\phi^{(m+1)}(k_1,...,k_{m+1})\right|\left|\xi^{(m)}(k_2,...,k_{m+1})\right| g_l(k_1) \tag{3.70}$$

with

$$\begin{aligned} g_1(k_1) &= \frac{(3k_1^2+2\mu^2)|f(k_1^2)|}{2\sqrt{2}k_1^2(k_1^2+\mu^2)^{\frac{5}{4}}} \\ g_2(k_2) &= \frac{|\partial_{|k_1|} f(k_1^2)|}{\sqrt{2}|k_1|(k_1^2+\mu^2)^{\frac{1}{4}}} \\ g_3(k_2) = g_4(k_2) &= \frac{|f(k_1^2)|}{\sqrt{2}|k_1|(k_1^2+\mu^2)^{\frac{1}{4}}} \end{aligned} \tag{3.71}$$

Note that we have chosen $f \in C_0^\infty(\mathbb{R}^3)$ and hence $\partial_{|k|} f \in C_0^\infty(\mathbb{R}^3)$. This yields again $g_l \in L^2(\mathbb{R}^3)$.

Using the Minkowski inequality and Cauchy-Schwarz, we get for $l=1,2,3$

$$\begin{aligned} \sum_{m=0}^{\infty} I_l^{(m)}(\phi,\xi) &\overset{\text{CS}}{\leq} \sum_{m=0}^{\infty} \left\|\xi^{(m)}(\bullet)\right\|_2 \underbrace{\left\|\int \mathrm{d^3k_1}\phi^{(m+1)}(k_1,\bullet) g_l(k_1)\right\|_2}_{\overset{\text{Minkowski}}{\leq} \int \mathrm{dk_1} \left\|\phi^{(m+1)}(k_1,\bullet)\right\|_2 |g_l(k_1)|} \\ &\overset{\text{CS}}{\leq} \|g_l\|_2 \sum_{m=0}^{\infty} \left\|\xi^{(m)}\right\|_2 \left\|\phi^{(m+1)}\right\|_2 \\ &\overset{\text{CS}}{\leq} \underbrace{\|g_l\|_2}_{<\infty} \underbrace{\Big(\sum_{m=0}^{\infty} \left\|\xi^{(m)}\right\|_2^2\Big)^{\frac{1}{2}}}_{\overset{\xi\in\mathcal{F}}{<}\infty} \underbrace{\Big(\sum_{m=0}^{\infty} \left\|\phi^{(m+1)}\right\|_2^2\Big)_2^{\frac{1}{2}}}_{\overset{\phi\in\mathcal{F}}{<}\infty} \\ &< \infty \end{aligned} \tag{3.72}$$

A similar calculation can be done for $l = 4$:

$$\begin{aligned}
\sum_{m=0}^{\infty} I_4^{(m)}(\phi,\xi) &\overset{CS}{\leq} \sum_{m=0}^{\infty} \left\|\xi^{(m)}(\bullet)\right\|_2 \underbrace{\left\|\int \mathrm{dk}_1 \partial_{|k_1|}\Big(\phi^{(m+1)}(k_1,\bullet)\Big) g_4(k_1)\right\|_2}_{\overset{\text{Minkowski}}{\leq} \int \mathrm{dk}_1 \left\|\partial_{|k_1|}\phi^{(m+1)}(k_1,\bullet)\right\|_2 |g_4(k_1)|} \\
&\overset{\text{CS}}{\leq} \|g_4\|_2 \sum_{m=0}^{\infty} \left\|\xi^{(m)}\right\|_2 \left\|\partial_{|k_1|}\phi^{(m+1)}(k_1,\bullet)\right\|_2 \\
&\overset{\text{CS}}{\leq} \underbrace{\|g_4\|_2}_{<\infty} \underbrace{\Big(\sum_{m=0}^{\infty} \left\|\xi^{(m)}\right\|_2^2\Big)^{\frac{1}{2}}}_{\overset{\xi\in\mathcal{F}}{<}\ \infty} \underbrace{\Big(\sum_{m=0}^{\infty} \left\|\partial_{|k_1|}\phi^{(m+1)}\right\|_2^2\Big)_2^{\frac{1}{2}}}_{\overset{\partial_{|k|}\phi\in\mathcal{F}}{<}\ \infty} \\
&< \infty
\end{aligned} \tag{3.73}$$

Here we have used that the first derivatives are again in the Fock space, this also contains the condition we needed to perform the integration by parts before.

Note that we can substitute the condition on the first derivative of $\phi^{(m)}$ by a condition on $\phi^{(m)}$, if we integrate by parts in the first step. Then, since $\left\|\partial_{|k|} g_4(k)\right\|_2 < \infty$ we just obtain the condition $\phi^{(m)}, \xi^{(n)} \in \mathrm{L}^2(\mathbb{R}^3)$ for all $n, m \in \mathbb{N}$. Nevertheless, in order to perform the integration by parts we need those functions to be continuosly differentiable.

Hence, we have proved the weak convergence up to first order in perturbation theory.

In the following, we want to prove it also for higher orders. This we will do step by step:
We already know:

$$U_I(t,t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} U_I^{(n)}(t,t_0) \tag{3.74}$$

with

$$U_I^{(n)}(t,t_0) = (-i)^n \int_{t_0}^{t} \mathrm{dt_1...dt_n} T\Big[V(t_1)...V(t_n)\Big] \tag{3.75}$$

We will proceed in the following way, we do the $t_1$ integration first, then the $t_2$ integration and so on. Meanwhile, we keep the $V(t_i)$ in the same order as before. This means, we can forget about the time ordering, since in this procedure $t_n \leq ... \leq t_1$ is valid at any step of the calculation. Hence

$U_I^{(n)}(t,t_0) =$

$$= (-i)^n \frac{\lambda^n}{(2\pi)^{\frac{3n}{2}}} \int \mathrm{d^3x_1...d^3x_n} \Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n) \int \mathrm{d^3k_1...d^3k_n} \gamma_{k_1}...\gamma_{k_n} \int_{t_0}^{t} \mathrm{dt_1...dt_n}$$
$$\times \left(a_{k_1} e^{-i\omega_{k_1} t_1} e^{ik_1\cdot x_1} + a_{k_1}^* e^{i\omega_{k_1} t_1} e^{-ik_1\cdot x_1}\right) ... \left(a_{k_n} e^{-i\omega_{k_n} t_n} e^{ik_n\cdot x_n} + a_{k_n}^* e^{i\omega_{k_n} t_n} e^{-ik_n\cdot x_n}\right)$$

$$= \frac{\lambda^n}{(2\pi)^{\frac{3n}{2}}} \int \mathrm{d^3x_1...d^3x_n} \Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n) \int \mathrm{d^3k_1...d^3k_n} \frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_n}}{\omega_{k_n}}$$
$$\times \left[\left(a_{k_1} e^{-i\omega_{k_1} t + ik_1\cdot x_1} - a_{k_1}^* e^{i\omega_{k_1} t - ik_1\cdot x_1}\right) - \left(a_{k_1} e^{-i\omega_{k_1} t_0 + ik_1\cdot x_1} - a_{k_1}^* e^{i\omega_{k_1} t_0 - ik_1\cdot x_1}\right)\right] ...$$
$$\times \left[\left(a_{k_n} e^{-i\omega_{k_n} t + ik_n\cdot x_n} - a_{k_n}^* e^{i\omega_{k_n} t - ik_n\cdot x_n}\right) - \left(a_{k_n} e^{-i\omega_{k_n} t_0 + ik_n\cdot x_n} - a_{k_n}^* e^{i\omega_{k_n} t_0 - ik_n\cdot x_n}\right)\right] \tag{3.76}$$

Using (3.48) in the same way as before, we get

$$\begin{aligned} U_I^{(n)}(t,t_0) = & \frac{\lambda^n}{(2\pi)^{\frac{3n}{2}}} \int \mathrm{d^3x_1...d^3x_n} \Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n) \int \mathrm{d^3k_1...d^3k_n} \frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_n}}{\omega_{k_n}} \\ & \times \Bigg[ \left(a_{k_1} e^{-i\omega_{k_1} t + ik_1\cdot x_1} - a_{k_1}^* e^{i\omega_{k_1} t - ik_1\cdot x_1}\right) \\ & + \frac{i\omega_{k_1}}{|k_1| t_0} \left(a_{k_1} e^{ik_1\cdot x_1} (\partial_{|k_1|} e^{-i\omega_{k_1} t_0}) + a_{k_1}^* e^{-ik_1\cdot x_1} (\partial_{|k_1|} e^{i\omega_{k_1} t_0})\right) \Bigg] ... \\ & \times \Bigg[ \left(a_{k_n} e^{-i\omega_{k_n} t + ik_n\cdot x_n} - a_{k_n}^* e^{i\omega_{k_n} t - ik_n\cdot x_n}\right) \\ & + \frac{i\omega_{k_n}}{|k_n| t_0} \left(a_{k_n} e^{ik_n\cdot x_n} (\partial_{|k_n|} e^{-i\omega_{k_n} t_0}) + a_{k_n}^* e^{-ik_n\cdot x_n} (\partial_{|k_n|} e^{i\omega_{k_n} t_0})\right) \Bigg] \end{aligned} \tag{3.77}$$

As before, we claim that $U_I(t,t_0)$ converges weakly to the desired evolution operator $T_f$, i.e. for any $|\phi\rangle, |\xi\rangle \in \mathcal{F}_{|0\rangle}$

$$\langle\phi| U_I(t,t_0) |\xi\rangle \to \exp\left\{ \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d^3x} \Psi^*(x)\Psi(x) \int \mathrm{d^3k} \frac{\gamma_k}{\omega_k} \left(a_k e^{ik\cdot x - i\omega_k} - a_k^* e^{-ik\cdot x + i\omega_k}\right)\right\} \tag{3.78}$$

as $t_0 \to -\infty$.

Looking at (3.77), this seems to be true if for each n all the terms containing $t_0$ vanish. Unfortunately, this is a necessary condition, but not a sufficient one. We need that the sum over all order n of the correction terms (i.e. the terms containing $t_0$) must be zero. Hence the correction must not increase to fast as n goes to infinity.
At first, let us make the following definition:

**Definition:** Let $C_n$ be all the terms in (3.77) which contain $t_0$, i.e. $C_n$ is the correction term of order n.

If we can show that $| \langle\phi| C_n |\xi\rangle |$ is vanishing for $t_0 \to \infty$ while the sum $\sum_{n=0}^{\infty} \frac{1}{n!} | \langle\phi| C_n |\xi\rangle |$ is finite, then our job is done.
We have already calculated

$$| \langle\phi| C_1 |\xi\rangle | \leq \frac{1}{t_0} \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3 x \psi^*(x)\psi(x) 2 \underbrace{|I(\phi,\xi)|}_{<\infty} \to 0 \text{ as } t_0 \to -\infty \tag{3.79}$$

in the section before.
Intuitively, it seems true that the correction coming from the second order is not bigger than the one coming from the first order squared. In order to proof this, we will estimate the correction term $| \langle\phi| C_n |\xi\rangle |$ ($|\phi\rangle, |\xi\rangle \in \mathcal{F}$ abitrary) from above for each order $n$, keep track of all these corrections and sum over them. Then we just have to show that this sum vanishes and we are finished with the proof.

First of all, note that by our defintion of $| \langle\phi| C_n |\xi\rangle |$ it already contains the $x_1, ..., x_n$ integrations and the prefactor $\frac{\lambda^n}{(2\pi)^{\frac{3n}{2}}}$. In order to shorten our calculations and for notational simplicity, let us make the following definitions:

**Definition:**

$$\frac{\lambda^n}{(2\pi)^{\frac{3n}{2}}} \int \mathrm{d}^3\mathrm{x}_1 ... \mathrm{d}^3\mathrm{x}_\mathrm{n} \Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n) C_n' := C_n \tag{3.80}$$

and

$$\begin{aligned} A_n(\phi,\xi) := \int \mathrm{d}^3\mathrm{k}_1 ... \mathrm{d}^3\mathrm{k}_\mathrm{n} \frac{\gamma_{k_1}}{\omega_{k_1}} ... \frac{\gamma_{k_n}}{\omega_{k_n}} \Bigg| \langle\phi| \Bigg[ & \left( a_{k_1} e^{-i\omega_{k_1} t + i k_1 \cdot x_1} - a_{k_1}^* e^{i\omega_{k_1} t - i k_1 \cdot x_1} \right) \\ & + \frac{i\omega_{k_1}}{|k_1| t_0} \left( a_{k_1} e^{i k_1 \cdot x_1} (\partial_{|k_1|} e^{-i\omega_{k_1} t_0}) + a_{k_1}^* e^{-i k_1 \cdot x_1} (\partial_{|k_1|} e^{i\omega_{k_1} t_0}) \right) \Bigg] ... \\ & \times \Bigg[ \left( a_{k_n} e^{-i\omega_{k_{n+1}} t + i k_n \cdot x_{n+1}} - a_{k_n}^* e^{i\omega_{k_n} t - i k_n \cdot x_n} \right) \\ & + \frac{i\omega_{k_n}}{|k_n| t_0} \left( a_{k_n} e^{i k_n \cdot x_n} (\partial_{|k_n|} e^{-i\omega_{k_n} t_0}) + a_{k_n}^* e^{-i k_n \cdot x_n} (\partial_{|k_n|} e^{i\omega_{k_n} t_0}) \right) \Bigg] \\ & - \Bigg[ \left( a_{k_1} e^{-i\omega_{k_1} t + i k_1 \cdot x_1} - a_{k_1}^* e^{i\omega_{k_1} t - i k_1 \cdot x_1} \right) ... \\ & \times \left( a_{k_n} e^{-i\omega_{k_n} t + i k_n \cdot x_n} - a_{k_n}^* e^{i\omega_{k_n} t - i k_n \cdot x_n} \right) \Bigg] \Bigg] |\xi\rangle \Bigg| \end{aligned} \tag{3.81}$$

Then for any $|\phi\rangle, |\xi\rangle \in \mathcal{F}$ and $n \in \mathbb{N}$, we know by the triangle inequality.

$$| \langle\phi| C_n' |\xi\rangle | \leq A_n(\phi,\xi) \tag{3.82}$$

and since $\Psi$ lives in our Fock space, the integrations over $x$ each give a finite result and hence do not make any problem. Thus, it suffices to estimate $A_n(\phi,\xi))$ from above. In particular, we need to show:

**Theorem 3.2.1** *Let* $|\phi\rangle, |\xi\rangle \in \mathcal{F}$, *then*

$$\sum_{n=1}^{\infty} \frac{1}{n!} |\langle\phi| C_n' |\xi\rangle| \leq \sum_{n=0}^{\infty} \frac{1}{n!} A_n(\phi,\xi) \to 0 \quad as\ t_0 \to -\infty \tag{3.83}$$

**Proof:** For $n=1$ we have already calculated the correction in the section above, i.e.

$$|\langle\phi| C_1' |\xi\rangle| \leq A_1(\phi,\xi) \leq \frac{2}{t_0} |I(\phi,\xi)| \tag{3.84}$$

Next, we estimate $|\langle\phi| C_n' |\xi\rangle|$ from above by looking carefully at all the different terms in it.

$$\begin{aligned} |\langle\phi| C_n' |\xi\rangle| = \Bigg| \int \mathrm{d^3k_1 ... d^3k_n} \frac{\gamma_{k_1}}{\omega_{k_1}} ... \frac{\gamma_{k_n}}{\omega_{k_n}} \langle\phi| \Bigg[ \Big( a_{k_1} e^{-i\omega_{k_1} t + ik_1\cdot x_1} - a^*_{k_1} e^{i\omega_{k_1} t - ik_1 \cdot x_1} \Big) \\ + \frac{i\omega_{k_1}}{|k_1| t_0} \Big( a_{k_1} e^{ik_1\cdot x_1} (\partial_{|k_1|} e^{-i\omega_{k_1} t_0}) + a^*_{k_1} e^{-ik_1\cdot x_1} (\partial_{|k_1|} e^{i\omega_{k_1} t_0}) \Big) \Bigg] ... \\ \times \Bigg[ \Big( a_{k_n} e^{-i\omega_{k_{n+1}} t + ik_n \cdot x_{n+1}} - a^*_{k_n} e^{i\omega_{k_n} t - ik_n\cdot x_n} \Big) \\ + \frac{i\omega_{k_n}}{|k_n| t_0} \Big( a_{k_n} e^{ik_n\cdot x_n} (\partial_{|k_n|} e^{-i\omega_{k_n} t_0}) + a^*_{k_n} e^{-ik_n\cdot x_n} (\partial_{|k_n|} e^{i\omega_{k_n} t_0}) \Big) \Bigg] \\ - \Bigg[ \Big( a_{k_1} e^{-i\omega_{k_1} t + ik_1\cdot x_1} - a^*_{k_1} e^{i\omega_{k_1} t - ik_1\cdot x_1} \Big) ... \\ \times \Big( a_{k_n} e^{-i\omega_{k_n} t + ik_n\cdot x_n} - a^*_{k_n} e^{i\omega_{k_n} t - ik_n \cdot x_n} \Big) \Bigg] \Bigg] |\xi\rangle \Bigg| \end{aligned} \tag{3.85}$$

Basically, there are $(2^n - 1)$ terms in this formula, since for each $m = 1,..,n$ there is one factor containing $t$ and one factor containing $t_0$. The $-1$ comes from the fact that if we do the multiplication, then there is exactly one term where only $t$ and no $t_0$ appears, but this term cancels with the last summand.

Now will try to estimate each of the other terms step by step:

There are $\binom{n}{1}$ terms, for which only one factor contains $t_0$.
At first, let us look at the term, where the factor containing $t_0$ stands on the very left. Then, by inserting the identity $\mathbb{I} = \sum_{i=0}^{\infty} \int \mathrm{d^3p_1 ... d^3p_i} |p_1...p_i\rangle \langle p_1 ... p_i|$, we get the

following estimation from above for this particular term:

$$\Bigg| \sum_{i=0}^{\infty} \int \mathrm{d^3p_1...d^3p_i} \int \mathrm{d^3k_1...d^3k_n} \frac{\gamma_{k_1}}{\omega_{k_1}} ... \frac{\gamma_{k_n}}{\omega_{k_n}}$$

$$\langle\phi| \left[ \frac{i\omega_{k_1}}{|k_1|t_0} \left( a_{k_1} e^{ik_1\cdot x_1} (\partial_{|k_1|} e^{-i\omega_{k_1} t_0}) + a^*_{k_1} e^{-ik_1\cdot x_1} (\partial_{|k_1|} e^{i\omega_{k_1} t_0}) \right) \right] |p_1...p_i\rangle$$

$$\langle p_1...p_i| \left[ \left( a_{k_2} e^{-i\omega_{k_2} t + ik_2\cdot x_2} - a^*_{k_2} e^{i\omega_{k_2} t - ik_2\cdot x_2} \right) ... \left( a_{k_n} e^{-i\omega_{k_n} t + ik_n\cdot x_n} - a^*_{k_n} e^{i\omega_{k_n} t - ik_n\cdot x_n} \right) \right] |\xi\rangle \Bigg|$$

$$\leq \sum_{i=0}^{\infty} \int \mathrm{d^3p_1...d^3p_i} 2\frac{1}{t_0} |I(\phi, p_1...p_i)| \cdot 2^{n-1} \frac{1}{t^{n-1}} |I_t(p_1...p_i, \xi)|^{n-1}$$

$$= \frac{2^n}{t_0 t^{n-1}} \sum_{i=0}^{\infty} \int \mathrm{d^3p_1...d^3p_i} |I(\phi, p_1...p_i)| \cdot |I_t(p_1...p_i, \xi)|^{n-1} \tag{3.86}$$

In the first step, we used nothing else than the definition (3.53) of $I(\cdot,\cdot)$ and $I_t(\cdot,\cdot)$ and again the triangle inequality. Remember that there are $\binom{n}{1}$ terms which all give this contribution.

Proceeding like this, we find that there are $\binom{n}{l}$ terms, for which $l$ factors contain $t_0$ and $n-l$ factors contain $t$. Each giving a contribution, which can be estimated from above by

$$\frac{2^n}{t_0^l t^{n-l}} \sum_{i=0}^{\infty} \int \mathrm{d^3p_1...d^3p_i} |I(\phi, p_1...p_i)|^l \cdot |I_t(p_1...p_i, \xi)|^{n-l} \tag{3.87}$$

If we keep track of all the contributions and sum them up, we obtain

$$\begin{aligned} |\langle\phi| C'_n |\xi\rangle| &\leq 2^n \sum_{l=1}^{n} \binom{n}{l} \frac{1}{|t_0|^l \cdot |t|^{n-l}} \sum_{i=0}^{\infty} \int \mathrm{d^3p_1...d^3p_i} |I(\phi, p_1...p_i)|^l \cdot |I_t(p_1...p_i, \xi)|^{n-l} \\ &= 2^n \sum_{i=0}^{\infty} \int \mathrm{d^3p_1...d^3p_i} \sum_{l=1}^{n} \binom{n}{l} \frac{1}{|t_0|^l \cdot |t|^{n-l}} |I(\phi, p_1...p_i)|^l \cdot |I_t(p_1...p_i, \xi)|^{n-l} \end{aligned} \tag{3.88}$$

using the binomial theorem we obtain

$$\begin{aligned} |\langle\phi| C'_n |\xi\rangle| &\leq 2^n \sum_{i=0}^{\infty} \int \mathrm{d^3p_1...d^3p_i} \left[ \underbrace{\left| \frac{|I(\phi, p_1...p_i)|}{|t_0|} + \frac{|I_t(p_1...p_i, \xi)|}{|t|} \right|^n}_{\leq \frac{|I(\phi,p_1...p_i)|^n}{|t_0|^n} + \frac{|I_t(p_1...p_i,\xi)|^n}{|t|^n}} - \frac{|I_t(p_1...p_i, \xi)|^n}{|t|^n} \right] \\ &\leq 2^n \sum_{i=0}^{\infty} \int \mathrm{d^3p_1...d^3p_i} \frac{|I(\phi, p_1...p_i)|^n}{|t_0|^n} \end{aligned} \tag{3.89}$$

Now, we have calculated the correction terms in each order, it remains to show that the sum over them still vanishes as $t_0 \to -\infty$.

$$\sum_{n=1}^{\infty} \frac{1}{n!} |\langle\phi| C'_n |\xi\rangle| \leq \sum_{n=0}^{\infty} \frac{1}{n!} 2^n \sum_{i=0}^{\infty} \int \mathrm{d}^3\mathrm{p}_1 ... \mathrm{d}^3\mathrm{p}_\mathrm{i} \frac{|I(\phi, p_1...p_i)|^n}{|t_0|^n} \tag{3.90}$$

In the section before, we calculated $|I(\phi, p_1...p_i)| < \infty$, $\forall |\phi\rangle, |p_1...p_i\rangle \in \mathcal{F}$.Futher, since $|\phi\rangle$ and $|p_1...p_i\rangle$ live in the Fock space, also

$$Z := \sum_{i=0}^{\infty} \int \mathrm{d}^3\mathrm{p}_1 ... \mathrm{d}^3\mathrm{p}_\mathrm{i} |I(\phi, p_1...p_i)| < \infty \tag{3.91}$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n!} |\langle\phi| C'_n |\xi\rangle| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \frac{2^n Z^n}{|t_0|^n} \leq \underbrace{\exp\left\{\frac{2 \overbrace{Z}^{<\infty}}{|t_0|}\right\}}_{\to 1} - 1 \to 0 \tag{3.92}$$

as $t_0 \to -\infty$. □

### 3.2.2 The massless case - Infrared problem

Another interesting situation is the case that the meson field has no mass. This is relevant for physical applications, since for example photons are massless. In this case, there may appear further difficulties, because then there is a singularity at $k = 0$, we have to take care of (Infrared problem).

The only point in the calculation above, where such an infrared problem could appear is in (3.72) and (3.73). Looking at (3.71), one can easily see that without a cutoff for small momenta (infrared cutoff), there are two different kinds of integrals, which are divergent in the small momentum regime: $\int_0^\infty \frac{\mathrm{d}|k|}{|k|^3}$ and $\int_0^\infty \frac{\mathrm{d}|k|}{|k|}$. The first integral diverges as $\frac{1}{|k|^2}$ for $|k| \to 0$ and the second one is diverging logarithmical in the same regime. This means in the massless case we have both: ultraviolet and infrared divergences.

As we will discuss later, the infrared divergence are just a representation problem. On the one hand, there are Maxwell equations, where absolutely no probabilities appear and we try to fit this in a second quantized form, i.e. we want to describe everything in a Fock space in order to obtain our usual probabilistic view, which we are used to.

We have to pay a price to combine these two pictures. Sending $t_0 \to -\infty$, our fields have "enough" time for a big change. For example the vacuum state is not in the original Fock space anymore after performing this limit. This happens, since the desire that everything has to be square integrable, which comes from a probabilistic view, is not implemented in the Maxwell equations. Maxwell equations allow more solutions than that. As mentioned before, there is a way out of this, since the only problem is really that our solution space is not big enough, hence we have to change our Fock space

if it is appropriate. We can do this by a unitary transformation.
However, this is just a short outlook and can not be fully understood at this point. Hence, we will discuss it more detailed in the end of section3.2.3 and finally completely in section 3.6.

### 3.2.3 Weak convergence without a cutoff - Ultraviolet problem

In this section, we will examine what happen if there is no cutoff, i.e. we examine the limit $f \to 1$. Of course, this limit is very problematic, since without a cutoff the Hamiltonian itself is not a well-defined object anymore. Still, we can try to at least make sense out of the matrix elements by implementing further conditions on the elements in the Fock space $|\phi\rangle \in \mathcal{F}$.
If we want the weak convergence still to hold, we still need the integrals to be finite, i.e. we need

$$\sum_{m=0}^{\infty} I_l^{(m)}(\phi, \xi) < \infty \tag{3.93}$$

In principle, we can proceed the same way as before, the only problem is that the cutoff guaranteed $g_l \in \mathrm{L}^2(\mathbb{R}^3)$ for $i = 1, 2, 3, 4$. In this scenario, this is not true anymore. Hence, we have to impose additional conditions on the sates $|\phi\rangle$ and $|\xi\rangle$. We do the calculations (3.72) and (3.73) again for each $l = 1, 2, 3, 4$ and in the limit $f \to 1$ in order to find such conditions.
For $l = 1$, one can easily check that for a small $\epsilon > 0$

$$g_1(k)\frac{1}{(k^2+\mu^2)^{\frac{1}{4}+\epsilon}} = \frac{3k^2+2\mu^2}{2\sqrt{2}k^2(k^2+\mu^2)^{\frac{5}{4}}}\frac{1}{(k^2+\mu^2)^{\frac{1}{4}+\epsilon}} = \frac{3k^2+2\mu^2}{2\sqrt{2}k^2(k^2+\mu^2)^{\frac{3}{2}+\epsilon}} \tag{3.94}$$

and hence

$$g_1(k)\frac{1}{(k^2+\mu^2)^{\frac{1}{4}+\epsilon}} \in \mathrm{L}^2(\mathbb{R}^3) \tag{3.95}$$

Then, we can repeat the calculation (3.72) with the following small modification

$$\begin{aligned}
\sum_{m=0}^{\infty} I_1^{(m)}(\phi,\xi) &\overset{\text{CS}}{\leq} \sum_{m=0}^{\infty} \left\|\xi^{(m)}(\bullet)\right\|_2 \underbrace{\left\|\int \mathrm{dk}_1 \phi^{(m+1)}(k_1,\bullet)g_1(k_1)\right\|_2}_{\overset{\text{Minkowski}}{\leq} \int \mathrm{dk}_1 \left\|\phi^{(m+1)}(k_1,\bullet)(k_1^2+\mu^2)^{\frac{1}{4}+\epsilon}\right\|_2 |g_1(k_1)(k_1^2+\mu^2)^{-\frac{1}{4}-\epsilon}|} \\
&\overset{\text{CS}}{\leq} \left\|g_1(\bullet)(\bullet^2+\mu^2)^{-\frac{1}{4}-\epsilon}\right\|_2 \sum_{m=0}^{\infty}\left\|\xi^{(m)}\right\|_2 \left\|\phi^{(m+1)}(\bullet,\ldots)(\bullet^2+\mu^2)^{\frac{1}{4}+\epsilon}\right\|_2 \\
&\overset{\text{CS}}{\leq} \underbrace{\left\|g_1(\bullet)(\bullet^2+\mu^2)^{-\frac{1}{4}-\epsilon}\right\|_2}_{<\infty} \underbrace{\Big(\sum_{m=0}^{\infty}\left\|\xi^{(m)}\right\|_2^2\Big)^{\frac{1}{2}}}_{\overset{\xi\in\mathcal{F}}{<}\infty} \underbrace{\Big(\sum_{m=0}^{\infty}\left\|\phi^{(m+1)}(\bullet,\ldots)(\bullet^2+\mu^2)^{\frac{1}{4}+\epsilon}\right\|_2^2\Big)^{\frac{1}{2}}}_{\overset{!}{<}\infty} \\
&< \infty
\end{aligned} \tag{3.96}$$

So basically, we need the following: For any $m \in \mathbb{N}$

$$k \to \phi^{(m+1)}(k, K) \cdot k^{\frac{1}{2}+\epsilon} \in \mathrm{L}^2(\mathbb{R}^3) \tag{3.97}$$

for all $K \in \mathbb{R}^{3m}$ and without singularities.

For $l = 2$, the integrals are obviously zero, since $f(k^2) = 1$ and hence $\partial_{|k|} f(k^2) = 0$.

For $l = 3$, we can do exactly the same calculation as before and obtain the condition: For any $m \in \mathbb{N}$

$$k \to \phi^{(m+1)}(k, K) \cdot k^{\frac{3}{2}+\epsilon} \in \mathrm{L}^2(\mathbb{R}^3) \tag{3.98}$$

for all $K \in \mathbb{R}^{3m}$ and without singularities.
Note that this condition implies the first one.

For $l = 4$, there are two possibilities, either we again do the calculation straight forward, which yields, again the second condition, but for the derivative or we integrate by parts and end up with exactly the first condition again. This one is included in the second one, once again.

One could argue that the whole model is not even well-defined without a cutoff function, however despite this fact, we still consider this case, since we actually can make sense out of matrix element of for example the dressing operator. Albeit, this only true for a few test functions and we have just specified the conditions the have to fulfill.

The origin of these ultraviolet divergences is the self-interaction of the charged particle with its own field, since the field is evaluated at the origin in the Maxwell equations, but as mentioned in the introduction the Maxwell equations are divergent at this point. This ultraviolet divergence already appears in classical calculations, whenever one assumes a point particle. When QED was developed, there was the hope that this ultraviolet divergence vanishes, since in quantum there position of the particles smear out like $|\psi|^2$ and therefore the concept of point particles gets a little bit weaker. Unfortunately, it turned out that there is still a ultraviolet divergence in QED, but it is "only" a logarithmical divergence. Some people would say this is a good achievement. The probabilistic view, which comes from quantum mechanics, makes the terms a little bit "less divergent", but in the end they are still divergent. Nevertheless, the ultraviolet divergence is not a particular problem of QED, but a problem which arises from the concept of point particles.

However, the additional conditions on our states in the Hilbert space are not the only problem which occurs if we have no cutoff. Another problem is, thatthat considering $f(k^2) = 1 \; \forall k \in \mathbb{R}^3$, the dressed vacuum state is not in our Fock space $\mathcal{F}_{|0\rangle}$. At first this sounds alarming, but luckily we are still able to make sense out of this. In the following we will try to understand where this problem comes from and how we can deal with it.

At first, let us remember what our Fock space looks like:

$$\mathcal{F}_{|0\rangle} = \overline{\sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int \mathrm{dk_1...dk_n} \phi^{(n)}(k_1, ..., k_n) a^*_{k_1} ... a^*_{k_n} |0\rangle} \tag{3.99}$$

with $\phi^{(n)} \in \mathrm{L}^2(\mathbb{R}^{3n})$ $\forall n \in \mathbb{N}$.
The dressed vacuum state is

$$|\Psi^{gs}\rangle = D_f |0\rangle = \exp\left\{ \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{dk} \frac{\gamma_k}{\omega_k} (a_k e^{ik\cdot x} - a^*_k e^{-ik\cdot x})) \right\} |0\rangle \tag{3.100}$$

Unfortunately, if we do not have a cutoff

$$\frac{\gamma_k}{\omega_k} = \frac{1}{\sqrt{2\omega_k^3}} = \frac{1}{\sqrt{2}(k^2 + \mu^2)^{\frac{3}{4}}} \notin \mathrm{L}^2(\mathbb{R}^{3n}) \tag{3.101}$$

and therefore

$$|\Psi^{gs}\rangle = D_f |0\rangle \notin \mathcal{F}_{|0\rangle} \tag{3.102}$$

But we could still construct a "new" Fock space

$$\mathcal{F}_{|\Psi^{gs}\rangle} = \overline{\sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int \mathrm{dk_1...dk_n} \phi^{(n)}(k_1, ..., k_n) b^*_{k_1} ... b^*_{k_n} |\Psi^{gs}\rangle} \tag{3.103}$$

with $\phi^{(n)} \in \mathrm{L}^2(\mathbb{R}^{3n})$ $\forall n \in \mathbb{N}$.
$b^*_k$ is then the appropriate "new" creation operator defined by the following Bogoliubov transformation

$$b_k = D^*_f a_k D_f \tag{3.104}$$

Since, $D_f$ is a unitary operator, the two Fock spaces $\mathcal{F}_{|0\rangle}$ and $\mathcal{F}_{|\Psi^{gs}\rangle}$ are unitarily equivalent by the transformation above. Clearly, $b_k$ annihilates the dressed vacuum state $|\Psi gs\rangle$, i.e. $b_k |\Psi^{gs}\rangle = 0$. Further, by the commutation relations it is easy to see that

$$a_k |\Psi^{gs}\rangle = a_k D_f |0\rangle = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} D_f |0\rangle = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} |\Psi^{gs}\rangle \tag{3.105}$$

Hence

$$b_k = a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} \tag{3.106}$$

Note that this equation cannot be understood as an operator identity, but only as an analytical identification.

It remains the question where does this problem come from? We are describing electromagnetic fields, which obey the Maxwell equations and we want to press them into a quantum mechanical description, i.e. we want to describe them in a probabilistic theory. Obviously, this probabilities need to be square integrable, i.e. in $L^2$. In Maxwells theory the are no probabilities. This is the origin of the infrared divergence.

As discussed before, the ultraviolet divergence occurs because of the fact the we consider point particles. Unfortunately, the ultraviolet divergence cannot be explained by just a bad representation. It is really inherited from classical electrodynamics. We always describe point charges and their self-interaction. In order to do this we have to evaluate the field along the trajectory of the charge, i.e. along the spacetime point, where the field is divergent.

In the end, the infrared problem is just a representational problem and can be resolved by choosing the right Fock space. Whereas, the ultraviolet problem is a problem inherited from classical electro dynamics and we kind of have to live with it.

The dressed vacuum state does not fulfill the conditions coming from this probabilistic description, but still if we describe everything in the "new" Fock space $\mathcal{F}_{|\Psi^{gs}\rangle}$, then all the excitations will fulfill them.

### 3.2.4 Weak convergence

This section is a summarization of the previous result and we write them down in a mathematical way.

**Definition:** Let $\kappa, \Lambda \in \mathbb{R}_0^+$ with $\Lambda > \kappa$, then we call $f_\kappa^\Lambda \in C_0^\infty(\mathbb{R}^3)$ with

$$supp(f_\kappa^\Lambda) = \mathcal{B}_\Lambda(0) \setminus \mathcal{B}_\kappa(0) \tag{3.107}$$

a cutoff function.
Note that by $\kappa = 0$, we mean $supp(f^\Lambda) = \mathcal{B}_\Lambda(0)$.

Then we can sum up the results we proven so far in the following theorem

**Theorem 3.2.2** *Let $f_\kappa^\Lambda \in C_0^\infty(\mathbb{R}^3)$ ($\kappa = 0$ is allowed) be a cutoff function and $|\xi\rangle, |\phi\rangle \in \mathcal{H}$ states in the Hilbert space, which has been defined before and the meson amplitudes $\phi^{(n)}(k_1, ..., k_n)$, $\xi^{(n)}(k_1, ..., k_n)$ be continuously differentiable functions in every argument. Further, let the mass of the meson field be non zero, i.e. $\mu \neq 0$. Then*

$$\langle\xi| U_I(t, t_0) |\phi\rangle \to \langle\xi| T_{f_\kappa^\Lambda} |\phi\rangle \quad as \quad t_0 \to -\infty \tag{3.108}$$

*where*

$$T_{f_\kappa^\Lambda} = \exp\left\{ \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{\omega_k} \left( a_k e^{ik\cdot x} - a_k^* e^{-ik\cdot x} \right) \right\} \tag{3.109}$$

*with $\gamma_k = \frac{f_\kappa^\Lambda}{\sqrt{2\omega_k}}$. Hence the evolution operator in the interaction picture converges weakly to operator $T_{f_\kappa^\Lambda}$.*

**Proof:** We have already proved the theorem in the previous sections. □

**Remark:** In the limit of a massless meson, i.e. $\mu \to 0$, in addition to the ultraviolet divergences there appear also infrared divergences. Regardless, we can always choose a cutoff function $f_\kappa^\Lambda$ with $\kappa > 0$, which causes the infrared divergences to vanish. This leads to the following corollary.

**Corollary 3.2.3** *Let $f_\kappa^\Lambda \in C_0^\infty(\mathbb{R}^3)$ be a cutoff function with $\kappa > 0$ and $|\xi\rangle, |\phi\rangle \in \mathcal{H}$ states in the Hilbert space, which we have defined before and the meson amplitudes $\phi^{(n)}(k_1, ..., k_n)$, $\xi^{(n)}(k_1, ..., k_n)$ be continuously differentiable functions in every argument. Further, let the meson field be massless, i.e. $\mu = 0$. Then*

$$\langle\xi| U_I(t, t_0) |\phi\rangle \to \langle\xi| T_{f_\kappa^\Lambda} |\phi\rangle \quad as \quad t_0 \to -\infty \tag{3.110}$$

*where*

$$T_{f_\kappa^\Lambda} = \exp\left\{ \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{\omega_k} \left( a_k e^{ik\cdot x} - a_k^* e^{-ik\cdot x} \right) \right\} \tag{3.111}$$

*with $\gamma_k = \frac{f_\kappa^\Lambda}{\sqrt{2\omega_k}}$. Hence the evolution operator in the interaction picture converges weakly to operator $T_{f_\kappa^\Lambda}$.*

**Proof:** This corollary is a direct consequence of the theorem above. □

We can make a similar statement without a cutoff, but we have this impose extra condition on our states in the Hilbert space then.

**Theorem 3.2.4** *Let $|\xi\rangle, |\phi\rangle \in \mathcal{H}$, i.e.*
*$|\xi\rangle = \left|\xi^{(nucleon)}\right\rangle \otimes \left|\xi^{(meson)}\right\rangle$ and $|\phi\rangle = \left|\phi^{(nucleon)}\right\rangle \otimes \left|\phi^{(meson)}\right\rangle$.*
*Further, let the meson states $\left|\xi^{(meson)}\right\rangle, \left|\phi^{(meson)}\right\rangle \in \mathcal{F}_{|0\rangle}$ be normalized product states in the meson Fock space with amplitudes*

$$\xi^{(n)}(k_1, ..., k_n) = \frac{1}{\sqrt{n!}} \prod_{i=1}^{n} h_{|\xi\rangle}(k_i) \quad \forall n \in \mathbb{N} \tag{3.112}$$

$$\phi^{(n)}(k_1, ..., k_n) = \frac{1}{\sqrt{n!}} \prod_{i=1}^{n} h_{|\phi\rangle}(k_i) \quad \forall n \in \mathbb{N} \tag{3.113}$$

*with $h_{|\xi\rangle}, h_{|\phi\rangle}, \nabla_k h_{|\xi\rangle}, \nabla_k h_{|\phi\rangle} \in L^2(\mathbb{R}^3)$ and suppose we can find constants $K > 0$ and $\epsilon > 0$, such that*

$$|h_i(k)| \le |k|^{-3-\epsilon} \quad \forall k \in \{k \in \mathbb{R}^3 : |k| \ge K\} \tag{3.114}$$

*for $i = |\xi\rangle, |\phi\rangle$. Then*

$$\langle\xi| U_I(t, t_0) |\phi\rangle \to \langle\xi| T |\phi\rangle \quad as \quad t_0 \to -\infty \tag{3.115}$$

*where*

$$T = \exp\left\{\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x)\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k}{\omega_k}\left(a_k e^{ik\cdot x} - a_k^* e^{-ik\cdot x}\right)\right\} \tag{3.116}$$

*with* $\gamma_k = \frac{1}{\sqrt{2\omega_k}}$. *Hence the evolution operator in the interaction picture converges weakly to operator* $T$.

**Proof:** The proof was already done in the previous sections. □

**Remark:** One may wonder, why weak convergence in the theorems above holds for all elements in the Hilbert space $\mathcal{H}$ (with maybe some additional conditions), since the model is just defined on the domain of $H$, which is a subspace of the Hilbert space, i.e. $\mathcal{D}(H) \subset \mathcal{H}$. At first, this sounds not right. However, Stone's theorem guarantees the existence time evolution $U(t, t_0)$ generated by the self-adjoint Hamiltonian $H$. Hence even though the model is not defined on the whole Hilbert space, we can still make sense out of the matrix element of the time evolution.

In the following, for notational simplicity, we will suppress the constants $\Lambda, \kappa$ whenever we write down a cutoff function. We just write $f$ instead of $f_\kappa^\Lambda$, but we know what is meant by this notation. Note that also in the beginning of this thesis we have used this simplified notation, but we have always used it in the sense of this section.

## 3.3 Results of this Toy model

This chapter should present the solution of the toy model and relate it to the results found by S. Schweber in [2].
So far, we have seen that in the weak limit the evolution operator is given by

$$U_I(t, -\infty) = \exp\left\{\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x)\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k}{\omega_k}\left(a_k e^{ik\cdot x - i\omega_k t} - a_k^* e^{-ik\cdot x + i\omega_k t}\right)\right\} \tag{3.117}$$

and for $t = 0$ this yields the operator

$$T_f := U_I(0, -\infty) = \exp\left\{\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x)\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k}{\omega_k}(a_k e^{ik\cdot x} - a_k^* e^{-ik\cdot x}))\right\} \tag{3.118}$$

As we have discussed before, this operator defines the dressed vacuum state of the model:

$$|\Psi^{gs}\rangle = T_f\,|0\rangle \tag{3.119}$$

which is not in our previous Fock space $\mathcal{F}_{|0\rangle}$ anymore, but we can define a "new" Fock space $\mathcal{F}_{|\Psi^{gs}\rangle}$ with "new" creation and annihilation operators

$$b_k := T_f^* a_k T_f \tag{3.120}$$

which are unitary equivalent. Then, in this picture also our Hamiltonian is transformed by the dressing operator. The "new" Hamiltonian, which is acting on our "new" Fock space $\mathcal{F}_{|\Psi^{gs}\rangle}$ can be obtained by applying (3.120) to (3.26) and to (3.27)

$$H' := T_f^* H T_f = T_f^* H_0 T_f + T_f^* V T_f = H_0' + V' \tag{3.121}$$

Remembering (3.37), we similarly get

$$\left[T_f, \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x)\right] = 0 \tag{3.122}$$

Applying this and using the unitarity of $T_f$, we get

$$\begin{aligned} H_0' &= T_f^* H_0 T_f \\ &= T_f^* \left( m_0 \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x) + \int \mathrm{d}^3\mathrm{k}\omega_k a_k^* a_k \right) T_f \\ &= m_0 \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x) + \int \mathrm{d}^3\mathrm{k}\omega_k T_f^* a_k^* T_f T_f^* a_k T_f \end{aligned} \tag{3.123}$$

We just have used (3.122) and the fact that $T_f$ is unitary, i.e. $T_f T_f^* = T_f^* T_f = \mathbb{I}$. Using the definition (3.120) we get the final result:

$$H_0' = m_0 \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x) + \int \mathrm{d}^3\mathrm{k}\omega_k b_k^* b_k \tag{3.124}$$

Similarly, (also by (3.122)) we obtain

$$\begin{aligned} V' &= T_f^* V T_f \\ &= T_f^* \left( \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k}\gamma_k (a_k e^{ik\cdot x} + a_k^* e^{-ik\cdot x}) \right) T_f \\ &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k}\gamma_k \left( T_f^* a_k T_f e^{ik\cdot x} + T_f^* a_k^* T_f e^{-ik\cdot x} \right) \\ &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k}\gamma_k \left( b_k e^{ik\cdot x} + b_k^* e^{-ik\cdot x} \right) \end{aligned} \tag{3.125}$$

We have to calculate $b_k$:

$$\begin{aligned} b_k &= T_f^* a_k T_f \\ &= T_f^* \sum_{n=0}^{\infty} \frac{\lambda^n}{(2\pi)^{\frac{3n}{2}}} \int \mathrm{d}^3\mathrm{x}_1 ... \mathrm{d}^3\mathrm{x}_\mathrm{n} \Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n) \int \mathrm{d}^3\mathrm{k}_1 ... \mathrm{d}^3\mathrm{k}_\mathrm{n} \frac{\gamma_{k_1}}{\omega_{k_1}} ... \frac{\gamma_{k_n}}{\omega_{k_n}} \\ &\cdot a_k \left( a_{k_1} e^{ik_1\cdot x_1} - a_{k_1}^* e^{-ik_1\cdot x_1} \right) ... \left( a_{k_n} e^{ik_n\cdot x_n} - a_{k_n}^* e^{-ik_n\cdot x_n} \right) \end{aligned} \tag{3.126}$$

From the commutation relations we know

$$\begin{aligned} a_k a_{k_i} &= a_{k_i} a_k \\ a_k a_{k_i}^* &= \left[ a_k, a_{k_i}^* \right] + a_{k_i}^* a_k = \delta^{(3)}(k - k_i) + a_{k_i}^* a_k \end{aligned} \tag{3.127}$$

and hence

$$a_k\left(a_{k_i}e^{ik_i\cdot x_i}-a^*_{k_i}e^{-ik_i\cdot x_i}\right)=\left(a_{k_i}e^{ik_i\cdot x_i}-a^*_{k_i}e^{-ik_i\cdot x_i}\right)a_k-e^{-ik_i\cdot x_i}\delta^{(3)}(k-k_i) \tag{3.128}$$

for $i=1,...,n$.

Apply (3.128) and bring $a_k$ all the way to the right step by step, then

$$\begin{aligned}
a_kT_f &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!(2\pi)^{\frac{3n}{2}}}\int \mathrm{d^3x_1...d^3x_n}\Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n)\int \mathrm{d^3k_1...d^3k_n}\frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_n}}{\omega_{k_n}}\\
&\cdot a_k\left(a_{k_1}e^{ik_1\cdot x_1}-a^*_{k_1}e^{-ik_1\cdot x_1}\right)...\left(a_{k_n}e^{ik_n\cdot x_n}-a^*_{k_n}e^{-ik_n\cdot x_n}\right)\\
&=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!(2\pi)^{\frac{3n}{2}}}\int \mathrm{d^3x_1...d^3x_n}\Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n)\int \mathrm{d^3k_1...d^3k_n}\frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_n}}{\omega_{k_n}}\\
&\cdot\left[\left(a_{k_1}e^{ik_1\cdot x_1}-a^*_{k_1}e^{-ik_1\cdot x_1}\right)a_k-e^{-ik_1\cdot x_1}\delta^{(3)}(k-k_1)\right]...\left(a_{k_n}e^{ik_n\cdot x_n}-a^*_{k_n}e^{-ik_n\cdot x_n}\right)\\
&=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!(2\pi)^{\frac{3n}{2}}}\Bigg[\int \mathrm{d^3x_1...d^3x_n}\Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n)\int \mathrm{d^3k_1...d^3k_n}\frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_n}}{\omega_{k_n}}\\
&\cdot\left(a_{k_1}e^{ik_1\cdot x_1}-a^*_{k_1}e^{-ik_1\cdot x_1}\right)a_k...\left(a_{k_n}e^{ik_n\cdot x_n}-a^*_{k_n}e^{-ik_n\cdot x_n}\right)\\
&-\int \mathrm{d^3x_1...d^3x_n}\Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n)e^{-ik\cdot x_1}\frac{\gamma_k}{\omega_k}\int \mathrm{d^3k_2...d^3k_n}\frac{\gamma_{k_2}}{\omega_{k_2}}...\frac{\gamma_{k_n}}{\omega_{k_n}}\\
&\cdot\left(a_{k_2}e^{ik_2\cdot x_2}-a^*_{k_2}e^{-ik_2\cdot x_2}\right)...\left(a_{k_n}e^{ik_n\cdot x_n}-a^*_{k_n}e^{-ik_n\cdot x_n}\right)\Bigg]
\end{aligned} \tag{3.129}$$

Doing this step n times using $[\Psi^*(x_i)\Psi(x_i), \Psi^*(x_j)\Psi(x_j)] = 0$ and relabeling smart, one obtains the following result:

$$
\begin{aligned}
a_k T_f = &\sum_{n=0}^{\infty} \frac{\lambda^n}{n!(2\pi)^{\frac{3n}{2}}} \Bigg[ \int \mathrm{d^3x_1...d^3x_n} \Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n) \int \mathrm{d^3k_1...d^3k_n} \frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_n}}{\omega_{k_n}} \\
&\cdot \left(a_{k_1} e^{ik_1\cdot x_1} - a^*_{k_1} e^{-ik_1\cdot x_1}\right) a_k... \left(a_{k_n} e^{ik_n\cdot x_n} - a^*_{k_n} e^{-ik_n\cdot x_n}\right) \\
&- \int \mathrm{d^3x_1...d^3x_n} \Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n) e^{-ik\cdot x_1} \frac{\gamma_k}{\omega_k} \int \mathrm{d^3k_2...d^3k_n} \frac{\gamma_{k_2}}{\omega_{k_2}}...\frac{\gamma_{k_n}}{\omega_{k_n}} \\
&\cdot \left(a_{k_2} e^{ik_2\cdot x_2} - a^*_{k_2} e^{-ik_2\cdot x_2}\right) ... \left(a_{k_n} e^{ik_n\cdot x_n} - a^*_{k_n} e^{-ik_n\cdot x_n}\right) \Bigg] \\
\\
&= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!(2\pi)^{\frac{3n}{2}}} \int \mathrm{d^3x_1...d^3x_n} \Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n) \int \mathrm{d^3k_1...d^3k_n} \frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_n}}{\omega_{k_n}} \\
&\cdot \left(a_{k_1} e^{ik_1\cdot x_1} - a^*_{k_1} e^{-ik_1\cdot x_1}\right) ... \left(a_{k_n} e^{ik_n\cdot x_n} - a^*_{k_n} e^{-ik_n\cdot x_n}\right) a_k \\
\\
&- \sum_{n=0}^{\infty} \frac{\lambda^n}{n!(2\pi)^{\frac{3n}{2}}} \left(\frac{\gamma_k}{\omega_k}\right) \int \mathrm{d^3x_1...d^3x_n} \Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n) \\
&\cdot \sum_{i=1}^{n} e^{-ik\cdot x_i} \int \mathrm{d^3k_1...d^3k_{i-1}d^3k_{i+1}...d^3k_n} \frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_{i-1}}}{\omega_{k_{i-1}}}\frac{\gamma_{k_{i+1}}}{\omega_{k_{i+1}}}...\frac{\gamma_{k_n}}{\omega_{k_n}} \\
&\cdot \left(a_{k_1} e^{ik_1\cdot x_1} - a^*_{k_1} e^{-ik_1\cdot x_1}\right) ... \left(a_{k_{i-1}} e^{ik_{i-1}\cdot x_{i-1}} - a^*_{k_{i-1}} e^{-ik_{i-1}\cdot x_{i-1}}\right) \\
&\cdot \left(a_{k_{i+1}} e^{ik_{i+1}\cdot x_{i+1}} - a^*_{k_{i+1}} e^{-ik_{i+1}\cdot x_{i+1}}\right) ... \left(a_{k_n} e^{ik_n\cdot x_n} - a^*_{k_n} e^{-ik_n\cdot x_n}\right) \\
\\
&= T_f a_k \\
&- \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} \int \mathrm{d^3x} \Psi^*(x)\Psi(x) e^{-ik\cdot x} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!(2\pi)^{\frac{3(n-1)}{2}}} \\
&\cdot \int \mathrm{d^3x_1...d^3x_{n-1}} \Psi^*(x_1)\Psi(x_1)...\Psi^*(x_{n-1})\Psi(x_{n-1}) \int \mathrm{d^3k_1...d^3k_{n-1}} \frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_{n-1}}}{\omega_{k_{n-1}}} \\
&\cdot \left(a_{k_1} e^{ik_1\cdot x_1} - a^*_{k_1} e^{-ik_1\cdot x_1}\right) ... \left(a_{k_{n-1}} e^{ik_{n-1}\cdot x_{n-1}} - a^*_{k_{n-1}} e^{-ik_{n-1}\cdot x_{n-1}}\right) \\
\\
&= T_f a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} \int \mathrm{d^3x} \Psi^*(x)\Psi(x) e^{-ik\cdot x}\ T_f \\
\\
&= T_f \left[ a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} \int \mathrm{d^3x} \Psi^*(x)\Psi(x) e^{-ik\cdot x} \right]
\end{aligned}
\tag{3.130}
$$

In the last step we again made use of (3.122).
Then

$$b_k = T_f^* a_k T_f = a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) e^{-ik\cdot x} \tag{3.131}$$

or

$$a_k = b_k + \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) e^{-ik\cdot x} \tag{3.132}$$

Finally, we are able to calculate the Hamiltonian (by substituting (3.131) in (3.125))

$$\begin{aligned}
V' &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k} \gamma_k \left( b_k e^{ik\cdot x} + c.c. \right) \\
&= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k} \gamma_k \left[ \left( a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} \int \mathrm{d}^3 y \Psi^*(y)\Psi(y) e^{-ik\cdot y} \right) e^{ik\cdot x} + c.c. \right] \\
&= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k} \gamma_k \left( a_k e^{ik\cdot x} + a_k^* e^{-ik\cdot x} \right) \\
&\quad - \frac{\lambda^2}{(2\pi)^3} \int \mathrm{d}^3\mathrm{x} \mathrm{d}^3 y \Psi^*(x)\Psi(x)\Psi^*(y)\Psi(y) \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k^2}{\omega_k} \left( e^{ik\cdot x} e^{-ik\cdot y} + e^{-ik\cdot x} e^{ik\cdot y} \right) \\
&= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k} \gamma_k \left( a_k e^{ik\cdot x} + a_k^* e^{-ik\cdot x} \right) \\
&\quad - \frac{\lambda^2}{(2\pi)^3} \int \mathrm{d}^3\mathrm{x} \mathrm{d}^3 y \Psi^*(x)\Psi(x)\Psi^*(y)\Psi(y) \int \mathrm{d}^3\mathrm{k} \frac{(f(k^2))^2}{\omega_k^2} e^{ik\cdot(x-y)}
\end{aligned} \tag{3.133}$$

Again the last step follows by $[\Psi^*(x)\Psi(x), \Psi^*(y)\Psi(y)] = 0$ and symmetry in $x, y$.

Similarly ,we obtain the free Hamiltonian by inserting (3.131) in (3.124)

$$H_0' = m_0 \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x) + \int \mathrm{d}^3\mathrm{k}\omega_k b_k^* b_k$$

$$= m_0 \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x)$$
$$+ \int \mathrm{d}^3\mathrm{k}\omega_k \left[a_k^* - \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\frac{\gamma_k}{\omega_k}\int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x)e^{ik\cdot x}\right]\left[a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\frac{\gamma_k}{\omega_k}\int \mathrm{d}^3\mathrm{y}\Psi^*(y)\Psi(y)e^{-ik\cdot y}\right]$$

$$= m_0 \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x) + \int \mathrm{d}^3\mathrm{k}\omega_k a_k^* a_k$$
$$- \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x)\int \mathrm{d}^3\mathrm{k}\gamma_k\left(a_k e^{ik\cdot x} + a_k^* e^{-ik\cdot x}\right)$$
$$+ \frac{\lambda^2}{(2\pi)^3}\int \mathrm{d}^3\mathrm{x}\mathrm{d}^3\mathrm{y}\Psi^*(x)\Psi(x)\Psi^*(y)\Psi(y)\int \mathrm{d}^3\mathrm{k}\frac{(f(k^2))^2}{2\omega_k^2}e^{ik\cdot x}e^{-ik\cdot y} \tag{3.134}$$

Hence

$$H' = H_0' + V'$$
$$H_0' = m_0 \int \mathrm{d}^3\mathrm{x}\Psi^*(x)\Psi(x) + \int \mathrm{d}^3\mathrm{k}\omega_k a_k^* a_k$$
$$V' = -\int \mathrm{d}^3\mathrm{x}\mathrm{d}^3\mathrm{y}\Psi^*(x)\Psi(x)V(x-y)\Psi^*(y)\Psi(y) \tag{3.135}$$

with

$$V(x-y) := \frac{\lambda^2}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{(f(k^2))^2}{2\omega_k^2}e^{ik\cdot x}e^{-ik\cdot y} \tag{3.136}$$

If we now consider the limit $f(k^2) \to 1$, i.e. the limit for point charges, we end up with the well-known Yukawa potential:

$$V(x-y) = \frac{\lambda^2}{2}\int \frac{\mathrm{d}^3\mathrm{k}}{(2\pi)^3}\frac{e^{ik\cdot(x-y)}}{k^2+\mu^2} = -\frac{\lambda^2}{8\pi}\frac{e^{-\mu|x-y|}}{|x-y|} \tag{3.137}$$

This means that the interaction is a Yukawa interaction and for $\mu \to 0$ it becomes just a Coulomb potential. This is exactly the result we would have expected, since in this model we described the interaction between a meson and a (charged) nucleon. The limit $\mu \to 0$ makes sense if for example we want to describe the interaction between photons and charged nucleons, since the photons are described by the meson field, but they have no mass. i.e. $\mu = 0$.

Clearly, $T_f = U_I(0, -\infty)$ diagonalizes the Hamiltonian. There is a physical interpretation for this: If we start with one "naked" charged particle and no mesons, i.e. the meson vacuum and evolve in time (for an infinite long period), then the charge gets dressed with its appropriate Coulomb field. One calls this relaxation to the ground state.

## 3.4 What happens to the initial data

In the introduction, we have seen that in the classical case our system forgets about the initial data (if we send the initial time point infinitely to the past) and only remembers the retarded field, namely the Liénard-Wiechert field. We would expect a similar phenomena in the quantized case.

We will do this for the special case, where we have just one nucleon at a fixed spatial point. As we have discussed before, the Hamiltonian looks like the following then:

$$H = \underbrace{m_0 + \int \mathrm{d}^3\mathrm{k}\,\omega_k a_k^* a_k}_{=H_0} + \underbrace{\frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\,\gamma_k \left(a_k e^{ik\cdot x} + a_k^* e^{-ik\cdot x}\right)}_{=\lambda\varphi_0(x)} \tag{3.138}$$

Mathematically, we define this operator exactly in way as we did before. The only difference is that we consider only one fermion and therefore we remove the fermion creation and annihilation operators. Hence also the Hilbert space changes, i.e. there is only the meson Fock space left: $\mathcal{H} := \mathcal{F}_{(meson)} = \mathcal{F}$.
The domain of $H$ is then $\mathcal{D}(H) := \mathcal{D}(n)$. Everything else is defined as before.
In the interaction picture, we know that

$$\varphi_t(x) = e^{iH_0t}\varphi_0(x)e^{-iH_0t} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\,\gamma_k \left(a_k e^{ik\cdot x - i\omega_k t} + a_k^* e^{-ik\cdot x + i\omega_k t}\right) \tag{3.139}$$

So in the end we would like to show that the expectation value of $\varphi_t(x)$ again looks like the Yukawa potential.

The states in our Fock space $\mathcal{F}$ are of the form:

$$|\xi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int \mathrm{d}^3\mathrm{k}_1 ... \mathrm{d}^3\mathrm{k}_\mathrm{n}\, \xi^{(n)}(k_1, ..., k_n) a_{k_1}^* ... a_{k_n}^* |0\rangle \tag{3.140}$$

Now, we can start with calculating the matrix elements, as we will need them in the following.
Remember that we have already computed

$$\langle\phi| a_k^* |\xi\rangle = \sum_{n=0}^{\infty} \sqrt{n+1} \int \mathrm{d}^3\mathrm{k}_1 ... \mathrm{d}^3\mathrm{k}_\mathrm{n}\, \overline{\phi^{(n+1)}(k, k_1, ..., k_n)}\xi^{(n)}(k_1, ..., k_n) \tag{3.141}$$

and

$$\langle\phi| a_k |\xi\rangle = \sum_{n=0}^{\infty} \sqrt{n+1} \int \mathrm{d}^3\mathrm{k}_1 ... \mathrm{d}^3\mathrm{k}_\mathrm{n}\, \overline{\phi^{(n)}(k_1, ..., k_n)}\xi^{(n+1)}(k, k_1, ..., k_n) \tag{3.142}$$

Hence

$$\langle\phi|\,\varphi_t(x)\,|\xi\rangle$$
$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\sum_{n=0}^{\infty}\sqrt{n+1}\int \mathrm{d^3k}\int \mathrm{d^3k_1...d^3k_n}\gamma_k$$
$$\cdot\left[\overline{\phi^{(n)}(k_1,...,k_n)}\xi^{(n+1)}(k,k_1,...,k_n)e^{ik\cdot x-i\omega_k t}+\overline{\phi^{(n+1)}(k,k_1,...,k_n)}\xi^{(n)}(k_1,...,k_n)e^{-ik\cdot x+i\omega_k t}\right] \tag{3.143}$$

It is easy to check that for $\mu = 0$ this matrix elements and therefore also the expectation value fulfills the free wave equation, i.e.

$$\Box\,\langle\xi|\,\varphi_t(x)\,|\xi\rangle = 0 \tag{3.144}$$

This should be intuitiv, since the interaction is not switched on yet, i.e. this is the free field. In the next step, we switch on the interaction, i.e. we want to compute $\lim_{t_0\to-\infty}\langle\xi_{t_0}|\,U(t_0,t)\varphi_t(x)U(t,t_0)\,|\xi_{t_0}\rangle$, where $U(t,t_0)=e^{-iH(t-t_0)}$ now is the full time evolution (not in the interaction picture anymore), i.e. (by the definition of the interaction picture)

$$U_I(t,t_0) = e^{iH_0t}e^{-iH(t-t_0)}e^{-iH_0t_0} \tag{3.145}$$
$$\Rightarrow U(t,t_0) = e^{-iH_0t}U_I(t,t_0)e^{iH_0t_0} \tag{3.146}$$

First of all, we notice that similar to the previous chapter (just without the $\int \mathrm{d^3x}\Psi^*(x)\Psi(x)$ factor), we get the following evolution operator:

$$\begin{aligned}U_I(t,t_0) = &\sum_{n=0}^{\infty}\frac{\lambda^n}{(2\pi)^{\frac{3n}{2}}}\int \mathrm{d^3k_1...d^3k_n}\frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_n}}{\omega_{k_n}}\\
&\times\left[\left(a_{k_1}e^{-i\omega_{k_1}t+ik_1\cdot x}-a^*_{k_1}e^{i\omega_{k_1}t-ik_1\cdot x}\right)-\left(a_{k_1}e^{-i\omega_{k_1}t_0+ik_1\cdot x}-a^*_{k_1}e^{i\omega_{k_1}t_0-ik_1\cdot x}\right)\right]...\\
&\times\left[\left(a_{k_n}e^{-i\omega_{k_n}t+ik_n\cdot x}-a^*_{k_n}e^{i\omega_{k_n}t-ik_n\cdot x}\right)-\left(a_{k_n}e^{-i\omega_{k_n}t_0+ik_n\cdot x}-a^*_{k_n}e^{i\omega_{k_n}t_0-ik_n\cdot x}\right)\right]\end{aligned} \tag{3.147}$$

and also analogously as before, we know that $U_I(t,t_0)$ weakly converges to

$$D_f(t) = \exp\left\{\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d^3k}\frac{\gamma_k}{\omega_k}\left(a_ke^{-i\omega_kt}-a^*_ke^{i\omega_kt}\right)\right\} \tag{3.148}$$

as $t_0\to-\infty$ and at $x=0$.
Note that by calculating $D_f$ at $x=0$, we have chosen the coordinate system in a way that the nucleon is fixed at $x=0$.

Further, let $|\chi_n\rangle$ be a basis of our Fock space $\mathcal{F}$, then $\mathbb{I}=\sum_{n=0}^{\infty}|\chi_n\rangle\langle\chi_n|$. Next, we will use dominated convergence in order to exchange lim and $\sum$. For now we will assume that dominated convergence holds, but we will show it later. Then we use the weak convergence

$$\begin{aligned}
&\lim_{t_0\to-\infty} \langle\xi_{t_0}| U(t_0,t)\varphi_0(x)U(t,t_0) |\xi_{t_0}\rangle \\
&= \lim_{t_0\to-\infty} \langle\xi_{t_0}| e^{-iH_0t_0}U_I(t_0,t) \underbrace{e^{iH_0t}\varphi_0(x)e^{-iH_0t}}_{=\varphi_t(x)} U_I(t,t_0)e^{iH_0t_0} |\xi_{t_0}\rangle \\
&= \sum_{n,n',m,m'=0}^{\infty} \lim_{t_0\to-\infty} \langle\xi_{t_0}| e^{-iH_0t_0} |\chi'_n\rangle \langle\chi'_n| U_I(t_0,t) |\chi_n\rangle \langle\chi_n| \varphi_t(x) |\chi_m\rangle \langle\chi_m| U_I(t,t_0) |\chi'_m\rangle \langle\chi'_m| e^{iH_0t_0} |\xi_{t_0}\rangle \\
&= \sum_{n,n',m,m'=0}^{\infty} \lim_{t_0\to-\infty} \langle\xi_{t_0}| e^{-iH_0t_0} |\chi'_n\rangle \langle\chi'_n| D_f^*(t) |\chi_n\rangle \langle\chi_n| \varphi_t(x) |\chi_m\rangle \langle\chi_m| D_f(t) |\chi'_m\rangle \langle\chi'_m| e^{iH_0t_0} |\xi_{t_0}\rangle \\
&= \lim_{t_0\to-\infty} \langle\xi_{t_0}| e^{-iH_0t_0} D_f^*(t)\varphi_t(x)D_f(t)e^{iH_0t_0} |\xi_{t_0}\rangle \qquad (3.149)
\end{aligned}$$

Let us do a little side computation:

$$\begin{aligned}
a_kD_f(t) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!(2\pi)^{\frac{3n}{2}}} \int \mathrm{d^3k_1...d^3k_n} \frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_n}}{\omega_{k_n}} \\
&\cdot a_k \left(a_{k_1}e^{-i\omega_{k_1}t} - a_{k_1}^*e^{+i\omega_{k_1}t}\right)...\left(a_{k_n}e^{-i\omega_{k_n}t} - a_{k_n}^*e^{+i\omega_{k_n}t}\right) \\
&= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!(2\pi)^{\frac{3n}{2}}} \int \mathrm{d^3k_1...d^3k_n} \frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_n}}{\omega_{k_n}} \\
&\cdot \left(a_{k_1}e^{-i\omega_{k_1}t} - a_{k_1}^*e^{i\omega_{k_1}t}\right)...\left(a_{k_n}e^{-i\omega_{k_n}t} - a_{k_n}^*e^{i\omega_{k_n}t}\right) a_k \\
&- \sum_{n=0}^{\infty} \frac{n\lambda^n}{n!(2\pi)^{\frac{3n}{2}}} \frac{\gamma_k}{\omega_k} e^{-ik\cdot x+i\omega_kt} \int \mathrm{d^3k_1...d^3k_{n-1}} \frac{\gamma_{k_1}}{\omega_{k_1}}...\frac{\gamma_{k_{n-1}}}{\omega_{k_{n-1}}} \\
&\cdot \left(a_{k_1}e^{-i\omega_{k_1}t} - a_{k_1}^*e^{i\omega_{k_1}t}\right)...\left(a_{k_{n-1}}e^{-i\omega_{k_{n-1}}t} - a_{k_{n-1}}^*e^{i\omega_{k_{n-1}}t}\right) \\
&= D_f(t)a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} e^{i\omega_kt} D_f(t) \qquad (3.150)
\end{aligned}$$

and also by building the conjugate

$$D_f^*(t)a_k^* = a_k^*D_f^*(t) - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} e^{-i\omega_kt} D_f^*(t) \qquad (3.151)$$

Therefore we obtain

$$D_f^*(t)a_k^*D_f(t) = a_k^* - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} e^{-i\omega_kt} \qquad (3.152)$$

$$D_f^*(t)a_kD_f(t) = a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} e^{i\omega_kt} \qquad (3.153)$$

This yields

$$
\begin{aligned}
D_f^*(t)\varphi_t(x)D_f(t) = &\frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k\left(D_f^*(t)a_kD_f(t)e^{ik\cdot x-i\omega_kt}+D_f^*(t)a_k^*D_f(t)e^{-ik\cdot x+i\omega_kt}\right)\\
=&\frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k\left(a_ke^{ik\cdot x-i\omega_kt}+a_k^*e^{-ik\cdot x+i\omega_kt}\right)\\
&-\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k}\left(e^{ik\cdot x}+e^{-ik\cdot x}\right)\\
=&\varphi_t(x)-\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{|f(k^2)|^2}{2\omega_k^2}\left(e^{ik\cdot x}+e^{-ik\cdot x}\right)
\end{aligned}
\tag{3.154}
$$

Hence

$$
\begin{aligned}
&\lim_{t_0\to-\infty}\langle\xi_{t_0}|\,U(t_0,t)\varphi_0(x)U(t,t_0)\,|\xi_{t_0}\rangle = \langle\xi_{t_0}|\,e^{-iH_0t_0}D_f^*(t)\varphi_t(x)D_f(t)e^{iH_0t_0}\,|\xi_{t_0}\rangle\\
&=\lim_{t_0\to-\infty}\langle\xi_{t_0}|\,e^{-iH_0t_0}\varphi_t(x)e^{iH_0t_0}\,|\xi_{t_0}\rangle\underbrace{-\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{|f(k^2)|^2}{2\omega_k^2}\left(e^{ik\cdot x}+e^{-ik\cdot x}\right)}_{=\Phi_{\text{Yukawa}}(x)}\\
&=\underbrace{\lim_{t_0\to-\infty}\langle\xi_{t_0}|\,\varphi_{t-t_0}(x)\,|\xi_{t_0}\rangle}_{\text{free field}}+\Phi_{\text{Yukawa}}(x)
\end{aligned}
\tag{3.155}
$$

We can see that this expectation value has two parts. The first one is just the one belonging to the free field (it solves the free wave equation), the second one is again the Yukawa potential we have already computed (for $f(k^2)\to 1$):

$$
\Phi_{\text{Yukawa}}(x) = -\frac{\lambda}{4\pi}\frac{e^{-\mu|x|}}{|x|} \tag{3.156}
$$

**Theorem 3.4.1** *For $\mu = 0$ we can see that $\Phi_{Yukawa}(x)$ is a particular solution of the inhomogeneous wave equation, i.e.*

$$
\Box\Phi_{Yukawa}(x) = \lambda\delta^{(3)}(x) \tag{3.157}
$$

**Remark:** This implies

$$
\Box\lim_{t_0\to-\infty}\langle\xi_{t_0}|\,U(t_0,t)\varphi_0(x)U(t,t_0)\,|\xi_{t_0}\rangle = \lambda\delta^{(3)}(x) \tag{3.158}
$$

**Proof:** Let $\mu = 0$, then

$$
\begin{aligned}
\Phi_{\text{Yukawa}}(x) &= -\frac{\lambda}{4\pi}\frac{1}{|x|} = \frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{1}{k^2}e^{ik\cdot x}\\
\Rightarrow \Box\Phi_{\text{Yukawa}}(x) &= \frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{1}{k^2}\underbrace{\Box e^{ik\cdot x}}_{=k^2e^{ik\cdot x}} = \frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}e^{ik\cdot x} = \lambda\delta^{(3)}(x)
\end{aligned}
\tag{3.159}
$$

□

This is exactly what we would have expected, since this just gives the Coulomb potential for $\mu \to 0$.
Also one can see that for the particular solution, i.e. the interaction part, the initial field is completely irrelevant, only the free field contains the initial data.

One could ask the following question: Can we find a condition on $|\xi\rangle$, such that the free field vanishes? In order to examine this question let us have a further look at

$$\begin{aligned}
&\langle\xi|\,\varphi_t(x)\,|\xi\rangle \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\sum_{n=0}^{\infty}\sqrt{n+1}\int \mathrm{d}^3\mathrm{k}\int \mathrm{d}^3\mathrm{k}_1...\mathrm{d}^3\mathrm{k}_\mathrm{n}\gamma_k\Big[\overline{\xi^{(n)}(k_1,...,k_n)}\xi^{(n+1)}(k,k_1,...,k_n)e^{ik\cdot x - i\omega_k t} + c.c.\Big]
\end{aligned} \tag{3.160}$$

There is kind of a natural assumption we could make, i.e. we could suppose that $|\xi\rangle$ is a product state

$$\xi^{(n)}(k_1,...,k_n) = \frac{1}{\sqrt{n!}}\prod_{i=1}^{n} h(k_i) \tag{3.161}$$

Then

$$\xi^{(n+1)}(k,k_1,...,k_n) = \frac{1}{\sqrt{n+1}}\xi^{(n)}(k_1,...,k_n)h(k) \tag{3.162}$$

and by using $\langle\xi|\xi\rangle = 1$, i.e. $\sum_{n=0}^{\infty}\int \mathrm{d}^3\mathrm{k}_1...\mathrm{d}^3\mathrm{k}_\mathrm{n}|\xi^{(n)}(k_1,...,k_n)|^2$, we get

$$\begin{aligned}
\langle\xi|\,\varphi_t(x)\,|\xi\rangle &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k\left[h(k)e^{ik\cdot x - i\omega_k t} + c.c.\right]\underbrace{\sum_{n=0}^{\infty}\int \mathrm{d}^3\mathrm{k}_1...\mathrm{d}^3\mathrm{k}_\mathrm{n}|\xi^{(n)}(k_1,...,k_n)|^2}_{=1} \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k\left[h(k)e^{ik\cdot x - i\omega_k t} + \overline{h(k)}e^{-ik\cdot x + i\omega_k t}\right]
\end{aligned} \tag{3.163}$$

**Remark:** In order to state the following theorem, we introduce a notation to denote the cutoff function. Let $A$ be a integral operator, then by writing $A^{\kappa,\Lambda}$, we denote the same integral operator but with a cutoff function $f_\kappa^\Lambda$

Hence we can state the following theorem

**Theorem 3.4.2** *Let $f = f_\kappa^\Lambda$ be a cutoff function, such that all the expectation values are well-defined. Then*

$$\lim_{t_0\to-\infty}\langle\xi_{t_0}|\,U(t_0,t)\varphi_0^{\kappa,\Lambda}(x)U(t,t_0)\,|\xi_{t_0}\rangle = \lim_{t_0\to-\infty}\langle\xi_{t_0}|\,\varphi_{t-t_0}^{\kappa,\Lambda}(x)\,|\xi_{t_0}\rangle + \Phi_{Yukawa}^{\kappa,\Lambda}(x) \tag{3.164}$$

*where $U(t_0,t)$ is the full time evolution operator and $|\xi_{t_0}\rangle$ is the initial data at $t_0 \to -\infty$ and*

$$\Phi_{Yukawa}^{\kappa,\Lambda}(x) := -\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{|f_\kappa^\Lambda(k^2)|^2}{2\omega_k^2}\left(e^{ik\cdot x} + e^{-ik\cdot x}\right) \tag{3.165}$$

*Further, suppose* $|\xi_{t_0}\rangle \in \mathcal{F}$ *is a product state in the Fock space with amplitudes*

$$\xi_{t_0}^{(n)}(k_1, ..., k_n) = \frac{1}{\sqrt{n!}} \prod_{i=1}^{n} h(k_i) \quad \forall n \in \mathbb{N} \tag{3.166}$$

*with* $h : \mathbb{R}^3 \to \mathbb{C}$ *and let one of the following be true:*

1. *The cutoff function has compact support, i.e.* $f_\kappa^\Lambda \in C_0^\infty(\mathbb{R}^3)$ *and let* $h \in C^2(\mathbb{R}^3)$
2. *Let* $f_{\frac{1}{\Lambda}}^{\Lambda}$ *be an arbitrary cutoff function and let* $h, \nabla h, \Delta h \in L^2(\mathbb{R}^3)$*. Suppose we can find* $k_0 > 0$ *and* $\epsilon > 0$*, such that*

$$|h(k)| \leq |k|^{-3-\epsilon} \quad \forall k \in \{k \in \mathbb{R}^3 : |k| \geq k_0\} \tag{3.167}$$

*Then the initial data vanishes, for any* $\Lambda > \kappa > 0$

$$\lim_{t_0 \to -\infty} \langle \xi_{t_0} | \, \varphi_{t-t_0}^{\kappa;\Lambda}(x) \, | \xi_{t_0} \rangle = 0 \;\; \forall x \in \mathbb{R}^3 \tag{3.168}$$

*Further, in the second case*

$$\lim_{\Lambda \to \infty} \lim_{t_0 \to -\infty} \langle \xi_{t_0} | \, \varphi_{t-t_0}^{\frac{1}{\Lambda};\Lambda}(x) \, | \xi_{t_0} \rangle = 0 \;\; \forall x \in \mathbb{R}^3 \tag{3.169}$$

**Remark:** This means the initial data has only a contribution to the free field and for $h(k)$ decaying fast enough, even this term vanishes. This is similar to the classical case, i.e. the Liénard-Wiechert fields.

Previously, we shown the weak convergence of $U_I(t, t_0)$ to $T_f$ for $t_0 \to -\infty$ holds even without a cutoff function for exactly the conditions on the meson states in the Fock space, we have found in the theorem above.
Note that, in principle the Hamiltonian is only defined on the $\mathcal{D}(H)$. Nevertheless, the theorem above holds even for a bigger subset of $\mathcal{H}$, defined by the conditions in the theorem. This is no contradiction, since Stones theorem tells us that $U(t, t_0)$ is defined on the whole Hilbert space.

Physically, this theorem tells us that we can start with any field (which fulfills our conditions) and the system converges into the ground state.

**Proof:** We have already proved the first part of the theorem before. Either $f \in C_0^\infty(\mathbb{R}^3)$ or the conditions on $h$ guarantees the weak convergence of $U_I(t, t_0)$ in the limit $t_0 \to -\infty$. The proof of the second part goes very similar to the one for the Liénard-Wiechert fields, which we have done in the introduction.
Let us make the following definition

$$A_t(x) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \frac{|f(k^2)|}{\sqrt{2\omega_k}} h(k) e^{ik\cdot x - i\omega_k(t-t_0)} \tag{3.170}$$

Assume that $|\xi_{t_0}\rangle$ is a product state, then we know

$$| \langle \xi_{t_0}| \varphi_{t-t_0}(x) |\xi_{t_0}\rangle | = |A_t(x) + \overline{A_t(x)}| \leq 2|A_t(x)| \tag{3.171}$$

Therefore, we have to estimate $A_t(x)$ similarly as we have done it in the introduction for the LWFs.
It is easy to check that for $\mu = 0$, $A_t(x)$ fulfills the homogeneous wave equation, i.e.

$$\Box A_t(x) = 0 \tag{3.172}$$

Similar to the calculation we have done in the introduction, we know that a solution of the homogeneous wave equation can be written as

$$A_t(x) = \left(G_{t-t_0} \star \dot{A}_{t_0}\right)(x) + \partial_t \left(G_{t-t_0} \star A_{t_0}\right)(x) \tag{3.173}$$

where $\star$ denotes the convolution and $G_t$ is the fundamental propagator defined in the introduction, i.e.

$$G_t(x) = \frac{1}{4\pi|t|}\left[\delta(|x| - t) + \delta(|x| + t)\right] \tag{3.174}$$

From the definition of $A_t(x)$, we get the boundary conditions

$$\begin{aligned} A_{t_0}(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \frac{|f(k^2)|^2}{\sqrt{2\omega_k}} h(k) e^{ik\cdot x} \\ \dot{A}_{t_0}(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \frac{-i|f(k^2)|^2\sqrt{\omega_k}}{\sqrt{2}} h(k) e^{ik\cdot x} \end{aligned} \tag{3.175}$$

Then we can estimate $A_t(x)$ by estimating both terms individually

$$\begin{aligned} &\partial_t \left(G_{t-t_0} \star A_{t_0}\right)(x) \\ &= \partial_t \left\{ \frac{1}{4\pi|t-t_0|} \int {}^3\mathrm{dy} \left[\delta(|x| - t + t_0) + \delta(|x| + t - t_0)\right] A_{t_0}(x-y) \right\} \\ &= \partial_t \left\{ \frac{1}{4\pi|t-t_0|} \int \mathrm{d}\Omega \int_0^\infty \mathrm{d|y|}|y|^2 \left[\delta(|x| - t + t_0) + \delta(|x| + t - t_0)\right] A_{t_0}(x-y) \right\} \\ &= \partial_t \left\{ \frac{|t-t_0|}{4\pi} \int \mathrm{d}\Omega \left[A_{t_0}(x - |t-t_0|e_r) + A_{t_0}(x + |t-t_0|e_r)\right] \right\} \\ &= \partial_t \left\{ \frac{|t-t_0|}{2\pi} \int \mathrm{d}\Omega A_{t_0}(x - |t-t_0|e_r) \right\} \\ &= \partial_t \left\{ \frac{|t-t_0|}{2\pi} \int_{\partial\mathcal{B}_1(x)} \mathrm{d}\Omega A_{t_0}(|t-t_0|e_r) \right\} \\ &= \frac{1}{2\pi} \int_{\partial\mathcal{B}_1(x)} \mathrm{d}\Omega \left\{ \pm A_{t_0}(|t-t_0|e_r) + |t-t_0|\dot{A}_{t_0}(|t-t_0|e_r) \right\} \end{aligned} \tag{3.176}$$

where $e_r$ is the unit vector in r direction, $d\Omega$ is the angular measure in spherical coordinates and $\partial\mathcal{B}_1(x)$ is the unit sphere around $x \in \mathbb{R}^3$.
Analogously for the other term

$$\begin{aligned} |A_t(x)| &= \left| \frac{1}{2\pi} \int_{\partial\mathcal{B}_1(x)} \mathrm{d}\Omega \left\{ \pm A_{t_0}(|t-t_0|e_r) + 2|t-t_0| \dot{A}_{t_0}(|t-t_0|e_r) \right\} \right| \\ &\leq \frac{1}{2\pi} \int_{\partial\mathcal{B}_1(x)} \mathrm{d}\Omega \left\{ |A_{t_0}(|t-t_0|e_r)| + 2|t-t_0||\dot{A}_{t_0}(|t-t_0|e_r)| \right\} \end{aligned} \tag{3.177}$$

The integral $\int_{\partial\mathcal{B}_1(x)} \mathrm{d}\Omega$ is of order one. Therefore it suffices to show that

$$|A_{t_0}(|t-t_0|e_r)| + 2|t-t_0||\dot{A}_{t_0}(|t-t_0|e_r)| \to 0 \quad \text{as} \quad t_0 \to -\infty \tag{3.178}$$

This can be reformulated by the following

$$|A_{t_0}(x)| + 2|x||\dot{A}_{t_0}(x)| \to 0 \quad \text{as} \quad \forall x \in \mathbb{R}^3 \quad \text{and} \quad |x| \to 0 \tag{3.179}$$

Now we have to estimate both terms from above. We know

$$\begin{aligned} A_{t_0}(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \gamma_k h(k) e^{ik\cdot x} \\ \dot{A}_{t_0}(x) &= \frac{-i}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \gamma_k \omega_k h(k) e^{ik\cdot x} \end{aligned} \tag{3.180}$$

We will use a similar trick as we have done before

$$\nabla_k e^{ik\cdot x} = ix e^{ik\cdot x} \tag{3.181}$$

$$\Rightarrow \Delta_k e^{ik\cdot x} = -x^2 e^{ik\cdot x} \tag{3.182}$$

Then by using the first Green's identity, we obtain

$$\begin{aligned} A_{t_0}(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \gamma_k h(k) e^{ik\cdot x} \\ &= \frac{-1}{(2\pi)^{\frac{3}{2}} x^2} \int \mathrm{d}^3\mathrm{k} \gamma_k h(k) \Delta_k e^{ik\cdot x} \\ &= \underbrace{\text{boundary term}}_{=0} + \frac{1}{(2\pi)^{\frac{3}{2}} x^2} \int \mathrm{d}^3\mathrm{k} \nabla_k \left( \gamma_k h(k) \right) \cdot \nabla_k e^{ik\cdot x} \\ &= \frac{i}{(2\pi)^{\frac{3}{2}} x^2} x \cdot \int \mathrm{d}^3\mathrm{k} e^{ik\cdot x} \nabla_k \left( \gamma_k h(k) \right) \end{aligned} \tag{3.183}$$

The boundary term vanishes since the field decays for large momenta. In order to be able to use Green's identity, we need $\gamma_k h(k)$ to be continuously differentiable. This is guaranteed by each of the two conditions in theorem.
Using $|a \cdot b| = |a||b| \cos\theta \leq |a||b|$ and the triangle inequality, we get

$$|A_{t_0}(x)| \leq \frac{1}{(2\pi)^{\frac{3}{2}} |x|} \int \mathrm{d}^3\mathrm{k} |\nabla_k \left( \gamma_k h(k) \right)| \tag{3.184}$$

In the case of a cutoff, we have that $\nabla_k(\gamma_k h(k)) \in C_0^\infty$ has compact support and hence the statement is trivial. In the case without an cutoff, we use the assumption that $h(k)$ has no singularities and there is a $k_0 > 0$ and $\epsilon > 0$, such that

$$|h(k)| \leq |k|^{-3-\epsilon} \quad \forall k \in \{k \in \mathbb{R}^3 : |k| \geq k_0\} \tag{3.185}$$

Then without a cutoff, i.e. $f(k^2) = 1$, we have for large momenta

$$\begin{aligned} &\gamma_k \sim |k|^{-\frac{1}{2}} \\ \Rightarrow |\nabla_k(\gamma_k h(k))| &\sim \nabla_k |k|^{-\frac{1}{2}} |k|^{-3-\epsilon} \sim |k|^{-\frac{9}{2}-\epsilon} \end{aligned} \tag{3.186}$$

Hence in the ultraviolet regime the integral behaves like

$$\int \mathrm{d^3k} |\nabla_k(\gamma_k h(k))| \sim \int_{k_0}^\infty \mathrm{d|k|} |k|^{-\frac{5}{2}-\epsilon} < \infty \tag{3.187}$$

In the infrared regime there is also no divergence, therefore the whole integral is finite. So we end up with

$$|A_{t_0}(x)| \leq const. \cdot \frac{1}{|x|} \tag{3.188}$$

Now, we want to estimate the second term. In order to do this, we again apply the first Greens identity and after that just the integration by parts formula for vector fields. Again we can argue that in both formulas the boundary terms vanish. Then

$$\begin{aligned} \dot{A}_{t_0}(x) &= \frac{-i}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d^3k} \gamma_k \omega_k h(k) e^{ik\cdot x} \\ &= \frac{i}{(2\pi)^{\frac{3}{2}} x^2} \int \mathrm{d^3k} \gamma_k \omega_k h(k) \Delta_k e^{ik\cdot x} \\ &= \underbrace{\text{boundary term}}_{=0} + \frac{-i}{(2\pi)^{\frac{3}{2}} x^2} \int \mathrm{d^3k} \nabla_k(\gamma_k \omega_k h(k)) \cdot \nabla_k e^{ik\cdot x} \\ &= \underbrace{\text{boundary term}}_{=0} + \frac{i}{(2\pi)^{\frac{3}{2}} x^2} \int \mathrm{d^3k} e^{ik\cdot x} \Delta_k(\gamma_k \omega_k h(k)) \end{aligned} \tag{3.189}$$

In order to use this fromula, we need $\gamma_k \omega_k h(k)$ to be twice continuously differentiable. This is guaranteed by each of the two conditions in theorem.
The triangle inequality yields

$$|x||\dot{A}_{t_0}(x)| \leq \frac{1}{(2\pi)^{\frac{3}{2}} |x|} \int \mathrm{d^3k} |\Delta_k(\gamma_k \omega_k h(k))| \tag{3.190}$$

Analogously as before, we can show that the integral is finite: If we have a cutoff, the statement is again trivial, since $\Delta_k(\gamma_k \omega_k h(k))$ has compact support and in the other case we can use the assumption. Then by assumption there is no infrared divergence and for large momenta we know

$$\gamma_k \omega_k h(k) \sim |k|^{-\frac{1}{2}} |k||k|^{-3-\epsilon} = |k|^{-\frac{5}{2}-\epsilon}$$
$$\Rightarrow |\Delta_k (\gamma_k h(k))| \sim |k|^{-\frac{9}{2}-\epsilon} \tag{3.191}$$

Hence the integral is again finite and we get

$$|x||\dot{A}_{t_0}(x)| \leq const.' \cdot \frac{1}{|x|} \tag{3.192}$$

Together with (3.190) this proves (3.179) and hence the claim. □

**Remark:** Since, we only consider elements in the Fock space, we have $h \in L^2(\mathbb{R}^3)$, further, we used that also $\nabla h, \Delta h \in L^2(\mathbb{R}^3)$ should hold. We know that a square integrable function whose derivative is also square integrable must decay in its argument. We can also say that $h(k)$ has to decay faster than $\frac{1}{|k|^{\frac{3}{2}}}$, since $\|h\|_2 < \infty$ must hold. Basically, this condition suffices to show that the initial field vanishes, but without a cutoff, we need a stronger condition in order to show the convergence for $t_0 \to -\infty$. The theorem tells us that in this QFT model the initial data vanishes in any case. This seems a little bit surprising, since in classical electrodynamics, the initial field only vanishes if the they decay is space, at least a little bit, but the decay is already implemented in the fact that we are only using square integrable functions.
For any element of the Hilbert space, which fulfills the condition for the weak convergence without a cutoff, we can show by a Paley-Wiener argument that $h(x)$ is decaying arbitrary fast in $x$. Hence it is a Schwarz function.

$$|h(x)| = \Big| \int \mathrm{d}^3 k e^{-ik\cdot x} h(k) \Big| \sim \Big| \int \mathrm{d}|k| e^{-i|k||x|} |k|^{2-3-\epsilon} \Big| \sim \frac{1}{|x|^n} \underbrace{\int \mathrm{d}|k||k|^{2-3-n-\epsilon}}_{<\infty \forall n \in \mathbb{N}} \tag{3.193}$$

Similar, the free field is a Schwarz function.

## 3.5 Dressed vacuum state

One could ask the question, how many photons we need in order to dress the vacuum $|0\rangle$ with its appropriate field. Or in other words, what is the expectation value of the number operator $\hat{N} := \int \mathrm{dk} a_k^* a_k$ in the dressed vacuum state $\left|\Psi^{(gs)}\right\rangle := D_f |0\rangle$. The following theorem will tell us that in the case of a massless field, we need infinitely many photons. This should not be too surprising, since we already know that $\left|\Psi^{(gs)}\right\rangle$ and $|0\rangle$ do not live in the same Fock space. The fact that we need infinitely many photon in order to dress the vacuum with its field is exactly the reason for this.

**Theorem 3.5.1** *Let $\hat{N}$ be the number operator and $\left|\Psi^{(gs)}\right\rangle$ the dressed vacuum state. Further let $\mu > 0$, then*

$$\left\langle \Psi^{(gs)} \middle| \hat{N} \middle| \Psi^{(gs)} \right\rangle = \infty \tag{3.194}$$

**Proof:** The proof is just a calculation and using the fact that we know how $a_k$ and $D_f$ commute and then that the dressed vacuum state is normalized (at least in the new Fock space)

$$\begin{aligned}\left\langle \Psi^{(gs)} \right| \hat{N} \left| \Psi^{(gs)} \right\rangle &= \int \mathrm{d}^3\mathrm{k}\, \langle 0| D_f^* a_k^* a_k D_f |0\rangle \\ &= \int \mathrm{d}^3\mathrm{k}\, \langle 0| \left[D_f^*, a_k^*\right][a_k, D_f] |0\rangle \\ &= \frac{\lambda^2}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k^2}{\omega_k^2} \underbrace{\langle 0| D_f^* D_f |0\rangle}_{=1(\text{normalization})} \\ &= \frac{\lambda^2}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \frac{1}{2\omega_k^3} \end{aligned} \tag{3.195}$$

Transforming to spherical coordinates yields

$$\left\langle \Psi^{(gs)} \right| \hat{N} \left| \Psi^{(gs)} \right\rangle = \frac{\lambda^2}{(2\pi)^2} \int_0^\infty \mathrm{d}|\mathrm{k}| \frac{|k|^2}{(|k|^2 + \mu^2)^{\frac{3}{2}}} \to \infty \tag{3.196}$$

□

**Remark:** Note that in the limit of a massless field, i.e. $\mu \to 0$, there is not only a ultraviolet divergence, which we somehow have to accept, since it is already there in classical physics, but in this case the integral (3.196) is also divergent in the infrared regime. We have already discussed the origin of the infrared divergence. The theorem above indicates that the divergence in the infrared regime causes the fact that the dressed vacuum state is not in the same Fock space than the vacuum state.

## 3.6 Charge with constant velocity

In the model discussed in the previous chapter, we looked at a nucleon (charge) at a fixed point, i.e. the particle is not able to move. In this chapter we will generalize this constraint a little bit by allowing the charge to move along a given classical trajectory. In particular, we consider a charge moving on a straight line with constant velocity $v$. This is of course still a pretty big simplification, since we only consider a classical motion and even here just the simplest motion one could image. Actually, we would like to describe a charge, which moves according to a quantum mechanical equation. In this case $x$ would be an operator and this would complicate the calculations a lot. However, we proceed step by step and hence we start with examining this "easy" case.

Actually, the difference to our first model is not too big. Basically, the step from a charge at a fixed point to a charge with constant velocity is just a change of the reference frame and hence the results stay pretty much the same. This change of the reference frame can be described by a Lorentz boost with constant velocity $v$ , which we denote by $\Lambda_v$.

$$x \to x' = x - vt \tag{3.197}$$

Then the fields are just the Lorentz transformation of the fields we had before, i.e.

$$\Psi(x) \to \Psi(x') = \Psi(\Lambda_v^{-1}x) \tag{3.198}$$

Despite nothing really new is happening in this chapter, we can still show a nice theorem, which shows us that we need to change the Fock space.
Again, we would like to have a dressing operator similar to the one in the chapter before. This can be done relatively easily by just doing the same derivation as in the $v = 0$ case. One ends up with the following expression for the evolution operator

$$\begin{aligned} &U_I^v(t,t_0) \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!(2\pi)^{\frac{3n}{2}}} \int \mathrm{d}^3\mathrm{x}_1...\mathrm{d}^3\mathrm{x}_\mathrm{n} \Psi^*(x_1)\Psi(x_1)...\Psi^*(x_n)\Psi(x_n) \int \mathrm{d}^3\mathrm{k}_1...\mathrm{d}^3\mathrm{k}_\mathrm{n} \frac{\gamma_{k_1}}{\omega'_{k_1}}...\frac{\gamma_{k_n}}{\omega'_{k_n}} \\ &\times \left[\left(a_{k_1}e^{-i\omega'_{k_1}t+ik_1\cdot x_1} - a^*_{k_1}e^{i\omega'_{k_1}t-ik_1\cdot x_1}\right) - \left(a_{k_1}e^{-i\omega'_{k_1}t_0+ik_1\cdot x_1} - a^*_{k_1}e^{i\omega'_{k_1}t_0-ik_1\cdot x_1}\right)\right]... \\ &\times \left[\left(a_{k_n}e^{-i\omega'_{k_n}t+ik_n\cdot x_n} - a^*_{k_n}e^{i\omega'_{k_n}t-ik_n\cdot x_n}\right) - \left(a_{k_n}e^{-i\omega'_{k_n}t_0+ik_n\cdot x_n} - a^*_{k_n}e^{i\omega'_{k_n}t_0-ik_n\cdot x_n}\right)\right] \end{aligned} \tag{3.199}$$

where $\omega'_k := \omega_k - k \cdot v$.
Similarly to the static case, we could show that the terms including $t_0$ are vanishing in the limit $t_0 \to -\infty$. Hence

**Definition:**

$$T_f^v := \lim_{t_0 \to -\infty} U_I^v(t,t_0) \tag{3.200}$$

is the dressing operator, which we have already defined in the previous chapter boosted with a constant velocity $v$. Similar as in the case $v = 0$ one can show that

$$T_f^v := \exp\left\{\frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x} \Psi^*(x)\Psi(x) \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{\omega_k - k\cdot v} \left(a_k e^{ik\cdot x - i\omega'_k t} - a^*_k e^{-ik\cdot x + i\omega'_k t}\right)\right\} \tag{3.201}$$

Nevertheless, in this chapter we are not interested in the dynamics of the nucleons, since the have a fixed path with $v = const$. We choose the coordinate system, such that at $t = 0$, the particle is at $x = 0$. Then the corresponding dressing operator looks like the following:

$$D_f^v := \exp\left\{\frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{\omega_k - k\cdot v} (a_k - a^*_k)\right\} \tag{3.202}$$

This means we get the same results as we had before, except we have to replace

$$\omega_k \to \omega'_k := \omega_k - v \cdot k \tag{3.203}$$

everywhere in the dressing operator. Then the field looks like the following

$$\Phi^{(v)}(x) = -\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k - v\cdot k}\left(e^{ik\cdot x} + e^{-ik\cdot x}\right) \tag{3.204}$$

Later, we will show that this is a boosted Coulomb potential and thus the result fits well into our intuition. Similarly, as in the static case, there might be situations, where the dressing operator maps into a different Fock space.
We can construct the following Fock space, similar to the previous chapter:

**Definition:** Let $|0\rangle_v$ be the "new" vacuum state, i.e. the no meson state belonging to the "new" annihilation operator

$$c_k^v := a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\frac{\gamma_k}{\omega_k - k\cdot v} \tag{3.205}$$

and

$$c_k^v |0\rangle_v = 0 \tag{3.206}$$

Then, we can proof the following Lemma:

**Lemma 3.6.1** *Let $f = f_\kappa^\Lambda \in C_0^\infty(\mathbb{R}^3)$ be an appropriate cutoff function, then*

$$|0\rangle_v = D_f^v |0\rangle \tag{3.207}$$

*We call $|0\rangle_v$ the dressed vacuum state boosted with velocity $v$.*

**Proof:** Use

$$a_k\left(a_{k'} - a_{k'}^*\right) = \left(a_{k'} - a_{k'}^*\right)a_k + \delta^{(3)}(k-k') \tag{3.208}$$

and obtain

$$\begin{aligned}
&a_k D_f^v |0\rangle \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!(2\pi)^{\frac{3n}{2}}}\int \mathrm{d}^3\mathrm{k}_1...\mathrm{d}^3\mathrm{k}_\mathrm{n}\frac{\gamma_{k_1}}{\omega_{k_1} - k_1\cdot v}...\frac{\gamma_{k_n}}{\omega_{k_n} - k_n\cdot v}a_k\left(a_{k_1} - a_{k_1}^*\right)...\left(a_{k_n} - a_{k_n}^*\right)|0\rangle \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!(2\pi)^{\frac{3n}{2}}}\int \mathrm{d}^3\mathrm{k}_1...\mathrm{d}^3\mathrm{k}_\mathrm{n}\frac{\gamma_{k_1}}{\omega_{k_1} - k_1\cdot v}...\frac{\gamma_{k_n}}{\omega_{k_n} - k_n\cdot v}\left(a_{k_1} - a_{k_1}^*\right)a_k...\left(a_{k_n} - a_{k_n}^*\right)|0\rangle \\
&+ \sum_{n=1}^{\infty}\frac{\lambda^n}{n!(2\pi)^{\frac{3n}{2}}}\frac{\gamma_k}{\omega_k - k\cdot v}\int \mathrm{d}^3\mathrm{k}_2...\mathrm{d}^3\mathrm{k}_\mathrm{n}\frac{\gamma_{k_2}}{\omega_{k_2} - k_2\cdot v}...\frac{\gamma_{k_n}}{\omega_{k_n} - k_n\cdot v}\left(a_{k_2} - a_{k_2}^*\right)...\left(a_{k_n} - a_{k_n}^*\right)|0\rangle \\
&= ... \\
&= 0 + \sum_{n=1}^{\infty}\frac{n\lambda^n}{n!(2\pi)^{\frac{3n}{2}}}\frac{\gamma_k}{\omega_k - k\cdot v}\int \mathrm{d}^3\mathrm{k}_1...\mathrm{d}^3\mathrm{k}_{\mathrm{n-1}}\frac{\gamma_{k_1}}{\omega_{k_1} - k_1\cdot v}...\frac{\gamma_{k_{n-1}}}{\omega_{k_{n-1}} - k_{n-1}\cdot v}\left(a_{k_1} - a_{k_1}^*\right)... \\
&\left(a_{k_{n-1}} - a_{k_{n-1}}^*\right)|0\rangle \\
&= \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\frac{\gamma_k}{\omega_k - k\cdot v}D_f^v|0\rangle
\end{aligned} \tag{3.209}$$

Hence

$$0 = \left[ a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k - k \cdot v} \right] D_f^v |0\rangle = c_k^v D_f^v |0\rangle = c_k^v |0\rangle_v \tag{3.210}$$

□

Then, the Fock space in this setting can be constructed like the following

**Definition:** $\mathcal{F}_{|0\rangle_v}$ is the Fock space with the dressed vacuum state $|0\rangle_v$ and annihilation/creation operators $c_k^v$, $c_k^{v*}$. The rest is defined as before.

Then we can state the following theorem:

**Theorem 3.6.2** *Let $f = f^\Lambda \in C_0^\infty(\mathbb{R}^3)$ be an ultraviolet cutoff and let the meson field be massless, i.e. $\mu = 0$. Then $|0\rangle_v \notin \mathcal{F}_{|0\rangle}$*

**Lemma 3.6.3** *Let $v < 1$ and $f = f_\kappa^\Lambda \in C_0^\infty(\mathbb{R}^3)$ be an appropriate cutoff function, then*

$$|0\rangle_v = D_f^v |0\rangle = \frac{1}{Z} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int \mathrm{d^3k_i} \frac{\gamma_{k_i}}{\omega_{k_i} - k_i \cdot v} a_{k_i}^* |0\rangle \tag{3.211}$$

*where $Z$ is just the normalization.*

**Proof:** We can do the following ansatz:

$$|0\rangle_v = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int \mathrm{d^3k_1 ... d^3k_n} u_v^{(n)}(k_1, ..., k_n) a_{k_1}^* ... a_{k_n}^* |0\rangle \tag{3.212}$$

together with

$$c_k^v |0\rangle_v = \left[ a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k - k \cdot v} \right] |0\rangle_v = 0 \tag{3.213}$$

this gives

$$\sqrt{n+1} u_v^{(n+1)}(k, k_1, ..., k_n) = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k - k \cdot v} u_v^{(n)}(k_1, ..., k_n) \tag{3.214}$$

whence

$$u_v^{(n)}(k_1, ..., k_n) = \frac{1}{\sqrt{n!}} \prod_{i=1}^{n} \left[ \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_{k_i}}{\omega_{k_i} - k_i \cdot v} \right] u_v^{(0)} \tag{3.215}$$

□

**Proof of the Theorem:** (Theorem 3.6.2) We will prove the theorem by contradiction. Assume that $|0\rangle_v \in \mathcal{F}_{|0\rangle}$, then there is a normalization constant $C := \langle 0|_v \, |0\rangle_v < \infty$. Let us just calculate with the help of the preceding lemma

$$C := \langle 0|_v \, |0\rangle_v = \sum_{n=0}^{\infty} \int \mathrm{d^3k_1 ... d^3k_n} |u_v^{(n)}(k_1, ..., k_n)|^2$$
$$= |u_v^{(0)}|^2 \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{n!(2\pi)^{3n}} \prod_{i=1}^{n} \int \mathrm{d^3k_i} \frac{\gamma_{k_i}^2}{(\omega_{k_i} - k_i \cdot v)^2}$$
$$= |u_v^{(0)}|^2 \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{n!(2\pi)^{3n}} \prod_{i=1}^{n} 2\pi \int_{-1}^{1} \mathrm{dz_i} \frac{1}{(1 - |v|z_i)^2} \int_{\kappa}^{\Lambda} \mathrm{d|k_i|} \frac{1}{|k_i|} \tag{3.216}$$

In the last step, we have used that the meson mass is zero $\mu = 0$. If we remove the infrared cutoff, i.e. $\kappa \to 0$, then the $\int \mathrm{d|k_i|}$ integral is logarithmically divergent in the infrared regime. This is a contradiction to the assumption. □

**Remark:** One could ask, why we just worry about the infrared divergence and still allow an ultraviolet cutoff. The answer is that the ultraviolet divergence already arises in the classical computations. It comes from the fact that we consider point-particles and hence the self-interaction term evaluates the field exactly at its divergent point. This means, we kind of have to accept the ultraviolet divergence, whenever considering an interacting model of point-particles.
However, the infrared divergence is a new type of divergence. On the one hand, there is the Fock space description, i.e. every state, we consider, has to be square integrable. This arises from the fact that in quantum mechanics one interprets the square of the wave function $|\psi|^2$ as a probability density and thus, in order to make sense out of this interpretation, we need $\int |\psi|^2 = 1$, or in other words, we need $\psi$ to be square integrable. On the other hand, there are Maxwell equations and they allow more solutions than just the ones, we can fit into one Fock space. Hence, if we try to combine these two theories, it is clear that one Fock space is not enough to describe everything. It seems reasonable that evolving an element of a Fock space over an infinitely long time period, the resulting state might be in another Fock space. This is exactly the situation, which is described by the theorem above.
One could say that the infrared divergence is a homemade problem and it can be solved by changing the Fock space appropriately at every time step. There are already some good results on such a change of the Fock space, for example by A. Pizzo in [11] and [12].
This representational problem already came along in this work earlier, but here it is the point, where it can be fully understood.

So far, we have seen that, without a infrared cutoff and in the massless case, the operator $D_f^v$ can not be an unitary transformation between the two Fock spaces $\mathcal{F}_{|0\rangle}$ and $\mathcal{F}_{|0\rangle_v}$ $(v \neq 0)$, where $|0\rangle$ is the vacuum state, annihilated by $a_k$ and $|0\rangle_v$ is the dressed vacuum state, defined above. In the following, we will show that the same holds for the two Fock spaces $\mathcal{F}_{|0\rangle_v}$ and $\mathcal{F}_{|0\rangle_{v'}}$ $(v \neq v')$.

Nevertheless, there is a natural candidate for such an operator and in certain situation, it is also well-defined. Remember the dressing operator in the static case (and for $x = 0$)

$$D_f := \exp\left\{ \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{\omega_k} (a_k - a_k^*) \right\} \tag{3.217}$$

transforming from $\mathcal{F}_{|0\rangle}$ to $\mathcal{F}_{D_f|0\rangle}$. In this situation, the relation between the corresponding creation-/annihilation operators was

$$b_k = a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k} \tag{3.218}$$

namely just a shift by $-\frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k}$. Looking at $D_f$ one can see that the amplitudes of the modes in $D_f$ are determined by the "shift" of the annihilation operators.
Similarly, we observe that

$$\begin{aligned} c_k^{v'} &= a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k - k \cdot v'} \\ &= a_k - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k - k \cdot v} + \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k - k \cdot v} - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k - k \cdot v'} \\ &= c_k^v + \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k - k \cdot v} - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k - k \cdot v'} \end{aligned} \tag{3.219}$$

and hence we can construct the following transformation:

**Definition:** Let one of the following conditions hold

- $\mu \neq 0$ and $f = f^\Lambda$ be an appropriate ultraviolet cutoff
- $\mu = 0$ and $f = f_\kappa^\Lambda$ with $\kappa > 0$

then we define the following bounded transformation operator

$$\begin{aligned} L_{v \to v'} :\ & \mathcal{F}_{|0\rangle_v} \to \mathcal{F}_{|0\rangle_{v'}} \\ & |\Phi\rangle \to L_{v \to v'} |\Phi\rangle = \exp\left\{ \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \gamma_k \left( \frac{1}{\omega_k - k \cdot v'} - \frac{1}{\omega_k - k \cdot v} \right) (a_k - a_k^*) \right\} |\Phi\rangle \end{aligned} \tag{3.220}$$

**Remark:** $L_{v \to v'}$ is well defined, since for $\mu \neq 0$ and $f$ being an appropriate ultraviolet cutoff or also for $\mu = 0$ and $f$ being an infrared and ultraviolet cutoff, we have

$$\int \mathrm{d}^3\mathrm{k} \gamma_k^2 \left| \frac{1}{\omega_k - k \cdot v} - \frac{1}{\omega_k - k \cdot v'} \right|^2 < \infty \tag{3.221}$$

and therefore this operator is bounded and hence it can be unitarily implemented on the Fock space.

**Theorem 3.6.4** *Let* $\mu = 0$ *and* $f = f^\Lambda$ *be an ultraviolet cutoff. Further, let* $\mathcal{F}_{|0\rangle_v}$ *and* $\mathcal{F}_{|0\rangle_{v'}}$ *be two Fock spaces with the dressed vacuum states* $|0\rangle_v$, $|0\rangle_{v'}$ *and creation-/annihilation operators* $c_k^{v*}, c_k^v$ *and* $c_k^{v'*}, c_k^{v'}$, *which have been defined before.*
*Then, for any* $v, v' \in \mathbb{R}^3$ *with* $v \neq v'$, *the following operator*

$$L_{v \to v'} = \exp\left\{\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k\left(\frac{1}{\omega_k - k\cdot v'} - \frac{1}{\omega_k - k\cdot v}\right)(a_k - a_k^*)\right\} \tag{3.222}$$

*(defined before) is the natural candidate for an unitary transformation between the two Fock spaces, but in this setting it is not unitary.*

**Proof:** The proof is pretty simple after all the work we have already done.
If $L_{v\to v'}$ wants to have a chance to be bounded, it clearly has to fulfill the following condition:

$$\int \mathrm{d}^3\mathrm{k}\gamma_k^2\left|\frac{1}{\omega_k - k\cdot v} - \frac{1}{\omega_k - k\cdot v'}\right|^2 < \infty \tag{3.223}$$

Therefore, let us just calculate at first for $\mu = 0$, i.e. $\omega_k = |k|$

$$\begin{aligned}
&\int \mathrm{d}^3\mathrm{k}\gamma_k^2\left|\frac{1}{\omega_k - k\cdot v} - \frac{1}{\omega_k - k\cdot v'}\right|^2 \\
&= \int \mathrm{d}^3\mathrm{k}\frac{|f_\kappa^\Lambda(k^2)|^2}{2|k|^3}\left|\frac{1}{(1-|v|\cos\theta_1)} - \frac{1}{(1-|v'|\cos\theta_2)}\right|^2 \\
&= \int_\kappa^\Lambda \mathrm{d}|\mathrm{k}|\frac{1}{2|k|}\underbrace{\int_0^{2\pi}\mathrm{d}\varphi\int_0^\pi \mathrm{d}\theta\sin\theta\frac{(|v|\cos\theta_1 - |v'|\cos\theta_2)^2}{(1-|v|\cos\theta_1)^2(1-|v'|\cos\theta_2)^2}}_{:=A(v,v')}
\end{aligned} \tag{3.224}$$

where $\theta_1$ is the angle between $v$ and $k$ and $\theta_2$ is the angle between $v'$ and $k$ respectively.

Now, we can easily see that for $v = v'$ $A(v, v') = 0$ and hence the whole integral is zero, but for $v \neq v'$, we obatin $A(v, v') = \text{const.} \neq 0$ that means without an infrared cutoff, the integral is logarithmically divergent, i.e. $L_{v\to v'}$ has no chance to be bounded and therefore it is not unitary. □

**Remark:** In other words, the theorem tells us that the transformation, we would like to have, $L_{v\to v'}$ can not be unitarily implemented on a Fock space.
Here we can see again that the infrared divergence shows us the fact that there are not enough states in one Fock space in order to describe all solutions of the Maxwell equations. in particular, even two massless charges with different velocities cannot be described in the same Fock space, if we do not allow for an infrared cutoff.

**Remark:** There is a little inconsistency in the notation, we have used here. Writing

$$L_{v \to v'} = \exp\left\{\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k\left(\frac{1}{\omega_k - k\cdot v} - \frac{1}{\omega_k - k\cdot v'}\right)(a_k - a_k^*)\right\} \tag{3.225}$$

then $L_{v\to v'}$ is expressed in the Fock space $\mathcal{F}_{|0\rangle}$, but the theorem above tells us that for $\mu \neq 0$ and $v \neq v'$ the Fock spaces $\mathcal{F}_{|0\rangle}$, $\mathcal{F}_{|0\rangle_v}$ and $\mathcal{F}_{|0\rangle_{v'}}$ are not even unitary equivalent. Thus, the expression above is kind of formal. What we really mean is, the transformation from $\mathcal{F}_{|0\rangle_v}$ into $\mathcal{F}_{|0\rangle_{v'}}$. This transformation is defined by

$$L_{v\to v'} : \mathcal{F}_{|0\rangle_v} \to \mathcal{F}_{|0\rangle_{v'}} \tag{3.226}$$

$$c_k^v \to c_k^{v'} + \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k - k\cdot v'} - \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\omega_k - k\cdot v} \tag{3.227}$$

$$|0\rangle_v \to |0\rangle_{v'} \tag{3.228}$$

**Theorem 3.6.5** *Let $f = f_\kappa^\Lambda \in C_0^\infty(\mathbb{R}^3)$ be an appropriate cutoff function, in particular, for $\mu = 0$, we need an infrared cutoff, i.e. $\kappa > 0$.*
*Then, similarly as in the $v = 0$ case, we find for the time evolved expectation value of the field*

$$\lim_{t_0\to-\infty} \langle\xi_{t_0}|\, U(t_0,t)\varphi_0(x)U(t,t_0)\,|\xi_{t_0}\rangle = \lim_{t_0\to-\infty} \underbrace{\langle\xi_{t_0}|\,\varphi_{t-t_0}(x)\,|\xi_{t_0}\rangle}_{\text{free field}} + \Phi^{(v)}(x) \tag{3.229}$$

*with*

$$\Phi^{(v)}(t,x) = -\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k} \frac{\gamma_k^2}{\omega_k - k\cdot v}\left(e^{ik\cdot(x-vt)} + e^{-ik\cdot(x-vt)}\right) \tag{3.230}$$

*The free field part vanishes for the same conditions as in the static case.*

**Proof:** The proof goes exactly as for the $v = 0$ case. □

**Remark:** The first term is again just the free field, and it vanishes for the same conditions as in the $v = 0$ case. Only the second term differs from the $v = 0$ case, hence it needs further examination and we need to find an connection to classical physics.

**Lemma 3.6.6**

$$\Phi^{(v)}(t,x) = -\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k} \frac{|f(k^2)|^2}{2\left(\omega_k^2 - (k\cdot v)^2\right)}\left(e^{ik\cdot(x-vt)} + e^{-ik\cdot(x-vt)}\right) \tag{3.231}$$

**Proof:**

$$\begin{aligned}
\Phi^{(v)}(t,x) =& -\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k} \frac{\gamma_k^2}{\omega_k - k\cdot v}\left(e^{ik\cdot(x-vt)} + e^{-ik\cdot(x-vt)}\right) \\
=& -\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k} \frac{|f(k^2)|^2}{2\omega_k\left(\omega_k - k\cdot v\right)}\left(e^{ik\cdot(x-vt)} + e^{-ik\cdot(x-vt)}\right) \\
=& -\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k} \frac{|f(k^2)|^2\left(\omega_k + k\cdot v\right)}{2\omega_k\left(\omega_k^2 - (k\cdot v)^2\right)}\left(e^{ik\cdot(x-vt)} + e^{-ik\cdot(x-vt)}\right) \\
=& -\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k} \frac{|f(k^2)|^2}{2\left(\omega_k^2 - (k\cdot v)^2\right)}\left(e^{ik\cdot(x-vt)} + e^{-ik\cdot(x-vt)}\right) \\
& -\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k} \frac{|f(k^2)|^2\left(k\cdot v\right)}{2\omega_k\left(\omega_k^2 - (k\cdot v)^2\right)}\left(e^{ik\cdot(x-vt)} + e^{-ik\cdot(x-vt)}\right)
\end{aligned} \tag{3.232}$$

It remains to show that the second integral vanishes. This can be done by a symmetry argument. Note that the integrant is odd for the substitution $k \to -k$:
$\int \mathrm{d}^3\mathrm{k} \frac{f(k^2)(k\cdot v)}{2\omega_k\left(\omega_k^2-(k\cdot v)^2\right)}\left(e^{ik\cdot(x-vt)}+e^{-ik\cdot(x-vt)}\right) = -\int \mathrm{d}^3\mathrm{k} \frac{f(k^2)(k\cdot v)}{2\omega_k\left(\omega_k^2-(k\cdot v)^2\right)}\left(e^{ik\cdot(x-vt)}+e^{-ik\cdot(x-vt)}\right)$
and hence, we conclude
$\int \mathrm{d}^3\mathrm{k} \frac{f(k^2)(k\cdot v)}{2\omega_k\left(\omega_k^2-(k\cdot v)^2\right)}\left(e^{ik\cdot(x-vt)}+e^{-ik\cdot(x-vt)}\right) = 0$. This completes the proof. □

**Theorem 3.6.7** *In the limit $f \to 1$, $\Phi^{(v)}$ solves the inhomogeneous wave equation with a delta source at the position of the nucleon, i.e.*

$$\left(\Box + \mu^2\right)\Phi^{(v)} = \lambda\delta^{(3)}(x - vt) \tag{3.233}$$

**Remark:** This is exactly the equation a classical field would fulfill, if we assume that the photon has mass $\mu$ and the charge moves along the trajectory $q(t) = vt$. Hence the expectation value of the field in our model exactly agrees with Maxwell theory.

**Proof:** We will prove the statement in the following way: First we find a solution for

$$\left(\Box + \mu^2\right)\varphi(t, x) = \lambda\delta^{(3)}(x - vt) \tag{3.234}$$

and then we will see that it agrees with $\Phi^{(v)}$. This we will do with the Greens function technique. The Greens function $G(t, x)$ for this problem solves

$$\left(\Box + \mu^2\right)G(t, x) = \delta(t)\delta^{(3)}(x) \tag{3.235}$$

Then, the solution can be calculated by

$$\varphi(t, x) = \int \mathrm{d}^4y G(t - y^0, x - y)f(y^0, y) = (G \star f)(t, x) \tag{3.236}$$

with $f(t, x) = \lambda\delta^{(3)}(x - vt)$. Note that here $\star$ denotes a four-convolution. Define the Fourier transform in Minkowski space $\mathcal{M}$ by

$$g(x^0, x) = \frac{1}{(2\pi)^4}\int \mathrm{dk}^0 \int \mathrm{d}^3\mathrm{k} e^{ik^0t-ik\cdot x}\hat{g}(k^0, k) \tag{3.237}$$

$$\hat{g}(k^0, k) = \int \mathrm{dx}^0 \int \mathrm{d}^3\mathrm{x} e^{-ik^0t+ik\cdot x}g(x^0, x) \tag{3.238}$$

where $g$ is a function with arguments $(x^0, x) \in \mathcal{M}$ and $\hat{g}$ is its Fourier transform. Note that we have chosen the metric to be $\eta_{\mu\nu} = diag(1, -1, -1, -1)$.
Then

$$\varphi(t, x) = \frac{1}{(2\pi)^2}\int \mathrm{dk}^0 \int \mathrm{d}^3\mathrm{k} e^{ik^0t-ik\cdot x}\widehat{(G \star f)}(k^0, k) \tag{3.239}$$

From the convolution theorem we know that

$$\widehat{(G \star f)}(k^0, k) = \hat{G}(k^0, k)\hat{f}(k^0, k) \tag{3.240}$$

Hence

$$\varphi(t,x) = \frac{1}{(2\pi)^4}\int \mathrm{dk}^0 \int \mathrm{d}^3\mathrm{k} e^{ik^0 t - ik\cdot x}\hat{G}(k^0,k)\hat{f}(k^0,k) \tag{3.241}$$

The Fourier components of the Greens function are given by

$$\hat{G}(k^0,k) = \frac{1}{-(k^0)^2 + k^2 + \mu^2} \tag{3.242}$$

and

$$\begin{aligned}\hat{f}(k^0,k) &= \int \mathrm{dx}^0 \int \mathrm{d}^3\mathrm{x} e^{-ik^0x^0 + ik\cdot x} f(x^0,x)\\ &= \lambda \int \mathrm{dx}^0 \int \mathrm{d}^3\mathrm{x} e^{-ik^0x^0 + ik\cdot x}\delta^{(3)}(x - vx^0)\\ &= \lambda \int \mathrm{dx}^0 e^{ix^0(k\cdot v - k^0)}\\ &= 2\pi\lambda\delta(k\cdot v - k^0)\end{aligned} \tag{3.243}$$

Then

$$\begin{aligned}\varphi(t,x) &= \frac{\lambda}{(2\pi)^3}\int \mathrm{dk}^0 \int \mathrm{d}^3\mathrm{k} e^{ik^0t - ik\cdot x}\frac{1}{-(k^0)^2 + k^2 + \mu^2}\delta(k\cdot v - k^0)\\ &= \frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k} e^{ik\cdot vt - ik\cdot x}\frac{1}{\omega_k^2 - (k\cdot v)^2}\end{aligned} \tag{3.244}$$

By symmetry, we can show that Then

$$\varphi(t,x) = \frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k} e^{-ik\cdot vt + ik\cdot x}\frac{1}{\omega_k^2 - (k\cdot v)^2} \tag{3.245}$$

is a solution as well. This proves the claim. □

## 3.7 Conclusion

The main result of this chapter is the calculation of the second-quantized field in this particular model. We found that considering a given classical trajectory, the field is just the same as the classical field. In addition to that, there is a free field, which solves the homogeneous wave equation. In this semi-classical model, all quantum effects are encoded in the free field. However, similar as in the classical case, given some initial data at $t_0$, the free field vanishes in the limit $t_0 \to -\infty$ for certain conditions.

# 4 Interaction between a scalar field and a spinless fermion field with a quantum mechanical motion

This chapter examines the same model as in the previous chapter, but now we drop the restriction of the nucleus being fixed at a point. Instead, we want the dynamics of the nucleon to be determined by a quantum mechanical equation. We still consider only one free fermion coupled to the scalar meson field and we assume the nucleon velocity to be in a non-relativistic regime. The Hamiltonian looks like the following:

$$H = H_{kin} + H_0 + V \tag{4.1}$$

$$H_{kin} = \frac{\hat{p}^2}{2m_0} \tag{4.2}$$

$$H_0 = \int \mathrm{d}^3\mathrm{k}\,\omega_k a_k^* a_k \tag{4.3}$$

$$V = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\,\gamma_k (a_k e^{ik\cdot\hat{x}} + a_k^* e^{-ik\cdot\hat{x}}) \tag{4.4}$$

with everything else defined as before. The only difference is the additional $H_{kin}$ term, which gives the dynamics of the (free) fermion and that $\hat{x} := \begin{pmatrix}\hat{x}_1 & \hat{x}_2 & \hat{x}_3\end{pmatrix}^T$ and $\hat{p} := \begin{pmatrix}\hat{p}_1 & \hat{p}_2 & \hat{p}_3\end{pmatrix}^T$ are now operators and not only numbers. This notation is a little bit inconsistent, since $\hat{x}$, $\hat{p}$ and also $a_k$, $a_k^*$ are operators and we denote some of them with a hat and the other ones not. We will still stick to this notation, because it makes clear that $\hat{x}$ and $\hat{p}$, which where just number in the old model are operator now.
Note that $a_k$, $a_k^*$ fulfill the usual commutation relations

$$[a_k, a_{k'}^*] = \delta^{(3)}(k - k') \tag{4.5}$$

$$[a_k^*, a_{k'}^*] = 0 \tag{4.6}$$

$$[a_k, a_{k'}] = 0 \tag{4.7}$$

and

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \tag{4.8}$$

Next, we define this model mathematically

**Definition:** We call

$$\mathcal{H} := L^2(\mathbb{R}^3, \mathbb{C}) \otimes \mathcal{F} \tag{4.9}$$

the Hilbert space of our system, where

$$\mathcal{F} := \bigoplus_{j=0}^{\infty} \mathcal{F}^j, \quad \mathcal{F}^0 := \mathbb{C}, \quad \mathcal{F}^{j\geq 1} := \bigodot_{l=1}^{j} L^2(\mathbb{R}^3, \mathbb{C}, d^3k) \tag{4.10}$$

Hence, an element $\psi \in \mathcal{H}$ is a sequence of functions $\left\{\psi_{(fermion)} \otimes \psi^{(n)}_{(meson)}\right\}$ on $\mathbb{R}^{3+3n}$ with $\|\Psi\| < \infty$, where $\|\cdot\| := \sqrt{\langle\cdot|\cdot\rangle}$ is the norm induced by the scalar product of $\mathcal{H}$ defined in the following:
Let $\psi = \psi_{(fermion)} \otimes \psi_{(meson)}, \xi = \xi_{(fermion)} \otimes \xi_{(meson)} \in \mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}) \otimes \mathcal{F}$, then the scalar product of these elements is given by

$$\langle\psi|\xi\rangle := \sum_{n=0}^{\infty} \int \mathrm{d}^3x \overline{\psi_{(fermion)}(x)} \xi_{(fermion)}(x) \int \mathrm{d}^3k_1 ... \mathrm{d}^3k_n \overline{\psi^{(n)}_{(meson)}(k_1, ..., k_n)} \xi^{(n)}_{(meson)}(k_1, ..., k_n) \tag{4.11}$$

where $\psi^{(n)}_{(meson)}, \xi^{(n)}_{(meson)}$ are symmetric in their arguments (bosons).
Further, we define the meson annihilation and creation operators $a_k, a^*_k$ as before

$$(a_k\psi) := \psi_{(fermion)} \otimes \left(a_k\psi_{(meson)}\right), \quad (a^*_k\psi) := \psi_{(fermion)} \otimes \left(a^*_k\psi_{(meson)}\right) \tag{4.12}$$

with

$$\begin{aligned} \left(a_k\psi_{(meson)}\right)^{(n)}(k_1, ..., k_n) &:= \sqrt{n+1}\, \psi^{(n+1)}_{(meson)}(k, k_1, ..., k_n) \\ \left(a^*_k\psi_{(meson)}\right)^{(n)}(k_1, ..., k_n) &:= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta^{(3)}(k - k_i) \psi^{(n-1)}_{(meson)}(k_1, ..., k_{i-1}, k_{i+1}, ..., k_n) \end{aligned} \tag{4.13}$$

Hence, for any element $|\psi\rangle \in \mathcal{H}$, we can write

$$|\psi\rangle = |\psi_{fermion}\rangle \otimes \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int \mathrm{d^3k_1...d^3k_n} \psi^{(n)}(k_1, ..., k_n) a^*_{k_1} ... a^*_{k_n} |0\rangle \tag{4.14}$$

where $|0\rangle$ is the vacuum state of $\mathcal{F}$ and $\langle 0|0\rangle = 1$. Formally, this is implemented by the commutation relations mentioned before.
Further, let n be the meson particle number operator defined by

$$(n\psi) := \psi_{(fermion)} \otimes \left(n\psi_{(meson)}\right), \quad \left(n\psi_{(meson)}\right)^{(n)} := n\psi^{(n)}_{(meson)} \tag{4.15}$$

on the domain $\mathcal{D}(n)$ of all $\psi \in \mathcal{H}$, such that $\left\{n\psi^{(n)}\right\}$ are again in $\mathcal{H}$.
Further, the position and the momentum operator just act the fermion part:

$$(\hat{x}\psi) := \left(\hat{x}\psi_{(fermion)}\right) \otimes \psi_{(meson)}, \quad (\hat{p}\psi) := \left(\hat{p}\psi_{(fermion)}\right) \otimes \psi_{(meson)} \tag{4.16}$$

and there are defined in the usual way.

In the following, talking about states in the Hilbert space, we always mean the normalized ones. Everything else is defined as in the previous chapter.

Then, we write down the following well-defined Hamiltonian

**Definition:** Let $f \in C_0^\infty$ be a cutoff function, $\omega_k = \sqrt{k^2 + \mu^2}$ and $\gamma_k = \frac{f(k^2)}{\sqrt{2\omega_k}}$, then we define the Hamiltonian by

$$H := \frac{\hat{p}^2}{2m_0} + \int \mathrm{d^3k}\omega_k a_k^* a_k + \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d^3k}\gamma_k (a_k e^{ik\cdot\hat{x}} + a_k^* e^{-ik\cdot\hat{x}}) \tag{4.17}$$

$$= H_{kin} + H_0 + V \tag{4.18}$$

The domain of this Hamiltonian is

$$\mathcal{D}(H) = \mathcal{D}(H_{kin}) \cap \mathcal{D}(n) \tag{4.19}$$

where

- $\mathcal{D}(H_{kin})$ contains all $\psi \in \mathcal{H}$, such that $\left\{H_{kin}\psi^{(n)}\right\}$ are again in $\mathcal{H}$
- $\mathcal{D}(n)$ contains all $\psi \in \mathcal{H}$, such that $\left\{n\psi^{(n)}\right\}$ are again in $\mathcal{H}$

## 4.1 Time evolution - Dressing operator

Next, we want to find the time evolution operator for this model. In order to find this operator, we follow a similar procedure as in the previous chapter. First, we calculate the evolution operator in the interaction picture. Conceptually, this computation is absolutely analogous to the semi-classical case, but along the way there appear further difficulties, since $\hat{x}$ and $\hat{p}$ do not commute. We know

$$i\partial_t U_I(t, t_0) = V(t)U_I(t, t_0) \tag{4.20}$$

with

$$V(t) = U_0^*(t)VU_0(t) \tag{4.21}$$

Here $U_0(t) := e^{-i(H_{kin}+H_0)t}$ defines the free time evolution.
Again, this yields

$$U_I(t, t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} U_I^{(n)}(t, t_0) \tag{4.22}$$

with

$$U_I^{(n)}(t, t_0) = (-i)^n \int_{t_0}^{t} \mathrm{dt_1 ... dt_n} T\Big[V(t_1)...V(t_n)\Big] \tag{4.23}$$

This means we have to calculate $V(t)$:

$$\begin{aligned} V(t) &= U_0^*(t) V U_0(t) \\ &= e^{i(H_{kin}+H_0)t} V e^{-i(H_{kin}+H_0)t} \\ &= e^{iH_{kin}t} e^{iH_0 t} V e^{-iH_0 t} e^{-iH_{kin}t} \\ &= e^{iH_{kin}t} \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \left( a_k e^{ik\cdot\hat{x} - i\omega_k t} + a_k^* e^{-ik\cdot\hat{x} + i\omega_k t} \right) e^{-iH_{kin}t} \\ &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \left( a_k e^{iH_{kin}t} e^{ik\cdot\hat{x}} e^{-iH_{kin}t} e^{-i\omega_k t} + c.c. \right) \end{aligned} \tag{4.24}$$

In the first step, we have used the Baker-Campbell-Hausdorff formula with $[H_0, H_{kin}] = 0$ and in the second step, we plugged in the result for $e^{iH_0 t} V e^{-iH_0 t}$, which we have already computed in the previous chapter. It remains to compute $e^{iH_{kin}t} e^{ik\cdot\hat{x}} e^{-iH_{kin}t}$. Use that $e^{-iH_{kin}t}$ is an unitary operator, i.e. $e^{-iH_{kin}t} e^{iH_{kin}t} = \mathbb{I}$. Then

$$\begin{aligned} e^{iH_{kin}t} e^{ik\cdot\hat{x}} e^{-iH_{kin}t} &= \sum_{n=0}^{\infty} \frac{i^n}{n!} e^{iH_{kin}t} (k\cdot\hat{x})^n e^{-iH_{kin}t} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \underbrace{e^{iH_{kin}t}(k\cdot\hat{x})e^{-iH_{kin}t} ... e^{iH_{kin}t}(k\cdot\hat{x})e^{-iH_{kin}t}}_{\text{n times}} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(k\cdot\hat{x}(t)\right)^n = e^{ik\cdot\hat{x}(t)} \end{aligned} \tag{4.25}$$

where

$$\hat{x}(t) = e^{iH_{kin}t} \hat{x} e^{-iH_{kin}t} = \hat{x} + e^{iH_{kin}t} \left[ \hat{x}, e^{-iH_{kin}t} \right] \tag{4.26}$$

Use $[\hat{x}_i, F(\hat{p}_i)] = i\partial_{\hat{p}_i} F(\hat{p}_i)$ and $e^{-iH_{kin}t} = e^{-i\frac{\hat{p}_1^2}{2m_0}t} e^{-i\frac{\hat{p}_2^2}{2m_0}t} e^{-i\frac{\hat{p}_3^2}{2m_0}t}$ in order to get

$$\left[ \hat{x}(t), e^{-iH_{kin}t} \right] = \frac{\hat{p}t}{m_0} e^{-iH_{kin}t} \tag{4.27}$$

Then

$$\hat{x}(t) = \hat{x} + \frac{\hat{p}t}{m_0} \tag{4.28}$$

This yields

$$V(t) = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \left( a_k e^{ik\cdot\hat{x} - i(\omega_k - \frac{k\cdot\hat{p}}{m_0})t} + a_k^* e^{-ik\cdot\hat{x} + i(\omega_k - \frac{k\cdot\hat{p}}{m_0})t} \right) \tag{4.29}$$

Let us start with computing the first order of $U_I(t, t_0)$:

$$\begin{aligned} U_I^{(1)}(t, t_0) &= -i \int_{t_0}^{t} \mathrm{dt}_1 V(t_1) \\ &= -i \int_{t_0}^{t} \mathrm{dt}_1 \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \left( a_k e^{ik\cdot\hat{x} - i(\omega_k - \frac{k\cdot\hat{p}}{m_0})t} + a_k^* e^{-ik\cdot\hat{x} + i(\omega_k - \frac{k\cdot\hat{p}}{m_0})t} \right) \end{aligned} \tag{4.30}$$

This is seems to be pretty much the same as in the static case. Nevertheless, performing the $\int_{t_0}^{t} \mathrm{d}t_1$ integration is not as simple, since formally we have to integrate over an exponential of an operator in this case. We know $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$ is a number, hence

$$[\hat{x}, [\hat{x}, \hat{p}]] = 0 \text{ and } [\hat{p}, [\hat{x}, \hat{p}]] = 0 \tag{4.31}$$

Then, by the Baker-Campbell-Hausdorff formula, we know that

$$\begin{aligned} e^{ik\cdot\hat{x}-i(\omega_k-\frac{k\cdot\hat{p}}{m_0})t} &= e^{-i\omega_k t} e^{i\frac{k\cdot\hat{p}}{m_0}t} e^{ik\cdot\hat{x}} e^{-\frac{1}{2}\left[i\frac{\hat{p}t}{m_0}, ik\cdot\hat{x}\right]} \\ &= e^{-i\omega_k t} e^{i\frac{k\cdot\hat{p}}{m_0}t} e^{ik\cdot\hat{x}} e^{-\frac{t}{2m_0}[k\cdot\hat{x}, k\cdot\hat{p}]} \\ &= e^{-i\omega_k t} e^{i\frac{k\cdot\hat{p}}{m_0}t} e^{ik\cdot\hat{x}} e^{\frac{-ik^2 t}{2m_0}} \\ &= e^{i\left(\frac{k\cdot\hat{p}}{m_0}-\omega_k-\frac{k^2}{2m_0}\right)t} e^{ik\cdot\hat{x}} \end{aligned} \tag{4.32}$$

Hence, we obtain

$$\begin{aligned} \int \mathrm{dt} e^{ik\cdot\hat{x}-i(\omega_k-\frac{k\cdot\hat{p}}{m_0})t} &= -i\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)^{-1} e^{i\left(\frac{k\cdot\hat{p}}{m_0}-\omega_k-\frac{k^2}{2m_0}\right)t} e^{ik\cdot\hat{x}} + const. \\ &= -i\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)^{-1} e^{ik\cdot\hat{x}-i(\omega_k-\frac{k\cdot\hat{p}}{m_0})t} + const. \end{aligned} \tag{4.33}$$

where

$$\frac{1}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} := \left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)^{-1} \tag{4.34}$$

is the inverse operator of $\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)$.
Putting everything together, we end up with

$$\begin{aligned} U_I^{(1)}(t, t_0) = -\frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d^3k} \frac{\gamma_k}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} \Big( a_k e^{ik\cdot\hat{x}-i(\omega_k-\frac{k\cdot\hat{p}}{m_0})t} - a_k^* e^{-ik\cdot\hat{x}+i(\omega_k-\frac{k\cdot\hat{p}}{m_0})t} \\ - a_k e^{ik\cdot\hat{x}-i(\omega_k-\frac{k\cdot\hat{p}}{m_0})t_0} + a_k^* e^{-ik\cdot\hat{x}+i(\omega_k-\frac{k\cdot\hat{p}}{m_0})t_0} \Big) \end{aligned} \tag{4.35}$$

**Remark:** In general, it is not clear, if the inverse of $\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)$ exists. In order to check, if this operator is invertible, it is in principle enough to show that it is non-zero everywhere (in momentum space). Physically, it is obviously a good assumption that we restrict ourself to velocities less than the speed of light, i.e. $v = \frac{|p|}{m_0} < 1$. Then

$$\frac{p\cdot k}{m_0} = \frac{|p|}{m_0}|k|\cos\theta < |k| \le \omega_k \le \omega_k + \frac{k^2}{2m_0} \tag{4.36}$$

$$\Rightarrow \left(\frac{k\cdot p}{m_0} - \omega_k - \frac{k^2}{2m_0}\right) < 0 \tag{4.37}$$

Hence, for velocities below the speed of light, this operator is invertible.

Similarly to the previous chapter, we can show that for $t_0 \to -\infty$, $U_I^{(1)}(t, t_0)$ weakly converges to

$$-\frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} \left(a_k e^{ik\cdot\hat{x} - i(\omega_k - \frac{k\cdot\hat{p}}{m_0})t} - a_k^* e^{-ik\cdot\hat{x} + i(\omega_k - \frac{k\cdot\hat{p}}{m_0})t}\right) \tag{4.38}$$

i.e. in the weak limit all terms containing $t_0$ vanish. We can extend this to all orders. This yields the dressing operator

**Definition:**

$$\begin{aligned} D_f(\hat{x}, t) :&= \lim_{t_0 \to -\infty} U_I(t, t_0) \\ &= \exp\left\{-\frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} \left(a_k e^{ik\cdot\hat{x}(t) - i\omega_k t} - a_k^* e^{-ik\cdot\hat{x}(t) + i\omega_k t}\right)\right\} \end{aligned} \tag{4.39}$$

where $\hat{x}(t) = \hat{x} + \frac{\hat{p}t}{m_0}$ and by $\lim\limits_{t_0 \to -\infty}$ we mean the weak limit.

**Remark:** In principle, we would have to prove the weak convergence in the definition above. Though, this proof would be very similar to the one we have done in the previous chapter. We would not learn something new out of it. Therefore we skip this proof.
Further, note that even though the $\frac{k^2}{2m_0}$ looks like a non-relativistic energy, it also appears, if we consider a relativistic dispersion.

In the following we will need to calculate commutators like $[a_k, D_f(\hat{x}, t)]$. Hence, we state the following

**Lemma 4.1.1** *Let* $|\xi\rangle, |\phi\rangle \in \mathcal{H}$. *Then*

$$\langle\xi| [a_k, D_f(\hat{x}, t)] |\phi\rangle = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \langle\xi| D_f(\hat{x}, t) \frac{\gamma_k}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} e^{-ik\cdot\hat{x}(t) + i\omega_k t} |\phi\rangle + \mathcal{O}(\lambda^2) \tag{4.40}$$

**Remark:** As in the previous chapter, we would like to establish an exact identity for this commutator. Unfortunately, this is not possible in this situation. We cannot separate the motion of the charge from the dynamics of the meson field, since it interacts with each other all the time. This comes from the way we have written down the interaction term in the Hamiltonian.

In the end, we can still make similar statements as in the previous chapter, but these statement only hold up to first order. This may still be a good approximation, since the coupling constant $\lambda$ is very small. Further, the only effects we are neglecting by doing so, are radiation back-reaction and similar things. However, the Coulomb field is just first order in $\lambda$ anyways. Nevertheless, we are not able to consider for example the loss of energy of the nucleon due to radiation. These effects are very small though.
In the semi-classical case, this problem did not arise, since there the trajectory of the particle was given, hence the model did not include effects like back-coupling anyways.

**Proof:** Let us define

$$Z := \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}' \frac{\gamma_{k'}}{\left(\frac{k'\cdot\hat{p}}{m_0} - \omega_{k'} - \frac{k'^2}{2m_0}\right)} \left(a_{k'} e^{ik'\cdot\hat{x}(t) - i\omega_{k'}t} - a^*_{k'} e^{-ik'\cdot\hat{x}(t) + i\omega_{k'}t}\right) \tag{4.41}$$

Then, we know for any operator $T$

$$D^*_f(\hat{x},t) T D_f(\hat{x},t) = e^{\lambda Z} T e^{-\lambda Z} = T + \frac{\lambda}{1!}[Z,T] + \frac{\lambda^2}{2!}[Z,[Z,T]] + ... \tag{4.42}$$

Now, set $T := a_k$, and calculate

$$\begin{aligned} [Z, a_k] &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}' \frac{\gamma_{k'}}{\left(\frac{k'\cdot\hat{p}}{m_0} - \omega_{k'} - \frac{k'^2}{2m_0}\right)} \left([a_{k'}, a_k] e^{ik'\cdot\hat{x}(t) - i\omega_{k'}t} - [a^*_{k'}, a_k] e^{-ik'\cdot\hat{x}(t) + i\omega_{k'}t}\right) \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{\gamma_k}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} e^{-ik\cdot\hat{x}(t) + i\omega_k t} \end{aligned} \tag{4.43}$$

Then, the equation above reads like

$$D^*_f(\hat{x},t) a_k D_f(\hat{x},t) = a_k + \lambda[Z, a_k] + \mathcal{O}(\lambda^2) \tag{4.44}$$

On the other hand

$$D^*_f(\hat{x},t) a_k D_f(\hat{x},t) = a_k + D^*_f(\hat{x},t)[a_k, D_f(\hat{x},t)] \tag{4.45}$$

Use that $D_f(\hat{x},t)$ is unitary and obtain

$$[a_k, D_f(\hat{x},t)] = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} D_f(\hat{x},t) \frac{\gamma_k}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} e^{-ik\cdot\hat{x}(t) + i\omega_k t} + \mathcal{O}(\lambda^2) \tag{4.46}$$

This proves the claim. □

**Remark:** We would need $[Z,[Z,a_k]] = 0$, for an exact equality in the theorem. In the following, we get only first order effect, like the emission of free photons. But we neglect higher order terms and therefore effects like back-coupling straight from beginning.

## 4.2 Fields and initial data

Again, as in the static case, we would like to know a little bit more about the field, i.e. we want to calculate the expectation value of the field in the time evolved state $|\xi_t\rangle = \lim_{t_0\to-\infty} U(t,t_0)\,|\xi_{t_0}\rangle$, where $|\xi_{t_0}\rangle \in L^2(\mathbb{R}^3)\otimes\mathcal{F}$ is the given initial data at $t_0 \to -\infty$ and $U(t,t_0)$ is the full time evolution from $t_0$ to $t$. We already know the time evolution in the interaction picture, therefore we can easily obtain the full time evolution by

$$U(t,t_0) = e^{-i(H_0+H_{kin})t} U_I(t,t_0) e^{i(H_0+H_{kin})t_0} =: U_0(t) U_I(t,t_0) U^*_0(t_0) \tag{4.47}$$

**Definition:**

$$U_0(t) := e^{-i(H_0+H_{kin})t} \tag{4.48}$$

is called the free time evolution.

We know that the interaction Hamiltonian looks like

$$V(t) = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \left(a_k e^{ik\cdot\hat{x}(t)-i\omega_k t} + a_k^* e^{-ik\cdot\hat{x}(t)+i\omega_k t}\right) \tag{4.49}$$

Therefore we define the following field in the interaction picture

**Definition:**

$$\varphi_0(x) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \left(a_k e^{ik\cdot x} + a_k^* e^{-ik\cdot x}\right) \tag{4.50}$$

with $x \in \mathbb{R}^3$ and

$$\begin{aligned} \varphi_t(x) &:= U_0^*(t)\varphi_t(x)U_0(t) \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \left(a_k e^{ik\cdot x-i\omega_k t} + a_k^* e^{-ik\cdot x+i\omega_k t}\right) \end{aligned} \tag{4.51}$$

**Theorem 4.2.1** *Let $f = f_\kappa^\Lambda$ be an appropriate cutoff function and let $|\xi_{t_0}\rangle = \left|\xi_{t_0}^{(x)}\right\rangle \otimes \left|\xi_{t_0}^{(\mathcal{F})}\right\rangle \in L^2(\mathbb{R}^3) \otimes \mathcal{F}$ be the initial state at $t_0 \to -\infty$. Then*

$$\begin{aligned} \lim_{t_0\to-\infty} \langle\xi_t|\, \varphi_0^{\kappa,\Lambda}(x)\, |\xi_t\rangle &= \lim_{t_0\to-\infty} \langle\xi_{t_0}|\, U(t_0,t)\varphi_0^{\kappa,\Lambda}(x)U(t,t_0)\, |\xi_{t_0}\rangle \\ &= \underbrace{\lim_{t_0\to-\infty} \left\langle\xi_{t_0}^{(\mathcal{F})}\right| \varphi_{t-t_0}^{\kappa,\Lambda}(x) \left|\xi_{t_0}^{(\mathcal{F})}\right\rangle}_{\textit{free field}} + \lim_{t_0\to-\infty} \left\langle\xi_{t_0}^{(x)}\right| \hat{\Phi}_{t-t_0}^{\kappa,\Lambda}(x) \left|\xi_{t_0}^{(x)}\right\rangle \end{aligned} \tag{4.52}$$

*with the field operator*

$$\hat{\Phi}_t^{\kappa,\Lambda}(x) := \frac{\lambda}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k}\left( \frac{\gamma_k^2}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} e^{ik\cdot(x-\hat{x}(t))} + c.c.\right) + \mathcal{O}(\lambda^2) \tag{4.53}$$

*where c.c. denotes the conjugated.*
*Further, suppose $|\xi_{t_0}\rangle \in L^2(\mathbb{R}^3) \otimes \mathcal{F}$ is a normalized product state, i.e.*

$$\begin{aligned} |\xi_{t_0}\rangle &= \left|\xi_{t_0}^{(x)}\right\rangle \otimes \left|\xi_{t_0}^{(\mathcal{F})}\right\rangle \\ &= \left|\xi_{t_0}^{(x)}\right\rangle \otimes \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \prod_{i=1}^{n} \int \mathrm{d}^3\mathrm{k_i}\xi_{t_0}^{(n)}(k_1,...,k_n) a_{k_i}\, |0\rangle \end{aligned} \tag{4.54}$$

*with $\left|\xi_{t_0}^{(x)}\right\rangle \in L^2(\mathbb{R}^3)$ beeing normalized, i.e. $\left\langle\xi_{t_0}^{(x)}\middle|\xi_{t_0}^{(x)}\right\rangle = \left\langle\xi_{t_0}^{(\mathcal{F})}\middle|\xi_{t_0}^{(\mathcal{F})}\right\rangle = 1$ and*

$$\xi_{t_0}^{(n)}(k_1,...,k_n) = \frac{1}{\sqrt{n!}} \prod_{i=1}^{n} h(k_i) \quad \forall n \in \mathbb{N} \tag{4.55}$$

*and let one of the following be true*

1. *Let* $f = f_\kappa^\Lambda \in C_0^\infty(\mathbb{R}^3)$

2. *Let* $f = f_{\frac{1}{\Lambda}}^\Lambda$ *be an arbitrary cutoff function and let* $h, \nabla h, \Delta h \in L^2(\mathbb{R}^3)$*. Further, suppose we can find* $k_0 > 0$ *and* $\epsilon > 0$*, such that*

$$|h(k)| \leq |k|^{-3-\epsilon} \quad \forall k \in \{k \in \mathbb{R}^3 : |k| \geq k_0\} \tag{4.56}$$

*Then, for any* $\Lambda > \kappa > 0$

$$\lim_{t_0 \to -\infty} \langle \xi_{t_0} | \varphi_{t-t_0}^{\kappa,\Lambda}(x) | \xi_{t_0} \rangle = 0 \ \forall x \in \mathbb{R}^3 \tag{4.57}$$

*In the second case*

$$\lim_{\Lambda \to \infty} \lim_{t_0 \to -\infty} \langle \xi_{t_0} | \varphi_{t-t_0}^{\frac{1}{\Lambda},\Lambda}(x) | \xi_{t_0} \rangle = 0 \ \forall x \in \mathbb{R}^3 \tag{4.58}$$

**Remark:** This is theorem is very similar to the one in the static case. In fact, it is exactly the same except the space, we are looking at is $L^2(\mathbb{R}^3) \otimes \mathcal{F}$ (instead of $\mathcal{F}$) and the field $\Phi_t(x)$ is a different one.

**Proof:**

$$\begin{aligned}
\lim_{t_0 \to -\infty} \langle \xi_t | \varphi_0^{\kappa,\Lambda}(x) | \xi_t \rangle &= \lim_{t_0 \to -\infty} \langle \xi_{t_0} | U(t_0,t)\varphi_0^{\kappa,\Lambda}(x)U(t,t_0) | \xi_{t_0} \rangle \\
&= \lim_{t_0 \to -\infty} \langle \xi_{t_0} | U_0(t_0)U_I(t_0,t)U_0^*(t)\varphi_0^{\kappa,\Lambda}(x)U_0(t)U_I(t,t_0)U_0^*(t_0) | \xi_{t_0} \rangle \\
&= \lim_{t_0 \to -\infty} \langle \xi_{t_0} | U_0(t_0)U_I(t_0,t)\varphi_t^{\kappa,\Lambda}(x)U_I(t,t_0)U_0^*(t_0) | \xi_{t_0} \rangle \\
&= \lim_{t_0 \to -\infty} \langle \xi_{t_0} | U_0(t_0)D_f^*(\hat{x},t)\varphi_t^{\kappa,\Lambda}(x)D_f(\hat{x},t)U_0^*(t_0) | \xi_{t_0} \rangle
\end{aligned} \tag{4.59}$$

We have used that $U_I(t,t_0)$ converges weakly to $D_f(\hat{x},t)$ as $t_0 \to -\infty$, this is guaranteed by one of the two conditions in the theorem.
Let us continue with the calculation:

$$\begin{aligned}
&D_f^*(\hat{x},t)\varphi_t^{\kappa,\Lambda}(x)D_f(\hat{x},t) \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \Big( D_f^*(\hat{x},t)a_k D_f(\hat{x},t)e^{ik\cdot x - i\omega_k t} + D_f^*(\hat{x},t)a_k^* D_f(\hat{x},t)e^{-ik\cdot x + i\omega_k t} \Big) \\
&= \varphi_t^{\kappa,\Lambda}(x) \\
&+ \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \Big( D_f^*(\hat{x},t)[a_k, D_f(\hat{x},t)]e^{ik\cdot x - i\omega_k t} + \Big[D_f^*(\hat{x},t), a_k^*\Big] D_f(\hat{x},t)e^{-ik\cdot x + i\omega_k t} \Big)
\end{aligned} \tag{4.60}$$

We know for matrix elements

$$[a_k, D_f(\hat{x},t)] = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} D_f(\hat{x},t) \frac{\gamma_k}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} e^{-ik\cdot\hat{x}(t) + i\omega_k t} + \mathcal{O}(\lambda^2) \tag{4.61}$$

and hence

$$\left[D_f^*(\hat{x},t),a_k^*\right] = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} e^{ik\cdot\hat{x}(t)-i\omega_k t} \frac{\gamma_k}{\left(\frac{k\cdot\hat{p}}{m_0}-\omega_k-\frac{k^2}{2m_0}\right)} D_f^*(\hat{x},t) + \mathcal{O}(\lambda^2) \tag{4.62}$$

Apply this and obtain

$$\begin{aligned} &D_f^*(\hat{x},t)\varphi_t^{\kappa,\Lambda}(x)D_f(\hat{x},t) \\ &= \varphi_t^{\kappa,\Lambda}(x) + \frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\left(\frac{\gamma_k^2}{\left(\frac{k\cdot\hat{p}}{m_0}-\omega_k-\frac{k^2}{2m_0}\right)} e^{ik\cdot(x-\hat{x}(t))} + c.c.\right) + \mathcal{O}(\lambda^2) \end{aligned} \tag{4.63}$$

Hence, we end up with

$$\begin{aligned} &\lim_{t_0\to-\infty} \langle\xi_t|\,\varphi_0^{\kappa,\Lambda}(x)\,|\xi_t\rangle \\ &= \lim_{t_0\to-\infty} \langle\xi_{t_0}|\,U(t_0,t)\varphi_0^{\kappa,\Lambda}(x)U(t,t_0)\,|\xi_{t_0}\rangle \\ &= \lim_{t_0\to-\infty} \left(\langle\xi_{t_0}|\,U_0(t_0)\varphi_t^{\kappa,\Lambda}(x)U_0^*(t_0)\,|\xi_{t_0}\rangle + \langle\xi_{t_0}|\,U_0(t_0)\hat{\Phi}_t(x)U_0^*(t_0)\,|\xi_{t_0}\rangle\right) \\ &= \lim_{t_0\to-\infty} \left(\langle\xi_{t_0}|\,\varphi_{t-t_0}^{\kappa,\Lambda}(x)\,|\xi_{t_0}\rangle + \langle\xi_{t_0}|\,U_0(t_0)\hat{\Phi}_t^{\kappa,\Lambda}(x)U_0^*(t_0)\,|\xi_{t_0}\rangle\right) \end{aligned} \tag{4.64}$$

with

$$\hat{\Phi}_t^{\kappa,\Lambda}(x) = \frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\left(\frac{\gamma_k^2}{\left(\frac{k\cdot\hat{p}}{m_0}-\omega_k-\frac{k^2}{2m_0}\right)} e^{ik\cdot(x-\hat{x}(t))} + c.c.\right) + \mathcal{O}(\lambda^2) \tag{4.65}$$

and $\hat{x}(t) = U_0^*(t)\hat{x}U_0(t)$

$$\begin{aligned} &U_0(t_0)\hat{\Phi}_t^{\kappa,\Lambda}(x)U_0^*(t_0) \\ &= \frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\left(\frac{\gamma_k^2}{\left(\frac{k\cdot\hat{p}}{m_0}-\omega_k-\frac{k^2}{2m_0}\right)} e^{ik\cdot(x-\hat{x}(t-t_0))} + c.c.\right) + \mathcal{O}(\lambda^2) \\ &= \frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\left(\frac{\gamma_k^2}{\left(\frac{k\cdot\hat{p}}{m_0}-\omega_k-\frac{k^2}{2m_0}\right)} e^{ik\cdot(x-\hat{x}-\frac{\hat{p}}{m_0}(t-t_0))} + c.c.\right) + \mathcal{O}(\lambda^2) \\ &=: \hat{\Phi}_{t-t_0}^{\kappa,\Lambda}(x) \end{aligned} \tag{4.66}$$

Note that

$$\begin{aligned} \left|\xi_{t_0}^{(x)}\right\rangle &= |\xi_x\rangle \otimes \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\prod_{i=1}^{n}\int \mathrm{d}^3\mathrm{k_i}\,\xi_{t_0}^{(n)}(k_1,...,k_n)a_{k_i}\,|0\rangle \\ &= \left|\xi_{t_0}^{(x)}\right\rangle \otimes \left|\xi_{t_0}^{(\mathcal{F})}\right\rangle \end{aligned} \tag{4.67}$$

So $\hat{x}$ only acts on the $\left|\xi_{t_0}^{(x)}\right\rangle$ term and $a_k, a_k^*$ only act on the $\left|\xi_{t_0}^{(\mathcal{F})}\right\rangle$ terms.
Then, we obtain

$$\lim_{t_0\to-\infty}\langle\xi_{t_0}|\,\varphi_{t-t_0}^{\kappa,\Lambda}(x)\,|\xi_{t_0}\rangle = \lim_{t_0\to-\infty}\left\langle\xi_{t_0}^{(x)}\right| \otimes \left\langle\xi_{t_0}^{(\mathcal{F})}\right|\varphi_{t-t_0}^{\kappa,\Lambda}(x)\left|\xi_{t_0}^{(x)}\right\rangle \otimes \left|\xi_{t_0}^{(\mathcal{F})}\right\rangle \tag{4.68}$$

Note that $\varphi_{t-t_0}(x)$ contains no $\hat{x}$ operator and $\left\langle \xi_{t_0}^{(x)} \middle| \xi_{t_0}^{(x)} \right\rangle = 1$ in order to obtain

$$\lim_{t_0 \to -\infty} \langle \xi_{t_0} | \, \varphi_{t-t_0}^{\kappa,\Lambda}(x) \, | \xi_{t_0} \rangle = \lim_{t_0 \to -\infty} \left\langle \xi_{t_0}^{(\mathcal{F})} \middle| \, \varphi_{t-t_0}^{\kappa,\Lambda}(x) \, \middle| \xi_{t_0}^{(\mathcal{F})} \right\rangle \tag{4.69}$$

Similarly,

$$\lim_{t_0 \to -\infty} \langle \xi_{t_0} | \, \hat{\Phi}_{t-t_0}^{\kappa,\Lambda}(x) \, | \xi_{t_0} \rangle = \lim_{t_0 \to -\infty} \left\langle \xi_{t_0}^{(x)} \right| \otimes \left\langle \xi_{t_0}^{(\mathcal{F})} \right| \, \hat{\Phi}_{t-t_0}^{\kappa,\Lambda}(x) \, \left| \xi_{t_0}^{(x)} \right\rangle \otimes \left| \xi_{t_0}^{(\mathcal{F})} \right\rangle \tag{4.70}$$

Again, note that $\hat{\Phi}_t(x))$ contains no $a_k, a_k^*$ operators and $\left\langle \xi_{t_0}^{(\mathcal{F})} \middle| \xi_{t_0}^{(\mathcal{F})} \right\rangle = 1$ in order to obtain

$$\lim_{t_0 \to -\infty} \langle \xi_{t_0} | \, \hat{\Phi}_{t-t_0}^{\kappa,\Lambda}(x) \, | \xi_{t_0} \rangle = \lim_{t_0 \to -\infty} \left\langle \xi_{t_0}^{(x)} \middle| \, \hat{\Phi}_{t-t_0}^{\kappa,\Lambda}(x) \, \middle| \xi_{t_0}^{(x)} \right\rangle \tag{4.71}$$

Putting everything together, we obtain

$$\lim_{t_0 \to -\infty} \langle \xi_t | \, \varphi_0^{\kappa,\Lambda}(x) \, | \xi_t \rangle = \lim_{t_0 \to -\infty} \left\langle \xi_{t_0}^{(\mathcal{F})} \middle| \, \varphi_{t-t_0}^{\kappa,\Lambda}(x) \, \middle| \xi_{t_0}^{(\mathcal{F})} \right\rangle + \lim_{t_0 \to -\infty} \left\langle \xi_{t_0}^{(x)} \middle| \, \hat{\Phi}_{t-t_0}(x) \, \middle| \xi_{t_0}^{(x)} \right\rangle \tag{4.72}$$

This proves the first part of the theorem and reduces the second part exactly to the one, which we have already proved for the static case. □

So far so good, but now until now we do not know how the field $\Phi_t(x)$ look like. In the following we want to examine this field a little bit further.
First we notice that in the limit $m_0 \to \infty$, i.e. in the static case, we formally get

$$\begin{aligned}
\hat{\Phi}_t(x) &= \frac{\lambda}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \left( \frac{\gamma_k^2}{\left( \frac{k \cdot \hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0} \right)} e^{ik\cdot(x-\hat{x}(t))} + c.c. \right) + \mathcal{O}(\lambda^2) \\
&\to -\frac{\lambda}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k^2}{\omega_k} \left( e^{ik\cdot(x-\hat{x}(t))} + e^{-ik\cdot(x-\hat{x}(t))} \right) + \mathcal{O}(\lambda^2) \\
&= \frac{\lambda}{4\pi} \frac{e^{-\mu|x-\hat{x}(t)|}}{|x-\hat{x}(t)|} + \mathcal{O}(\lambda^2)
\end{aligned} \tag{4.73}$$

as $m_0 \to \infty$. This is again the Yukawa potential (up to second order in $\lambda$). Hence in this limit we get the same result as before, which is a good indication that our calculations are right.

If we want to calculate the expectation value of the field operator given some initial state $\left| \xi_{t_0}^{(x)} \right\rangle$, it is useful to describe everything in the momentum space.

**Definition:** Let us define everything in the momentum space:

$$\left| \xi_{t_0}^{(x)} \right\rangle = \int \mathrm{d}^3\mathrm{p} \, \Xi(p) \, | p \rangle \tag{4.74}$$

with $\Xi \in L^2(\mathbb{R}^3)$ being the initial data at $t_0 \to -\infty$.

$$\hat{x} = i\nabla_p \Rightarrow \hat{x}(t) = i\nabla_p + \frac{pt}{m_0} \tag{4.75}$$

is the position operator in momentum space and

$$\begin{aligned}\Phi_t(x) &:= \left\langle \xi_{t_0}^{(x)} \middle| \hat{\Phi}_t(x) \middle| \xi_{t_0}^{(x)} \right\rangle \\ &= \frac{\lambda}{(2\pi)^3} \int \mathrm{d^3p} \int \mathrm{d^3k} \overline{\Xi(p)} \left( \frac{\gamma_k^2}{\left(\frac{k\cdot p}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} e^{ik\cdot(x - i\nabla_p - \frac{pt}{m_0})} + c.c. \right) \Xi(p) + \mathcal{O}(\lambda^2)\end{aligned} \tag{4.76}$$

Now, in the momentum space we can calculate the field for an arbitrary initial data $\Xi \in L^2(\mathbb{R}^3)$. Using the Baker-Campbell-Hausdorff again, yields

$$\begin{aligned}e^{ik\cdot(x - i\nabla_p - \frac{pt}{m_0})} &= e^{ik\cdot x} e^{k\cdot\nabla_p} e^{-ik\cdot\frac{pt}{m_0}} e^{-\frac{1}{2}\left[k\cdot\nabla_p, -ik\cdot\frac{pt}{m_0}\right]} \\ &= e^{ik\cdot x + i\frac{k^2}{2m_0}t} e^{k\cdot\nabla_p} e^{-ik\cdot\frac{pt}{m_0}}\end{aligned} \tag{4.77}$$

We also know

$$e^{k\cdot\nabla_p} f(p) = f(p+k) \tag{4.78}$$

Hence

$$\begin{aligned}e^{ik\cdot(x - i\nabla_p - \frac{pt}{m_0})}\Xi(p) &= e^{ik\cdot x + i\frac{k^2}{2m_0}t} e^{k\cdot\nabla_p}\left(e^{-ik\cdot\frac{pt}{m_0}}\Xi(p)\right) = e^{ik\cdot x + i\frac{k^2}{2m_0}t} e^{-ik\cdot\frac{(p+k)t}{m_0}}\Xi(p+k) \\ &= e^{ik\cdot(x - \frac{p}{m_0}t) - i\frac{k^2}{2m_0}t}\Xi(p+k)\end{aligned} \tag{4.79}$$

Then

$$\Phi_t(x) = \frac{\lambda}{(2\pi)^3} \int \mathrm{d^3pd^3k} \frac{\gamma_k^2}{\left(\frac{k\cdot p}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} \left( \overline{\Xi(p)}\Xi(p+k) e^{ik\cdot(x - \frac{p}{m_0}t) - i\frac{k^2}{2m_0}t} + c.c. \right) + \mathcal{O}(\lambda^2) \tag{4.80}$$

**Remark:** In principle, we could calculate the field with this formula, given some initial wave function. Though, this integral is implicit and $\Xi$-dependent and one does not see immediately a connection to the classical Liénard-Wiechert fields. This is due to the fact that it is impossible the separate the field from the nucleon in this model.
In the classical computation, we calculated the LWF given some initial data. In the case of a quantum mechanical motion of the charge, this is not possible anymore. Every time the nucleon emits a meson (photon), it gets a kick in the momentum space. Hence, we can not specify a trajectory of the charged particle.

Basically, Theorem 4.2.1 gives a very similar result as we have found in the semi- classical

case, however, considering initial data $|\xi_{t_0}\rangle$ at time $t_0$ and calculate the limit $t_0 \to -\infty$, the resulting expectation value will be zero in most of the cases. This is because of the fact that the time evolution belonging to the kinetic part of the Hamiltonian kinetic part of the Hamiltonian $e^{-iH_{kin}t}$ pushes down the wave function of the nucleon and the field gets just zero in this limit (point-wise). For example, a gaussian wave package evolved in time with the free propagator over a infinitely long period just tends to zero point-wise. In order to get a useful result, one has to kind exclude this effect. One could for example consider initial states likes $e^{-iH_{kin}t_0} |\xi_{t_0}\rangle$. However, it will be necessary to do pursue a scattering theory.
This problem did not arise in the semi-classical case, since there the trajectory of the nucleon was given and not determined by the Hamiltonian.

## 4.3 Connection to Maxwell theory

In the end, we want to describe moving charges, i.e. we would expect that we can find a connection between our result and Maxwell's theory. From Maxwell equations we know that the potential $f(t,x)$ belonging to the electromagnetic field of a single charge should fulfill the wave equation with a delta source term at the position of the charge.

$$\Box f(t,x) \sim \delta^{(3)}\left(x(t)\right) \tag{4.81}$$

The following theorem makes this connection implicitly

**Theorem 4.3.1** *Let $\varphi_0(x)$ be the field we have defined above and $\varphi_t^{full}(x) := U(t_0,t)\varphi_0(x)U(t,t_0)$ the full time evolved field.*
*The full time evolution is given by*

$$U(t,t_0) = e^{-iH(t-t_0)} \tag{4.82}$$

*Then*

$$\Box\varphi_t^{full}(x) = \lambda\delta^{(3)}(x-\hat{x}(t)) \tag{4.83}$$

**Remark:** Note that if we consider just matrix elements $\varphi_t^{full}(x)$ converges weakly to $\varphi_{t-t_0}(x) + \hat{\Phi}_t(x)$ as $t_0 \to -\infty$. We also know $\Box\varphi_{t-t_0}(x) = 0$. Hence

$$\Box \langle\eta| \hat{\Phi}_t(x) |\eta\rangle' = \lambda\delta^{(3)}(x-\hat{x}(t)) \tag{4.84}$$

for $|\eta\rangle, |\eta\rangle' \in L^2(\mathbb{R}^3)\otimes\mathcal{F}$.
In this sense, the theorem states that the field operator $\hat{\Phi}_t(x)$ fulfills the inhomogeneous wave equation.

**Proof:**

$$\begin{aligned}\Box\varphi_t^{full}(x) &= \Box\left(e^{iH(t-t_0)}\varphi_0(x)e^{-iH(t-t_0)}\right) \\ &= (\partial_t^2 - \Delta)\left(e^{iH(t-t_0)}\varphi_0(x)e^{-iH(t-t_0)}\right) \end{aligned} \tag{4.85}$$

Let us start with calculating the time derivative

$$\begin{aligned}\partial_t\left(e^{iH(t-t_0)}\varphi_0(x)e^{-iH(t-t_0)}\right) &= e^{iH(t-t_0)}i[H,\varphi_0(x)]e^{-iH(t-t_0)} \\ &= e^{iH(t-t_0)}i\left([H_{kin},\varphi_0(x)] + [H_0,\varphi_0(x)] + [V,\varphi_0(x)]\right)e^{-iH(t-t_0)}\end{aligned} \tag{4.86}$$

$\varphi_0(x)$ does not contain any $\hat{x}$ operators, hence $[H_{kin},\varphi_0(x)] = 0$. We also find $[V,\varphi_0(x)] = 0$, since $V$ and $\varphi_0(x)$ are essentially the same except that in $\varphi_0(x)$, $x$ is just a number and in $V$, $\hat{x}$ is an operator. It remains to compute the third commutator:

$$\begin{aligned}i[H_0,\varphi_0(x)] &= i\frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\mathrm{d}^3\mathrm{k}'\omega_k\gamma_{k'}\left(e^{ik'\cdot x}[a_k^*a_k,a_{k'}] + e^{-ik'\cdot x}[a_k^*a_k,a_{k'}^*]\right) \\ &= i\frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\mathrm{d}^3\mathrm{k}'\omega_k\gamma_{k'}\left(-\delta^{(3)}(k-k')a_ke^{ik'\cdot x} + \delta^{(3)}(k-k')a_k^*e^{-ik'\cdot x}\right) \\ &= -i\frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\omega_k\gamma_k\left(a_ke^{ik\cdot x} - a_k^*e^{-ik\cdot x}\right) \\ &= \dot{\varphi}_0(x)\end{aligned} \tag{4.87}$$

Hence

$$\partial_t\left(e^{iH(t-t_0)}\varphi_0(x)e^{-iH(t-t_0)}\right) = e^{iH(t-t_0)}\dot{\varphi}_0(x)e^{-iH(t-t_0)} \tag{4.88}$$

Similarly, we compute the second time derivative

$$\begin{aligned}\partial_t^2\left(e^{iH(t-t_0)}\varphi_0(x)e^{-iH(t-t_0)}\right) &= e^{iH(t-t_0)}i[H,\dot{\varphi}_0(x)]e^{-iH(t-t_0)} \\ &= e^{iH(t-t_0)}i\left([H_{kin},\dot{\varphi}_0(x)] + [H_0,\dot{\varphi}_0(x)] + [V,\dot{\varphi}_0(x)]\right)e^{-iH(t-t_0)}\end{aligned} \tag{4.89}$$

Again, we have $[H_{kin},\varphi_0(x)] = 0$, but know the other two commutators are non zero.

$$\begin{aligned}i[H_0,\dot{\varphi}_0(x)] &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\mathrm{d}^3\mathrm{k}'\omega_k\omega_{k'}\gamma_{k'}\left(e^{ik'\cdot x}[a_k^*a_k,a_{k'}] - e^{-ik'\cdot x}[a_k^*a_k,a_{k'}^*]\right) \\ &= -\frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\mathrm{d}^3\mathrm{k}'\omega_k\omega_{k'}\gamma_{k'}\left(\delta^{(3)}(k-k')a_ke^{ik'\cdot x} + \delta^{(3)}(k-k')a_k^*e^{-ik'\cdot x}\right) \\ &= -\frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\omega_k^2\gamma_k\left(a_ke^{ik\cdot x} + a_k^*e^{-ik\cdot x}\right) \\ &= \ddot{\varphi}_0(x)\end{aligned} \tag{4.90}$$

and

$$\begin{aligned} i[V,\dot{\varphi}_0(x)] &= \frac{\lambda}{(2\pi)^3}\int \mathrm{d^3k d^3k'}\omega_k\gamma_k\gamma_{k'}\left([a_k,a^*_{k'}]e^{ik\cdot x}e^{-ik'\cdot\hat{x}} - [a^*_k,a_{k'}]e^{-ik\cdot x}e^{ik'\cdot\hat{x}}\right) \\ &= \frac{\lambda}{(2\pi)^3}\int \mathrm{d^3k d^3k'}\omega_k\gamma_k\gamma_{k'}\delta^{(3)}(k-k')\left(e^{ik\cdot x}e^{-ik'\cdot\hat{x}} + e^{-ik\cdot x}e^{ik'\cdot\hat{x}}\right) \\ &= \frac{\lambda}{(2\pi)^3}\int \mathrm{d^3k}\omega_k\gamma_k^2\left(e^{ik\cdot x}e^{-ik\cdot\hat{x}} + e^{-ik\cdot x}e^{ik\cdot\hat{x}}\right) \\ &= \frac{\lambda}{(2\pi)^3}\int \mathrm{d^3k}e^{ik\cdot(x-\hat{x})} = \lambda\delta^{(3)}(x-\hat{x}) \end{aligned} \tag{4.91}$$

Hence

$$\begin{aligned} \partial_t^2\left(e^{iH(t-t_0)}\varphi_0(x)e^{-iH(t-t_0)}\right) &= e^{iH(t-t_0)}\ddot{\varphi}_0(x)e^{-iH(t-t_0)} + e^{iH(t-t_0)}\lambda\delta^{(3)}(x-\hat{x})e^{-iH(t-t_0)} \\ &= e^{iH(t-t_0)}\ddot{\varphi}_0(x)e^{-iH(t-t_0)} + \lambda\delta^{(3)}(x-\hat{x}(t)) \end{aligned} \tag{4.92}$$

We know $[\Delta, H] = 0$, hence

$$\Delta\left(e^{iH(t-t_0)}\varphi_0(x)e^{-iH(t-t_0)}\right) = e^{iH(t-t_0)}\Delta\varphi_0(x)e^{-iH(t-t_0)} \tag{4.93}$$

Putting everything together yields

$$\begin{aligned} \partial_t^2\left(e^{iH(t-t_0)}\varphi_0(x)e^{-iH(t-t_0)}\right) &= e^{iH(t-t_0)}\left(\ddot{\varphi}_0(x) - \Delta\varphi_0(x)\right)e^{-iH(t-t_0)} + \lambda\delta^{(3)}(x-\hat{x}(t)) \\ &= \lambda\delta^{(3)}(x-\hat{x}(t)) \end{aligned} \tag{4.94}$$

In the last step we have used that $\varphi_t(x)$ is a solution of the free wave equation, i.e. $\Box\varphi_t(x) = 0$. □

## 4.4 Dressed vacuum state

In this model we have to define the the dressed one-particle vacuum state in slightly more complicated way, since the space in $L^2(\mathbb{R}^3)\otimes\mathcal{F}$ in this case. Hence, product states is this space looks like $\left|\xi^{(x)}\right\rangle \otimes \left|\xi^{(\mathcal{F})}\right\rangle$.

**Definition:** Let $|0\rangle \in \mathcal{F}$ be the usual vacuum, which is annihilated by the annihilation oeprator, i.e. $a_k|0\rangle = 0$, then

(i)

$$|0\rangle_\xi := \left|\xi^{(x)}\right\rangle \otimes |0\rangle \in L^2(\mathbb{R}^3)\otimes\mathcal{F} \tag{4.95}$$

is called the one-particle vacuum state of the system.

(ii)

$$\left|\xi^{(gs)}\right\rangle := D_f(\hat{x},t)\,|0\rangle_\xi \tag{4.96}$$

is called the dressed one-particle vacuum state of the system.

(iii)

$$\hat{N} := \int \mathrm{d}^3\mathrm{k} a_k^* a_k =: \int \mathrm{dk} \hat{n}_k \tag{4.97}$$

is called the number operator of the system.

One could ask the question how many photons we need in order to dress the vacuum appropriately? For photons we set $\mu = 0$. The following theorem tells us that we need infinitely many of them.

**Theorem 4.4.1** *Let $\hat{N}$ be the number operator and $\left|\xi^{(gs)}\right\rangle$ of the system. Also let $\mu = 0$ and assume that there is no cutoff, i.e. $f = 1$. Then*

$$\left\langle \xi^{(gs)} \right| \hat{N} \left| \xi^{(gs)} \right\rangle = \infty + \mathcal{O}(\lambda^3) \tag{4.98}$$

**Proof:**

$$\begin{aligned}
&\left\langle \xi^{(gs)} \right| \hat{N} \left| \xi^{(gs)} \right\rangle \text{ă} \\
&= \int \mathrm{d}^3\mathrm{k} \left\langle 0 \right|_\xi D_f^*(\hat{x}, t) a_k^* a_k D_f(\hat{x}, t) \left| 0 \right\rangle_\xi \\
&= \int \mathrm{d}^3\mathrm{k} \left\langle 0 \right|_\xi \left[ D_f^*(\hat{x}, t), a_k^* \right] \left[ a_k, D_f(\hat{x}, t) \right] \left| 0 \right\rangle_\xi \\
&= \frac{\lambda^2}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \left\langle 0 \right|_\xi e^{ik\cdot\hat{x}(t) - i\omega_k t} \frac{\gamma_k}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} D_f^*(\hat{x}, t) D_f(\hat{x}, t) \frac{\gamma_k}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)} e^{-ik\cdot\hat{x}(t) + i\omega_k t} \left| 0 \right\rangle_\xi \\
&+ \mathcal{O}(\lambda^3) \\
&= \frac{\lambda^2}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \left\langle 0 \right|_\xi e^{ik\cdot\hat{x}(t)} \frac{\gamma_k^2}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)^2} e^{-ik\cdot\hat{x}(t)} \left| 0 \right\rangle_\xi + \mathcal{O}(\lambda^3) \\
&= \frac{\lambda^2}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \left\langle \xi^{(x)} \right| e^{ik\cdot\hat{x}(t)} \frac{\gamma_k^2}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)^2} e^{-ik\cdot\hat{x}(t)} \left| \xi^{(x)} \right\rangle + \mathcal{O}(\lambda^3)
\end{aligned} \tag{4.99}$$

In the first step, we have used that $a_k \left|0\right\rangle = 0$, in the second step, we have used the theorem from before and finally that the vacuum $\left|0\right\rangle$ is normalized.
Use that $\hat{x} = i\partial_p$ in momentum space, hence

$$\begin{aligned}
e^{ik\cdot\hat{x}} \frac{\gamma_k^2}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)^2} &= e^{-k\cdot\nabla_p} \frac{\gamma_k^2}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{k^2}{2m_0}\right)^2} \\
&= \frac{\gamma_k^2}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{3k^2}{2m_0}\right)^2} e^{ik\cdot\hat{x}}
\end{aligned} \tag{4.100}$$

Hence

$$\left\langle \xi^{(gs)} \right| \hat{N} \left| \xi^{(gs)} \right\rangle = \frac{\lambda^2}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \left\langle \xi^{(x)} \right| \frac{\gamma_k^2}{\left(\frac{k\cdot\hat{p}}{m_0} - \omega_k - \frac{3k^2}{2m_0}\right)^2} \left| \xi^{(x)} \right\rangle + \mathcal{O}(\lambda^3) \tag{4.101}$$

Going to momentum space yields

$$\left\langle \xi^{(gs)} \right| \hat{N} \left| \xi^{(gs)} \right\rangle = \frac{\lambda^2}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k}\mathrm{d}^3\mathrm{p} |\xi(p)|^2 \frac{\gamma_k^2}{\left(\frac{k \cdot p}{m_0} - \omega_k - \frac{3k^2}{2m_0}\right)^2} + \mathcal{O}(\lambda^3) \tag{4.102}$$

We are interested in the case $\mu = 0$, hence

$$\left\langle \xi^{(gs)} \right| \hat{N} \left| \xi^{(gs)} \right\rangle = \frac{\lambda^2}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k}\mathrm{d}^3\mathrm{p} |\xi(p)|^2 \frac{1}{|k|^3 \left(\frac{|p|\cos\theta}{m_0} - 1 - \frac{3|k|}{2m_0}\right)^2} + \mathcal{O}(\lambda^3) \tag{4.103}$$

where $\theta$ is the angle between $k$ and $p$. Now, we switch to spherical coordinates.

$$\begin{aligned}
&\left\langle \xi^{(gs)} \right| \hat{N} \left| \xi^{(gs)} \right\rangle \\
&= \frac{\lambda^2}{(2\pi)^2} \int \mathrm{d}^3\mathrm{p} |\xi(p)|^2 \int_0^\infty \mathrm{d|k|} \int_0^\pi \mathrm{d}(\cos\theta) \frac{1}{|k| \left(\frac{|p|\cos\theta}{m_0} - 1 - \frac{3|k|}{2m_0}\right)^2} + \mathcal{O}(\lambda^3) \\
&= \frac{\lambda^2}{(2\pi)^2} \int \mathrm{d}^3\mathrm{p} |\xi(p)|^2 \int_0^\infty \mathrm{d|k|} \frac{1}{|k|} \left[ -\frac{m_0}{|p|} \frac{1}{\left(\frac{|p|\cos\theta}{m_0} - 1 - \frac{3|k|}{2m_0}\right)} \right]_{\theta=0}^{\pi} + \mathcal{O}(\lambda^3) \\
&= \frac{\lambda^2}{(2\pi)^2} \int \mathrm{d}^3\mathrm{p} |\xi(p)|^2 \frac{m_0}{|p|} \int_0^\infty \mathrm{d|k|} \frac{1}{|k|} \left[ \frac{1}{\left(\frac{|p|}{m_0} - 1 - \frac{3|k|}{2m_0}\right)} - \frac{1}{\left(-\frac{|p|}{m_0} - 1 - \frac{3|k|}{2m_0}\right)} \right] + \mathcal{O}(\lambda^3) \\
&= \frac{\lambda^2}{(2\pi)^2} \int \mathrm{d}^3\mathrm{p} |\xi(p)|^2 \frac{m_0}{|p|} \int_0^\infty \mathrm{d|k|} \frac{1}{|k|} \left[ \frac{2\frac{|p|}{m_0}}{\left(-\frac{|p|}{m_0} - 1 - \frac{3|k|}{2m_0}\right)\left(\frac{|p|}{m_0} - 1 - \frac{3|k|}{2m_0}\right)} \right] + \mathcal{O}(\lambda^3) \\
&= -\frac{\lambda^2}{2\pi^2} \int \mathrm{d}^3\mathrm{p} |\xi(p)|^2 \int_0^\infty \mathrm{d|k|} \frac{1}{|k|} \frac{1}{\frac{p^2}{m_0^2} - \left(1 + \frac{3|k|}{2m_0}\right)^2} + \mathcal{O}(\lambda^3)
\end{aligned} \tag{4.104}$$

Substitute $y := \frac{3|k|}{2m_0}$ then $\mathrm{d}y = \frac{3\mathrm{d|k|}}{2m_0}$. Then

$$\left\langle \xi^{(gs)} \right| \hat{N} \left| \xi^{(gs)} \right\rangle = \frac{\lambda^2}{2\pi^2} \int \mathrm{d}^3\mathrm{p} |\xi(p)|^2 \underbrace{\int_0^\infty \frac{\mathrm{d}y}{y} \frac{1}{(1+y)^2 - \frac{p^2}{m_0^2}}}_{:=B} + \mathcal{O}(\lambda^3) \tag{4.105}$$

From the assumption that the charge is in a non-relativistic regime, i.e. $\frac{|p|}{m_0} = v < 1$, it follows that, the integrand in $B$ is always positive and so is $B$ itself.

$$B = \underbrace{\int_0^z \frac{\mathrm{d}y}{y} \frac{1}{(1+y)^2 - \frac{p^2}{m_0^2}}}_{:=B_1} + \underbrace{\int_z^\infty \frac{\mathrm{d}y}{y} \frac{1}{(1+y)^2 - \frac{p^2}{m_0^2}}}_{:=B_2} \tag{4.106}$$

with $z := \sqrt{1 + \frac{p^2}{m_0}} - 1$. Then, for $y \leq z$, we know $\frac{1}{(1+y)^2 - \frac{p^2}{m_0^2}} \geq 1$ and hence

$$B_1 \geq \int_0^z \mathrm{d}y \frac{1}{y} \tag{4.107}$$

with is logarithmically divergent. Since $B, B_1, B_2 \geq 0$, we conclude $B = B_1 + B_2 \geq B_1$. This means, there is definitely an infrared divergence in $B$.

Note that, we could similarly show the existence of an ultraviolet divergence. All the divergences are logarithmical, this is the same kind of divergence, as we found in the semi-classical case. □

This is basically the same result, we found in the semi-classical situation. As discussed before, the infrared divergence indicates the representational problem and the ultraviolet divergence is due to the fact that we are considering point-particles.

## 4.5 Momentum operator

In this chapter we will introduce the momentum operator

**Definition:** Let

$$\hat{P} := \hat{p} + p_f \tag{4.108}$$

$$p_f := \int \mathrm{d}^3\mathrm{k} k a_k^* a_k \tag{4.109}$$

Then we call $\hat{P}$ the total momentum operator and $p_f$ the momentum operator of the field.

We would expect that the momentum is conserved in our model. This can be shown in the following theorem:

**Theorem 4.5.1** *Let $\hat{P}$ be the total momentum operator defined above, then*

$$\left[\hat{P}, H\right] = 0 \tag{4.110}$$

*Hence, the total momentum is conserved.*

**Proof:** We know that $\hat{P}$ is the generator of spatial translations. Therefore

$$e^{id\cdot\hat{P}} H_{kin} e^{-id\cdot\hat{P}} = H_{kin} \tag{4.111}$$

since $\left[\hat{P}, \hat{p}\right] = 0$ and

$$e^{id\cdot\hat{P}} H_0 e^{-id\cdot\hat{P}} = H_0 \tag{4.112}$$

since $\left[\hat{P}, H_0\right] = 0$ and

$$e^{id\cdot\hat{P}} a_k e^{-id\cdot\hat{P}} = a_k + \frac{id}{1!} \cdot [P, a_k] + \frac{i^2 d^2}{2!} \cdot [P, [P, a_k]] + ... \tag{4.113}$$

$$[P, a_k] = [p_f, a_k] = \int \mathrm{d}^3\mathrm{q} q \left[a_q^* a_q, a_k\right] = -k a_k \tag{4.114}$$

Hence

$$[P, ...[P, a_k]...] = (-k)^n a_k \tag{4.115}$$

Then

$$e^{id\cdot\hat{P}} a_k e^{-id\cdot\hat{P}} = a_k \left(1 + \frac{-ik\cdot d}{1!} + \frac{(-ik\cdot d)^2}{2!} + ...\right) = a_k e^{-ik\cdot d} \tag{4.116}$$

Similar

$$\begin{aligned} e^{id\cdot\hat{P}} e^{ik\cdot\hat{x}} e^{-id\cdot\hat{P}} &= e^{id\cdot\hat{p}} e^{ik\cdot\hat{x}} e^{-id\cdot\hat{p}} = e^{ik\cdot\hat{x}} + \frac{id}{1!}\cdot\left[\hat{p}, e^{ik\cdot\hat{x}}\right] + \frac{i^2d^2}{2!}\cdot\left[\hat{p}, \left[\hat{p}, e^{ik\cdot\hat{x}}\right]\right] + ... \\ &= e^{ik\cdot\hat{x}} e^{ik\cdot d} \end{aligned} \tag{4.117}$$

Hence

$$e^{id\cdot\hat{P}} V e^{-id\cdot\hat{P}} = \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \left(e^{id\cdot\hat{P}} a_k e^{ik\cdot\hat{x}} e^{-id\cdot\hat{P}} + e^{id\cdot\hat{P}} a_k^* e^{-ik\cdot\hat{x}} e^{-id\cdot\hat{P}}\right) = V \tag{4.118}$$

Putting everything together

$$e^{id\cdot\hat{P}} H e^{-id\cdot\hat{P}} = H \tag{4.119}$$

This implies

$$\left[\hat{P}, H\right] = 0 \tag{4.120}$$

□

**Remark:** The total momentum $\hat{P}$ is conserved, but the momentum of the nucleon $\hat{p}$ is not, since

$$\begin{aligned} [\hat{p}, H] = [\hat{p}, V] &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \left(a_k\left[\hat{p}, e^{ik\cdot\hat{x}}\right] + a_k^*\left[\hat{p}, e^{-ik\cdot\hat{x}}\right]\right) \\ &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k k \left(a_k e^{ik\cdot\hat{x}} - a_k^* e^{-ik\cdot\hat{x}}\right) \neq 0 \end{aligned} \tag{4.121}$$

in general.

## 4.6 Conclusion

In this chapter we considered the interaction between a scalar field and a spinless fermion field with a free (quantum mechanical) dispersion relation. This was supposed to be the simplest case of a quantum mechanical motion. The results found in Theorem 4.2.1 can be interpreted similarly as in the semi-classical case. Nevertheless, they only hold up to first order in the coupling constant $\lambda$. Hence, they neglect higher order effects like radiation back-reaction. Unfortunately, having this at hand, it is not as easy to actually calculate the field for some given initial data $|\xi_{t_0}\rangle$ at time $t_0$ and in the limit $t_0 \to -\infty$, since the free time evolution, due to the dispersion of the nucleon, pushes down the wave function of the nucleon and the resulting field is just zero. In order to get useful results, one would have to use scattering theory and remove this effect. For example, one could consider initial states like $e^{-iH_{kin}t_0}\,|\xi_{t_0}\rangle$.

However, we proceed with another situation, namely, a fermion in quantum mechanical harmonic oscillator. We know that the eigenstates of the harmonic oscillator are bound states. Therefore, in this case, we can hopefully evaluate the field even without considering the problem, we had in this chapter.

# 5 Charge in a harmonic oscillator

## 5.1 Setting and definition

Actually, we would like to calculate the second quantized field for a given trajectory of the charge. Nevertheless, this causes troubles in this setting. It is not possible to separate the motion of the charge from the dynamics of the field. Whenever a photon with momentum $k$ is absorbed, the charge gets a kick in momentum space, exactly by this amount $k$. This is implemented already in the Hamiltonian. The best we can do, is to give suitable initial data and calculate how the system evolves.

Considering a charge, which moves according to equations of a quantum mechanical harmonic oscillator and then calculate the expectation value of the field operator in the eigenstates of the harmonic oscillator (and linear combinations of them), might give better result, since the eigenstates of the harmonic oscillator are bound states. Therefore, the free time evolution does hopefully not destroy the initial data, as it did, when we considered a free particle.
As we have already seen, we can only to this up to second order in $\lambda$, but still this is a good approximation, since the coupling constant $\lambda$ is very small. This will give us a good intuition about the second quantized field.
For simplicity, let us assume that the angular frequency $\omega$ of the harmonic oscillator is the same in every spacial direction. Also note that $\omega$ is just the angular frequency of the harmonic oscillator and has nothing to do with the modes $\omega_k$ of the meson field.
Then, the Hamiltonian for this situation looks like the following

$$\begin{aligned} H &= H_{osc} + H_0 + V \\ H_{osc} &= \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega \hat{x}^2 \\ H_0 &= \int \mathrm{d}^3\mathrm{k}\, \omega_k a_k^* a_k \\ V &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\, \gamma_k \left( a_k e^{ik\cdot\hat{x}} + a_k^* e^{-ik\cdot\hat{x}} \right) \end{aligned} \tag{5.1}$$

Note that again $\hat{x}$ and $\hat{p}$ is a short hand notation and it has three spatial components.
The domain of this Hamiltonian is

$$\mathcal{D}(H) := \mathcal{D}(H_{osc}) \cap \mathcal{D}(n) \tag{5.2}$$

where the domain $\mathcal{D}(H_{osc})$ contains all $\psi \in \mathcal{H}$, such that $\left\{ H_{osc}\psi^{(n)} \right\}$ are again in $\mathcal{H}$ and everything else is defined as before.

Calculations get easier if we diagonalize $H_{osc}$. This can be done by introducing the well-known ladder operators for the quantum mechanical harmonic oscillator:

$$\hat{b}_i = \sqrt{\frac{m\omega}{2}}\left(\hat{x}_i + \frac{i}{m\omega}\hat{p}_i\right)$$
$$\hat{b}_i^* = \sqrt{\frac{m\omega}{2}}\left(\hat{x}_i - \frac{i}{m\omega}\hat{p}_i\right) \tag{5.3}$$

and therefore

$$\hat{x}_i = \sqrt{\frac{1}{2m\omega}}\left(\hat{b}_i + \hat{b}_i^*\right)$$
$$\hat{p}_i = i\sqrt{\frac{m\omega}{2}}\left(\hat{b}_i - \hat{b}_i^*\right) \tag{5.4}$$

Exploiting the canonical commutations relations, we obtain

$$[b_i, b_j] = \left[b_i^*, b_j^*\right] = 0 \tag{5.5}$$
$$\left[b_i, b_j^*\right] = \delta_{ij} \tag{5.6}$$

Then

$$H_{osc} = \omega\left(\sum_{j=1}^{3} b_j^* b_j + \frac{3}{2}\right) \tag{5.7}$$

This yields the following

**Definition:**

$$H = H_{osc} + H_0 + V$$
$$H_{osc} = \omega\left(\sum_{j=1}^{3} b_j^* b_j + \frac{3}{2}\right)$$
$$H_0 = \int \mathrm{d}^3\mathrm{k}\omega_k a_k^* a_k$$
$$V = \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k\left(a_k e^{\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}+\hat{b}^*\right)} + a_k^* e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}+\hat{b}^*\right)}\right) \tag{5.8}$$

Note that $\hat{b}$ actually has three entries, i.e. $\hat{b} := \begin{pmatrix}\hat{b}_1 & \hat{b}_2 & \hat{b}_3\end{pmatrix}^T$ and same for $\hat{b}_i^*$.

## 5.2 Dressing operator

Again, we start with the calculation of the time evolution operator in the interaction picture. It is given by

$$U_I(t, t_0) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}\mathrm{dt}_1 ... \int_{t_0}^{t}\mathrm{dt}_\mathrm{n} T\left[V(t_1)...V(t_n)\right] \tag{5.9}$$

Remember

$$U_0(t) = e^{-i(H_{osc}+H_0)t} = e^{-iH_0t}e^{-iH_{osc}t} \tag{5.10}$$

This is true, since $[H_{osc}, H_0] = 0$ and

$$\begin{aligned}
V(t) &= U_0^*(t)VU_0(t) = e^{iH_{osc}t}e^{iH_0t}Ve^{-iH_0t}e^{-iH_{osc}t} \\
&= e^{iH_{osc}t}\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k \left(a_k e^{\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}+\hat{b}^*\right)-i\omega_k t} + a_k^* e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}+\hat{b}^*\right)+i\omega_k t}\right) e^{-iH_{osc}t} \\
&= \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k \left(a_k e^{-i\omega_k t}e^{iH_{osc}t}e^{\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}+\hat{b}^*\right)}e^{-iH_{osc}t} + c.c.\right) \\
&= \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k \Bigg(a_k e^{-i\omega_k t}e^{\frac{i}{\sqrt{2m\omega}}k\cdot\left(e^{iH_{osc}t}\hat{b}e^{-iH_{osc}t}+e^{iH_{osc}t}\hat{b}^*e^{-iH_{osc}t}\right)} \\
&+ a_k^* e^{i\omega_k t}e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(e^{iH_{osc}t}\hat{b}e^{-iH_{osc}t}+e^{iH_{osc}t}\hat{b}^*e^{-iH_{osc}t}\right)}\Bigg)
\end{aligned} \tag{5.11}$$

This means, we have to calculate

$$\begin{aligned}
e^{iH_{osc}t}\hat{b}_i e^{-iH_{osc}t} &= e^{i\omega t\sum_{j=1}\hat{b}_j^*\hat{b}_j}\hat{b}_i e^{-i\omega t\sum_{l=1}\hat{b}_l^*\hat{b}_l} \\
&= e^{i\omega t\hat{b}_i^*\hat{b}_i}\hat{b}_i e^{-i\omega t\hat{b}_i^*\hat{b}_i} \\
&= \hat{b}_i + \frac{i\omega t}{1!}\left[\hat{b}_i^*\hat{b}_i, \hat{b}_i\right] + \frac{(i\omega t)^2}{2!}\left[\hat{b}_i^*\hat{b}_i, \left[\hat{b}_i^*\hat{b}_i, \hat{b}_i\right]\right] + ...
\end{aligned} \tag{5.12}$$

Use

$$\left[\hat{b}_i^*\hat{b}_i, \hat{b}_i\right] = -\hat{b}_i \tag{5.13}$$

and obtain

$$\begin{aligned}
e^{iH_{osc}t}\hat{b}_i e^{-iH_{osc}t} &= \hat{b}_i\left(1 + \frac{-i\omega t}{1!} + \frac{(-i\omega t)^2}{2!} + ...\right) \\
&= e^{-i\omega t}\hat{b}_i
\end{aligned} \tag{5.14}$$

and similar

$$e^{iH_{osc}t}\hat{b}_i^* e^{-iH_{osc}t} = e^{i\omega t}\hat{b}_i^* \tag{5.15}$$

Then

$$V(t) = \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k \Bigg(a_k e^{\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega t}+\hat{b}^*e^{i\omega t}\right)-i\omega_k t} + a_k^* e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega t}+\hat{b}^*e^{i\omega t}\right)+i\omega_k t}\Bigg) \tag{5.16}$$

Unfortunately, the time integral of $V(t)$ is not easy to solve (as it was in the other cases before). Therefore, we will not calculate it explicitly at this point.
Further, already in the case without a potential for the fermion field, we were not able to calculate the field exactly. We only calculated it up to first order in the coupling constant $\lambda$. Here we will do the same. We will only consider terms up to first order in the coupling constant $\lambda$ from the beginning, then

**Definition:**

$$
\begin{aligned}
&U_I(t,t_0)\\
&= 1 - i\int_{t_0}^{t} \mathrm{dt}' V(t') + \mathcal{O}(\lambda^2)\\
&= 1 - i\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int_{t_0}^{t}\mathrm{dt}'\int \mathrm{d^3k}\gamma_k\left(a_k e^{\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega t'}+\hat{b}^* e^{i\omega t'}\right)-i\omega_k t'} + a_k^* e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega t'}+\hat{b}^* e^{i\omega t'}\right)+i\omega_k t'}\right)\\
&+ \mathcal{O}(\lambda^2)
\end{aligned}
\tag{5.17}
$$

In order to calculate the fields, we will again need to know how $a_k$ and $U_I(t,t_0)$ commute with each other.

**Lemma 5.2.1** *Let* $|\xi\rangle, |\phi\rangle \in \mathcal{H}$

$$
\langle\xi|\,[a_k, U_I(t,t_0)]\,|\phi\rangle = -i\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int_{t_0}^{t}\mathrm{dt}'\gamma_k\,\langle\xi|\,e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega t'}+\hat{b}^* e^{i\omega t'}\right)+i\omega_k t'}\,|\phi\rangle + \mathcal{O}(\lambda^2)
\tag{5.18}
$$

**Proof:** Together with the definition of $U_I(t,t_0)$, we obtain

$$
\begin{aligned}
&[a_k, U_I(t,t_0)]\\
&= -i\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int_{t_0}^{t}\mathrm{dt}'\int \mathrm{d^3k}'\left[a_k, \gamma_{k'}\left(a_{k'} e^{\frac{i}{\sqrt{2m\omega}}k'\cdot\left(\hat{b}e^{-i\omega t'}+\hat{b}^* e^{i\omega t'}\right)-i\omega_{k'} t'} + c.c.\right)\right]\\
&+ \mathcal{O}(\lambda^2)\\
&= -i\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int_{t_0}^{t}\mathrm{dt}'\int \mathrm{d^3k}'\gamma_{k'} e^{-\frac{i}{\sqrt{2m\omega}}k'\cdot\left(\hat{b}e^{-i\omega t'}+\hat{b}^* e^{i\omega t'}\right)+i\omega_{k'} t'}[a_k, a_{k'}^*] + \mathcal{O}(\lambda^2)\\
&= -i\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int_{t_0}^{t}\mathrm{dt}'\gamma_k e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega t'}+\hat{b}^* e^{i\omega t'}\right)+i\omega_k t'} + \mathcal{O}(\lambda^2)
\end{aligned}
\tag{5.19}
$$

□

**Remark:** This implies

$$
\langle\xi|\,[U_I(t_0,t), a_k^*]\,|\phi\rangle = i\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int_{t_0}^{t}\mathrm{dt}'\gamma_k\,\langle\xi|\,e^{\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega t'}+\hat{b}^* e^{i\omega t'}\right)-i\omega_k t'}\,|\phi\rangle + \mathcal{O}(\lambda^2)
\tag{5.20}
$$

for any $|\xi\rangle, |\phi\rangle \in \mathcal{H}$.

As in the previous chapters, we proceed with calculating the expectation value of the field.

## 5.3 Expectation values of the field and initial data

Remember the definition of the field:

$$\varphi_0(x) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \left(a_k e^{ik\cdot x} + a_k^* e^{-ik\cdot x}\right) \tag{5.21}$$

and

$$\begin{aligned}\varphi_t(x) &:= U_0^*(t)\varphi_t(x)U_0(t)\\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k}\gamma_k \left(a_k e^{ik\cdot x - i\omega_k t} + a_k^* e^{-ik\cdot x + i\omega_k t}\right)\end{aligned} \tag{5.22}$$

This is still true, since the is no $\hat{x}$ or $\hat{p}$ operator in $\varphi_0(x)$ and hence it commutes with $H_{osc}$.

**Theorem 5.3.1** *Let* $|\xi_{t_0}\rangle = \left|\xi_{t_0}^{(x)}\right\rangle \otimes \left|\xi_{t_0}^{(\mathcal{F})}\right\rangle \in L^2(\mathbb{R}^3) \otimes \mathcal{F}$ *be the normalized initial state at* $t_0 \in \mathbb{R}$ *and* $f = f_\kappa^\Lambda$ *be an appropriate cutoff function. Then*

$$\begin{aligned}\langle \xi_t | \varphi_0^{\kappa,\Lambda}(x) | \xi_t\rangle &= \langle \xi_{t_0} | U(t_0,t)\varphi_0^{\kappa,\Lambda}(x)U(t,t_0) | \xi_{t_0}\rangle\\ &= \underbrace{\left\langle \xi_{t_0}^{(\mathcal{F})}\right| \varphi_{t-t_0}^{\kappa,\Lambda}(x) \left|\xi_{t_0}^{(\mathcal{F})}\right\rangle}_{\text{free field}} + \left\langle \xi_{t_0}^{(x)}\right| \hat{\Phi}_{t,t_0}^{\kappa,\Lambda}(x) \left|\xi_{t_0}^{(x)}\right\rangle\end{aligned} \tag{5.23}$$

*with the field operator*

$$\begin{aligned}\hat{\Phi}_{t,t_0}^{\kappa,\Lambda}(x) := &-\frac{i\lambda}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k}\gamma_k^2 \int_{t_0}^t \mathrm{d}t' \left(e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega(t'-t_0)} + \hat{b}^* e^{i\omega(t'-t_0)}\right) + i\omega_k(t'-t)} e^{ik\cdot x} - c.c.\right)\\ &+ \mathcal{O}(\lambda^2)\end{aligned} \tag{5.24}$$

*where c.c. denotes the conjugated.*

**Remark:** This is similar to the theorem in the case without a harmonic potential, but we consider an initial state at some finite time $t_0 \in \mathbb{R}$ instead of the limit $t_0 \to -\infty$. As we will see later, this is reasonable, since if the charge evolves in time over an infinite time period, radiation has infinitely long time to run away and we do not see it anymore.

**Proof:** The proof is again very similar to the previous cases, but without taking the limit $t_0 \to -\infty$

$$\begin{aligned}\langle \xi_t | \varphi_0^{\kappa,\Lambda}(x) | \xi_t\rangle &= \langle \xi_{t_0} | U(t_0,t)\varphi_0^{\kappa,\Lambda}(x)U(t,t_0) | \xi_{t_0}\rangle\\ &= \langle \xi_{t_0} | U_0(t_0)U_I(t_0,t)U_0^*(t)\varphi_0^{\kappa,\Lambda}(x)U_0(t)U_I(t,t_0)U_0^*(t_0) | \xi_{t_0}\rangle\\ &= \langle \xi_{t_0} | U_0(t_0)U_I(t_0,t)\varphi_t^{\kappa,\Lambda}(x)U_I(t,t_0)U_0^*(t_0) | \xi_{t_0}\rangle\end{aligned} \tag{5.25}$$

where $U(t,t_0)$ is the full time evolution from $t_0$ to t. Also

$$\begin{aligned}
&U_I(t_0,t)\varphi_t^{\kappa,\Lambda}(x)U_I(t,t_0)\\
&=\frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k\Big(U_I(t_0,t)a_kU_I(t,t_0)e^{ik\cdot x-i\omega_k t}+U_I(t_0,t)a_k^*U_I(t,t_0)e^{-ik\cdot x+i\omega_k t}\Big)\\
&=\varphi_t^{\kappa,\Lambda}(x)\\
&+\frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathrm{k}\gamma_k\Big(U_I(t_0,t)[a_k,U_I(t,t_0)]e^{ik\cdot x-i\omega_k t}+[U_I(t_0,t),a_k^*]U_I(t,t_0)e^{-ik\cdot x+i\omega_k t}\Big)
\end{aligned} \tag{5.26}$$

The statement in the theorem is only first order in $\lambda$, hence we do not need to consider higher order terms in this proof. From the previous lemma we know that both commutators are already of order $\lambda$, therefore we only need to consider the 0-th order of $U_I$. All other terms are $\mathcal{O}(\lambda^2)$. Then

$$\begin{aligned}
&U_I(t_0,t)\varphi_t^{\kappa,\Lambda}(x)U_I(t,t_0)\\
&=\varphi_t(x)-\frac{i\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\gamma_k\Bigg(\int_{t_0}^t \mathrm{d}t'\gamma_k e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega t'}+\hat{b}^*e^{i\omega t'}\right)+i\omega_k t'}e^{ik\cdot x-i\omega_k t}\\
&-\int_{t_0}^t \mathrm{d}t'\gamma_k e^{\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega t'}+\hat{b}^*e^{i\omega t'}\right)-i\omega_k t'}e^{-ik\cdot x+i\omega_k t}\Bigg)+\mathcal{O}(\lambda^3)\\
&=\varphi_t(x)-\frac{i\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\gamma_k^2\int_{t_0}^t \mathrm{d}t'\Big(e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega t'}+\hat{b}^*e^{i\omega t'}\right)+i\omega_k t'}e^{ik\cdot x-i\omega_k t}-c.c.\Big)\\
&+\mathcal{O}(\lambda^2)
\end{aligned} \tag{5.27}$$

Then, similar as in the proof of the theorem without a harmonic potential

$$\begin{aligned}
\langle\xi_t|\,\varphi_0^{\kappa,\Lambda}(x)\,|\xi_t\rangle &= \langle\xi_{t_0}|\,\varphi_{t-t_0}^{\kappa,\Lambda}(x)\,|\xi_{t_0}\rangle+\langle\xi_{t_0}|\,\hat{\Phi}_{t,t_0}^{\kappa,\Lambda}(x)\,|\xi_{t_0}\rangle\\
&=\left\langle\xi_{t_0}^{(\mathcal{F})}\middle|\,\varphi_{t-t_0}^{\kappa,\Lambda}(x)\,\middle|\xi_{t_0}^{(\mathcal{F})}\right\rangle+\left\langle\xi_{t_0}^{(x)}\middle|\,\hat{\Phi}_{t,t_0}^{\kappa,\Lambda}(x)\,\middle|\xi_{t_0}^{(x)}\right\rangle
\end{aligned} \tag{5.28}$$

where

$$\begin{aligned}
&\hat{\Phi}_{t,t_0}^{\kappa,\Lambda}(x)\\
&=U_0(t_0)\left(-\frac{i\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\gamma_k^2\int_{t_0}^t \mathrm{d}t'\Big(e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega t'}+\hat{b}^*e^{i\omega t'}\right)+i\omega_k t'}e^{ik\cdot x-i\omega_k t}-c.c.\Big)\right)U_0^*(t_0)\\
&+\mathcal{O}(\lambda^2)
\end{aligned} \tag{5.29}$$

We made use of the fact that the initial state is normalized and $\varphi$ only contains only operators acting on $\mathcal{F}$, whereas $\hat{\Phi}_{t,t_0}$ only contains operators acting on $L^2(\mathbb{R}^3)$.

The first term in the equation above is again the free field. This can be shown absolutely analogously as before. It remains to calculate the second term.

Note that $U_0(t) = e^{-iH_0 t}e^{-iH_{osc}t}$ and the $H_0$ part commutes with the bracket, since there is no $a_k$ or $a_k^*$ in it. Hence

$$\begin{aligned}
&\hat{\Phi}^{\kappa,\Lambda}_{t,t_0}(x)\\
=&e^{-i\omega t_0\sum_{j=1}^3 b_j^* b_j}\left(-\frac{i\lambda}{(2\pi)^3}\int \mathrm{d^3k}\gamma_k^2\int_{t_0}^t \mathrm{dt'}\left(e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega t'}+\hat{b}^*e^{i\omega t'}\right)+i\omega_k t'}e^{ik\cdot x-i\omega_k t}-c.c.\right)\right)e^{i\omega t_0\sum_{l=1}^3 b_l^* b_l}\\
&+\mathcal{O}(\lambda^2)\\
=&-\frac{i\lambda}{(2\pi)^3}\int \mathrm{d^3k}\gamma_k^2\int_{t_0}^t \mathrm{dt'}\left(e^{-\frac{i}{\sqrt{2m\omega}}k\cdot e^{-i\omega t_0\sum_{j=1}^3 b_j^* b_j}\left(\hat{b}e^{-i\omega t'}+\hat{b}^*e^{i\omega t'}\right)e^{i\omega t_0\sum_{l=1}^3 b_l^* b_l}+i\omega_k t'}e^{ik\cdot x-i\omega_k t}-c.c.\right)\\
&+\mathcal{O}(\lambda^2)\\
=&-\frac{i\lambda}{(2\pi)^3}\int \mathrm{d^3k}\gamma_k^2\int_{t_0}^t \mathrm{dt'}\left(e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega(t'-t_0)}+\hat{b}^*e^{i\omega(t'-t_0)}\right)+i\omega_k(t'-t)}e^{ik\cdot x}-c.c.\right)+\mathcal{O}(\lambda^2)
\end{aligned} \tag{5.30}$$

In the last line we have used (5.14) in order to obtain the desired result. □

Even though, we might loose some interesting terms in the limit $t_0 \to -\infty$, we can still formulate the theorem above also in this limit.

**Theorem 5.3.2** *Let $|\xi_{t_0}\rangle = \left|\xi_{t_0}^{(x)}\right\rangle \otimes \left|\xi_{t_0}^{(\mathcal{F})}\right\rangle \in L^2(\mathbb{R}^3)\otimes\mathcal{F}$ be the normalized initial state at $t_0 \to -\infty$ and $f = f_\kappa^\Lambda$ be an appropriate cutoff function. Then*

$$\begin{aligned}
\lim_{t_0\to-\infty}\langle\xi_t|\,\varphi_0(x)\,|\xi_t\rangle &= \lim_{t_0\to-\infty}\langle\xi_{t_0}|\,U(t_0,t)\varphi_0(x)U(t,t_0)\,|\xi_{t_0}\rangle\\
&= \underbrace{\lim_{t_0\to-\infty}\left\langle\xi_{t_0}^{(\mathcal{F})}\right|\varphi_{t-t_0}(x)\left|\xi_{t_0}^{(\mathcal{F})}\right\rangle}_{\text{free field}}+\lim_{t_0\to-\infty}\left\langle\xi_{t_0}^{(x)}\right|\hat{\Phi}_{t,t_0}(x)\left|\xi_{t_0}^{(x)}\right\rangle
\end{aligned} \tag{5.31}$$

*with the field operator*

$$\begin{aligned}
\hat{\Phi}_{t,t_0}(x) := &-\frac{i\lambda}{(2\pi)^3}\int \mathrm{d^3k}\gamma_k^2\int_{t_0}^t \mathrm{dt'}\left(e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega(t'-t_0)}+\hat{b}^*e^{i\omega(t'-t_0)}\right)+i\omega_k(t'-t)}e^{ik\cdot x}-c.c.\right)\\
&+\mathcal{O}(\lambda^2)
\end{aligned} \tag{5.32}$$

*where c.c. denotes the conjugated.*
*Further, suppose $|\xi_{t_0}\rangle \in L^2(\mathbb{R}^3)\otimes\mathcal{F}$ is a normalized product state, i.e.*

$$\begin{aligned}
|\xi_{t_0}\rangle &= \left|\xi_{t_0}^{(x)}\right\rangle \otimes \left|\xi_{t_0}^{(\mathcal{F})}\right\rangle\\
&= \left|\xi_{t_0}^{(x)}\right\rangle \otimes \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\prod_{i=1}^{n}\int \mathrm{d^3k_i}\xi_{t_0}^{(n)}(k_1,...,k_n)a_{k_i}\,|0\rangle
\end{aligned} \tag{5.33}$$

*with $\left|\xi_{t_0}^{(x)}\right\rangle \in L^2(\mathbb{R}^3)$ beeing normalized, i.e. $\left\langle\xi_{t_0}^{(x)}\middle|\xi_{t_0}^{(x)}\right\rangle = \left\langle\xi_{t_0}^{(\mathcal{F})}\middle|\xi_{t_0}^{(\mathcal{F})}\right\rangle = 1$ and*

$$\xi_{t_0}^{(n)}(k_1,...,k_n) = \frac{1}{\sqrt{n!}}\prod_{i=1}^{n}h(k_i)\quad \forall n\in\mathbb{N} \tag{5.34}$$

*and let one of the following be true*

1. *Let* $f = f_{\kappa}^{\Lambda} \in C_0^{\infty}(\mathbb{R}^3)$

2. *Let* $f = f_{\frac{1}{\Lambda}}^{\Lambda}$ *be an arbitrary cutoff function and let* $h, \nabla h, \Delta h \in L^2(\mathbb{R}^3)$. *Further, suppose we can find* $k_0 > 0$ *and* $\epsilon > 0$, *such that*

$$|h(k)| \leq |k|^{-3-\epsilon} \quad \forall k \in \{k \in \mathbb{R}^3 : |k| \geq k_0\} \tag{5.35}$$

*Then for any* $\Lambda > \kappa > 0$

$$\lim_{t_0 \to -\infty} \langle \xi_{t_0} | \, \varphi_{t-t_0}^{\kappa,\Lambda}(x) \, |\xi_{t_0}\rangle = 0 \;\; \forall x \in \mathbb{R}^3 \tag{5.36}$$

*In the second case*

$$\lim_{\Lambda \to \infty} \lim_{t_0 \to -\infty} \langle \xi_{t_0} | \, \varphi_{t-t_0}^{\frac{1}{\Lambda},\Lambda}(x) \, |\xi_{t_0}\rangle = 0 \;\; \forall x \in \mathbb{R}^3 \tag{5.37}$$

**Proof:** The first part of the theorem is a direct consequence from the previous theorem. The prove of the fact that the free field vanishes in the weak limit $t_0 \to -\infty$ for the conditions given in the theorem, is exactly the same as in the previous cases. □

In the following, we calculate some expectation values of the field operator. We start with calculating the matrix elements.
It is well known that the eigenstates $|n\rangle := |n_1, n_2, n_3\rangle \in L^2(\mathbb{R}^3)$ of the three dimension quantum harmonic oscillator are given by the following relation

$$H_{osc} \, |n\rangle = \left(n + \frac{3}{2}\right) \omega \, |n\rangle \tag{5.38}$$

with $n := n_1 + n_2 + n_3$ and $n_1, n_2, n_3 \in \mathbb{N}$.
We can construct them by the ladder operators $\hat{b}$ and $\hat{b}^*$:

$$\hat{b}_i \, |n\rangle = \sqrt{n} \, |n-1\rangle \tag{5.39}$$

$$\hat{b}_i^* \, |n\rangle = \sqrt{n+1} \, |n+1\rangle \tag{5.40}$$

This states form an orthonormal basis of $L^2(\mathbb{R}^3)$, hence it is sufficient to consider the matrix elements $\langle m| \, \hat{\Phi}_{t,t_0}(x) \, |n\rangle$ with $|n\rangle, |m\rangle \in L^2(\mathbb{R}^3)$. Then

$$\begin{aligned} &\langle m| \, \hat{\Phi}_{t,t_0}(x) \, |n\rangle \\ &= -\frac{i\lambda}{(2\pi)^3} \int \mathrm{d^3k} \gamma_k^2 \int_{t_0}^{t} \mathrm{dt'} \Bigg( \langle m| \, e^{-\frac{i}{\sqrt{2m\omega}} k \cdot \left(\hat{b} e^{-i\omega(t'-t_0)} + \hat{b}^* e^{i\omega(t'-t_0)}\right)} \, |n\rangle \, e^{ik\cdot x + i\omega_k(t'-t)} \\ &- \langle m| \, c.c. \, |n\rangle \Bigg) + \mathcal{O}(\lambda^2) \end{aligned} \tag{5.41}$$

We know $[b,[b,b^*]] = [b^*,[b,b^*]] = 0$, then by Baker-Campbell-Hausdorff

$$\begin{aligned}
&\langle m| \, e^{\pm\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega(t'-t_0)}+\hat{b}^*e^{i\omega(t'-t_0)}\right)} \, |n\rangle \\
&= \langle m| \, e^{\pm\frac{i}{\sqrt{2m\omega}}k\cdot\hat{b}^*e^{i\omega(t'-t_0)}} e^{\pm\frac{i}{\sqrt{2m\omega}}k\cdot\hat{b}e^{-i\omega(t'-t_0)}} e^{\frac{1}{2}\left[-\frac{i}{\sqrt{2m\omega}}k\cdot\hat{b}e^{-i\omega(t'-t_0)},-\frac{i}{\sqrt{2m\omega}}k\cdot\hat{b}^*e^{i\omega(t'-t_0)}\right]} \, |n\rangle \\
&= \langle m| \, e^{\pm\frac{i}{\sqrt{2m\omega}}k\cdot\hat{b}^*e^{i\omega(t'-t_0)}} e^{\pm\frac{i}{\sqrt{2m\omega}}k\cdot\hat{b}e^{-i\omega(t'-t_0)}} e^{-\sum_{i,j=1}^{3}\frac{k_ik_j}{4m\omega}\left[\hat{b}_i,\hat{b}_j^*\right]} \, |n\rangle \\
&= e^{-\frac{k^2}{4m\omega}} \langle m| \, e^{\pm\frac{i}{\sqrt{2m\omega}}k\cdot\hat{b}^*e^{i\omega(t'-t_0)}} e^{\pm\frac{i}{\sqrt{2m\omega}}k\cdot\hat{b}e^{-i\omega(t'-t_0)}} \, |n\rangle \\
&= e^{-\frac{k^2}{4m\omega}} \sum_{g=0}^{\infty}\sum_{h=0}^{\infty} \frac{(\pm i)^{g+h}e^{-i\omega(t'-t_0)(g-h)}}{g!h!(2m\omega)^{\frac{g+h}{2}}} \langle m| \, (k\cdot\hat{b}^*)^h(k\cdot\hat{b})^g \, |n\rangle \\
&= e^{-\frac{k^2}{4m\omega}} \sum_{g=0}^{\infty}\sum_{h=0}^{\infty} \frac{(\pm i)^{g+h}e^{-i\omega(t'-t_0)(g-h)}}{g!h!(2m\omega)^{\frac{g+h}{2}}} \sum_{j_1,\ldots,j_g=1}^{3}\sum_{l_1,\ldots,l_g=1}^{3} k_{l_1}\ldots k_{l_h}k_{j_1}\ldots k_{j_g} \langle m| \, b_{l_1}^*\ldots b_{l_h}^* b_{j_1}\ldots b_{j_g} \, |n\rangle
\end{aligned} \tag{5.42}$$

From the construction of the eigenstates we know for $g \le n$ and $h \le m$:

$$\begin{aligned}
\langle m| \, b_{l_1}^*\ldots b_{l_h}^* b_{j_1}\ldots b_{j_g} \, |n\rangle &= \sqrt{m(m-1)\ldots(m-h)}\sqrt{n(n-1)\ldots(n-g)} \, \langle m-h|n-g\rangle \\
&= \sqrt{m(m-1)\ldots(m-h)}\sqrt{n(n-1)\ldots(n-g)}\delta_{h,g+m-n}
\end{aligned} \tag{5.43}$$

and for $g > n$ or $h > m$:

$$\langle m| \, b_{l_1}^*\ldots b_{l_h}^* b_{j_1}\ldots b_{j_g} \, |n\rangle = 0 \tag{5.44}$$

Hence

$$\begin{aligned}
&\langle m| \, e^{\pm\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega(t'-t_0)}+\hat{b}^*e^{i\omega(t'-t_0)}\right)} \, |n\rangle \\
&= e^{-\frac{k^2}{4m\omega}} \sum_{g=0}^{n}\sum_{h=0}^{m} \frac{(\pm i)^{g+h}e^{-i\omega(t'-t_0)(g-h)}}{g!h!(2m\omega)^{\frac{g+h}{2}}} \sqrt{m(m-1)\ldots(m-h)}\sqrt{n(n-1)\ldots(n-g)}\delta_{h,g+m-n} \\
&\cdot \sum_{j_1,\ldots,j_g=1}^{3}\sum_{l_1,\ldots,l_g=1}^{3} k_{l_1}\ldots k_{l_h}k_{j_1}\ldots k_{j_g}
\end{aligned} \tag{5.45}$$

Let us consider some special cases, e.g. $m = n = 0$. Then

$$\langle 0| \, e^{\pm\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega(t'-t_0)}+\hat{b}^*e^{i\omega(t'-t_0)}\right)} \, |0\rangle = e^{-\frac{k^2}{4m\omega}} \tag{5.46}$$

Hence, we can calculate the expectation value of the field operator in the ground state of the harmonic oscillator:

$$\begin{aligned}
\langle 0| \, \hat{\Phi}_{t,t_0}(x) \, |0\rangle &= -\frac{i\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\gamma_k^2 \int_{t_0}^{t}\mathrm{dt}'\left(e^{-\frac{k^2}{4m\omega}}e^{ik\cdot x+i\omega_k(t'-t)} - c.c.\right) + \mathcal{O}(\lambda^2) \\
&= -\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k}\left(e^{-\frac{k^2}{4m\omega}}e^{ik\cdot x}\left(1-e^{i\omega_k(t_0-t)}\right) + c.c.\right) + \mathcal{O}(\lambda^2)
\end{aligned} \tag{5.47}$$

Substitute $k_i \to -k_i$ for all $i = 1, 2, 3$ in the conjugated terms and use a symmetry argument in order to get

$$\langle 0| \hat{\Phi}_{t,t_0}(x) |0\rangle = -\frac{2\lambda}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k^2}{\omega_k} e^{-\frac{k^2}{4m\omega}} e^{ik\cdot x} \left(1 - \cos\left(\omega_k(t-t_0)\right)\right) + \mathcal{O}(\lambda^2) \tag{5.48}$$

Similarly, as in the very first chapter, we can show that the term containing $t_0$ vanish in the limit $t_0 \to -\infty$. Let us integrate this term by parts for $\mu = 0$ in order to see this:

$$\begin{aligned}
I :=& \int \mathrm{d}^3\mathrm{k} \frac{\gamma_k^2}{\omega_k} e^{-\frac{k^2}{4m\omega}} e^{ik\cdot x} e^{i\omega_k(t_0-t)} = \pi \int_0^\infty \mathrm{d}|\mathrm{k}| e^{-\frac{k^2}{4m\omega}} e^{i|k|(t_0-t)} \int_{-1}^1 \mathrm{d}u e^{i|k||x|u} \\
=& \frac{2\pi}{|x|} \int_0^\infty \mathrm{d}|\mathrm{k}| e^{-\frac{k^2}{4m\omega}} e^{i|k|(t_0-t)} \sin\left(|k||x|\right) \\
=& \frac{2\pi}{|x|} \left[\frac{1}{i(t_0-t)} e^{-\frac{k^2}{4m\omega}} e^{i|k|(t_0-t)} \sin\left(|k||x|\right)\right]_{|k|=0}^\infty \\
&- \frac{2\pi}{i|x|(t_0-t)} \int_0^\infty \mathrm{d}|\mathrm{k}| e^{i|k|(t_0-t)} \left(e^{-\frac{k^2}{4m\omega}} |x| \cos\left(|k||x|\right) - \frac{|k|}{2m\omega} e^{-\frac{k^2}{4m\omega}} e^{i|k|(t_0-t)} \sin\left(|k||x|\right)\right) \\
=& -\frac{2\pi}{i(t_0-t)} \int_0^\infty \mathrm{d}|\mathrm{k}| e^{i|k|(t_0-t)} e^{-\frac{k^2}{4m\omega}} \cos\left(|k||x|\right) \\
&+ \frac{2\pi}{i|x|(t_0-t)} \int_0^\infty \mathrm{d}|\mathrm{k}| e^{i|k|(t_0-t)} \frac{|k|}{2m\omega} e^{-\frac{k^2}{4m\omega}} e^{i|k|(t_0-t)} \sin\left(|k||x|\right)
\end{aligned} \tag{5.49}$$

Using the triangle inequality yields

$$\begin{aligned}
|I| \leq& \frac{2\pi}{|t_0-t|} \int_0^\infty \mathrm{d}|\mathrm{k}| \left| e^{i|k|(t_0-t)} e^{-\frac{k^2}{4m\omega}} \cos\left(|k||x|\right)\right| \\
&+ \frac{2\pi}{|x||t_0-t|} \int_0^\infty \mathrm{d}|\mathrm{k}| \left| e^{i|k|(t_0-t)} \frac{|k|}{2m\omega} e^{-\frac{k^2}{4m\omega}} e^{i|k||t_0-t|} \sin\left(|k||x|\right)\right| \\
\leq& \frac{2\pi}{|t_0-t|} \underbrace{\int_0^\infty \mathrm{d}|\mathrm{k}| e^{-\frac{k^2}{4m\omega}}}_{<\infty} + \frac{2\pi}{|x||t_0-t|} \underbrace{\int_0^\infty \mathrm{d}|\mathrm{k}| \frac{|k|}{2m\omega} e^{-\frac{k^2}{4m\omega}}}_{<\infty} \\
&\to 0 \quad \text{as} \quad t_0 \to -\infty
\end{aligned} \tag{5.50}$$

The integrals which are remaining in the end are just gaussian integrals and thus they are finite. (This is well-known).
Hence, only the terms without $t_0$ survive in this limit and we end up with

$$\begin{aligned}
\lim_{t_0\to-\infty} \langle 0| \hat{\Phi}_{t,t_0}(x) |0\rangle &= -\frac{\lambda}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \frac{2\gamma_k^2}{\omega_k} e^{-\frac{k^2}{4m\omega}} e^{ik\cdot x} + \mathcal{O}(\lambda^2) \\
&= -\frac{\lambda}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \frac{1}{k^2+\mu^2} e^{-\frac{k^2}{4m\omega}} e^{ik\cdot x} + \mathcal{O}(\lambda^2)
\end{aligned} \tag{5.51}$$

Note that the Fourier transformation of a Gaussian wave package is well-known, i.e.

$$e^{-\frac{k^2}{4m\omega}} = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}} \int \mathrm{d}^3 y e^{-ik\cdot y} e^{-m\omega y^2} \tag{5.52}$$

Then

$$\lim_{t_0\to-\infty}\langle 0|\,\hat{\Phi}_{t,t_0}(x)\,|0\rangle = \int \mathrm{d}^3y\,\underbrace{\left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}}e^{-m\omega y^2}}_{=|\Psi_0(y)|^2}\underbrace{\left(-\frac{\lambda}{(2\pi)^3}\right)\int \mathrm{d}^3\mathrm{k}\frac{1}{k^2+\mu^2}e^{ik\cdot(x-y)}}_{=\Phi_{\mathrm{Yukawa}}(x-y)}+\mathcal{O}(\lambda^2)$$
$$= \left(|\Psi_0|^2 \star \Phi_{\mathrm{Yukawa}}\right)(x) + \mathcal{O}(\lambda^2) \tag{5.53}$$

where $\star$ denotes the convolution and $\Phi_n(x) := \langle x|\,|n\rangle \in L^2(\mathbb{R}^3)$ are the eigenfunctions of the quantum harmonic oscillator in position space. It is easy to check that

$$\Psi_0(x) = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{4}} e^{-\frac{1}{2}m\omega x^2} \tag{5.54}$$

is normalized and fulfills

$$H_{osc}\Psi_0 = 0 \Rightarrow \left(\frac{-\Delta}{2m}+\frac{1}{2}m\omega x^2\right)\Psi_0(x) = \frac{3}{2}\omega \tag{5.55}$$

Hence, it is the ground state of the quantum harmonic oscillator.
If we want to describe photon, we have to consider the limit $\mu \to 0$, i.e. a massless meson field. In this limit, the Yukawa potential just becomes the Coulomb potential.
We can state the following theorem as a result:

**Theorem 5.3.3** *The expectation value of the field operator in the ground state of the quantum harmonic oscillator $|0\rangle$ is given by the convolution of the Yukawa potential with the square of the wave function of the ground state, i.e.*

$$\lim_{t_0\to-\infty}\langle 0|\,\hat{\Phi}_{t,t_0}(x)\,|0\rangle = \int \mathrm{d}^3y\,\underbrace{\left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}}e^{-m\omega y^2}}_{=|\Psi_0(y)|^2}\underbrace{\left(-\frac{\lambda}{(2\pi)^3}\right)\int \mathrm{d}^3\mathrm{k}\frac{1}{k^2+\mu^2}e^{ik\cdot(x-y)}}_{=\Phi_{Yukawa}(x-y)}+\mathcal{O}(\lambda^2)$$
$$= \left(|\Psi_0|^2 \star \Phi_{Yukawa}\right)(x) + \mathcal{O}(\lambda^2) \tag{5.56}$$

*where*

$$\Psi_0(x) := \langle x|\,|0\rangle = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{4}} e^{-\frac{1}{2}m\omega x^2} \tag{5.57}$$

*is the normalized ground state wave function of the quantum harmonic oscillator.*

**Proof:** We have already deduced the result step by step, i.e. the theorem is already proved. □

**Remark:** The obtained result is exactly what we would have expected. Up to order $\lambda^2$, we just get the classical solution (the Yukawa potential) convolved with the ground state wave function. This means that due to the uncertainty relation the field smears out a little bit, but for the ground state expectation value this is the only difference to

the classical case.
However, considering the limit $t_0 \to -\infty$, we might loose informations about possible radiation terms. Physically, we can understand this in the following: If we prepare a quantum mechanical system in a way that it emits radiation at an initial time $t_0$ and then let it evolve in time, then the radiation runs away with the speed of light. Hence, if the system evolves in time over an infinitely long period, the radiation runs infinitely far away and thus we can not see it anymore.
Therefore, it might be interesting to state a similar theorem without considering this limit. We will do this in the following.

**Theorem 5.3.4** *The expectation value of the field operator in the ground state of the quantum harmonic oscillator $|0\rangle$ is given by the convolution of the potential with the square of the wave function of the ground state, i.e.*

$$\langle 0| \hat{\Phi}_{t,t_0}(x) |0\rangle = \left(|\Psi_0|^2 \star \Phi_{Yukawa}\right)(x) + \left(|\Psi_0|^2 \star \Phi_{00}\right)(x) + \mathcal{O}(\lambda^2) \tag{5.58}$$

*where*

$$\Psi_0(x) := \langle x| |0\rangle = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{4}} e^{-\frac{1}{2}m\omega x^2} \tag{5.59}$$

*is the normalized ground state wave function of the quantum harmonic oscillator and*

$$\Phi_{00}(x) := \frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k} e^{ik\cdot x}\cos\left(\omega_k(t-t_0)\right) \tag{5.60}$$

**Proof:** From (5.48) we know

$$\langle 0| \hat{\Phi}_{t,t_0}(x) |0\rangle = -\frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k} e^{-\frac{k^2}{4m\omega}} e^{ik\cdot x}\left(1-\cos\left(\omega_k(t-t_0)\right)\right) + \mathcal{O}(\lambda^2) \tag{5.61}$$

We have already shown that the first term is the convolution between the square of the ground state wave function and the Yukawa potential. Together with (5.52), we obtain

$$\begin{aligned}
&\langle 0| \hat{\Phi}_{t,t_0}(x) |0\rangle \\
&= \left(|\Psi_0|^2 \star \Phi_{\text{Yukawa}}\right)(x) \\
&+ \frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k}\left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}}\int \mathrm{d}^3 y e^{-ik\cdot y} e^{-m\omega y^2} e^{ik\cdot x}\cos\left(\omega_k(t-t_0)\right) + \mathcal{O}(\lambda^2) \\
&= \left(|\Psi_0|^2 \star \Phi_{\text{Yukawa}}\right)(x) \\
&+ \frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3 y |\Psi_0|^2(y)\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k} e^{ik\cdot(x-y)}\cos\left(\omega_k(t-t_0)\right) + \mathcal{O}(\lambda^2) \\
&= \left(|\Psi_0|^2 \star \Phi_{\text{Yukawa}}\right)(x) + \left(|\Psi_0|^2 \star \Phi_{00}\right)(x) + \mathcal{O}(\lambda^2)
\end{aligned} \tag{5.62}$$

with

$$\Phi_{00}(x) := \frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k} e^{ik\cdot x}\cos\left(\omega_k(t-t_0)\right) \tag{5.63}$$

□

This result is nice, since it agrees with our intuition and somehow tells us that our calculations are right. It even yields a term, which could maybe be interpreted as a radiation term. However, we continue with calculating some other expectation values in order to get a better understanding of this.

The first idea would be to consider not just the expectation value in the ground state of the harmonic oscillator but in a mixed state, e.g. in a mixture of the ground state and the first excitation, i.e. $|\Psi_{mix}\rangle := |0\rangle + |1\rangle$.
It is clear that if we calculate $\Phi_{mix}(x) := \lim_{t_0\to-\infty} \langle\Psi_{mix}|\,\hat{\Phi}_{t,t_0}(x)\,|\Psi_{mix}\rangle$, there will appear terms like $\Phi_{01}(x) := \langle 0|\,\hat{\Phi}_{t,t_0}(x)\,|1\rangle$. These "cross terms" are not static anymore, hence we would expect that there occurs radiation as well. Therefore we start calculating these cross terms:
From (5.42) we obtain

$$\begin{aligned}\langle 0|\, e^{\pm\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega(t'-t_0)}+\hat{b}^*e^{i\omega(t'-t_0)}\right)}\,|1\rangle &= \pm i e^{-\frac{k^2}{4m\omega}}\frac{e^{-i\omega(t'-t_0)}}{(2m\omega)^{\frac{1}{2}}}\sum_{j=1}^{3}k_j\,\langle 0|\,b_j\,|1\rangle \\ &= \pm i e^{-\frac{k^2}{4m\omega}}\frac{e^{-i\omega(t'-t_0)}}{(2m\omega)^{\frac{1}{2}}}(k_1+k_2+k_3) \qquad (5.64)\end{aligned}$$

and together with (5.41) this yields

$$\begin{aligned}&\langle 0|\,\hat{\Phi}_{t,t_0}(x)\,|1\rangle \\ &= -\frac{\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\,\gamma_k^2 e^{-\frac{k^2}{4m\omega}}\frac{\sum_{j=1}^3 k_j}{\sqrt{2m\omega}}\int_{t_0}^{t}\mathrm{d}t'\left(e^{ik\cdot x-i\omega_k t}e^{i\omega_k t'}e^{-i\omega(t'-t_0)} - e^{-ik\cdot x+i\omega_k t}e^{-i\omega_k t'}e^{-i\omega(t'-t_0)}\right) \\ &+\mathcal{O}(\lambda^2) \\ &= \frac{i\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k^2-\omega^2}e^{-\frac{k^2}{4m\omega}}\frac{\sum_{j=1}^3 k_j}{\sqrt{2m\omega}} \\ &\cdot\left((\omega_k+\omega)e^{ik\cdot x-i\omega(t-t_0)} - (\omega_k+\omega)e^{ik\cdot x-i\omega_k(t-t_0)} - (\omega_k-\omega)e^{-ik\cdot x-i\omega(t-t_0)} + (\omega_k-\omega)e^{ik\cdot x+i\omega_k(t-t_0)}\right) \\ &+\mathcal{O}(\lambda^2) \qquad (5.65)\end{aligned}$$

Substitute $k_i \to -k_i, i = 1,2,3$ in the terms containing $e^{-ik\cdot x}$ and obtain

$$\begin{aligned}&\langle 0|\,\hat{\Phi}_{t,t_0}(x)\,|1\rangle \\ &= \frac{2i\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k^2-\omega^2}e^{-\frac{k^2}{4m\omega}}\frac{\sum_{j=1}^3 k_j}{\sqrt{2m\omega}}e^{ik\cdot x}\left(\omega_k e^{-i\omega(t-t_0)} - \omega_k\cos(\omega_k(t-t_0)) + 2i\omega\sin(\omega_k(t-t_0))\right) \\ &+\mathcal{O}(\lambda^2) \qquad (5.66)\end{aligned}$$

We have already seen that considering the limit $t_0 \to -\infty$ might destroy the radiation terms. Therefore, if we want to see a radiation term, we have to consider a time evolution over finite time interval $[t_0, t] \in \mathbb{R}$.

Similarly as for the ground state, we can rewrite the above expression in terms of the eigenfunctions of the harmonic oscillator. We know the ground state of the harmonic oscillator $\Psi_0$ and also its first excitation $\Psi_1$:

$$\Psi_0(x) := \langle x| \, |0\rangle = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{4}} e^{-\frac{1}{2}m\omega x^2}$$
$$\Psi_1(x) := \langle x| \, |1\rangle = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{4}} e^{-\frac{1}{2}m\omega x^2}\sqrt{2m\omega}\,(x_1 + x_2 + x_3) \tag{5.67}$$

with $x \in \mathbb{R}^3$. Then

$$(\Psi_0 \cdot \Psi_1)(x) = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}} e^{-m\omega x^2}\sqrt{2m\omega}\,(x_1 + x_2 + x_3) \tag{5.68}$$

Performing a Fourier transformation yields

$$\begin{aligned}(\Psi_0 \cdot \Psi_1)(k) &= \left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}}\sqrt{2m\omega}\,\underbrace{\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3x e^{-m\omega x^2}(x_1 + x_2 + x_3)\,e^{-ik\cdot x}}_{=-i\frac{k_1+k_2+k_3}{(2m\omega)^{\frac{5}{2}}}e^{-\frac{k^2}{4m\omega}}} \\ &= -i\left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}}\frac{k_1 + k_2 + k_3}{(2m\omega)^2}e^{-\frac{k^2}{4m\omega}}\end{aligned} \tag{5.69}$$

Then

$$\frac{k_1 + k_2 + k_3}{(2m\omega)^{\frac{1}{2}}}e^{-\frac{k^2}{4m\omega}} = i\left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}}\sqrt{2m\omega}\int d^3x e^{-m\omega x^2}(x_1 + x_2 + x_3)\,e^{-ik\cdot x} \tag{5.70}$$

Plugging this into (5.66), we obtain

$$\begin{aligned}&\langle 0|\,\hat{\Phi}_{t,t_0}(x)\,|1\rangle \\ &= -\frac{2\lambda}{(2\pi)^3}\int d^3k\frac{\gamma_k^2}{\omega_k^2 - \omega^2}\left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}}\sqrt{2m\omega}\int d^3y e^{-m\omega y^2}(y_1 + y_2 + y_3)\,e^{ik\cdot(x-y)} \\ &\cdot\left(\omega_k e^{-i\omega(t-t_0)} - \omega_k\cos(\omega_k(t - t_0)) + 2i\omega\sin(\omega_k(t - t_0))\right) + \mathcal{O}(\lambda^2) \\ &= ((\Psi_0 \cdot \Psi_1) \star \Phi_{01})(x) + \mathcal{O}(\lambda^2)\end{aligned} \tag{5.71}$$

where $\star$ denotes again the convolution and

$$\Phi_{01}(x) := -\frac{2\lambda}{(2\pi)^3}\int d^3k\frac{\gamma_k^2}{\omega_k^2 - \omega^2}e^{ik\cdot x}\left(\omega_k e^{-i\omega(t-t_0)} - \omega_k\cos(\omega_k(t - t_0)) + 2i\omega\sin(\omega_k(t - t_0))\right) \tag{5.72}$$

This result fits well into our picture, since we get again just an potential $\Phi_{01}$ which smears out with the eigenfunctions of the quantum harmonic oscillator.

From now on, we consider the case $\mu = 0$. We can sum up the results in the following theorem:

**Theorem 5.3.5** *The matrix element of the field operator between the ground state of the quantum harmonic oscillator $|0\rangle$ and its first excitation $|1\rangle$ is given by the convolution of the potential $\Phi_{01}$ with the product of the two wave functions of the ground state and the first excitation, i.e.*

$$\langle 0|\,\hat{\Phi}_{t,t_0}(x)\,|1\rangle = \left((\Psi_0 \cdot \Psi_1) \star \Phi_{01}\right)(x) + \mathcal{O}(\lambda^2) \tag{5.73}$$

*where*

$$\begin{aligned}\Psi_0(x) &:= \langle x|\,|0\rangle = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{4}} e^{-\frac{1}{2}m\omega x^2} \\ \Psi_1(x) &:= \langle x|\,|1\rangle = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{4}} e^{-\frac{1}{2}m\omega x^2}\sqrt{2m\omega}\,(x_1+x_2+x_3)\end{aligned} \tag{5.74}$$

*with $x \in \mathbb{R}^3$ are the normalized ground state wave function and the normalized wave function of the first excitation of the quantum harmonic oscillator and*

$$\Phi_{01}(x) := -\frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k^2-\omega^2}e^{ik\cdot x}\left(\omega_k e^{-i\omega(t-t_0)} - \omega_k\cos(\omega_k(t-t_0)) + 2i\omega\sin(\omega_k(t-t_0))\right) \tag{5.75}$$

**Proof:** We have already deduced the result step by step, i.e. the theorem is already proved. □

Next, we are interested in the expectation value of the field in the first excitation state, i.e.

$$\begin{aligned}&\langle 1|\,\hat{\Phi}_{t,t_0}(x)\,|1\rangle \\ &= -\frac{i\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\gamma_k^2\int_{t_0}^{t}\mathrm{d}t'\left(\langle 1|\,e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega(t'-t_0)}+\hat{b}^*e^{i\omega(t'-t_0)}\right)}\,|1\rangle\,e^{ik\cdot x+i\omega_k(t'-t)} - c.c.\right) \\ &+ \mathcal{O}(\lambda^2)\end{aligned} \tag{5.76}$$

with (5.42)

$$\begin{aligned}&\langle 1|\,e^{-\frac{i}{\sqrt{2m\omega}}k\cdot\left(\hat{b}e^{-i\omega(t'-t_0)}+\hat{b}^*e^{i\omega(t'-t_0)}\right)}\,|1\rangle \\ &= e^{-\frac{k^2}{4m\omega}}\sum_{g=0}^{1}\sum_{h=0}^{1}\frac{(-i)^{g+h}e^{-i\omega(t'-t_0)(g-h)}}{g!h!(2m\omega)^{\frac{g+h}{2}}}\sum_{j_1,\dots,j_g=1}^{3}\sum_{l_1,\dots,l_g=1}^{3}k_{l_1}\dots k_{l_h}k_{j_1}\dots k_{j_g}\,\langle 1|\,b_{l_1}^*\dots b_{l_h}^*b_{j_1}\dots b_{j_g}\,|1\rangle \\ &= e^{-\frac{k^2}{4m\omega}}\left(1-\frac{1}{2m\omega}\sum_{j=1}^{3}\sum_{l=1}^{3}k_jk_l\,\langle 1|\,b_j^*b_l\,|1\rangle\right) \\ &= e^{-\frac{k^2}{4m\omega}}\left(1-\frac{1}{2m\omega}\sum_{j,l=1}^{3}k_jk_l\right)\end{aligned} \tag{5.77}$$

This yields

$$\begin{aligned}
&\langle 1|\,\hat{\Phi}_{t,t_0}(x)\,|1\rangle = \\
&= -\frac{i\lambda}{(2\pi)^3}\int \mathrm{d^3k}\gamma_k^2 e^{-\frac{k^2}{4m\omega}}\left(1-\frac{1}{2m\omega}\sum_{j,l=1}^{3}k_j k_l\right)\left[e^{ik\cdot x-i\omega_k t}\int_{t_0}^{t}\mathrm{dt}' e^{i\omega_k t'} - c.c.\right] + \mathcal{O}(\lambda^2) \\
&= -\frac{\lambda}{(2\pi)^3}\int \mathrm{d^3k}\frac{\gamma_k^2}{\omega_k} e^{-\frac{k^2}{4m\omega}}\left(1-\frac{1}{2m\omega}\sum_{j,l=1}^{3}k_j k_l\right)\left[e^{ik\cdot x-i\omega_k t}e^{i\omega_k t'} + c.c.\right]_{t'=t_0}^{t} + \mathcal{O}(\lambda^2) \\
&= -\frac{\lambda}{(2\pi)^3}\int \mathrm{d^3k}\frac{\gamma_k^2}{\omega_k} e^{-\frac{k^2}{4m\omega}}\left(1-\frac{1}{2m\omega}\sum_{j,l=1}^{3}k_j k_l\right)\left[e^{ik\cdot x} - e^{ik\cdot x-i\omega_k(t-t_0)} + c.c.\right] + \mathcal{O}(\lambda^2) \\
&= -\frac{\lambda}{(2\pi)^3}\int \mathrm{d^3k}\frac{\gamma_k^2}{\omega_k} e^{-\frac{k^2}{4m\omega}}\left(1-\frac{1}{2m\omega}\sum_{j,l=1}^{3}k_j k_l\right)\left[e^{ik\cdot x} - e^{ik\cdot x-i\omega_k(t-t_0)} + c.c.\right] + \mathcal{O}(\lambda^2)
\end{aligned} \tag{5.78}$$

Substitute $k_i \to -k_i$ for all $i = 1, 2, 3$ in the conjugated terms and use a symmetry argument in order to get

$$\begin{aligned}
&\langle 1|\,\hat{\Phi}_{t,t_0}(x)\,|1\rangle \\
&= -\frac{2\lambda}{(2\pi)^3}\int \mathrm{d^3k}\frac{\gamma_k^2}{\omega_k} e^{-\frac{k^2}{4m\omega}}\left(1-\frac{1}{2m\omega}\sum_{j,l=1}^{3}k_j k_l\right) e^{ik\cdot x}\left[1-\cos\left(\omega_k(t-t_0)\right)\right] + \mathcal{O}(\lambda^2)
\end{aligned} \tag{5.79}$$

We know the first excitation state of the quantum harmonic oscillator:

$$\Psi_1(x) = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{4}} e^{-\frac{1}{2}m\omega x^2}\sqrt{2m\omega}(x_1+x_2+x_3) \tag{5.80}$$

and hence

$$|\Psi_1|^2(x) = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}} e^{-m\omega x^2} 2m\omega(x_1+x_2+x_3)^2 \tag{5.81}$$

We can calculate the Fourier transform

$$\begin{aligned}
|\Psi_1|^2(k) &= \left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}} 2m\omega\frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d^3x} e^{-ik\cdot x}e^{-m\omega x^2}(x_1+x_2+x_3)^2 \\
&= \left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}} 2m\omega\int \frac{\mathrm{d^3x}}{(2\pi)^{\frac{3}{2}}} e^{-ik\cdot x}e^{-m\omega x^2}\left(x_1^2+x_2^2+x_3^2+2x_1x_2+2x_1x_3+2x_2x_3\right)
\end{aligned} \tag{5.82}$$

We calculate step by step

$$\begin{aligned}
&\int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^{\frac{3}{2}}} e^{-ik\cdot x} e^{-m\omega x^2} x_1^2 \\
&= \int \frac{\mathrm{dx}_1}{(2\pi)^{\frac{1}{2}}} e^{-ik_1x_1} e^{-m\omega x_1^2} x_1^2 \int \frac{\mathrm{dx}_2}{(2\pi)^{\frac{1}{2}}} e^{-ik_2x_2} e^{-m\omega x_2^2} \int \frac{\mathrm{dx}_3}{(2\pi)^{\frac{1}{2}}} e^{-ik_3x_3} e^{-m\omega x_3^2} \\
&= \frac{1}{(2m\omega)^{\frac{1}{2}}} e^{-\frac{k_3^2}{4m\omega}} \frac{1}{(2m\omega)^{\frac{1}{2}}} e^{-\frac{k_2^2}{4m\omega}} \int \frac{\mathrm{dx}_1}{(2\pi)^{\frac{1}{2}}} e^{-ik_1x_1} e^{-m\omega x_1^2} x_1^2 \\
&= \frac{1}{(2m\omega)^{\frac{1}{2}}} e^{-\frac{k_3^2}{4m\omega}} \frac{1}{(2m\omega)^{\frac{1}{2}}} e^{-\frac{k_2^2}{4m\omega}} \frac{1}{(2m\omega)^{\frac{3}{2}}} e^{-\frac{k_1^2}{4m\omega}} \left(1 - \frac{k_1^2}{2m\omega}\right) \\
&= \frac{1}{(2m\omega)^{\frac{5}{2}}} e^{-\frac{k^2}{4m\omega}} \left(1 - \frac{k_1^2}{2m\omega}\right) \qquad (5.83)
\end{aligned}$$

Hence

$$\int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^{\frac{3}{2}}} e^{-ik\cdot x} e^{-m\omega x^2} \left(x_1^2 + x_2^2 + x_3^2\right) = \frac{1}{(2m\omega)^{\frac{7}{2}}} e^{-\frac{k^2}{4m\omega}} \left(2m\omega - k_1^2 - k_2^2 - k_3^2\right) \qquad (5.84)$$

And similar

$$\begin{aligned}
&\int \frac{\mathrm{d}^3\mathrm{x}}{(2\pi)^{\frac{3}{2}}} e^{-ik\cdot x} e^{-m\omega x^2} 2x_1x_2 \\
&= 2\int \frac{\mathrm{dx}_1}{(2\pi)^{\frac{1}{2}}} e^{-ik_1x_1} e^{-m\omega x_1^2} x_1 \int \frac{\mathrm{dx}_2}{(2\pi)^{\frac{1}{2}}} e^{-ik_2x_2} e^{-m\omega x_2^2} x_2 \int \frac{\mathrm{dx}_3}{(2\pi)^{\frac{1}{2}}} e^{-ik_3x_3} e^{-m\omega x_3^2} \\
&= 2\frac{1}{(2m\omega)^{\frac{1}{2}}} e^{-\frac{k_3^2}{4m\omega}} \frac{-i}{(2m\omega)^{\frac{3}{2}}} k_2 e^{-\frac{k_2^2}{4m\omega}} \frac{-i}{(2m\omega)^{\frac{3}{2}}} k_1 e^{-\frac{k_1^2}{4m\omega}} \\
&= -\frac{1}{(2m\omega)^{\frac{7}{2}}} e^{-\frac{k^2}{4m\omega}} 2k_1k_2 \qquad (5.85)
\end{aligned}$$

Putting everything together, we obtain

$$\begin{aligned}
|\Psi_1|^2(k) &= \left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}} 2m\omega \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{x} e^{-ik\cdot x} e^{-m\omega x^2} (x_1 + x_2 + x_3)^2 \\
&= \left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}} 2m\omega \frac{1}{(2m\omega)^{\frac{5}{2}}} e^{-\frac{k^2}{4m\omega}} \left(1 - \frac{1}{2m\omega}(k_1 + k_2 + k_3)^2\right) \qquad (5.86)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
e^{-\frac{k^2}{4m\omega}} \left(1 - \frac{1}{2m\omega}(k_1 + k_2 + k_3)^2\right) &= 2m\omega \left(\frac{m\omega}{\pi}\right)^{\frac{3}{2}} \int \mathrm{d}^3\mathrm{x} e^{-ik\cdot x} e^{-m\omega x^2} (x_1 + x_2 + x_3)^2 \\
&= \int \mathrm{d}^3\mathrm{x} e^{-ik\cdot x} |\Psi_1|^2(x) \qquad (5.87)
\end{aligned}$$

Plugging this into (5.79) yields

$$\begin{aligned}
&\langle 1|\,\hat{\Phi}_{t,t_0}(x)\,|1\rangle \\
&= -\frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3y|\Psi_1|^2(y)\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k}e^{ik\cdot(x-y)}\left[1-\cos\left(\omega_k(t-t_0)\right)\right]+\mathcal{O}(\lambda^2) \\
&= \left(|\Psi_1|^2\star\Phi_{\mathrm{Yukawa}}\right)(x)+\left(|\Psi_1|^2\star\Phi_{11}\right)(x)+\mathcal{O}(\lambda^2)
\end{aligned} \tag{5.88}$$

where

$$\Phi_{11}(x) := \frac{2\lambda^2}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k}e^{ik\cdot x}\cos\left(\omega_k(t-t_0)\right) \tag{5.89}$$

Note that $\Phi_{11}=\Phi_{00}$ This can be summarized in the following theorem

**Theorem 5.3.6** *The expectation value of the field operator in the first excitation state of the quantum harmonic oscillator $|1\rangle$ is given by the convolution of the potential $\Phi_{Yukawa}+\Phi_{00}$ with the square of the wave function.*

$$\langle 1|\,\hat{\Phi}_{t,t_0}(x)\,|1\rangle = \left(|\Psi_1|^2\star\Phi_{Yukawa}\right)(x)+\left(|\Psi_1|^2\star\Phi_{00}\right)(x)+\mathcal{O}(\lambda^2) \tag{5.90}$$

*where*

$$\begin{aligned}
\Psi_0(x) &:= \langle x|\,|0\rangle = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{4}}e^{-\frac{1}{2}m\omega x^2} \\
\Psi_1(x) &:= \langle x|\,|1\rangle = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{4}}e^{-\frac{1}{2}m\omega x^2}\sqrt{2m\omega}\,(x_1+x_2+x_3)
\end{aligned} \tag{5.91}$$

*with $x\in\mathbb{R}^3$ is the normalized wave function of the first excitation of the quantum harmonic oscillator and*

$$\Phi_{00}(x) : = \frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k}e^{ik\cdot x}\cos\left(\omega_k(t-t_0)\right) \tag{5.92}$$

**Proof:** We have already done the proof by deducing the result step by step. □

Finally, we can make a statement about the expectation value of the field operator in the mixed state $|\Psi_{mix}\rangle := |0\rangle + |1\rangle$

**Theorem 5.3.7** *Let $|\Psi_{mix}\rangle := \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ be the normalized mixed state out of the ground state and the first excitation of the quantum harmonic oscillator, where*

$$\begin{aligned}
\Psi_0(x) &:= \langle x|\,|0\rangle = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{4}}e^{-\frac{1}{2}m\omega x^2} \\
\Psi_1(x) &:= \langle x|\,|1\rangle = \left(\frac{m\omega}{\pi}\right)^{\frac{3}{4}}e^{-\frac{1}{2}m\omega x^2}\sqrt{2m\omega}\,(x_1+x_2+x_3)
\end{aligned} \tag{5.93}$$

*with $x \in \mathbb{R}^3$ are the normalized ground state wave function and the normalized wave function of the first excitation of the quantum harmonic oscillator. Then*

$$\begin{aligned}\langle\Psi_{mix}|\,\hat{\Phi}_{t,t_0}(x)\,|\Psi_{mix}\rangle &= \left(|\Psi_{mix}|^2 \star \Phi_{Yukawa}\right)(x) + \left(|\Psi_{mix}|^2 \star \Phi_{00}\right)(x)\\ &\quad + \left((\Psi_0\Psi_1) \star \mathrm{Re}\{\Phi_{01}\}\right)(x) + \mathcal{O}(\lambda^2)\end{aligned} \tag{5.94}$$

*with*

$$\begin{aligned}\Phi_{Yukawa}(x) &:= -\frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k}e^{ik\cdot x}\\ \Phi_{00}(x) &:= \frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k}e^{ik\cdot x}\cos\left(\omega_k(t-t_0)\right)\\ \mathrm{Re}\{\Phi_{01}(x)\} &:= -\frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2\omega_k}{\omega_k^2-\omega^2}e^{ik\cdot x}\Big(\cos(\omega(t-t_0)) - \cos(\omega_k(t-t_0))\Big)\end{aligned} \tag{5.95}$$

**Proof:** Actually, this theorem is a direct consequence of the previous ones. It is just summing up all the contributions.

$$\begin{aligned}&\langle\Psi_{mix}|\,\Phi_{t,t_0}(x)\,|\Psi_{mix}\rangle\\ &= \frac{1}{2}\langle 0|\,\Phi_{t,t_0}(x)\,|0\rangle + \frac{1}{2}\langle 1|\,\Phi_{t,t_0}(x)\,|1\rangle + \frac{1}{2}\langle 0|\,\Phi_{t,t_0}(x)\,|1\rangle + \frac{1}{2}\langle 1|\,\Phi_{t,t_0}(x)\,|0\rangle\\ &= \frac{1}{2}\left(|\Psi_0|^2 \star \Phi_{\text{Yukawa}}\right)(x) + \frac{1}{2}\left(|\Psi_0|^2 \star \Phi_{00}\right)(x)\\ &+ \frac{1}{2}\left(|\Psi_1|^2 \star \Phi_{\text{Yukawa}}\right)(x) + \frac{1}{2}\left(|\Psi_1|^2 \star \Phi_{00}\right)(x)\\ &+ \frac{1}{2}\left((\Psi_0\Psi_1) \star \Phi_{01}\right)(x) + \frac{1}{2}\left((\Psi_0\Psi_1) \star \overline{\Phi_{01}}\right)(x) + \mathcal{O}(\lambda^2)\end{aligned} \tag{5.96}$$

$|0\rangle$ and $|1\rangle$ are orthogonal, hence $|\Psi_{mix}|^2 = \frac{1}{2}|\Psi_0|^2 + \frac{1}{2}|\Psi_1|^2$.
Further, $\frac{1}{2}\Phi_{01} + \frac{1}{2}\overline{\Phi_{01}} = \mathrm{Re}\{\Phi_{01}\}$. Then

$$\begin{aligned}\langle\Psi_{mix}|\,\Phi_{t,t_0}(x)\,|\Psi_{mix}\rangle &= \left(|\Psi_{mix}|^2 \star \Phi_{\text{Yukawa}}\right)(x) + \left(|\Psi_{mix}|^2 \star \Phi_{00}\right)(x)\\ &\quad + \left((\Psi_0\Psi_1) \star \mathrm{Re}\{\Phi_{01}\}\right)(x) + \mathcal{O}(\lambda^2)\end{aligned} \tag{5.97}$$

And with (5.66)

$$\begin{aligned}&\mathrm{Re}\{\Phi_{01}(x)\} = \frac{1}{2}\left(\Phi_{01}(x) + \overline{\Phi_{01}(x)}\right)\\ &= -\frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2}{\omega_k^2-\omega^2}e^{ik\cdot x}\left(\frac{\omega_k}{2}\left(e^{-i\omega(t-t_0)} + e^{-i\omega(t-t_0)}\right) - \omega_k\cos(\omega_k(t-t_0))\right)\\ &= -\frac{2\lambda}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\frac{\gamma_k^2\omega_k}{\omega_k^2-\omega^2}e^{ik\cdot x}\Big(\cos(\omega(t-t_0)) - \cos(\omega_k(t-t_0))\Big)\end{aligned} \tag{5.98}$$

All the remaining terms are know from the previous theorems. □

We expect the expectation value of the field to fulfill the inhomogeneous wave equation with a delta source term at the position of the charge, independent of the state in which we evaluate this expectation value. At least, up the first excitation of the harmonic oscillator, this can be formulated in the following theorem.

**Theorem 5.3.8** *Let $|\Psi_{mix}\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ be the normalized mixed state out of the ground state and the first excitation of the quantum harmonic oscillator.*
*Then the in the limit $f \to 1$, we get*

$$\left(\Box_{(x,t)} + \mu^2\right) \Phi_{Yukawa}(x) = \lambda\delta^{(3)}(x) \tag{5.99}$$

$$\left(\Box_{(x,t)} + \mu^2\right) \Phi_{00}(x) = 0 \tag{5.100}$$

$$\left(\Box_{(x,t)} + \mu^2\right) \mathrm{Re}\{\Phi_{01}(x)\} = -\lambda\cos(\omega(t-t_0))\delta^{(3)}(x) \tag{5.101}$$

**Proof:** We already know that for $f \to 1$

$$\left(\Box_{(x,t)} + \mu^2\right) \Phi_{\mathrm{Yukawa}}(x) = \lambda\delta^{(3)}(x) \tag{5.102}$$

Similar, we calculate for $f \to 1$

$$\begin{aligned}
\left(\Box_{(x,t)} + \mu^2\right) \Phi_{00}(x) &= \frac{\lambda}{(2\pi)^3}\int \mathrm{d^3k}\frac{1}{\omega_k^2}\left(\Box_{(x,t)} + \mu^2\right)\left(e^{ik\cdot x}\cos\left(\omega_k(t-t_0)\right)\right) \\
&= \frac{\lambda}{(2\pi)^3}\int \mathrm{d^3k}\frac{1}{\omega_k^2}\underbrace{\left(k^2+\mu^2-\omega_k^2\right)}_{=0}\left(e^{ik\cdot x}\cos\left(\omega_k(t-t_0)\right)\right) \\
&= 0
\end{aligned} \tag{5.103}$$

Further, from Theorem 5.3.7 we conclude in the limit $f \to 1$

$$\begin{aligned}
&\left(\Box_{(x,t)} + \mu^2\right) \mathrm{Re}\{\Phi_{01}(x)\} \\
&= -\frac{2\lambda}{(2\pi)^3}\int \mathrm{d^3k}\frac{\gamma_k^2\omega_k}{\omega_k^2-\omega^2}\left(\Box_{(x,t)} + \mu^2\right)e^{ik\cdot x}\Big(\cos(\omega(t-t_0)) - \cos(\omega_k(t-t_0))\Big) \\
&= -\frac{2\lambda}{(2\pi)^3}\int \mathrm{d^3k}\frac{\gamma_k^2\omega_k}{\omega_k^2-\omega^2}e^{ik\cdot x}\cos(\omega(t-t_0))\left(-\omega^2+\mu^2+k^2\right) \\
&\quad+\frac{2\lambda}{(2\pi)^3}\int \mathrm{d^3k}\frac{\gamma_k^2\omega_k}{\omega_k^2-\omega^2}e^{ik\cdot x}\cos(\omega_k(t-t_0))\underbrace{\left(-\omega_k^2+\mu^2+k^2\right)}_{=0} \\
&= -\frac{\lambda}{(2\pi)^3}\int \mathrm{d^3k}e^{ik\cdot x}\cos(\omega(t-t_0)) \\
&= -\lambda\cos(\omega(t-t_0))\delta^{(3)}(x)
\end{aligned} \tag{5.104}$$

□

**Remark:** Note that $\left(\Box_{(x,t)} + \mu^2\right)$ commutes with $\star$ and hence, the statement above can be directly applied to the expectation values of the field via the theorems 5.3.4, 5.3.6 and 5.3.7.

## 5.4 Conclusion and interpretation

Calculating the expectation values in this section, we found that they are basically given by the convolution of the wave function of the nucleon with a potential. In general, the potential consists of different parts. There is always a Yukawa potential, which solves the inhomogeneous wave equation with a source at the position of the nucleon. Similarly to the classical LWFs, this can be associated with the the field attached the particle.
Further, the potential might contain terms like $\Phi_{00}$ and $\text{Re}\{\Phi_{01}\}$, defined in Theorem 5.3.7. These terms solve the homogeneous wave equation. This suggests that they can be interpreted as radiation terms.
The origin of these radiation terms is a different one. The $\Phi_{00}$ term already appears, if we calculate the expectation value of field in the ground state of the harmonic oscillator (Theorem 5.3.4). The reason for this is that starting with some initial data, the Yukawa potential builds up inside the light-cone. Hence, in a finite time interval it can not build up everywhere, the $\Phi_{00}$ term cuts of the part outside the light-cone. In the limit $t_0 \to -\infty$, this term vanishes and therefore the full Yukawa potential remains. The same term occurs, if we calculate the expectation value of the field in the first excitation of the harmonic oscillator (Theorem 5.3.6) and it can be interpreted in the same way. However, the term $\text{Re}\{\Phi_{01}\}$ only appears in the formula for the mixed state $|\Psi_{mix}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ in Theorem 5.3.7 and it is due to the oscillation of the particle.

# 6 Outlook

This work gave a first understanding of how we could find a second quantized analogue of the LWFs. However, we considered only some very specific situations and we made simplification like the negligence of pair-creation and spin. Further, in the case with quantum mechanical motion, the results, we have found, hold just up to first order in the coupling constant, i.e. we do not consider effects like radiation back-reaction.
Nevertheless, the results do fit well into our physical intuition and it seems possible to include these neglected effects in a next step.
Moreover, in the last chapter we found terms in the expectation value of the field, which suggest to be interpreted as radiation. Unfortunately, we still do not have an idea how we can see such radiation effects in for example a wave function.

# Notation

| | |
|---|---|
| $\hbar = c = 1$ | We work in natural units |
| $x, y, v, p, k, E, B, j \in \mathbb{R}^3$ | Writing a spatial vector, a momentum/velocity vector, a electromagnetic field vector or a current vector we mean a three dimensional vector in $\mathbb{R}^3$. Nevertheless, for notational simplicity we do not denote these vector by bold symbols or something else, but it will be clear out of the context. |
| $\int \mathrm{d}^3x$ | $\int_{\mathbb{R}^3} \mathrm{d}^3x$ |
| $d\Omega$ | $r^2 \sin\theta d\varphi d\theta$ (angular measure in spherical coordinates) |
| $e_r$ | unit vector in radial direction in spherical coordinates |
| $\mathcal{B}_r(x)$ | compact ball around $x \in \mathbb{R}^3$ with radius $r > 0$ |
| $x \cdot y$ | Euclidean scalar product between two vectors $x, y \in \mathbb{R}^3$ |
| $x \times y$ | Cross product between two vectors $x, y \in \mathbb{R}^3$ |
| $\nabla$, $\nabla\cdot$, $\nabla\times$ | Gradient, divergence, curl |
| $\lvert\cdot\rvert$, $\lVert\cdot\rVert_2$ | Euclidean norm, 2-norm |
| $\mathbb{I}$ | Identity operator |
| $\mathcal{O}_{x\to y}(g(x))$ | $f(x) \in \mathcal{O}_{x\to y}(g(x))$ iff $\lim\limits_{x\to y} \frac{\lVert f(x)\rVert}{\lVert g(x)\rVert}$. Throughout this work we have not written $x \to y$ explicitly, but it is always clear from clear context, what it meant by the notation. |
| $L^2(\mathbb{R}^3)$ | We actually mean $L^2(\mathbb{R}^3, \mathbb{C}, \mathrm{d}^3x)$, the Hilbert space of square integrable functions $\mathbb{R}^3 \to \mathbb{C}$. |
| $C_0^\infty(\mathbb{R}^3)$ | We actually mean $C_0^\infty(\mathbb{R}^3, \mathbb{C}, \mathrm{d}^3x)$, the space of smooth functions $\mathbb{R}^3 \to \mathbb{C}$ with compact support. |
| $f \star g$ | $(f \star g)(x) = \int \mathrm{d}^3y f(x-y) g(y)$ is the convolution of two functions $f, g$ |

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# Acknowledgement

Hereby, I would like to thank my family, especially my parents, Margret Hänle and Manfred Hänle for their ongoing loving support, including their boundless financial help and encouragement throughout my life. Without their support, it would have been impossible for me to finish my education in any comparable way.

Moreover, I would like to express my deepest gratitude to my supervisor, Jun.-Prof. Dr. Dirk-André Deckert for his continuous support, for his patience, motivation and immense knowledge. He was my teacher in mathematics and physics over the past year. I am very grateful for the time he invested in me explaining and discussing mathematical and physical topics. His guidance helped me to get a better understanding of nature. I could not have imagined having a better advisor and mentor.

Last but not least, I want to thank all my friends and fellow students for stimulating discussions and funny times in between my studies during as well as common leisure activities in order to get new motivation for upcoming studies. Marco Smolla and Stephan Kigle, it was a pleasure to spend most of the lunch breaks with you discussing over physics and other things in the world.

# Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig und ohne Benutzung anderer als der von mir angegebenen Quellen und Hilfsmittel verfasst habe.

München, den