On Relativistic Interaction of Electric Charges and External Fields in Quantum Electrodynamics

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig-Maximilians-Universität München

eingereicht von
Markus Hartmut Nöth
06.08.2021

1. Gutachter/in: Dirk-André Deckert
2. Gutachter/in: Prof. Herbert Spohn
3. Gutachter/in: Prof. Roderich Tumulka

Tag der mündlichen Prüfung:

"Je n'ai fait celle-ci plus longue que parce que je n'ai pas eu le loisir de la faire plus courte."[79]

Abstract

The main subject of this thesis is the problem of introducing interactions into relativistic quantum mechanics. This problem has many facets, two of which will be discussed.

The first one deals with a recent relativistically invariant integral equation for multi-time wave functions by Lienert [64]. From a mathematical point of view this proposal is promising, since variants of it have been shown to be mathematically well-defined. In this thesis, firstly, previous results on existence and uniqueness of solutions of a variant of this equation for scalar particles are extended to include more realistic types of interaction.
Secondly, a proof of existence and uniqueness of solutions of another variant that allows to treat spin 1/2 particles is provided.

The second facet concerns interactions in the context of a variable number of particles. Following famous works of Dirac [22], Feynman [35] and Schwinger [95], we treat external electrodynamic fields in an otherwise free Quantum Field Theory of electrons. In previous results [86, 97, 75, 13, 15], candidates for the time evolution operator have been constructed in this setting. This construction is unique up to a phase, which may depend on the external field. This phase affects the charge current density and should thus be identified. In this work, this problem is addressed by a geometric, which was inspired by [90] and developed jointly with my supervisors, construction assuming a certain causality condition.
Secondly, a compact formula for the scattering operator in terms of the corresponding one-particle scattering operator is provided and shown to be well-defined, assuming certain conditions on the external field. This formula is used to show that the second quantized scattering operator is an analytic function of the external field in a certain sense.

Das Hauptthema dieser Arbeit sind die Schwierigkeiten die dabei auftreten Wechselwirkungen in die relativistische Quantenmechanik einzuführen. Dieses Problem weist viele Facetten auf. Zwei dieser Facetten werden bearbeitet.
Die erste handelt von einer kürzlich von Lienert [64] vorgestellten relativistisch invariante Integralgleichung für Wellenfunktionen für mehrere Zeitkoordinaten. Aus einer mathematischen Perspektive ist diese Vorgehensweise vielversprechend, denn für Varianten dieser Gleichung wurde bereits Wohldefiniertheit bewiesen. In dieser Arbeit werden zunächst bestehende Resultate über Existenz und Eindeutigkeit von Lösungen einer Variante dieser Gleichung für skalare Teilchen auf realistische Wechselwirkungen erweitert. Weiterhin wird ein erstes Resultat über Existenz und Eindeutigkeit von Lösungen für eine Variante der Gleichung für spin-1/2 bewiesen.
Die zweite Facette handelt von Wechselwirkung im Kontext veränderlicher Teilchenzahl. Wir behandeln die Theorie externer elektromagnetischer Felder in ansonsten freier Quantenfeldtheorie im Sinne der berühmten Arbeiten von Dirac [22], Feynman [35] and Schwinger [95]. Frühere Resultate [86, 97, 75, 13, 15] konstruierten Zeitentwicklungsoperatoren bis auf eine Phase eindeutig, welche vom externen Feld abhängen kann. Diese Phase beeinflusst die elektrische Stromdichte und sollte daher identifiziert werden. In dieser Arbeit wird dieses Problem durch eine geometrische Konstruktion, welche eine gewisse Kausalitätsbedingung vorraussetzt und inspiriert wurde durch [90] und gemeinsam mit meinen Betreuern entwickelt wurde, behandelt.
Anschließend wird eine kompakte Formel für den Streuoperator als Funktion des Einteilchenstreuoperators angegeben und unter gewissen Annahmen an das Feld wird dessen Wohldefiniertheit gezeigt. Anschließend wir diese Formel verwendet um zu zeigen, dass der Streuoperator in einem gewissen Sinne eine Analytische Funktion des Feldes ist.

Notes on Style

I want to follow the example of the textbooks I enjoy reading which invite the reader to follow jointly the line of arguments together with the text. So I will mostly be using the plural to refer to the reader as well as myself.

Furthermore, the right-hand side of equations will be referred to by the number of the equation or estimate in parentheses. These references will also be used inside other equations and estimates.

Contents

Chapter 1

Introduction

Interacting relativistic quantum physics in general and Quantum Field Theory (QFT) in particular is in a curious state. On the one hand it has been applied to predict the outcomes of experiments at particle accelerators such as the Large Hadron Collider with extraordinary success, on the other hand there is still no rigorous mathematical framework except for the free theory. Bearing this in mind, we might ask ourselves what kinds of interaction can be rigorously defined in a relativistic quantum mechanical setting at all. We are going to describe two possible approaches in detail.

1. The first kind of interaction to be discussed is introduced for a system of $N \in \mathbb{N}$ persistent particles. In order for the wave function of this system to transform covariantly with respect to Poincaré transforms we will consider it a function of N spacetime points $x_k \in \mathbb{R}^4, k = 1, \ldots, N$. This *multi-time* formulation goes back to Dirac [25]. It heavily inspired works that were essen-

tial for the development of Quantum Electrodynamics such as that of Tomonaga [102] and has influenced research since then, see e.g. [74, 103] and [68] for an overview. A natural way of introducing interaction in this setting would be to let the wave function solve N Dirac equations each minimally coupled to a multiplication operator; however, recent no-go theorems [82, 17] show that such systems are either not interacting, not Poincaré invariant or mutually inconsistent. The approach we will follow in chapter 2 bypasses those no-go theorem in [82, 17] by introducing only a single integral equation for the wave function of the particles. The equation under consideration will be of the type

$$\begin{aligned}\psi(x_1, x_2) = \psi^{\text{free}}(x_1, x_2) + i\,\frac{e_1 e_2}{4\pi}\int d^4x_1'\, d^4x_2' \qquad (1.1)\\ \times\, \mathcal{S}_1^{\text{ret}}(x_1 - x_1')\mathcal{S}_2^{\text{ret}}(x_2 - x_2')\,\gamma_1^{\mu}\gamma_{2,\mu}\delta((x_1' - x_2')^2)\psi(x_1', x_2'),\end{aligned}$$

of Lienert [64] for the wave function ψ of two spin $1/2$ particles. Subscripts 1 and 2 refer to quantities associated with the respective particle: e_k is the electric charge, the Dirac matrices γ_k^{α} are defined in (2.3) while ψ^{free} and $\mathcal{S}_k^{\text{ret}}$ refer to a solution and the retarded Greens function (2.5) of the free Dirac equation (2.8), respectively. Equations like (1.1) have, in fact, already been considered by Feynman [36, equation (4)] in a paper of fundamental importance to the development of quantum electrodynamics (QED) in writing down the "effect of exchange of one quantum[...] between two electrons". The main difference between (1.1) and what appeared in [36] is that Feynman only uses positive Fourier modes in time of the delta in the integral. Despite the fact that an equation close to (1.1) appeared very early in the development of QED and the similarity between (1.1) and the Bethe-Salpeter equation, not much is known about the mathematical properties of equation (1.1). Previous mathematical results about related equations are summarized in subsection 2.1.2.

Two main results of this thesis extend the current solution theory of equations of the same type as (1.1). The first of which, theorem 15, extends previous results on the existence and uniqueness of the dynamics of a version of equation (1.1) for spin-less particles to singular interaction along the light cone. The second of which, theorem 25, is the first result on existence and uniqueness of an equation of the type of (1.1) for spin $1/2$ particles. The non-Markovian nature of equations of this type and the fact that the delay inherent in them is not bounded makes them technically challenging. Section 2.1 provides an overview of the state of the art in this field.

2. The second kind of interaction we shall investigate is the interaction of an external electromagnetic field with an otherwise free quantum field representing spin $1/2$ particles. Even in this setting there are classical theorems by Ruijsenaars [86, 87] and Shale and Stinespring [97] that seem to prevent a dynamical mathematical description of the processes in question in the presence of magnetic fields. Judged by the timescales of this field a short time ago this obstacle has been overcome by abandoning the restriction to work in a static Fock space [13, 15, 14]. These results form the basis upon which we will build our analysis in chapter 3.

   Chapter 3 contains two further main results of this thesis in three theorems. The first, theorem 51 contains a construction which partially fixes the phase freedom of previous results. The phase is a relevant quantity here since it influences the current via Bogolyubov's formula. This theorem was difficult to obtain as many ideas of the literature had to be combined and adapted.

> For example, differential geometric concepts of [96] of fibre bundles, whose fibres are phases, lie at the heart of the proof of theorem 51. Theorem 84 provides a well-defined formula that gives the scattering operator of external field QED in terms of the one-particle scattering operator. This result is then used to prove theorem 85 which states that the second quantized scattering operator can be given in terms of a power series in the external field convergent on the finite particle subspace. This theorem may be useful in the future as it can support the derivation of rigorous error bounds on power series expansions often employed by physicists. This result is non-trivial as it relies on theorem 84 whose result had to be guessed first and whose proof relies on a version of the Shale-Stinespring theorem for our representation.

Sections 2.2 and 2.3 of chapter 2 are based on the preprint [66] and the successfully published paper [67], respectively. Both of these publications are the result of joint work with Matthias Lienert. The references are given again in the respective sections. The results in chapter 3 are the result of joint work with my supervisors and has not yet resulted in publications.

Chapter 2

Direct Interaction in Relativistic Quantum Mechanics

As discussed in the last chapter in the paragraph above equation (1.1), having interaction mediated by multiplication operators in a set of Dirac equations is not a viable option, as was proven recently [82, 17]. As mentioned, one alternative approach to this problem is to reformulate Dirac's equation as an integral equation of type (1.1) and explore the possibilities of interaction in that formulation. We will take a few steps in this direction in this chapter. It is based on the paper [67] and the preprint [66] which are a result of the joint work of Lienert and the author of this thesis. While these results fall short of establishing an empirically adequate relativistic quantum mechanical theory, they do provide self-consistent relativistic interacting quantum mechanical toy models in three spacial and one temporal dimension.

Nevertheless, also from a physical perspective these models may still be interesting, because they provide a tool to circumvent the ultraviolet problem due to self-interaction. From a mathematical perspective, the equation is not in Hamiltonian form and involves temporal integrals, hence the well-developed theory of one-parameter unitary groups cannot be directly applied. In fact, there are only few mathematical results about equations of this type.
We will first give a heuristic derivation of this type of equation, briefly review some relevant mathematical results that have been established in the past, and finally discuss the new results of the paper [67] and the preprint [66].

2.1 Overview

2.1.1 Heuristic Derivation

In order to motivate the subject of our study, we will now closely follow the heuristic derivation of equation (1.1) given in [64]. This section is organized as follows: We start by reformulating Dirac's equation for a single particle as an integral equation. The reformulated version is then extended to two particles in a Poincaré invariant manner. Extending the equation is conveniently done in the framework of multi-time wave functions.
Dirac's equation for one particle subject to an external potential V takes the form

$$i\partial_t \phi(t,\vec{x}) = \big(H^{\text{free}} + V(t,\vec{x})\big)\phi(t,\vec{x}), \tag{2.1}$$

here ϕ denotes a potential $\mathbb{C}^4$-valued solution, $\vec{x} \in \mathbb{R}^3, t \in \mathbb{R}$ and H^{free} is the Hamiltonian associated with a free Dirac particle. The latter acts on wave functions as

$$H^{\text{free}}\phi = -i\gamma^0\vec{\gamma}\cdot \operatorname{grad}\phi + m\gamma^0\phi, \tag{2.2}$$

where $\vec{\gamma} = (\gamma^1, \gamma^2, \gamma^3)$ and $\operatorname{grad}\phi$ denotes the gradient of ϕ with respect to the non-temporal coordinates. The matrices $\gamma^\alpha \in \mathbb{C}^{4\times 4}$ fulfil the anti-commutation relation

$$\forall \alpha, \beta \in \{0,1,2,3\} : \{\gamma^\alpha, \gamma^\beta\} := \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\boldsymbol{\eta}^{\alpha\beta}, \tag{2.3}$$

where $\boldsymbol{\eta}$ is the Minkowski metric. We will work with the $(+,-,-,-)$ metric signature and the standard Dirac representation of this algebra. Squared four dimensional vectors always refer to the Minkowski square, meaning for all $a \in \mathbb{C}^4$, $a^2 := a^\alpha a_\alpha = \left(a^0\right)^2 - \vec{a}\cdot\vec{a}$. Small arrows denote three-dimensional vectors, for $a \in \mathbb{C}^4$ we denote by $\vec{a} := (a^1, a^2, a^3)^T$. In the following, a slashed four vector denotes

$$\not{a} := a_\alpha \gamma^\alpha, \tag{2.4}$$

where Einstein's summation convention is used. We will be working in units where $\hbar = 1 = c$, i.e. Planck's constant and the speed of light are set to one.
We denote by $\mathcal{S}^{\text{ret}}$ the retarded Green's function of the non-interacting Dirac equation, that is, the distribution $\mathcal{S}^{\text{ret}}$ satisfies

$$\left(i\partial_t - H^{\text{free}}\right)\mathcal{S}^{\text{ret}} = \delta^4, \tag{2.5}$$

$$\operatorname{supp}\mathcal{S}^{\text{ret}} \subseteq \mathbb{R}_0^+ \times \mathbb{R}^3, \tag{2.6}$$

in a suitable weak sense. Here δ^4 denotes the Dirac measure in four dimensions concentrated on the origin. This allows to recast (2.1) in terms of the following integral equation

$$\phi(t,\vec{x}) = \phi^{\text{free}}(t,\vec{x}) + \int_{t^0}^{\infty} d\tau \int d^3\vec{y}\; \mathcal{S}^{\text{ret}}(t-\tau, \vec{x}-\vec{y}) V(\tau,\vec{y})\phi(\tau,\vec{y}), \tag{2.7}$$

where ϕ^{free} denotes the solution of the non-interacting equation

$$\left(i\partial_t - H^{\text{free}}\right)\phi^{\text{free}} = 0, \tag{2.8}$$

subject to the initial condition $\phi^{\text{free}}(t_0) = \phi_0, t_0 \in \mathbb{R}$ which for the present purpose we think of as a sufficiently regular square integrable function. Analogously, we may recast the two particle Dirac equation including a $\mathbb{C}^{16\times 16}$-valued interaction potential V to be specified later

$$i\partial_t \phi(t,\vec{x}_1,\vec{x}_2) = \left(H_1^{\text{free}} + H_2^{\text{free}} + V(t,\vec{x}_1,\vec{x}_2)\right)\phi(t,\vec{x}_1,\vec{x}_2), \tag{2.9}$$

subject to the initial condition $\phi(t_0) = \phi_0, t_0 \in \mathbb{R}$ sufficiently regular and square integrable, into the integral equation

$$\phi(t,\vec{x}_1,\vec{x}_2) = \phi^{\text{free}}(t,\vec{x}_1,\vec{x}_2) + \int_{t_0}^{\infty} dt' \int d^3\vec{x}_1'\, d^3\vec{x}_2'\, \mathcal{S}_1^{\text{ret}}(t-t',\vec{x}_1-\vec{x}_1') \\ \times \mathcal{S}_2^{\text{ret}}(t-t',\vec{x}_2-\vec{x}_2')V(t',\vec{x}_1',\vec{x}_2')\phi(t',\vec{x}_1',\vec{x}_2'), \tag{2.10}$$

where again, $\phi^{\text{free}}(t)$ solves (2.9) for $V = 0$ with the boundary condition $\phi^{\text{free}}(t_0) = \phi_0$ and $\mathcal{S}_k^{\text{ret}}$ is the retarded Green's function of the free Dirac equation of particle number k. That is the distribution $\mathcal{S}_k^{\text{ret}}$ satisfies

$$\left(i\partial_t - H_k^{\text{free}}\right)\mathcal{S}_k^{\text{ret}} = \delta^4, \tag{2.11}$$

$$\operatorname{supp} \mathcal{S}_k^{\text{ret}}(t,\vec{x}_k) \subset \mathbb{R}_0^+ \times \mathbb{R}^3, \tag{2.12}$$

in a suitably weak sense, where

$$H_k^{\text{free}} = -i\gamma_k^0\vec{\gamma}_k \cdot \text{grad}_k + m_k\gamma^0, \tag{2.13}$$

with

$$\gamma_1^\mu = \gamma^\mu \otimes 1 \tag{2.14}$$

$$\gamma_2^\mu = 1 \otimes \gamma^\mu \tag{2.15}$$

and $1 \in \mathbb{C}^{4\times 4}$ denotes the identity matrix and grad_k is the gradient with respect to the non-temporal coordinates of the k-th particle and $m_k \in \mathbb{R}_0^+$ the respective mass. Here, it is crucial to notice that the

Green's function of the free two particle Dirac equation factorizes into a product of two Green's functions of the Dirac equation for one particle.

Since equation (2.10) contains only one temporal variable but six spatial ones, there is no unitary operator implementing Lorentz boots on such wave functions and hence it is not a relativistically covariant equation. In order to find a relativistically covariant equation, we will directly generalize the one-particle equation (2.7) to two particles instead of generalizing to two particles at the level of (2.1). Before we do so let us first rewrite equation (2.7) it in a more suggestive way:

$$\psi(x) = \psi^{\text{free}}(x) + \int d^4x'\, \mathcal{S}^{\text{ret}}(x - x')V(x')\psi(x'), \tag{2.16}$$

where non-bold letters denote elements of Minkowski spacetime, and we replaced ϕ by ψ in order to emphasize the change to a relativistic notation. Furthermore, we replaced the lower bound in the temporal integral domain by $-\infty$ in order to render the total domain of integral Poincaré invariant, which implies a change of the initial condition. Please note that solutions of equation (2.16) have a conserved current leading to a conserved integral over space-like Cauchy surfaces reducing integrals over $|\psi|^2$ for equal time hypersurfaces.

Equation (2.16) suggests the following generalization for two particles

$$\begin{aligned}\psi(x_1, x_2) &= \psi^{\text{free}}(x_1, x_2) \\ &\quad + \int d^4x_1'\, d^4x_2'\, \mathcal{S}_1^{\text{ret}}(x_1 - x_1')\mathcal{S}_2^{\text{ret}}(x_2 - x_2')K(x_1', x_2')\psi(x_1', x_2'),\end{aligned} \tag{2.17}$$

where we integrate over all of $\mathbb{R}^8$ and ψ^{free} is a solution of the free Dirac equation both in x_1 and x_2 and their respective spinor indices:

$$D_1\psi^{\text{free}}(x_1, x_2) = \gamma_1^0(i\partial_{t_1} - H_1^{\text{free}})\psi^{\text{free}}(x_1, x_2) = 0, \tag{2.18}$$

$$D_2\psi^{\text{free}}(x_1, x_2) = \gamma_2^0(i\partial_{t_2} - H_2^{\text{free}})\psi^{\text{free}}(x_1, x_2) = 0. \tag{2.19}$$

Nomenclature 1. *The class of equations*

$$\psi = \psi^{\text{free}} + A^K\psi \tag{2.20}$$

where the linear operator A^K *obeys*

$$D_1 D_2 (A^K\psi)(x_1, x_2) = K(x_1, x_2)\psi(x_1, x_2), \tag{2.21}$$

while the free parts ψ^{free} *are* $\mathbb{C}^{16}$*-valued and obey the restrictions* (2.18) *and* (2.19) *and tempered distributions* $K \in (\mathcal{S}(\mathbb{R}^8) \otimes \mathbb{C}^{16\times 16})'$ *for* $\mathbb{C}^{16}$*-valued wave functions* ψ *will be referred to as* spin-1/2 delay-equation*.* *The distribution* K *will be referred to as* interaction kernel*.*
Similarly, the class of equations

$$\psi = \psi^{\text{free}} + A^K\psi \tag{2.22}$$

where ψ *and* ψ^{free} *are* $\mathbb{C}-$*valued and* ψ^{free} *and* A^K *obey*

$$(\Box_{x_1} + m_1^2)\psi(x_1, x_2) = 0, \tag{2.23}$$

$$(\Box_{x_2} + m_2^2)\psi(x_1, x_2) = 0, \tag{2.24}$$

$$(\Box_{x_1} + m_1^2)(\Box_{x_2} + m_2^2)(A^K\psi)(x_1, x_2) = K(x_1, x_2)\psi(x_1, x_2) \tag{2.25}$$

where $m_k \in \mathbb{R}^+$ *is the mass and* $\Box_{x_k}$ *is the d'Alembert operator acting on the spacetime coordinates of particle* $k \in \{1, 2\}$ *will be called* spin-0 delay-equation *with* interaction kernel $K \in (\mathcal{S}(\mathbb{R}^8))'$*.*

In the motivation we gave A^K as the convolution of $K\psi$ with the retarded Green's function. In nomenclature 1 we widened the class also to other choices. An optimal choice in the sense of empirical adequacy of the interaction kernel is not yet known. However, a simple way of ensuring Poincaré invariance of spin-1/2 or spin-0 delay-equation is to let K only depend on the squared Minkowski distance $(x_1 - x_2)^2$ and choose a Poincaré covariant form of A^K. A choice that incorporates interaction along light-like distances, i.e. $(x_1 - x_2)^2 = 0$ and has a correct non-relativistic limit [64, section 3.6] is given by

$$K(x_1, x_2) = i\,\frac{e_1 e_2}{4\pi}\gamma_1^\mu \gamma_{2,\mu}\;\delta((x_1 - x_2)^2), \tag{2.26}$$

for the spin-1/2 case and

$$K(x_1, x_2) = \frac{\lambda}{4\pi}\delta((x_1 - x_2)^2), \tag{2.27}$$

with $\lambda \in \mathbb{R}$ for the spin-0 case, where δ is the one-dimensional Dirac measure. The model defined the spin-1/2 delay-equation with A^K given by the convolution with the retarded Green's function in each particle's coordinates and interaction kernel given by (2.26) shows some resemblance of Wheeler-Feynman electrodynamics for the following reason: there are only particles but no electrodynamic field, the particles interact with each other along light-like distances, and the particles do not interact with themselves.

Summarizing, we gave a heuristic line of arguments motivating the class of spin-1/2 and spin-0 delay-equations and a special choice of interaction kernel (2.26) and (2.27), respectively. In this thesis we will study a spin-1/2 delay-equation with regular interaction kernel and a spin-0 delay-equation incorporating (2.27). As mentioned in the beginning of this section, there are only very few results on spin-1/2 and spin-0 delay-equations. The reason for this is that for these types of equations the theory of one-parameter unitary groups cannot be applied. Furthermore, the integrals on the right-hand side of these equations involve the delayed wave function with arbitrarily large delay. As for the theory of delayed differential equations, it is interesting to study existence and uniqueness of solutions as well as dependence on initial data. We will review the mathematical results which are for our purposes most relevant in the next subsection before we move on to the new results of this thesis and [66, 67].

2.1.2 Previous Mathematical Results on Directly Interacting Particles

In this subsection we summarize previous important mathematical existence results on spin-0 delay-equations introduced in nomenclature 1 as well as generalizations of them to curved spacetime. To the best of my knowledge there are no mathematical existence results on spin-1/2 delay-equations prior to the one of Lienert and myself in [67]. The results we are going to cover are taken from [64] and [70]. Mentioned below are only the theorems that are about a four dimensional spacetime; however, there are also results concerning lower dimensions; the interested reader is referred to [64, 70]. Since not much is known about the mathematical properties of spin-1/2 and spin-0 delay-equations the first results on spin-0 delay-equations will be subject to the following modifications:

(A) The physical motivation for equation (2.22) was given for ψ defined on all of Minkowski spacetime. All the rigorous results concerning vanishing curvature so far are about the domain $\mathbb{R}_0^+ \times \mathbb{R}^3 =: \mathbb{M}_0^+$. That is, there is a beginning in time. This modification has technical reasons. However, it may be justified on physical grounds as current cosmological models also have a beginning in time. In order to give this reasoning additional weight the existence and uniqueness result was also proved on Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. In sections 2.2 and 2.3 we provide results on this spacetime for the same reason. In the Klein-Gordon case the form of A^K is then given by Duhamel's principle [33] i.e.

$$A^K\psi(x_1, x_2) = \int_0^{x_2^0} d{x'_1}^0 \int_0^{x_2^0} d{x'_2}^0 u(x_1, x_2; {x'_1}^0, {x'_2}^0), \qquad (2.28)$$

where $u(\cdot,\cdot;{x_1'}^0,{x_2'}^0)$ is a solution to the homogeneous Klein-Gordon equation in the first and second arguments on the domain $\{(x_1,x_2)\in\mathbb{R}^8 \mid x_1^0 \geqslant {x_1'}^0 \wedge x_2^0 \geqslant {x_2'}^0\}$ subject to the boundary conditions

$$u(x_1,x_2;{x_1'}^0,{x_2'}^0)|_{x_j^0={x_j'}^0} = 0, \tag{2.29}$$

$$\partial_{x_j^0} u(x_1,x_2;{x_1'}^0,{x_2'}^0)|_{x_j^0=x_j^{0\prime}} = U_j(x_1,x_2;{x_{\tilde{j}}'}^0)|_{x_j^0=x_j^{0\prime}}, \tag{2.30}$$

for $j\in\{1,2\}$ and $\tilde{j}=3-j$, where U_j is again a solution of the Klein-Gordon equation with boundary conditions:

$$(\Box_{x_{\tilde{j}}} + m_{\tilde{j}}^2)U_j(x_1,x_2;x_{\tilde{j}}^{\prime 0}) = 0, \tag{2.31}$$

$$U_j(x_1,x_2;{x_{\tilde{j}}'}^0)|_{x_j^0=x_j^0} = 0, \tag{2.32}$$

$$\partial_{x_{\tilde{j}}^0} U_j(x_1,x_2;{x_1'}^0,{x_2'}^0)|_{x_{\tilde{j}}^0={x_{\tilde{j}}'}^0} = K(x_1,x_2)\psi(x_1,x_2). \tag{2.33}$$

Plugging in the formula for solutions to the Klein-Gordon equation in terms of the propagator and initial conditions and using that the propagator agrees with the retarded Green's function on the domain [10, 11, 12] of integration yields

$$A^K\psi(x_1,x_2) = \int_0^{x_2^0} d{x_1'}^0 \int_0^{x_2^0} d{x_2'}^0 \int d^3\vec{x}_1' \int d^3\vec{x}_2' \tag{2.34}$$
$$G_1^{\text{ret}}(x_1')G_2^{\text{ret}}(x_2')K(x_1-x_1',x_2-x_2')\psi(x_1-x_1',x_2-x_2'),$$

the expressions for the operator given in the theorems below and also in section 2.2. Note that for regular enough $K\psi$ one can exchange differentiation and integration and the properties of the propagator to directly verify

$$(\Box_{x_1}+m_1^2)(\Box_{x_2}+m_2^2)(A^K\psi)(x_1,x_2) = K(x_1,x_2)\psi(x_1,x_2). \tag{2.35}$$

In section 2.3 we will modify this expression to make it a solution of the inhomogeneous Dirac equation.

(B) The interaction kernel K is replaced by various classes of less singular objects. These classes do not include the singular interaction kernel proportional to $\delta((x_1 - x_2)^2)$ motivated in the last subsection. This modification is purely technical, and we do not justify it. In section 2.3, where we treat Dirac particles, we will also use a rather regular interaction kernel compared to (2.26). The new result about Klein-Gordon particles presented in section 2.2 employs the fully singular $\delta((x_1 - x_2)^2)$ kernel.

The space $\mathcal{B}$ to be defined below provides the solution sense for the following existence and uniqueness results.

Definition 1. *For $T > 0$, we define the Bochner space $\mathcal{B} := L^\infty([0,T]^2, L^2(\mathbb{R}^6, \mathbb{C}))$, where $L^\infty(X,Y)$ and $L^2(X,Y)$ are the spaces of essentially bounded and square integrable functions from X to Y, respectively. That is, the Bochner space is the space of measurable functions from $[0,T]^2$ to the space of measurable square integrable functions from $\mathbb{R}^6$ to $\mathbb{C}$ with essentially finite L^2 norm, i.e. $\forall f \in \mathcal{B}$:*

$$\operatorname*{ess\,sup}_{(t_1,t_2)\in[0,T]^2} \|f(t_1,t_2,\cdot)\|_2^2 := \inf_{\substack{\sigma\subset[0,T]^2\\ \lambda(\sigma)=0}} \sup_{(t_1,t_2)\in[0,T]^2\setminus\sigma} \int_{\mathbb{R}^6} d^6x |f(t_1,t_2,x)|^2 < \infty, \tag{2.36}$$

where λ is the Lebesgue measure.

Theorem 2 (theorem 3.4 ($d = 3$) of [72]). *Let $T > 0, \lambda \in \mathbb{C}$, for every essentially bounded $K : \mathbb{R}^8 \to \mathbb{C}$ and every $\psi^{\text{free}} \in \mathcal{B}$ the equation*

$$\begin{aligned}\psi(t_1,\vec{x}_1,t_2,\vec{x}_2) = \psi^{\text{free}}(t_1,\vec{x}_1,t_2,\vec{x}_2) + \frac{\lambda}{(4\pi)^2}\int d^3\vec{x}_1' d^3\vec{x}_2' \\ \times \frac{H(t_1-|\vec{x}_1-\vec{x}_1'|)}{|\vec{x}_1-\vec{x}_1'|}\frac{H(t_2-|\vec{x}_2-\vec{x}_2'|)}{|\vec{x}_2-\vec{x}_2'|} \\ \times K(t_1-|\vec{x}_1-\vec{x}_1'|,\vec{x}_1',t_2-|\vec{x}_2-\vec{x}_2'|,\vec{x}_2') \\ \times \psi(t_1-|\vec{x}_1-\vec{x}_1'|,\vec{x}_1',t_2-|\vec{x}_2-\vec{x}_2'|,\vec{x}_2')\end{aligned}$$

has a unique solution $\psi \in \mathcal{B}$, where H is the Heaviside step function.

Theorem 3 (theorem 3.5 of [72]). *Let $T > 0, \lambda \in \mathbb{C}$, for every essentially bounded $f : \mathbb{R}^8 \to \mathbb{C}$ and every $\psi^{\text{free}} \in \mathcal{B}$ the equation*

$$\begin{aligned}\psi(t_1,\vec{x}_1,t_2,\vec{x}_2) = \psi^{\text{free}}(t_1,\vec{x}_1,t_2,\vec{x}_2) + \frac{\lambda}{(4\pi)^2}\int d^3\vec{x}_1' d^3\vec{x}_2' \\ \times \frac{H(t_1-|\vec{x}_1-\vec{x}_1'|)}{|\vec{x}_1-\vec{x}_1'|}\frac{H(t_2-|\vec{x}_2-\vec{x}_2'|)}{|\vec{x}_2-\vec{x}_2'|} \\ \times \frac{f(t_1-|\vec{x}_1-\vec{x}_1'|,\vec{x}_1',t_2-|\vec{x}_2-\vec{x}_2'|,\vec{x}_2')}{|\vec{x}_1'-\vec{x}_1|} \\ \times \psi(t_1-|\vec{x}_1-\vec{x}_1'|,\vec{x}_1',t_2-|\vec{x}_2-\vec{x}_2'|,\vec{x}_2'),\end{aligned}$$

has a unique solution $\psi \in \mathcal{B}$.

The next results are about the open FLRW spacetime. There are also results about the closed FLRW universe which we omit here. The reader is referred to [70, theorem 4.3]. We have to introduce some notation before we can present the next results. In order to do so we follow [66, sec 3.3].
We consider particles on a flat (FLRW) spacetime $\mathcal{M}$ which admits a global coordinate chart $x \mapsto (\eta, \vec{x}) \in \mathbb{R}^+ \times \mathbb{R}^3$. The metric g in these coordinates at the point x is given by

$$g_x(v_1, v_2) = a^2(\eta)(v_1^0 v_2^0 - \vec{v}_1 \cdot \vec{v}_2)$$

for all tangent vectors $v_1, v_2 \in T_x\mathcal{M}$, i.e. the metric at every point is a multiple of the Minkowski metric. The global time coordinate η is called conformal time, and the scale function $a : \mathbb{R}^+ \to \mathbb{R}$ is continuous with $\lim_{\eta\to 0} a(\eta) = 0$ and $a(\eta) > 0$ for all η. In this spacetime the free wave equation takes the form

$$(\Box_g - R/6)\,\chi(x) = 0, \tag{2.37}$$

where R denotes the Ricci scalar and the Laplace Beltrami operator acts on scalar functions χ on $\mathcal{M}$ as

$$\Box_g \chi = \frac{1}{\sqrt{|\det g|}}\partial_\alpha \left(\sqrt{|\det g|} g^{\alpha,\beta}\partial_\beta \chi\right). \tag{2.38}$$

The retarded and symmetric Green's functions of equation (2.37) are given by

$$G^{\mathrm{ret}}_{\mathcal{M}}(x,x') = \frac{1}{4\pi}\frac{1}{a(\eta)a(\eta')}\frac{\delta(\eta-\eta'-|\vec{x}-\vec{x}\,'|)}{|\vec{x}-\vec{x}\,'|} \tag{2.39}$$

$$G^{\mathrm{sym}}_{\mathcal{M}}(x,x') = \frac{1}{4\pi}\frac{1}{a(\eta)a(\eta')}\delta((\eta-\eta')^2-|\vec{x}-\vec{x}\,'|^2). \tag{2.40}$$

This form may be derived exploiting the conformal equivalence of FLRW spacetime and Minkowski spacetime, see [70, 58] for details. The generalization of (2.22) to FLRW spacetime is straightforward: ψ becomes a scalar function on $\mathcal{M}\times\mathcal{M}$, one exchanges the Minkowski spacetime volume element with

$$dV(x) = a^4(\eta)\,d\eta\,d^3\vec{x}, \tag{2.41}$$

where the one-forms on the right-hand side are the canonical ones in these coordinates and the product of one-forms is to be understood as a wedge product. Equation (2.41) gives the invariant 4-volume form on $\mathcal{M}$. As in the Minkowski case, the interaction kernel is given by the symmetric Green's function. With this, the generalization of equation (2.22) with kernel (2.27) turns into:

$$\psi(x,y) = \psi^{\text{free}}(x,y) + \lambda \int_{\mathcal{M}\times\mathcal{M}} dV(x)\, dV(y)\; G^{\text{ret}}_{\mathcal{M}}(x,x') G^{\text{ret}}_{\mathcal{M}}(y,y') \times G^{\text{sym}}_{\mathcal{M}}(x',y') \psi(x',y'). \tag{2.42}$$

For regular and only weakly singular interaction kernels $K(x', y')$ instead of $G^{\text{sym}}(x', y')$, the problem of existence and uniqueness of solutions of this equation has been treated in [70]:

Theorem 4 (theorem 4.1 of [70]). *Let $T > 0$, $\lambda \in \mathbb{C}$. Furthermore, let $a : [0,\infty) \to [0,\infty)$ be a continuous function with $a(0) = 0$ and $a(\eta) > 0$ for $\eta > 0$, and $\tilde{K} : ([0,\infty) \times \mathbb{R}^3)^2 \to \mathbb{C}$ be essentially bounded. Then for every ψ^{free} with $a(\eta_1)a(\eta_2)\psi^{\text{free}} \in \mathcal{B}$, the respective integral equation on the 4−dimensional flat FLRW universe with scale function $a(\eta)$:*

$$\begin{aligned}\psi(\eta_1, \vec{x}_1, \eta_2, \vec{x}_2) = \psi^{\text{free}}(\eta_1, \vec{x}_1, \eta_2, \vec{x}_2) + \frac{\lambda}{(4\pi)^2 a(\eta_1) a(\eta_2)} \\ \times \int d\vec{x}'_1 d\vec{x}'_2 a^2(\eta_1 - |\vec{x}_1 - \vec{x}'_1|) a^2(\eta_2 - |\vec{x}_2 - \vec{x}'_2|) \\ \times \frac{H(\eta_1 - |\vec{x}_1 - \vec{x}'_1|)}{|\vec{x}_1 - \vec{x}'_1|} \frac{H(\eta_2 - |\vec{x}_2 - \vec{x}'_2|)}{|\vec{x}_2 - \vec{x}'_2|} \\ \times \tilde{K}(\eta_1 - |\vec{x}_1 - \vec{x}'_1|, \vec{x}'_1, \eta_2 - |\vec{x}_2 - \vec{x}'_2|, \vec{x}'_2) \\ \times \psi(\eta_1 - |\vec{x}_1 - \vec{x}'_1|, \vec{x}'_1, \eta_2 - |\vec{x}_2 - \vec{x}'_2|, \vec{x}'_2)\end{aligned} \tag{2.43}$$

has a unique solution ψ for $a(\eta_1)a(\eta_2)\psi \in \mathcal{B}$.

Theorem 5 (theorem 4.2 of [70]). *Let $f : ([0,\infty] \times \mathbb{R}^3)^2 \to \mathbb{C}$ be a bounded function. Then, under the same assumptions as in theorem 4* *but with*

$$\tilde{K}(\eta_1, \vec{x}_1, \eta_2, \vec{x}_2) = \frac{f(\eta_1, \vec{x}_1, \eta_1, \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|}, \tag{2.44}$$

the integral equation (2.43) *has a unique solution* ψ *for* $a(\eta_1)a(\eta_2)\psi \in \mathcal{B}$*.*

2.2 Singular light cone interactions of spin-less particles

This section is based on the preprint [66] which is the result of joint work with Matthias Lienert. We prove existence and uniqueness of solutions of the spin-0 delay-equation (2.22) where the solution of the inhomogeneous Klein-Gordon equation in the operator A^K is given by the expression in the previous subsection 2.1.2 obtained by Duhamel's principle to accommodate solutions on $\mathbb{M}_0^+$ and the interaction kernel given by (2.27). In contrast to the results of subsection 2.1.2 the equation will not be subject to the assumption (B). Additionally, we will extend the result to an arbitrary number of particles. This extension is not the only possible one; however, we choose the extension called most promising in [64]. In order to justify the treatment on the half-space $\mathbb{M}_0^+$, i.e. the cut-off in time, we extend the one-particle result to the FLRW spacetime, where the cut-off appears naturally.

2.2.1 Overview

This subsection is structured as follows. In subsection 2.2.2 it is shown how to precisely define the integral operator A^K in equation (2.22), i.e. by giving meaning to the delta distributions. Subsection 2.2.3 contains our main results: theorem 14 shows that in the case of massless particles solutions grow at most exponentially in time. Our main result is theorem 15, an existence and uniqueness theorem for the full (massive) case.

Subsection 2.2.3.2 deals with generalizing this existence and uniqueness theorem to N scalar particles; the corresponding theorem, theorem 16, is a direct consequence of theorem 15. To the best of the knowledge of Lienert and the author, this is the first rigorous result about a multi-time integral equation for N-particles.

In Subsection 2.2.3.3 we show, as discussed above, also in this case it is possible to extend the analysis from $\mathbb{M}_0^+$ to FLRW spacetime. That is, we show the equivalent result of [70] for singular light cone interactions. The respective existence and uniqueness theorem is theorem 18.

Subsection 2.2.4 contains the proofs.

2.2.2 Precise formulation of equation (2.22)

In the following, we show how to precisely define the integral operator in spin-0 delay-equation (2.22) with simplifying assumption (A) and interaction kernel (2.27).

It is necessary to take special care of the definition of the integral operator as it contains certain combinations (convolutions and products) of distributions (the Green's functions). First, our strategy is to consider the integral operator acting on Schwartz space $\mathcal{S} := \mathcal{S}((\mathbb{M}_0^+)^2)$ where its action can be defined straightforwardly. Later it will be shown that it is bounded on test functions with respect to a suitably chosen weighted norm. This will make it possible to linearly extend the integral operator to the completion of $\mathcal{S}$ with respect to that norm.

The retarded Green's function of the Klein-Gordon equation with mass $m \in \mathbb{R}_0^+$ is given by:

$$G^{\mathrm{ret}}(x) \;=\; \frac{1}{4\pi|\vec{x}|}\delta(x^0 - |\vec{x}|) - \frac{m}{4\pi}H(x^0 - |\vec{x}|)\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} \qquad (2.45)$$

where J_1 is the Bessel function of the first kind of order one. Then, with $K(x,y) = \frac{\lambda}{4\pi}\delta((x-y)^2)$, our integral equation (2.22) on $(\mathbb{M}_0^+)^2$ turns into:

$$\psi \;=\; \psi^{\text{free}} + A\psi \tag{2.46}$$

where $A = A_0 + A_1 + A_2 + A_{12}$ and

$$\begin{aligned}
(A_0\psi)(x,y) \; =& \frac{\lambda}{(4\pi)^3}\int_0^{x^0} x'^0 \int_{\mathbb{R}^3} d^3\vec{x}' \int_0^{y^0} dy'^0 \int_{\mathbb{R}^3} \\
& \times \frac{\delta(x^0 - x'^0 - |\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}\frac{\delta(y^0 - y'^0 - |\vec{y}-\vec{y}'|)}{|\vec{y}-\vec{y}'|} \\
& \times \delta((x'-y')^2)\psi(x',y'), \qquad (2.47)\\
(A_1\psi)(x,y) \; =& -\frac{\lambda\, m_1}{(4\pi)^3}\int_0^{x^0} dx'^0 \int d^3\vec{x}' \int_0^{y^0} dy'^0 \int d^3\vec{y}' \\
& \times H(x^0 - x'^0 - |\vec{x}-\vec{x}'|)\frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \\
& \times \frac{\delta(y^0 - y'^0 - |\vec{y}-\vec{y}'|)}{|\vec{y}-\vec{y}'|}\delta((x'-y')^2)\psi(x',y') \qquad (2.48)\\
(A_2\psi)(x,y) \; =& -\frac{\lambda\, m_2}{(4\pi)^3}\int_0^{x^0} dx'^0 \int d^3\vec{x}' \int_0^{y^0} dy'^0 \int d^3\vec{y}' \\
& \times \frac{\delta(x^0 - x'^0 - |\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|} H(y^0 - y'^0 - |\vec{y}-\vec{y}'|) \\
& \times \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}}\delta((x'-y')^2)\psi(x',y') \qquad (2.49)\\
(A_{12}\psi)(x,y) \; =& \frac{\lambda\, m_1 m_2}{(4\pi)^3}\int_0^{x^0} dx'^0 \int d^3\vec{x}' \int_0^{y^0} dy'^0 \int d^3\vec{y}' \\
& \times H(x^0 - x'^0 - |\vec{x}-\vec{x}'|)\frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}}
\end{aligned}$$

$$\times H(y^0 - y'^0 - |\vec{y} - \vec{y}'|)\frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}}$$
$$\times \delta((x'-y')^2)\psi(x',y'). \tag{2.50}$$

We now manipulate these expressions in a heuristic way such that the end results can be given a precise meaning on test functions. Let $\psi \in \mathcal{S}$.

2.2.2.1 Rigorous definition of A_0.

We consider the massless term A_0 first which is also the most singular term. Using the delta distributions to dauntlessly eliminate the integration over x'^0 and y'^0 results in:

$$(A_0\psi)(x,y) = \frac{\lambda}{(4\pi)^3}\int_{B_{x^0}(\vec{x})} d^3\vec{x}' \int_{B_{y^0}(\vec{y})} d^3\vec{y}'$$
$$\times \frac{\delta((x^0 - y^0 - |\vec{x}'| + |\vec{y}'|)^2 - |\vec{x} - \vec{y} + \vec{x}' - \vec{y}'|^2)}{|\vec{x}'||\vec{y}'|}$$
$$\times \psi(x+x', y+y')|_{x'^0 = -|\vec{x}'|,\, y'^0 = -|\vec{y}'|}, \tag{2.51}$$

Note that no complications arose because of the finite size of the integration domain on the two time dimensions. There is still one more delta distribution left. We choose to use it to eliminate $|\vec{x}'| =: r$. It is convenient to introduce the vector

$$b = x - y - (-|\vec{y}'|, \vec{y}'). \tag{2.52}$$

Then, the argument of the delta distribution can be written as:

$$(b^0 - |\vec{x}'|)^2 - |\vec{b} + \vec{x}'|^2. \tag{2.53}$$

This expression has a root in r for

$$r = r^* := \frac{1}{2}\frac{b^2}{b^0 + |\vec{b}|\cos\vartheta} \tag{2.54}$$

where ϑ is the angle between $\vec{b}$ and $\vec{x}'$. Of course, r^* inherits the restrictions of the range of r, thus, a valid root must fulfil

$$0 < r^* < x^0. \tag{2.55}$$

The requirement $0 < r^*$ can be satisfied in two cases, either $b^2 > 0$ and $b^0 > 0$, or $b^2 < 0$ and $\cos\vartheta < -\frac{b^0}{|\vec{b}|}$. Using these restrictions, the condition $r^* < x^0$ can be converted into a restriction of the domain of integration in ϑ:

$$\begin{aligned}
& \frac{1}{2}\frac{b^2}{b^0 + |\vec{b}|\cos\vartheta} \; < \; x^0 \\
\Longleftrightarrow \quad & \operatorname{sgn}(b^2) b^2 \; < \; 2x^0 \operatorname{sgn}(b^2)(b^0 + |\vec{b}|\cos\vartheta) \\
\Longleftrightarrow \quad & \frac{|b^2|}{2x^0|\vec{b}|} - \frac{\operatorname{sgn}(b^2) b^0}{|\vec{b}|} \; < \; \operatorname{sgn}(b^2)\cos\vartheta \\
\Longleftrightarrow \quad & \begin{cases} \cos\vartheta > \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}, & \text{for } b^2 > 0 \\ \cos\vartheta < \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}, & \text{for } b^2 < 0. \end{cases}
\end{aligned} \tag{2.56}$$

In case of $b^2 < 0$, the new restriction on $\cos\vartheta$ is stricter than $\cos\vartheta < -\frac{b^0}{|\vec{b}|}$; we thus use it to replace the latter. We evaluate the delta distribution using spherical coordinates in $\vec{y}'$ and the usual rule

$$\delta(f(z)) = \sum_{z^*: f(z^*)=0} \frac{\delta(z - z^*)}{|f'(z^*)|}, \tag{2.57}$$

where $f(r) = (b^0 - r)^2 - (\vec{b} + x')^2 = -(r - r^*)2(b^0 + |\vec{b}|\cos\vartheta)$. The result is an expression for $A_0\psi$ which does not contain distributions any more:

$$(A_0\psi)(x,y) = \frac{\lambda}{(4\pi)^3} \int_{B_{y^0}(\vec{y})} d^3\vec{y}' \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta \; \frac{|b^2|}{4(b^0 + |\vec{b}|\cos\vartheta)^2 |\vec{y}'|}$$
$$\left(1_{b^2>0} 1_{b^0>0} 1_{\cos\vartheta > \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}} + 1_{b^2<0} 1_{\cos\vartheta < \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}} \right) \psi(x+x', y+y'), \tag{2.58}$$

still subject to $x'^0 = -r^* = -|\vec{x}'|, y'^0 = -|\vec{y}'|$. The different cases for b have been implemented through the various indicator functions.

Definition 6. *We define the operator $A_0 : \mathcal{S} \to \mathcal{B}_{\boldsymbol{w}}, \psi \mapsto A_0\psi$ according to (2.58), where $\mathcal{B}_{\boldsymbol{w}}$ is defined in (2.65) and equality is to be read as equality in $\mathcal{B}_{\boldsymbol{w}}$. Well-definedness and boundedness will be shown in theorem 14 and 15.*

2.2.2.2 Rigorous definition of A_1.

Next, we turn to the definition of A_1, starting from the heuristic expression (2.48). We first split up the delta distribution of the interaction kernel according to (2.57). Then we use $\delta(y^0 - y'^0 - |\vec{y} - \vec{y}'|)$ to eliminate y'^0 $(= y^0 - |\vec{y} - \vec{y}'|)$. Note that the order of these two steps does not matter. This yields:

$$(A_1\psi)(x,y) = -\frac{\lambda m_1}{2(4\pi)^3} \int_0^\infty dx'^0 \int d^3\vec{x}' \int d^3\vec{y}' \; H(x^0 - x'^0 - |\vec{x} - \vec{x}'|)$$
$$\times \frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \frac{H(y^0 - |\vec{y} - \vec{y}'|)}{|\vec{y} - \vec{y}'|} \frac{1}{|\vec{x}' - \vec{y}'|}$$
$$\left[\delta(x'^0 - y^0 + |\vec{y} - \vec{y}'| - |\vec{x}' - \vec{y}'|) + \delta(x'^0 - y^0 + |\vec{y} - \vec{y}'| + |\vec{x}' - \vec{y}'|) \right]$$
$$\times \psi(x', y^0 - |\vec{y} - \vec{y}'|, \vec{y}'). \tag{2.59}$$

Finally, we use the remaining delta distribution to eliminate x'^0. We obtain:

$$
\begin{aligned}
(A_1\psi)(x,y) = -\frac{\lambda\, m_1}{2(4\pi)^3} \int d^3\vec{x}' \int d^3\vec{y}' \; \frac{H(y^0 - |\vec{y}-\vec{y}'|)}{|\vec{y}-\vec{y}'|} \frac{1}{|\vec{x}'-\vec{y}'|} \\
\Bigg[H(x'^0) H(x^0 - x'^0 - |\vec{x}-\vec{x}'|) \\
\times \frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \psi(x',y') \Big|_{\substack{y'^0 = y^0 - |\vec{y}-\vec{y}'|, \\ x'^0 = y^0 - |\vec{y}-\vec{y}'| + |\vec{x}'-\vec{y}'|}} \\
+ \; H(x'^0) H(x^0 - x'^0 - |\vec{x}-\vec{x}'|) \\
\times \frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} \psi(x',y') \Big|_{\substack{y'^0 = y^0 - |\vec{y}-\vec{y}'|, \\ x'^0 = y^0 - |\vec{y}-\vec{y}'| - |\vec{x}'-\vec{y}'|}} \Bigg]. \qquad (2.60)
\end{aligned}
$$

Note that the domain of integration is effectively finite due to the Heaviside functions.

Definition 7. *We define the operator $A_1 : \mathcal{S} \to \mathcal{B}_{\boldsymbol{w}}, \psi \mapsto A_1\psi$ according to (2.60), equality is to be read as equality in $\mathcal{B}_{\boldsymbol{w}}$. Well-definedness and boundedness will be shown in theorem 15.*

2.2.2.3 Rigorous definition of A_2.

Starting from (2.49), the analogous steps as for A_1 yield:

$$
\begin{aligned}
(A_2\psi)(x,y) = -\frac{\lambda m_2}{2(4\pi)^3} \int d^3\vec{x}' \int d^3\vec{y}' \frac{H(x^0 - |\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|} \frac{1}{|\vec{x}'-\vec{y}'|} \\
\Bigg[H(y'^0) H(y^0 - y'^0 - |\vec{y}-\vec{y}'|) \\
\times \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}} \psi(x',y') \Big|_{\substack{x'^0 = x^0 - |\vec{x}-\vec{x}'|, \\ y'^0 = x^0 - |\vec{x}-\vec{x}'| + |\vec{x}'-\vec{y}'|}}
\end{aligned}
$$

$$+ H(y'^0)H(y^0 - y'^0 - |\vec{y} - \vec{y}'|) \times \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}}\psi(x',y')\Big|_{\substack{x'^0 = x^0 - |\vec{x}-\vec{x}'|,\\ y'^0 = x^0 - |\vec{x}-\vec{x}'| - |\vec{x}'-\vec{y}'|}}\Bigg]. \tag{2.61}$$

Definition 8. *We define the operator $A_2 : \mathcal{S} \to \mathcal{B}_{\boldsymbol{w}}, \psi \mapsto A_2\psi$ according to (2.61), equality is to be read as equality in $\mathcal{B}_{\boldsymbol{w}}$. Well-definedness and boundedness will be shown in theorem 15.*

2.2.2.4 Rigorous definition of A_{12}.

Here, we start with (2.50). We change variables $(\vec{x}', \vec{y}') \mapsto (\vec{x}', \vec{z} = \vec{x}' - \vec{y}')$ (Jacobi determinant = 1), with the goal of using the remaining delta distribution to eliminate $|\vec{z}| = |\vec{x}' - \vec{y}'|$ in mind. We find:

$$(A_{12}\psi)(x,y) = \frac{\lambda m_1 m_2}{(4\pi)^3}\int_0^\infty dx'^0 \int d^3\vec{x}' \int_0^\infty dy'^0 \int d^3\vec{z} H(x^0 - x'^0 - |\vec{x} - \vec{x}'|) \times \frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}} H(y^0 - y'^0 - |\vec{y} - \vec{x}' + \vec{z}|) \times \frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}} \delta((x'^0 - y'^0)^2 - |\vec{z}|^2)\psi(x',y')\Big|_{\vec{y}' = \vec{x}' - \vec{z}}. \tag{2.62}$$

Now we use spherical coordinates for $\vec{z}$ and eliminate $|\vec{z}|$ through the delta distribution, using

$$\delta((x'^0 - y'^0)^2 - |\vec{z}|^2) = \frac{1}{2|\vec{z}|}\delta(|x^{0\prime} - y^{0\prime}| - |\vec{z}|). \tag{2.63}$$

This yields:

$$(A_{12}\psi)(x,y) = \frac{\lambda\, m_1 m_2}{2(4\pi)^3}\int_0^\infty dx'^0 \int d^3\vec{x}\,' \int_0^\infty dy'^0 \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta$$
$$\times \sin(\vartheta)|x'^0 - y'^0| H(x^0 - x'^0 - |\vec{x} - \vec{x}\,'|)\frac{J_1(m_1\sqrt{(x-x')^2})}{\sqrt{(x-x')^2}}$$
$$\times H(y^0 - y'^0 - |\vec{y} - \vec{x}\,' + \vec{z}|)\frac{J_1(m_2\sqrt{(y-y')^2})}{\sqrt{(y-y')^2}}\psi(x',y')\Big|_{\vec{y}\,'=\vec{x}\,'-\vec{z},\,|\vec{z}|=|x^{0\prime}-y^{0\prime}|}. \tag{2.64}$$

Note that the domain of integration is again effectively finite.

Definition 9. *We define the operator $A_{12} : \mathcal{S} \to \mathcal{B}_{\boldsymbol{w}}, \psi \mapsto A_{12}\psi$ according to (2.64), equality is to be read as equality in $\mathcal{B}_{\boldsymbol{w}}$. Well-definedness and boundedness will be shown in theorem 14 and 15.*

Definition 10. *Finally, $A : \mathcal{S} \to \mathcal{B}_{\boldsymbol{w}}$ is defined as the sum of the individual A operators $A = A_0 + A_1 + A_2 + A_{12}$.*

After we prove that A is well-defined, we will have collected rigorous definitions of the ingredients of equation (2.22) on test functions. However, this is not sufficient for our strategy of construction of solutions. For a fixed point argument we need A to be defined at least on the image $A(\mathcal{S})$, more convenient is to lift A to a Banach space.

2.2.2.5 Lifting A from test functions to a suitable Banach space.

In order to prove the existence and uniqueness of solutions of the integral equation $\psi = \psi^{\text{free}} + A\psi$, we will define the operator A not only on test functions but on a suitable Banach space which includes (at least) sufficiently many solutions ψ^{free} of the free multi-time Klein-Gordon equations, $(\Box_k + m_k^2)\psi^{\text{free}}(x_1, x_2) = 0,\ k = 1, 2$. We shall define this Banach space as the completion of $\mathcal{S}$ with respect to a suitable norm. A good choice which works well for the upcoming existence and uniqueness proofs is the class of weighted L^∞-norms.

Definition 11. *Let $\boldsymbol{w} : \mathbb{R}_0^+ \to \mathbb{R}^+$ be a monotonically increasing function such that $1/\boldsymbol{w}$ is bounded. Then our Banach space is given by the completion*

$$\mathcal{B}_{\boldsymbol{w}} = \overline{\mathcal{S}}^{\|\cdot\|_{\boldsymbol{w}}} \tag{2.65}$$

with respect to the norm $\|\cdot\|_{\boldsymbol{w}} : \mathcal{S} \to \mathbb{R}$,

$$\|\psi\|_{\boldsymbol{w}} := \operatorname*{ess\,sup}_{x,y \in \mathbb{M}_0^+} \frac{|\psi(x,y)|}{\boldsymbol{w}(x^0)\boldsymbol{w}(y^0)}. \tag{2.66}$$

Our next goal is to find a weight function $\boldsymbol{w}$ such that the operator A is not only bounded but even defines a contraction on $\mathcal{B}_{\boldsymbol{w}}$. By linear extension, it is sufficient to estimate $\|A\psi\|_{\boldsymbol{w}}$ on test functions $\psi \in \mathcal{S}$. Before we move on to the main results of this section and its proofs, we remark on the choice of space.

Remarks:

1. We have attempted to use a Bochner space, L^∞ in the times and L^2 in the space variables. However, we did not succeed in obtaining suitable estimates for that case. This might not be a problem in principle, but its treatment would require further technical innovation. More precisely, one would need to understand integral operators such as (2.58) whose kernel is in L^1 but not in L^2.

2. Nevertheless, our definition of $\mathcal{B}_{\boldsymbol{w}}$ contains a large class of free solutions of the Klein-Gordon equation. As the Klein-Gordon equation preserves boundedness, all bounded initial data for ψ^{free} lead to a free solution $\psi^{\text{free}} \in \mathcal{B}_{\boldsymbol{w}}$ which can be used as an input to our integral equation.

2.2.2.6 Rigorous formulation of N particle problem

While there are different possibilities to generalize the two-particle integral equation (2.22), we focus on the one advocated in [64] as the most promising. For

$$\psi : \left(\mathbb{M}_0^+\right)^N \to \mathbb{C}, \qquad (x_1, ..., x_N) \mapsto \psi(x_1, ..., x_N) \tag{2.67}$$

we consider the integral equation

$$\begin{aligned}\psi(x_1, ..., x_N) \;=\; & \psi^{\text{free}}(x_1, ..., x_N) + \frac{\lambda}{4\pi} \sum_{i,j=1,\ldots,N;\, i<j} \\ & \times \int_{\mathbb{M}_0^+} d^4x_i \int_{\mathbb{M}_0^+} d^4x_j\; G^{\text{ret}}(x_i - x_i')G^{\text{ret}}(x_j - x_j') \\ & \times \delta((x_i' - x_j')^2)\psi(x_1, ..., x_i, ..., x_j, ..., x_N).\end{aligned} \tag{2.68}$$

Here, ψ^{free} is again a solution of the free Klein-Gordon equations $(\Box_k + m_k^2)\phi(x_k)$ in each spacetime variable.
Equation (2.68) is written down in a heuristic way. Now we come to its rigorous version.

Definition 12. *Let $\psi \in \mathcal{S}\big((\mathbb{M}_0^+)^N\big)$ be a test function. Moreover, let $A^{(ij)}$ be the integral operator of definition 10 of the two-particle problem acting on the variables x_i and x_j instead of $x = x_1$ and $y = x_2$. We define the space ${}^{(N)}\mathcal{B}_{\boldsymbol{w}}$ as the completion of $\mathcal{S}\big((\mathbb{M}_0^+)^N\big)$ with respect to the norm*

$$\|\psi\|_{\boldsymbol{w}} \;=\; \operatorname*{ess\,sup}_{x_1,\ldots,x_N\in\mathbb{M}_0^+} \frac{|\psi|(x_1, ..., x_N)}{\boldsymbol{w}(x_1^0)\cdots\boldsymbol{w}(x_N^0)}, \tag{2.69}$$

where the function $\boldsymbol{w}$ is defined as before. Finally, we define $A : \mathcal{S}\big((\mathbb{M}_0^+)^N\big) \to {}^{(N)}\mathcal{B}_{\boldsymbol{w}}$ by its action

$${}^{(N)}A \;=\; \sum_{i,j=1,\ldots,N;\, i<j} A^{(ij)}. \tag{2.70}$$

As will be shown below, ${}^{(N)}A$ can be linearly extended to a bounded operator on the Banach space ${}^{(N)}\mathcal{B}_{\boldsymbol{w}}$. Then we take the equation

$$\psi \;=\; \psi^{\text{free}} + {}^{(N)}A\psi, \tag{2.71}$$

to be the rigorous version of (2.68) on ${}^{(N)}\mathcal{B}_{\boldsymbol{w}}$.

2.2.2.7 Rigorous formulation of the problem on FLRW spacetime

Recall the formulation of the spin-0 delay-equation (2.22) on FLRW spacetime, equation (2.42) which upon plugging in the expressions (2.39) and (2.40) for the Green's functions becomes

$$\begin{aligned}\psi(\eta_1,\vec{x}_1,\eta_2,\vec{x}_2) &= \psi^{\text{free}}(\eta_1,\vec{x}_1,\eta_2,\vec{x}_2) + \frac{\lambda}{(4\pi)^3}\frac{1}{a(\eta_1)a(\eta_2)}\\ &\int_0^{\eta_1} d\eta_1' \int d^3\vec{x}_1' \int_0^{\eta_2} d\eta_2' \int d^3\vec{x}_2' a^2(\eta_1')a^2(\eta_2')\\ &\times\ \frac{\delta(\eta_1-\eta_1'-|\vec{x}_1-\vec{x}_1'|)}{|\vec{x}_1-\vec{x}_1'|}\frac{\delta(\eta_2-\eta_2'-|\vec{x}_2-\vec{x}_2'|)}{|\vec{x}_2-\vec{x}_2'|}\\ &\times\ \delta((\eta_1'-\eta_2')^2-|\vec{x}_1'-\vec{x}_2'|^2)\psi(\eta_1',\vec{x}_1',\eta_2',\vec{x}_2'). \end{aligned}\tag{2.72}$$

Now let

$$\chi(\eta_1,\vec{x}_1,\eta_2) = a(\eta_1)a(\eta_2)\psi(\eta_1,\vec{x}_1,\eta_2) \tag{2.73}$$

and $\chi^{\text{free}}(\eta_1,\vec{x}_1,\eta_2) = a(\eta_1)a(\eta_2)\psi^{\text{free}}(\eta_1,\vec{x}_1,\eta_2)$. Then (2.72) is equivalent to:

$$\begin{aligned}\chi(\eta_1,\vec{x}_1,\eta_2,\vec{x}_2) &= \chi^{\text{free}}(\eta_1,\vec{x}_1,\eta_2,\vec{x}_2) + \frac{\lambda}{(4\pi)^3}\int_0^{\eta_1} d\eta_1' \int d^3\vec{x}_1'\\ &\times \int_0^{\eta_2} d\eta_2' \int d^3\vec{x}_2' \frac{\delta(\eta_1-\eta_1'-|\vec{x}_1-\vec{x}_1'|)}{|\vec{x}_1-\vec{x}_1'|}\frac{\delta(\eta_2-\eta_2'-|\vec{x}_2-\vec{x}_2'|)}{|\vec{x}_2-\vec{x}_2'|}\\ &\times a(\eta_1')a(\eta_2')\delta((\eta_1'-\eta_2')^2-|\vec{x}_1'-\vec{x}_2'|^2)\chi(\eta_1',\vec{x}_1',\eta_2',\vec{x}_2'). \end{aligned}\tag{2.74}$$

We can see that this equation has almost exactly the same form as the massless version of (2.22) on $\mathbb{M}_0^+$ (see (2.47)). The only difference is the additional appearance of the factor $a(\eta_1')a(\eta_2')$ inside the integrals.

Definition 13. *The operator $\widetilde{A}_0 : \mathcal{S} \to \mathcal{B}_{\boldsymbol{w}}$ is defined using coordinates $x = (\eta_1, \vec{x})$, $y = (\eta_2, \vec{y})$:*

$$(\widetilde{A}_0\chi)(x,y) = \frac{\lambda}{(4\pi)^3} \int_{B_{y^0}(\vec{y})} d^3\vec{y}' \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta \, \frac{|b^2|}{4(b^0 + |\vec{b}|\cos\vartheta)^2 |\vec{y}'|}$$
$$\times\, a(\eta_1 + \eta_1')a(\eta_2 + \eta_2')\chi(x + x', y + y')$$
$$\times \left(1_{b^2>0} 1_{b^0>0} 1_{\cos\vartheta > \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}} + 1_{b^2<0} 1_{\cos\vartheta < \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}} \right), \tag{2.75}$$

with $\eta_1' = -r^* = -|\vec{x}'|, \eta_2' = -|\vec{y}'|$. (Here, b and r^* are defined as in (2.52) and (2.54), respectively).

We take the equation

$$\chi = \chi^{\text{free}} + \widetilde{A}_0\chi \tag{2.76}$$

to be the rigorous version of equation (2.74).

2.2.3 Results

This subsection is structured as follows. Sec. 2.2.3.1 (which is about the two-particle case) contains the theorems about existence and uniqueness of solutions which is the main result of Sec. 2.2. Sec. 2.2.3.2 extends these results to the N-particle case and in Sec. 2.2.3.3 we show that a curved spacetime with a Big Bang singularity can provide a natural reason for a cut-off in time.

2.2.3.1 The two-particle case

Theorem 14 (Bounds for A_0 and $\boldsymbol{w}(t) = e^{\gamma t}$; existence of massless dynamics.)**.**

For any $\gamma > 0$, let $\boldsymbol{w}(t) = e^{\gamma t}$. Then A_0 can be linearly extended to a bounded operator on $\mathcal{B}_{\boldsymbol{w}}$ with norm

$$\|A_0\| \leqslant \frac{\lambda}{8\pi\gamma^2}. \tag{2.77}$$

Consequently, for all $\gamma > \sqrt{\frac{\lambda}{8\pi}}$, the integral equation $\psi = \psi^{\text{free}} + A_0\psi$ has a unique solution $\psi \in \mathcal{B}_{\boldsymbol{w}}$ for every $\psi^{\text{free}} \in \mathcal{B}_{\boldsymbol{w}}$.

Now we come to our main result.

Theorem 15 (Existence of dynamics in the massive case.)**.**
For any $\alpha > 0$, let

$$\boldsymbol{w}(t) = (1 + \alpha t^2)e^{\alpha t^2/2}. \tag{2.78}$$

Then A_0, A_1, A_2 and A_{12} can be linearly extended to bounded operators on $\mathcal{B}_{\boldsymbol{w}}$ with norms

$$\|A_0\| \leqslant \frac{\lambda}{32\pi}\frac{1}{\alpha}, \tag{2.79}$$

$$\|A_1\| \leqslant \frac{5\lambda\, m_1^2}{16\pi}\frac{1}{\alpha^2}, \tag{2.80}$$

$$\|A_2\| \leqslant \frac{5\lambda\, m_2^2}{16\pi}\frac{1}{\alpha^2}, \tag{2.81}$$

$$\|A_{12}\| \leqslant \frac{\lambda\, m_1^2 m_2^2}{80\pi}\frac{1}{\alpha^3}. \tag{2.82}$$

Consequently, for all $\alpha > 0$ with

$$\frac{\lambda}{8\pi\alpha}\left(\frac{1}{4} + \frac{5(m_1^2 + m_2^2)}{2}\frac{1}{\alpha} + \frac{m_1^2\, m_2^2}{10}\frac{1}{\alpha^2}\right) < 1, \tag{2.83}$$

the integral equation $\psi = \psi^{\text{free}} + A\psi$ has a unique solution $\psi \in \mathcal{B}_{\boldsymbol{w}}$ for every $\psi^{\text{free}} \in \mathcal{B}_{\boldsymbol{w}}$.

The proof can be found in Sec. 2.2.4.3.

Remarks: 1. *Comparison of theorem 14 and 15 in the massless case.* At the first glance, the result of theorem 14 looks stronger in the sense that for $\boldsymbol{w}(t) = e^{\gamma t}$, the estimate of $\|A_0\|$ goes with γ^{-2} while for $\boldsymbol{w}(t) = (1 + \alpha t^2)e^{\alpha t^2/2}$, the estimate of $\|A_0\|$ goes with α^{-1}. However, one should note that γ is the constant in front of t while α occurs in combination with t^2. Thus, if one wants to draw a comparison between these different cases at all, then it should be between γ and $\sqrt{\alpha}$. Of course, the main difference between the two theorems is the admitted growth rate of the solutions. In this regard, theorem 14 contains the stronger statement.

2. A *physically realistic value of* λ is $\frac{1}{137}$, the value of the fine structure constant. In that case, α need not even be particularly large in order for condition (2.83) to be satisfied.

3. *Initial value problem.* By the integral equation (2.22), we obtain that the solution ψ satisfies $\psi(0, \vec{x}, 0, \vec{y}) = \psi^{\text{free}}(0, \vec{x}, 0, \vec{y})$. If ψ^{free} is a solution of the free multi-time Klein-Gordon equations, then it is itself determined by initial data at $x_1^0, x_2^0 = 0$. (As the Klein-Gordon equation is of second order in time, these initial data include data for $\partial_{x^0}\psi$, $\partial_{y^0}\psi$ and $\partial_{x^0}\partial_{y^0}\psi$, see [77, chap. 5].) Thus, we find that ψ is determined by these data at $x_1^0, x_2^0 = 0$ as well. Note that for later times, ψ and ψ^{free} do not, in general, coincide and consequently a similar statement does not hold.

4. *Finite propagation speed.* The theorem implies that $\psi = \sum_{k=0}^{\infty} A^k \psi^{\text{free}}$. As $(A\psi^{\text{free}})(x, y)$ involves only values of ψ^{free} in $\text{past}(x) \times \text{past}(y)$ where $\text{past}(x)$ denotes the causal past of $x \in \mathbb{M}_0^+$ (see equations (2.58), (2.60), (2.61), (2.64)), so do $A^k\psi^{\text{free}}$ for all $k \in \mathbb{N}$ and ψ. Therefore, we obtain: if the initial data for

ψ^{free} at $x^0 = 0 = y^0$ are compactly supported in a region $R \subset (\{0\} \times \mathbb{R}^3)^2$, then for all Cauchy surfaces $\Sigma \subset \mathbb{M}_0^+$, $\psi|_{\Sigma\times\Sigma}$ is supported in the causally grown set $\text{Gr}(R,\Sigma) = \left(\bigcup_{(x,y)\in R} \text{future}(x) \times \text{future}(y)\right) \cap (\Sigma \times \Sigma)$ where future(x) stands for the causal future of $x \in \mathbb{M}_0^+$.

5. *Square integrable solutions.* As a consequence of the previous item, compactly supported and bounded initial data for ψ^{free} lead to a compactly supported and bounded solution ψ. In particular, this implies that $\psi(x^0, \cdot, y^0)$ lies in $L^2(\mathbb{R}^6)$ for all times $x^0, y^0 \geqslant 0$.

2.2.3.2 The N-particle case

With these preparations, we are ready to formulate the N-particle existence and uniqueness theorem.

Theorem 16 (Existence of dynamics for N particles.)**.**
For any $\alpha > 0$, let $\boldsymbol{w}(t) = (1+\alpha t^2)e^{\alpha t^2/2}$. Then the operator ${}^{(N)}A$ can be linearly extended to a bounded operator on ${}^{(N)}\mathcal{B}_{\boldsymbol{w}}$ with norm

$$\|{}^{(N)}A\| \leqslant \frac{\lambda}{8\pi\alpha} \sum_{i,j=1,\ldots,N;\, i<j} \left(\frac{1}{4} + \frac{5(m_i^2 + m_j^2)}{2} \frac{1}{\alpha} + \frac{m_i^2\, m_j^2}{10} \frac{1}{\alpha^2} \right). \quad (2.84)$$

If α is such that this expression is strictly smaller than one, the integral equation (2.71) has a unique solution $\psi \in {}^{(N)}\mathcal{B}_{\boldsymbol{w}}$ for every $\psi^{\text{free}} \in {}^{(N)}\mathcal{B}_{\boldsymbol{w}}$.

The proof follows straightforwardly from that of theorem 15 using

$$\|{}^{(N)}A\| \leqslant \sum_{i,j=1,\ldots,N;\, i<j} \|A^{(ij)}\|_{\boldsymbol{w}}. \quad (2.85)$$

For the norms of the operators $A^{(ij)}$, one can use the previous expressions as these operators act only as the identity on variables x_k with $k \notin \{i,j\}$.

Remark 17. *To the best of my knowledge, theorem 16 is the first result about the existence and uniqueness of solutions of multi-time integral equations for N particles. While for the present contraction argument the generalization to N particles has been straightforward, this is not the case for other works. For example, the Volterra iterations used in [72] become increasingly complicated with increasing particle number. For Dirac particles, a similar technique is used section 2.3. However, as the Dirac Green's functions contain distributional derivatives, one has to control weak derivatives of the solutions, and the number of such derivatives depends on N. That situation also does not allow for such a straightforward generalization to N particles as has been possible here.*

2.2.3.3 Result on FLRW Spacetime

Recall the rigorous definition 13 of the integral operator in the spin-0 delay-equation on FLRW spacetime (2.76). In terms of these, we can formulate the respective existence and uniqueness theorem:

Theorem 18 (Existence of dynamics for an open FLRW universe)**.** *Let $a : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous function with $a(0) = 0$ and $a(\eta) > 0$ for $\eta > 0$. Moreover, let*

$$\boldsymbol{w}(t) = \exp\left(\gamma \int_0^t d\tau\, a(\tau)\right). \tag{2.86}$$

Then, the operator $\widetilde{A}_0$ satisfies the following estimate:

$$\sup_{\chi \in \mathcal{S}\left((\mathbb{M}_0^+)^2\right)} \frac{\|\widetilde{A}_0 \chi\|_{\boldsymbol{w}}}{\|\chi\|_{\boldsymbol{w}}} \leqslant \frac{\lambda}{8\pi\gamma^2}. \tag{2.87}$$

$\widetilde{A}_0$ can be extended to a linear operator on $\mathcal{B}_{\boldsymbol{w}}$ which satisfies the same bound. Moreover, for $\gamma < \sqrt{\frac{\lambda}{8\pi}}$, the equation $\chi = \chi^{\text{free}} + \widetilde{A}_0 \chi$ has a unique solution $\chi \in \mathcal{B}_{\boldsymbol{w}}$ for every $\psi^{\text{free}} \in \mathcal{B}_{\boldsymbol{w}}$.

The proof can be found in Sec. 2.2.4.4.

Remarks: 1. *Manifest covariance.* The theorem shows the existence and uniqueness of solutions of the manifestly covariant integral equation (2.42). Our example of a particular FLRW spacetime thus achieves its goal of demonstrating that a cut-off in time can arise naturally in a cosmological context.

2. *Initial value problem.* As in the case of $\mathbb{M}_0^+$, the solution χ satisfies $\chi(0, \vec{x}, 0, \vec{y}) = \chi^{\text{free}}(0, \vec{x}, 0, \vec{y})$ where χ^{free} is determined by the solution ψ^{free} of the free conformal wave equation (2.37) in both spacetime variables. Since ψ^{free} is determined by initial data at $\eta_1 = 0 = \eta_2$, so are χ^{free} and χ.

3. *Behaviour of ψ towards the Big Bang singularity.* While the transformed wave function χ remains bounded for $\eta_1, \eta_2 \to 0$, the physical wave function $\psi(\eta_1, \vec{x}, \eta_2, \vec{y}) = \frac{1}{a(\eta_1)a(\eta_2)}\chi(\eta_1, \vec{x}, \eta_2, \vec{y})$ diverges like $\frac{1}{a(\eta_1)a(\eta_2)}$. This is to be expected, as the Klein-Gordon equation has a preserved "energy" (given by a certain spatial integral) and as the volume in $\vec{x}, \vec{y}$ contracts to zero towards the Big Bang.

4. *N-particle generalization.* As shown in Sec. 2.2.3.2 for the Minkowski half-space, it would also be possible to directly extend theorem 18 to N particles. To avoid duplication, we do not carry this out explicitly for the curved spacetime example here.

2.2.4 Proofs

For $t \geqslant 0$, we define the functions:

$$\boldsymbol{w}_0(t) = \boldsymbol{w}(t),$$

$$\text{and for } n \in \mathbb{N}: \quad \boldsymbol{w}_n(t) = \int_0^t dt' \, \boldsymbol{w}_{n-1}(t'). \tag{2.88}$$

Note that due to the properties of $\boldsymbol{w}$, the functions $\boldsymbol{w}_n$ are monotonically increasing for all $n \in \mathbb{N}$; furthermore, by definition, they satisfy $\boldsymbol{w}_n(0) = 0$.
The theorem gives explicit bounds for the operators A_0, A_1, A_2, A_{12} in terms of the functions $\boldsymbol{w}_n$ and is therefore together with Banach's fixed point theorem the main tool leading to the results of subsections 2.2.3, 2.2.3.2 and 2.2.3.3. The proof is the result of the next subsections.

Theorem 19 (Bounds of the integral operators on $\mathcal{S}$.)**.** *For all $\psi \in \mathcal{S}((\mathbb{M}_0^+)^2)$, the integral operators A_0, A_1, A_2, A_{12} satisfy the following bounds:*

$$\sup_{\psi \in \mathcal{S}((\mathbb{M}_0^+)^2)} \frac{\|A_0\psi\|_{\boldsymbol{w}}}{\|\psi\|_{\boldsymbol{w}}} \leqslant \frac{\lambda}{8\pi} \left(\sup_{t \geqslant 0} \frac{\boldsymbol{w}_1(t)}{\boldsymbol{w}(t)} \right)^2, \tag{2.89}$$

$$\begin{aligned} \sup_{\psi \in \mathcal{S}((\mathbb{M}_0^+)^2)} \frac{\|A_1\psi\|_{\boldsymbol{w}}}{\|\psi\|_{\boldsymbol{w}}} \leqslant \frac{\lambda\, m_1^2}{16\pi} \Bigg[& 3 \left(\sup_{t \geqslant 0} \frac{t\boldsymbol{w}_1(t)}{\boldsymbol{w}(t)} \right) \left(\sup_{t \geqslant 0} \frac{\boldsymbol{w}_2(t)}{\boldsymbol{w}(t)} \right) \\ & + 3 \left(\sup_{t \geqslant 0} \frac{\boldsymbol{w}_1(t)}{\boldsymbol{w}(t)} \right) \left(\sup_{t \geqslant 0} \frac{t\boldsymbol{w}_2(t)}{\boldsymbol{w}(t)} \right) \\ & + 2 \left(\sup_{t \geqslant 0} \frac{\boldsymbol{w}_1(t)}{\boldsymbol{w}(t)} \right) \left(\sup_{t \geqslant 0} \frac{\boldsymbol{w}_3(t)}{\boldsymbol{w}(t)} \right) \Bigg], \end{aligned} \tag{2.90}$$

$$\begin{aligned} \sup_{\psi \in \mathcal{S}((\mathbb{M}_0^+)^2)} \frac{\|A_2\psi\|_{\boldsymbol{w}}}{\|\psi\|_{\boldsymbol{w}}} \leqslant \frac{\lambda\, m_2^2}{16\pi} \Bigg[& 3 \left(\sup_{t \geqslant 0} \frac{t\boldsymbol{w}_1(t)}{\boldsymbol{w}(t)} \right) \left(\sup_{t \geqslant 0} \frac{\boldsymbol{w}_2(t)}{\boldsymbol{w}(t)} \right) \\ & + 3 \left(\sup_{t \geqslant 0} \frac{\boldsymbol{w}_1(t)}{\boldsymbol{w}(t)} \right) \left(\sup_{t \geqslant 0} \frac{t\boldsymbol{w}_2(t)}{\boldsymbol{w}(t)} \right) \\ & + 2 \left(\sup_{t \geqslant 0} \frac{\boldsymbol{w}_1(t)}{\boldsymbol{w}(t)} \right) \left(\sup_{t \geqslant 0} \frac{\boldsymbol{w}_3(t)}{\boldsymbol{w}(t)} \right) \Bigg], \end{aligned} \tag{2.91}$$

$$\sup_{\psi \in \mathcal{S}((\mathbb{M}_0^+)^2)} \frac{\|A_{12}\psi\|_{\boldsymbol{w}}}{\|\psi\|_{\boldsymbol{w}}} \leqslant \frac{\lambda\, m_1^2 m_2^2}{96\pi} \Bigg[\left(\sup_{t \geqslant 0} \frac{t^2\boldsymbol{w}_2(t)}{\boldsymbol{w}(t)} \right) \left(\sup_{t \geqslant 0} \frac{t\boldsymbol{w}_1(t)}{\boldsymbol{w}(t)} \right)$$

$$+\frac{1}{2}\left(\sup_{t\geqslant 0}\frac{t^2\boldsymbol{w}_3(t)}{\boldsymbol{w}(t)}\right)\left(\sup_{t\geqslant 0}\frac{\boldsymbol{w}_1(t)}{\boldsymbol{w}(t)}\right)\Bigg]. \tag{2.92}$$

In case these expressions are finite, A_0, A_1, A_2, A_{12} extend to linear operators all of on $\mathcal{B}_{\boldsymbol{w}}$ with the same norms. Our next task is to find suitable weight functions $\boldsymbol{w}$ such that this is actually the case. We begin with the massless case where already an exponential weight function leads to an estimate which remains finite after taking the supremum. The massive case is treated subsequently; it is a little more difficult since all the estimates for the operators A_0, A_1, A_2, A_{12} have to be finite at the same time. This requires a different choice of weight function (see theorem 15).

2.2.4.1 Proof of Theorem 19

The proof is divided into the proofs of the estimates (2.89), (2.90), (2.91) and (2.92), respectively. Here, (2.89) is the most singular and most difficult term which deserves the most attention.
Throughout this subsection, let $\psi \in \mathcal{S}((\mathbb{M}_0^+)^2)$.

2.2.4.1.1 Estimate of the massless term (2.89). We start with equation (2.58) and take the absolute value. Using, in addition, that

$$|\psi(x,y)| \;\leqslant\; \|\psi\|_{\boldsymbol{w}}\,\boldsymbol{w}(x^0)\boldsymbol{w}(y^0) \tag{2.93}$$

leads us to:

$$|A_0\psi|(x,y)\leqslant\frac{\lambda\|\psi\|_{\boldsymbol{w}}}{4(4\pi)^3}\int_{B_{y^0}(\vec{y})}d^3\vec{y}'\int_0^{2\pi}d\varphi\int_{-1}^{1}d\cos\vartheta\,\frac{|b^2|}{(b^0+|\vec{b}|\cos\vartheta)^2|\vec{y}'|}$$
$$\times\,\boldsymbol{w}(y^0-|\vec{y}'|)\boldsymbol{w}\left(x^0-\frac{1}{2}\frac{b^2}{b^2+|\vec{b}|\cos\vartheta}\right)$$

$$\times \left(1_{b^2>0} 1_{b^0>0} 1_{\cos\vartheta > \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}} + 1_{b^2<0} 1_{\cos\vartheta < \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}} \right). \tag{2.94}$$

Next, we observe that the fraction $\frac{|b^2|}{(b^0+|\vec{b}|\cos\vartheta)^2}$ is the derivative of the fraction which occurs in the argument of the second $\boldsymbol{w}$-function. Introducing $u = \cos\vartheta$ allows us to rewrite (2.94) as

$$\begin{aligned}
(2.94) = \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{8(4\pi)^2} \int_{B_{y^0}(\vec{y})} d^3\vec{y}' \int_{-1}^{1} du 2\,\mathrm{sgn}(b^2) \\
\times \partial_u \boldsymbol{w}_1 \left(x^0 - \frac{1}{2} \frac{b^2}{b^0 + |\vec{b}|u} \right) \boldsymbol{w}(y^0 - |\vec{y}'|) \frac{1}{|\vec{b}||\vec{y}'|} \\
\times \left(1_{b^2>0} 1_{b^0>0} 1_{u > \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}} + 1_{b^2<0} 1_{u < \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}} \right)
\end{aligned} \tag{2.95}$$

$$\begin{aligned}
= \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{4(4\pi)^2} \int_{B_{y^0}(\vec{y})} d^3\vec{y}' \int_{-1}^{1} du\ \partial_u \boldsymbol{w}_1 \left(x^0 - \frac{1}{2} \frac{b^2}{b^0 + |\vec{b}|u} \right) \boldsymbol{w}(y^0 - |\vec{y}'|) \\
\times \left(\underbrace{1_{b^2>0} 1_{b^0>0} 1_{u > \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}}}_{\boxed{1}} - \underbrace{1_{b^2<0} 1_{u < \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}}}_{\boxed{2}} \right) \frac{1}{|\vec{b}||\vec{y}'|}.
\end{aligned} \tag{2.96}$$

This form allows for a direct integration with respect to u. Before we integrate, we check for both terms $\boxed{1}$ and $\boxed{2}$ whether the conditions implicit in the characteristic functions can always be satisfied. (Otherwise, the respective term would not contribute any further, and we could drop it.) Unfortunately, this results in a tedious treatment in a case by case manner. Recall that $b = x - y - (-|\vec{y}'|, \vec{y}')$.
First, we check for term $\boxed{1}$ whether in the case $b^2 > 0, b^0 > 0$ it is true that $1 > \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}$ holds. (The comparison with 1 is due to the

upper range for u.) We compute

$$\begin{aligned} 1 > \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|} &\iff 2x^0|\vec{b}| + 2x^0 b^0 > b^2 \\ &\iff 2x^0(b^0 + |\vec{b}|) > (b^0 + |\vec{b}|)(b^0 - |\vec{b}|) \\ &\overset{b^2>0, b^0>0}{\iff} 2x^0 > b^0 - |\vec{b}| \\ &\iff x^0 + y^0 - |\vec{y}'| > -|\vec{b}|. \end{aligned} \tag{2.97}$$

Now because of $|\vec{y}'| < y^0$ we see that this inequality always holds true. Hence, the respective term in (2.96) contributes without further restrictions.

Next, we turn to the term $\boxed{2}$. Here we check whether $b^2 < 0$ implies $-1 < \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|}$ or whether extra conditions are needed. (The comparison with -1 is due to the lower bound for u.) A similar calculation yields

$$-1 < \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|} \iff -2x^0|\vec{b}| + 2x^0|\vec{b}| < b^2 \tag{2.98}$$

$$\iff 2x^0(b^0 - |\vec{b}|) < (b^0 - |\vec{b}|)(b^0 + |\vec{b}|) \tag{2.99}$$

$$\overset{b^2<0}{\iff} 2x^0 > b^0 + |\vec{b}|. \tag{2.100}$$

This inequality need not always hold, as we can increase $|\vec{b}|$ with respect to b^0 as much as we like, e.g., by picking $|\vec{x} - \vec{y}|$ large. Therefore, in this case, the respective term is only sometimes non-zero. We make this clear by including the characteristic function $1_{2x^0 > b^0 + |\vec{b}|}$.

Taking these considerations into account, we now carry out the u-

integration in (2.96):

$$
\begin{aligned}
|A_0\psi|(x,y) \;&\leqslant\; \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{4(4\pi)^2}\int_{B_{y^0}(\vec{y})} d^3\vec{y}'\;\frac{\boldsymbol{w}(y^0-|\vec{y}'|)}{|\vec{b}||\vec{y}'|}\\
&\times\left(1_{b^2>0,b^0>0}\left[\boldsymbol{w}_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0+|\vec{b}|}\right)\right.\right.\\
&\qquad\left.-\,\boldsymbol{w}_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0+|\vec{b}|\max(-1,\frac{b^2}{2x^0|\vec{b}|}-\frac{b^0}{|\vec{b}|})}\right)\right] \qquad (2.101)\\
&\quad-\,1_{b^2<0}1_{2x^0>b^0+|\vec{b}|}\left[\boldsymbol{w}_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0+|\vec{b}|\min(1,\frac{b^2}{2x^0|\vec{b}|}-\frac{b^0}{|\vec{b}|})}\right)\right.\\
&\qquad\left.\left.-\,\boldsymbol{w}_1\left(x^0-\frac{1}{2}\frac{b^2}{b^0-|\vec{b}|}\right)\right]\right). \qquad (2.102)
\end{aligned}
$$

The minima and maxima in this expression result from the indicator functions $1_{u>\frac{b^2}{2x^0|\vec{b}|}-\frac{b^0}{|\vec{b}|}}$ and $1_{u<\frac{b^2}{2x^0|\vec{b}|}-\frac{b^0}{|\vec{b}|}}$, respectively.

Our next step is to simplify the complicated fractions in (2.101) and (2.102) involving min and max. For (2.101) we use that $1/\max(a,b) = \min(1/a,1/b)$ whenever $a,b>0$ or $a,b<0$ holds. Therefore, we have:

$$
\begin{aligned}
\frac{1}{2}\frac{b^2}{b^0+|\vec{b}|\max\left(-1,\frac{b^2}{2x^0|\vec{b}|}-\frac{b^0}{|\vec{b}|}\right)} &= \frac{1}{2}\frac{b^2}{\max\left(b^0-|\vec{b}|,\frac{b^2}{2x^0}\right)}\\
=\frac{1}{2}\min\left(\frac{b^2}{b^0-|\vec{b}|},2x^0\right) &= \min\left(\frac{b^0+|\vec{b}|}{2},x^0\right).
\end{aligned}
$$

The fraction in (2.102) can be simplified by observing that

$$
b^0+|\vec{b}|\min\left(1,\frac{b^2}{2x^0|\vec{b}|}-\frac{b^0}{|\vec{b}|}\right) = \min\left(b^0+|\vec{b}|,\frac{b^2}{2x^0}\right) = \frac{b^2}{2x^0} \qquad (2.103)
$$

as the term contributes only for $b^2 < 0$ hence $b^0 + |\vec{b}| > 0 > b^2/(2x^0)$. Thus,

$$\frac{1}{2}\frac{b^2}{b^0 + |\vec{b}|\min(1, \frac{b^2}{2x^0|\vec{b}|} - \frac{b^0}{|\vec{b}|})} = x^0. \tag{2.104}$$

With these simplifications, we obtain for (2.101) and (2.102) (using $\boldsymbol{w}_1(0) = 0$):

$$\begin{aligned}
|A_0\psi|(x,y) \leqslant \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{4(4\pi)^2}\int_{B_{y^0}(\vec{y})} d^3\vec{y}'\ \frac{\boldsymbol{w}(y^0 - |\vec{y}'|)}{|\vec{b}||\vec{y}'|} \\
\times\Bigg(1_{b^2>0,b^0>0}\left[\boldsymbol{w}_1\left(x^0 - \frac{b^0 - |\vec{b}|}{2}\right) - \boldsymbol{w}_1\left(x^0 - \min\left(\frac{b^0 + |\vec{b}|}{2}, x^0\right)\right)\right] \\
-1_{b^2<0}\ 1_{2x^0>b^0+|\vec{b}|}\left[\boldsymbol{w}_1\left(x^0 - x^0\right) - \boldsymbol{w}_1\left(x^0 - \frac{b^0 + |\vec{b}|}{2}\right)\right]\Bigg) \\
= \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{4(4\pi)^2}\int_{B_{y^0}(\vec{y})} d^3\vec{y}'\frac{\boldsymbol{w}(y^0 - |\vec{y}'|)}{|\vec{b}||\vec{y}'|}1_{b^2>0,b^0>0} \\
\times\boldsymbol{w}_1\left(\frac{x^0 + y^0 - |\vec{y}'| + |\vec{b}|}{2}\right) \qquad (2.105) \\
- \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{4(4\pi)^2}\int_{B_{y^0}(\vec{y})} d^3\vec{y}'\ \frac{\boldsymbol{w}(y^0 - |\vec{y}'|)}{|\vec{b}||\vec{y}'|}1_{b^2>0,b^0>0} \\
\times\boldsymbol{w}_1\left(\max\left(\frac{x^0 + y^0 - |\vec{y}'| - |\vec{b}|}{2}, 0\right)\right) \qquad (2.106) \\
+ \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{4(4\pi)^2}\int_{B_{y^0}(\vec{y})} d^3\vec{y}'\ \frac{\boldsymbol{w}(y^0 - |\vec{y}'|)}{|\vec{b}||\vec{y}'|}1_{b^2<0}\ 1_{x^0+y^0-|\vec{y}'|>|\vec{b}|} \\
\times\boldsymbol{w}_1\left(\frac{x^0 + y^0 - |\vec{y}'| - |\vec{b}|}{2}\right). \qquad (2.107)
\end{aligned}$$

We now want to carry out as many of the remaining $\vec{y}'$-integrations as possible. In order to do so, we orient the coordinates such that $\vec{x}-\vec{y}$ is parallel to the $(\vec{y}')_3$ axis. Then the integrands in (2.105)- (2.107) are independent of the azimuthal angle φ of the respective spherical coordinate system (ρ, θ, φ) with standard conventions.
In order to perform the remaining angular and then the radial integral, we need to find out which boundaries for θ and r result from the characteristic functions. First we analyse for which arguments the maximum in (2.106) is greater than zero and therefore contributes to the integral (as $\boldsymbol{w}_1(0)=0$). We have:

$$\begin{aligned}
&\frac{x^0+y^0-|\vec{y}'|-|\vec{b}|}{2} \;>\; 0 \\
&\iff (x^0+y^0-|\vec{y}'|)^2 > |\vec{x}-\vec{y}|^2+|\vec{y}'|^2+2|\vec{y}'||\vec{x}-\vec{y}|\cos\theta \\
&\iff \cos\theta < \frac{(x^0+y^0)^2}{2|\vec{y}'||\vec{x}-\vec{y}|}-\frac{|\vec{x}-\vec{y}|}{2|\vec{y}'|}-\frac{x^0+y^0}{|\vec{x}-\vec{y}|} =: P_{x,y}(|\vec{y}'|). \qquad (2.108)
\end{aligned}$$

This calculation also helps to reformulate the second indicator function $1_{b^2<0}\ 1_{x^0+y^0-|\vec{y}'|>|\vec{b}|}$ in (2.107) (for which we have $b^2<0$). The condition $b^0>0$ in (2.105) and (2.106) is readily seen to be equivalent to

$$|\vec{y}'| > y^0-x^0. \qquad (2.109)$$

In order to perform the θ-integral we have to translate $b^2 \gtrless 0$ into conditions on θ. We have:

$$\begin{aligned}
b^2>0 &\iff (x^0-y^0+|\vec{y}'|)^2 \;>\; |\vec{x}-\vec{y}|^2+|\vec{y}'|^2+2|\vec{y}'||\vec{x}-\vec{y}|\cos\theta \\
&\iff \cos\theta \;<\; \frac{(x-y)^2}{2|\vec{y}'||\vec{x}-\vec{y}|}+\frac{x^0-y^0}{|\vec{x}-\vec{y}|} := K_{x-y}(|\vec{y}'|). \qquad (2.110)
\end{aligned}$$

With these considerations, we have extracted relatively simple conditions on the boundaries of the integrals in spherical coordinates. However, if different restrictions of the boundaries conflict with each other, it may happen that for some parameter values the domain of integration is the empty set. We check whether this is so term by term, focusing on the θ-integration first. For term (2.105), θ needs to satisfy $-1 < \cos\theta < \min(1, K_{x-y}(|\vec{y}'|))$, so we need to check whether $-1 < K_{x-y}(|\vec{y}'|)$ holds. We have:

$$\begin{aligned} &-1 < K_{x-y}(|\vec{y}'|) \\ &\iff -2|\vec{y}'||\vec{x}-\vec{y}| < (x-y)^2 + 2|\vec{y}'|(x^0-y^0) \\ &\iff 0 \ < \ (x-y)^2 + 2|\vec{y}'|(x^0-y^0+|\vec{x}-\vec{y}|) \\ &\iff \begin{cases} \frac{y^0-x^0+|\vec{x}-\vec{y}|}{2} < |\vec{y}'| & \text{for } |\vec{x}-\vec{y}| > y^0-x^0 \\ \frac{y^0-x^0+|\vec{x}-\vec{y}|}{2} > |\vec{y}'| & \text{for } |\vec{x}-\vec{y}| < y^0-x^0. \end{cases} \end{aligned} \tag{2.111}$$

Together with (2.109), we obtain the condition $y^0-x^0<|\vec{y}'|<\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2} < y^0-x^0$ in the second case which means that there is no contribution to the integral. For $y^0-x^0 = |\vec{x}-\vec{y}|$ we have $K_{x-y}(|\vec{y}'|) = -1$ so this case is also ruled out. So we focus on the first case,

$$\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2} < |\vec{y}'| \quad \text{and } |\vec{x}-\vec{y}| > y^0-x^0, \tag{2.112}$$

by including the characteristic function $1_{|\vec{x}-\vec{y}|>y^0-x^0}$ in the integral. Next, we turn to the radial integral. By comparing its upper limit $|\vec{y}'| < y^0$ and lower limit $(y^0-x^0+|\vec{x}-\vec{y}|)/2$, we find that the integral can only be non-zero for

$$y^0 + x^0 > |\vec{x}-\vec{y}|. \tag{2.113}$$

For equality the integral vanishes, because the integral domain, while not empty, is of measure zero. We make this clear by including the respective characteristic function.

2.2.4.1.1.1 Simplification of term (2.105). These considerations allow us to continue computing (2.105):

$$(2.105) = \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{4(4\pi)^2} 1_{y^0+x^0>|\vec{x}-\vec{y}|} \int_{\max(0,y^0-x^0)}^{y^0} d\rho \int_0^{2\pi} d\varphi\; 1_{\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}<\rho}$$
$$\times\, 1_{|\vec{x}-\vec{y}|>y^0-x^0} \int_{-1}^{\min(1,K_{x-y}(\rho))} d\cos\theta\; \frac{\rho\, \boldsymbol{w}(y^0-\rho)}{\sqrt{|\vec{x}-\vec{y}|^2+\rho^2+2|\vec{x}-\vec{y}|\rho\cos\theta}}$$
$$\times\, \boldsymbol{w}_1\left(\frac{x^0+y^0-\rho+\sqrt{|\vec{x}-\vec{y}|^2+\rho^2+2\rho|\vec{x}-\vec{y}|\cos\theta}}{2}\right). \tag{2.114}$$

Now we carry out the φ-integration and use the same trick for the θ-integral as for the ϑ-integral in the $\vec{x}\,'$-integration earlier. Moreover, we absorb some restrictions of ρ into the limits of the integrals. This yields:

$$(2.105) = \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{8(4\pi)} 1_{y^0+x^0>|\vec{x}-\vec{y}|>y^0-x^0} \int_{\max\left(0,y^0-x^0,\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}\right)}^{y^0} d\rho$$
$$\int_{-1}^{\min(1,K_{x-y}(\rho))} dw\; \frac{2\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|}$$
$$\times\, \partial_w \boldsymbol{w}_2\left(\frac{x^0+y^0-\rho+\sqrt{|\vec{x}-\vec{y}|^2+\rho^2+2\rho|\vec{x}-\vec{y}|w}}{2}\right)$$
$$= \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{4(4\pi)} 1_{x^0+y^0>|\vec{x}-\vec{y}|>y^0-x^0} \int_{\max\left(0,y^0-x^0,\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}\right)}^{y^0} d\rho\; \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|}$$
$$\times \left[\boldsymbol{w}_2\left(\frac{x^0+y^0-\rho+\sqrt{|\vec{x}-\vec{y}|^2+\rho^2+2\rho|\vec{x}-\vec{y}|\min(1,K_{x-y}(\rho))}}{2}\right)\right.$$
$$\left.-\boldsymbol{w}_2\left(\frac{x^0+y^0-\rho+||\vec{x}-\vec{y}|-\rho|}{2}\right)\right] \tag{2.115}$$

The square root can be simplified using the following identity:

$$\sqrt{|\vec{x}-\vec{y}|^2+\rho^2+2\rho|\vec{x}-\vec{y}|K_{x-y}(\rho)} \\ = \sqrt{\rho^2+(x^0-y^0)^2+2\rho(x^0-y^0)} = |x^0-y^0+\rho|. \tag{2.116}$$

Using this, we can effectively pull the minimum out of the square root. We obtain:

$$(2.105) = \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{x^0+y^0>|\vec{x}-\vec{y}|>y^0-x^0} \int_{\max\left(0,y^0-x^0,\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}\right)}^{y^0} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\ \times \left[\boldsymbol{w}_2\left(\frac{x^0+y^0-\rho+\min(|\vec{x}-\vec{y}|+\rho,|x^0-y^0+\rho|)}{2}\right)\right. \\ \left. -\boldsymbol{w}_2\left(\frac{x^0+y^0-\rho+||\vec{x}-\vec{y}|-\rho|}{2}\right)\right]. \tag{2.117}$$

Next, we subdivide the conditions in the first indicator function into two cases, (a) $(x-y)^2 \geqslant 0$ and (b) $(x-y)^2 < 0$. In case (a), the condition $|\vec{x}-\vec{y}| > y^0-x^0$ implies $x^0 > y^0$. This, in turn, yields $\max\left(0, y^0-x^0, \frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}\right) = 0$. Moreover, the condition $x^0+y^0 > |\vec{x}-\vec{y}|$ is automatically satisfied (note that $x^0, y^0 > 0$). In case (b), the condition $|\vec{x}-\vec{y}| > y^0-x^0$ is automatically satisfied. We find:

$$(2.105) = \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{(x-y)^2\geqslant 0, x^0>y^0} \int_0^{y^0} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\ \times \left[\boldsymbol{w}_2\left(\frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}\right) - \boldsymbol{w}_2\left(\frac{x^0+y^0-\rho+||\vec{x}-\vec{y}|-\rho|}{2}\right)\right] \\ + \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{(x-y)^2<0}\; 1_{x^0+y^0>|\vec{x}-\vec{y}|} \int_{\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}}^{y^0} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\ \times \left[\boldsymbol{w}_2\left(\frac{x^0+y^0-\rho+|x^0-y^0+\rho|}{2}\right)\right.$$

$$
\begin{aligned}
&\qquad - \boldsymbol{w}_2 \left(\frac{x^0 + y^0 - \rho + ||\vec{x} - \vec{y}| - \rho|}{2} \right) \Bigg] \\
&= \frac{\lambda \|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{(x-y)^2 \geqslant 0, x^0 > y^0} \int_0^{y^0} d\rho \, \frac{\boldsymbol{w}(y^0 - \rho)}{|\vec{x} - \vec{y}|} \\
&\times \left[\boldsymbol{w}_2 \left(\frac{x^0 + y^0 + |\vec{x} - \vec{y}|}{2} \right) \right. \\
&\qquad \left. - \boldsymbol{w}_2 \max \left(\frac{x^0 + y^0 - |\vec{x} - \vec{y}|}{2}, \frac{x^0 + y^0 + |\vec{x} - \vec{y}|}{2} - \rho \right) \right] \\
&+ \frac{\lambda \|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{(x-y)^2 < 0} \; 1_{x^0 + y^0 > |\vec{x} - \vec{y}|} \int_{\frac{y^0 - x^0 + |\vec{x} - \vec{y}|}{2}}^{y^0} d\rho \, \frac{\boldsymbol{w}(y^0 - \rho)}{|\vec{x} - \vec{y}|} \\
&\times \left[\boldsymbol{w}_2 \max \left(x^0, y^0 - \rho \right) \right. \\
&\qquad \left. - \boldsymbol{w}_2 \max \left(\frac{x^0 + y^0 - |\vec{x} - \vec{y}|}{2}, \frac{x^0 + y^0 + |\vec{x} - \vec{y}|}{2} - \rho \right) \right]. \qquad (2.118)
\end{aligned}
$$

Here and in the following we abbreviate $\boldsymbol{w}_2(\max(\cdots))$ as $\boldsymbol{w}_2 \max(\cdots)$, and similarly for the minimum. This ends the calculation of (2.105): we have arrived at an expression where no more exact calculations can be done, and further estimates are needed.

2.2.4.1.1.2 Simplification of term (2.106)**.** Next, we proceed with (2.106) similarly. In case the reader is not interested in the details of the calculation, the result can be found in (2.126).
The restrictions of the integration variables for (2.106) are the same as for (2.105), namely:

$$
\cos\theta < K_{x-y}(|\vec{y}\,'|) \qquad \text{from} \quad (2.110), \qquad (2.119)
$$

$$\frac{y^0 - x^0 + |\vec{x} - \vec{y}|}{2} < |\vec{y}\,'| \qquad \text{from } (2.112) \tag{2.120}$$

$$y^0 - x^0 < |\vec{x} - \vec{y}| < y^0 + x^0 \qquad \text{from } (2.112) \text{ and } (2.113). \tag{2.121}$$

The only difference is that from the maximum in (2.106), we obtain the additional restriction (2.108), i.e.

$$\cos\theta < P_{x,y}(|\vec{y}\,'|). \tag{2.122}$$

We need to check if there are new restrictions imposed by $P_{x,y}(|\vec{y}\,'|) > -1$. We compute

$$\begin{aligned} P_{x,y}(|\vec{y}\,'|) &> -1 \iff \\ \frac{(x^0+y^0)^2}{2|\vec{y}\,'||\vec{x}-\vec{y}|} - \frac{|\vec{x}-\vec{y}|}{2|\vec{y}\,'|} - \frac{x^0+y^0}{|\vec{x}-\vec{y}|} &> -1 \iff \\ |\vec{y}\,'| &< \frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}; \end{aligned} \tag{2.123}$$

however, the last inequality is already ensured by (2.121), $x^0 > 0$ and $|\vec{y}'| < y^0$. In order to be able to evaluate (2.106) further, we next plug the condition $\cos\theta < P_{x,y}(|\vec{y}\,'|)$ into the expression for $|\vec{b}|$. This yields (recall that we use spherical variables for $|\vec{y}\,'|$):

$$\begin{aligned} |\vec{b}| &= \sqrt{|\vec{x}-\vec{y}|^2 + \rho^2 + 2\rho|\vec{x}-\vec{y}|\cos\theta} \\ &< \sqrt{|\vec{x}-\vec{y}|^2 + \rho^2 + 2\rho|\vec{x}-\vec{y}|P_{x,y}(\rho)} \\ &= \sqrt{\rho^2 - 2\rho(x^0+y^0) + (x^0+y^0)^2} = x^0 + y^0 - \rho. \end{aligned} \tag{2.124}$$

With this, we perform for (2.106) the analogous calculation to (2.114)–

(2.117). This yields:

$$
\begin{aligned}
(2.106) &= \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{y^0-x^0<|\vec{x}-\vec{y}|<x^0+y^0} \int_{\max\left(0,y^0-x^0,\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}\right)}^{y^0} d\rho \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\
&\times \Bigg[\boldsymbol{w}_2\left(\frac{x^0+y^0-\rho-\min(|\vec{x}-\vec{y}|+\rho, |x^0-y^0+\rho|, x^0+y^0-\rho)}{2}\right) \\
&\quad -\boldsymbol{w}_2\left(\frac{x^0+y^0-\rho-||\vec{x}-\vec{y}|-\rho|}{2}\right)\Bigg] \\
&= \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{(x-y)^2\geqslant 0, x^0>y^0} \int_0^{y^0} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\
&\times \left[\boldsymbol{w}_2\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}-\rho\right)-\boldsymbol{w}_2\left(\frac{x^0+y^0-\rho-||\vec{x}-\vec{y}|-\rho|}{2}\right)\right] \\
&+ \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{(x-y)^2<0}\ 1_{x^0+y^0>|\vec{x}-\vec{y}|} \int_{\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}}^{y^0} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\
&\times \left[\boldsymbol{w}_2\left(\frac{x^0+y^0-\rho-|x^0-y^0+\rho|}{2}\right)-\boldsymbol{w}_2\left(\frac{x^0+y^0-\rho-||\vec{x}-\vec{y}|-\rho|}{2}\right)\right] \\
&= \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{(x-y)^2\geqslant 0, x^0>y^0} \int_0^{y^0} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\
&\times \Bigg[\boldsymbol{w}_2\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}-\rho\right) \qquad (2.125) \\
&\quad -\boldsymbol{w}_2\min\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}, \frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}-\rho\right)\Bigg] \\
&+ \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{(x-y)^2<0}\ 1_{x^0+y^0>|\vec{x}-\vec{y}|} \int_{\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}}^{y^0} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\
&\times \Bigg[\boldsymbol{w}_2\min\left(x^0, y^0-\rho\right)
\end{aligned}
$$

$$- \boldsymbol{w}_2 \min\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}, \frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}-\rho\right)\Bigg]. \tag{2.126}$$

This ends the calculation of (2.106).

2.2.4.1.1.3 Simplification of term (2.107). We next turn to (2.107). In case the reader is not interested in the details of the computation, the result can be found in (2.139). First we note that the restriction imposed by the first indicator function here is $\cos\theta > K_{x-y}(|\vec{y}'|)$ and the condition of the second indicator function is $\cos\theta < P_{x,y}(|\vec{y}'|)$. In order to to satisfy these conditions (and the restrictions of the regular range of integration) it is required that

$$\max(-1, K_{x-y}(|\vec{y}'|) < \cos\theta < \min(1, P_{x,y}(|\vec{y}'|)). \tag{2.127}$$

This leads us to ask which restrictions on $|\vec{y}'|$ are imposed by the conditions

$$K_{x-y}(|\vec{y}'|) \ < \ 1, \tag{2.128}$$
$$P_{x,y}(|\vec{y}'|) \ > \ -1, \tag{2.129}$$
$$K_{x-y}(|\vec{y}'|) \ < \ P_{x,y}(|\vec{y}'|). \tag{2.130}$$

These restrictions shall be computed next. With $|\vec{y}'| = \rho$, we find:

$$\begin{aligned}
&K_{x-y}(|\vec{y}'|) < 1 \\
&\iff \frac{(x-y)^2}{2\rho|\vec{x}-\vec{y}|} + \frac{x^0-y^0}{|\vec{x}-\vec{y}|} \ < \ 1 \\
&\iff (x-y)^2 \ < \ 2\rho(y^0-x^0+|\vec{x}-\vec{y}|) \\
&\iff \begin{cases} \rho > \frac{y^0-x^0-|\vec{x}-\vec{y}|}{2} & \text{for } |\vec{x}-\vec{y}| > x^0-y^0, \\ \rho < \frac{y^0-x^0-|\vec{x}-\vec{y}|}{2} & \text{for } |\vec{x}-\vec{y}| < x^0-y^0. \end{cases}
\end{aligned} \tag{2.131}$$

The second case in the last line is in conflict with $\rho > 0$, so we have to impose the first condition on (2.107). We continue with $P_{x,y}(\rho) > -1$.

$$\begin{aligned} &P_{x,y}(\rho) > -1 \\ \iff\ & \frac{(x^0+y^0)^2}{2\rho|\vec{x}-\vec{y}|} - \frac{|\vec{x}-\vec{y}|}{2\rho} - \frac{x^0+y^0}{|\vec{x}-\vec{y}|} \ > \ -1 \\ \iff\ & (x^0+y^0)^2 - |\vec{x}-\vec{y}|^2 \ > \ 2\rho(x^0+y^0-|\vec{x}-\vec{y}|) \\ \iff\ & \begin{cases} \rho < \frac{x^0+y^0+|\vec{x}-\vec{y}|}{2} & \text{for } x^0+y^0 > |\vec{x}-\vec{y}|, \\ \rho > \frac{x^0+y^0+|\vec{x}-\vec{y}|}{2} & \text{for } x^0+y^0 < |\vec{x}-\vec{y}|. \end{cases} \end{aligned} \tag{2.132}$$

The second case is in conflict with $\rho < y^0$, so we implement indicator functions corresponding only to the first case in (2.107). The third condition $K_{x-y}(\rho) < P_{x,y}(\rho)$ in fact does not impose any additional conditions. This can be seen as follows:

$$\begin{aligned} &K_{x-y}(\rho) \ < \ P_{x,y}(\rho) \\ \iff\ & \frac{(x-y)^2}{2\rho|\vec{x}-\vec{y}|} + \frac{x^0-y^0}{|\vec{x}-\vec{y}|} \ < \ \frac{(x^0+y^0)^2}{2\rho|\vec{x}-\vec{y}|} - \frac{|\vec{x}-\vec{y}|}{2\rho} - \frac{x^0+y^0}{|\vec{x}-\vec{y}|} \\ \iff\ & -2x^0y^0 + 4\rho x^0 \ < \ 2x^0y^0 \\ \iff\ & \rho < y^0, \end{aligned} \tag{2.133}$$

which always holds true.
Taking into account the computed restrictions, we arrive at:

$$\begin{aligned} (2.107) \overset{\cos\theta = w}{=} & \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{4(4\pi)^2} \int_0^{2\pi} d\varphi \int_0^{y^0} d\rho \int_{-1}^{1} dw\, 1_{K_{x-y}(\rho)<w<P_{x,y}(\rho)} \\ & \times 1_{\frac{y^0-x^0-|\vec{x}-\vec{y}|}{2}<\rho<\frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}} \frac{\boldsymbol{w}(y^0-\rho)\rho}{\sqrt{\rho^2+|\vec{x}-\vec{y}|^2+2\rho|\vec{x}-\vec{y}|w}} \\ & \times 1_{x^0-y^0<|\vec{x}-\vec{y}|<x^0+y^0} \boldsymbol{w}_1\left(\frac{x^0+y^0-\sqrt{\rho^2+|\vec{x}-\vec{y}|^2+2\rho|\vec{x}-\vec{y}|w}}{2}\right) \end{aligned}$$

$$= \frac{\lambda \|\psi\|_{\boldsymbol{w}} 2\pi}{4(4\pi)^2} 1_{x^0-y^0<|\vec{x}-\vec{y}|<x^0+y^0} \int_{\max\left(0, \frac{y^0-x^0-|\vec{x}-\vec{y}|}{2}\right)}^{\min\left(y^0, \frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}\right)} d\rho$$
$$\times \int_{\max(-1, K_{x-y}(\rho))}^{\min(1, P_{x,y}(\rho))} dw \, \frac{-2\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|}$$
$$\times \partial_w \boldsymbol{w}_2 \left(\frac{x^0+y^0-\sqrt{\rho^2+|\vec{x}-\vec{y}|^2+2\rho|\vec{x}-\vec{y}|w}}{2} \right)$$
$$= \frac{\lambda \|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{x^0-y^0<|\vec{x}-\vec{y}|<x^0+y^0} \int_{\max\left(0, \frac{y^0-x^0-|\vec{x}-\vec{y}|}{2}\right)}^{\min\left(y^0, \frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}\right)} d\rho \, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|}$$
$$\times \left[\boldsymbol{w}_2 \left(\frac{x^0+y^0-\rho-\sqrt{\rho^2+|\vec{x}-\vec{y}|^2+2\rho|\vec{x}-\vec{y}| \max(-1, K_{x-y}(\rho))}}{2} \right) \right.$$
$$\left. -\boldsymbol{w}_2 \left(\frac{x^0+y^0-\rho-\sqrt{\rho^2+|\vec{x}-\vec{y}|^2+2\rho|\vec{x}-\vec{y}| \min(1, P_{x,y}(\rho))}}{2} \right) \right]. \quad (2.134)$$

At this point, the expressions look quite formidable. We can, however, achieve significant simplifications by inserting the functional form of $K_{x,y}(\rho)$ and $P_{x,y}(\rho)$ as in (2.124) and (2.116). This yields:

$$(2.107) =$$
$$\frac{\lambda \|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{x^0-y^0<|\vec{x}-\vec{y}|<x^0+y^0} \int_{\max\left(0, \frac{y^0-x^0-|\vec{x}-\vec{y}|}{2}\right)}^{\min\left(y^0, \frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}\right)} d\rho \, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|}$$
$$\times \left[\boldsymbol{w}_2 \left(\frac{x^0+y^0-\rho-\max(||\vec{x}-\vec{y}|-\rho|, |x^0-y^0+\rho|)}{2} \right) \right.$$
$$\left. -\boldsymbol{w}_2 \left(\frac{x^0+y^0-\rho-\min(|\vec{x}-\vec{y}|+\rho, x^0+y^0-\rho)}{2} \right) \right] \quad (2.135)$$

Now we simplify the arguments of the $\boldsymbol{w}_2$-functions. For the first one, we have:

$$\begin{aligned} &x^0+y^0-\rho-\max(||\vec{x}-\vec{y}|-\rho|, |x^0-y^0+\rho|) \\ &= x^0+y^0-\rho-\max(|\vec{x}-\vec{y}|-\rho, \rho-|\vec{x}-\vec{y}|, x^0-y^0+\rho, y^0-\rho-x^0) \\ &= \min\big(x^0+y^0-|\vec{x}-\vec{y}|, x^0+y^0+|\vec{x}-\vec{y}|-2\rho, 2(y^0-\rho), 2x^0\big)\,. \end{aligned} \tag{2.136}$$

For the second one we get

$$x^0+y^0-\rho-\min(|\vec{x}-\vec{y}|+\rho, x^0+y^0-\rho) = \max(x^0+y^0-|\vec{x}-\vec{y}|-2\rho, 0). \tag{2.137}$$

Using this in (2.135), we find:

$$\begin{aligned} (2.107) = \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{x^0-y^0<|\vec{x}-\vec{y}|<x^0+y^0} \int_{\max\left(0,\frac{y^0-x^0-|\vec{x}-\vec{y}|}{2}\right)}^{\min\left(y^0,\frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}\right)} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\ \times \left[\boldsymbol{w}_2 \min\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}, \frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}-\rho, y^0-\rho, x^0\right)\right. \\ \left. -\boldsymbol{w}_2 \max\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}-\rho, 0\right)\right]. \end{aligned} \tag{2.138}$$

As in the consideration below (2.117), we split the expression into separate terms with $(x-y)^2 \gtrless 0$. Using $y^0 \gtrless x^0 + |\vec{x}-\vec{y}|$, we can simplify the expressions involving the minimum. This results in:

$$\begin{aligned} (2.107) = \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{(x-y)^2\geqslant 0, y^0>x^0} \int_{\frac{y^0-x^0-|\vec{x}-\vec{y}|}{2}}^{\frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\ \times \left[\boldsymbol{w}_2 \min\left(\frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}-\rho, x^0\right) - \boldsymbol{w}_2 \max\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}-\rho, 0\right)\right] \\ + \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{16\pi} 1_{(x-y)^2<0, |\vec{x}-\vec{y}|<x^0+y^0} \int_0^{y^0} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\ \times \left[\boldsymbol{w}_2 \min\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}, y^0-\rho\right) - \boldsymbol{w}_2 \max\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}-\rho, 0\right)\right]. \end{aligned} \tag{2.139}$$

This concludes the calculation of (2.107).

2.2.4.1.1.4 Summary of the first estimate. We have obtained the following bound for $|A_0\psi|(x,y)$:

$$\begin{aligned}
&\frac{16\pi}{\lambda\|\psi\|_{\boldsymbol{w}}}|A_0\psi|(x,y) \leqslant 1_{(x-y)^2\geqslant 0, x^0>y^0} \int_0^{y^0} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\
&\times \Bigg[\boldsymbol{w}_2\left(\frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}\right) \\
&\qquad -\boldsymbol{w}_2\max\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}, \frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}-\rho\right)\Bigg] \\
&+ 1_{(x-y)^2<0}\ 1_{x^0+y^0>|\vec{x}-\vec{y}|} \int_{\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}}^{y^0} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\
&\times \Bigg[\boldsymbol{w}_2\max\big(x^0, y^0-\rho\big) \\
&\qquad - \boldsymbol{w}_2\max\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}, \frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}-\rho\right)\Bigg] \\
&+ 1_{(x-y)^2\geqslant 0, x^0>y^0} \int_0^{y^0} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\
&\times \Bigg[\boldsymbol{w}_2\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}-\rho\right) \\
&\qquad - \boldsymbol{w}_2\min\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}, \frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}-\rho\right)\Bigg] \\
&+ 1_{(x-y)^2<0}\ 1_{x^0+y^0>|\vec{x}-\vec{y}|} \int_{\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}}^{y^0} d\rho\, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\
&\times \Bigg[\boldsymbol{w}_2\min\big(x^0, y^0-\rho\big)
\end{aligned}$$

$$
\begin{aligned}
&- \boldsymbol{w}_2 \min\left(\frac{x^0 + y^0 - |\vec{x}-\vec{y}|}{2}, \frac{x^0+y^0+ |\vec{x}-\vec{y}|}{2} - \rho\right) \Bigg] \\
&+ 1_{(x-y)^2 \geqslant 0, y^0 > x^0} \int_{\frac{y^0-x^0-|\vec{x}-\vec{y}|}{2}}^{\frac{x^0+y^0+|\vec{x}-\vec{y}|}{2}} d\rho \, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\
&\times \Bigg[\boldsymbol{w}_2 \min\left(\frac{x^0+y^0+ |\vec{x}-\vec{y}|}{2} - \rho, x^0\right) \\
&\quad - \boldsymbol{w}_2 \max\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2} - \rho, 0\right) \Bigg] \\
&+ 1_{(x-y)^2 < 0, |\vec{x}-\vec{y}| < x^0+y^0} \int_0^{y^0} d\rho \, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \\
&\times \Bigg[\boldsymbol{w}_2 \min\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2}, y^0-\rho\right) \\
&\quad - \boldsymbol{w}_2 \max\left(\frac{x^0+y^0-|\vec{x}-\vec{y}|}{2} - \rho, 0\right) \Bigg]. \qquad (2.140)
\end{aligned}
$$

In order to simplify the notation, we introduce the variables

$$
\xi^+ := \frac{x^0 + y^0 + |\vec{x} - \vec{y}|}{2}, \qquad (2.141)
$$

$$
\xi^- := \frac{x^0 + y^0 - |\vec{x} - \vec{y}|}{2}. \qquad (2.142)
$$

Moreover, we collect terms with the same indicator functions. This results in:

$$
\begin{aligned}
&\frac{16\pi}{\lambda \|\psi\|_{\boldsymbol{w}}} |A_0 \psi|(x,y) \leqslant \\
&1_{(x-y)^2<0, \xi^->0} \int_0^{y^0} d\rho \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \Big[\boldsymbol{w}_2 \min(\xi^-, y^0-\rho) - \boldsymbol{w}_2 \max(\xi^- - \rho, 0)
\end{aligned}
$$

$$+ 1_{\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}<\rho}\big(\boldsymbol{w}_2(x^0) + \boldsymbol{w}_2(y^0-\rho) - \boldsymbol{w}_2(\xi^-) - \boldsymbol{w}_2(\xi^+-\rho)\big)\Big] \tag{2.143}$$

$$+ 1_{(x-y)^2\geqslant 0, x^0>y^0} \int_0^{y^0} d\rho \, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|}\big[\boldsymbol{w}_2(\xi^+) + \boldsymbol{w}_2(\xi^- -\rho) - \boldsymbol{w}_2(\xi^-) - \boldsymbol{w}_2(\xi^+-\rho)\big] \tag{2.144}$$

$$+ 1_{(x-y)^2\geqslant 0, y^0>x^0} \int_{(y^0-x^0-|\vec{x}-\vec{y}|)/2}^{\xi^+} d\rho \, \frac{\boldsymbol{w}(y^0-\rho)}{|\vec{x}-\vec{y}|} \times \big[\boldsymbol{w}_2 \min(\xi^+-\rho, x^0) - \boldsymbol{w}_2 \max(\xi^- -\rho, 0)\big]. \tag{2.145}$$

This estimate is an important stepping stone in the proof. Except for special weight functions, the resulting expressions are too complicated to be computed explicitly. We therefore continue with further estimates. The main difficulty in these estimates is that the $1/|\vec{x}-\vec{y}|$ singularity in the expressions needs to be compensated by the integrand and that this cancellation needs to be preserved by the respective estimate. Fortunately, the mean value theorem turns out suitable to provide such estimates.

2.2.4.1.1.5 Simplification of (2.143)- (2.145). First, we note that since $\boldsymbol{w}, \boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are monotonously increasing and since $\xi^- \leqslant \xi^+$, we have in (2.144):

$$\boldsymbol{w}_2(\xi^- - \rho) - \boldsymbol{w}_2(\xi^+ - \rho) \leqslant 0. \tag{2.146}$$

As the remaining terms in (2.144) still vanish in the limit $|\vec{x}-\vec{y}| \to 0$, we may replace this difference by zero to obtain a suitable estimate. Similarly, a brief calculation shows that we have $\xi^+ > y^0$ for $(x-y)^2 < 0$. It follows that:

$$\boldsymbol{w}_2(y^0 - \rho) - \boldsymbol{w}_2(\xi^+ - \rho) < 0. \tag{2.147}$$

We shall use this in (2.143).
Further simplifications can be obtained using the mean value theorem. We begin with the expression in the square brackets in (2.145). The mean value theorem then implies that there is a $\chi \in [\max(\xi^- - \rho, 0), \min(\xi^+ - \rho, x^0)]$ such that

$$\begin{aligned} &\boldsymbol{w}_2 \min(\xi^+ - \rho, x^0) - \boldsymbol{w}_2 \max(\xi^- - \rho, 0) \\ &\qquad = \big[\min(\xi^+ - \rho, x^0) - \max(\xi^- - \rho, 0)\big] \boldsymbol{w}_1(\chi). \end{aligned} \tag{2.148}$$

Therefore, we have:

$$\begin{aligned} &\boldsymbol{w}_2 \min(\xi^+ - \rho, x^0) - \boldsymbol{w}_2 \max(\xi^- - \rho, 0) \\ &\quad \leqslant \ \min(\xi^+ - \xi^-, \xi^+ - \rho, x^0 - \xi^- + \rho, x^0)\, \boldsymbol{w}_1 \min(\xi^+ - \rho, x^0) \\ &\quad \leqslant \ |\vec{x} - \vec{y}|\, \boldsymbol{w}_1 \min(\xi^+ - \rho, x^0) \ \leqslant \ |\vec{x} - \vec{y}|\, \boldsymbol{w}_1(x^0). \end{aligned} \tag{2.149}$$

Note that the factor $|\vec{x} - \vec{y}|$ exactly compensates the $1/|\vec{x} - \vec{y}|$ singularity. This is the main reason the mean value theorem is so useful here.
Analogously we find for the expression in the square bracket in the first line of (2.143):

$$\begin{aligned} &\boldsymbol{w}_2 \min(\xi^-, y^0 - \rho) - \boldsymbol{w}_2 \max(\xi^- - \rho, 0) \\ &\quad \leqslant \ \big[\min(\xi^-, y^0 - \rho) - \max(\xi^- - \rho, 0)\big] \boldsymbol{w}_1 \min(\xi^-, y^0 - \rho) \\ &\quad = \ \min(\rho, \xi^-, y^0 - \xi^-, y^0 - \rho)\, \boldsymbol{w}_1 \min(\xi^-, y^0 - \rho) \\ &\quad \leqslant \ (y^0 - \xi^-)\, \boldsymbol{w}_1 \min(\xi^-, y^0 - \rho) \\ &\quad \leqslant \ |\vec{x} - \vec{y}|\, \boldsymbol{w}_1 \min(\xi^-, y^0 - \rho), \end{aligned} \tag{2.150}$$

where we have used that the further restriction of that term, $(x-y)^2 < 0$, implies $|\vec{x} - \vec{y}| > |x^0 - y^0| \geqslant y^0 - x^0$.

With these considerations, we obtain a rougher but simpler estimate than (2.143)- (2.145):

$$\frac{16\pi}{\lambda\|\psi\|_{\boldsymbol{w}}}|A_0\psi|(x,y)$$
$$\leqslant \ 1_{(x-y)^2<0,\xi^->0}\int_0^{y^0} d\rho\ \boldsymbol{w}(y^0-\rho)\Big[\boldsymbol{w}_1\min(\xi^-,y^0-\rho) \tag{2.151}$$
$$+\,1_{\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}<\rho}\frac{\boldsymbol{w}_2(x^0)-\boldsymbol{w}_2(\xi^-)}{|\vec{x}-\vec{y}|}\Big] \tag{2.152}$$
$$+\,1_{(x-y)^2\geqslant 0,x^0>y^0}\frac{\boldsymbol{w}_2(\xi^+)-\boldsymbol{w}_2(\xi^-)}{|\vec{x}-\vec{y}|}\int_0^{y^0} d\rho\ \boldsymbol{w}(y^0-\rho) \tag{2.153}$$
$$+\,1_{(x-y)^2\geqslant 0,y^0>x^0}\,\boldsymbol{w}_1(x^0)\int_{\frac{y^0-x^0-|\vec{x}-\vec{y}|}{2}}^{\xi^+} d\rho\ \boldsymbol{w}(y^0-\rho). \tag{2.154}$$

Next, we continue estimating these terms separately so that only expressions without integrals remain.

2.2.4.1.1.6 Further estimate of (2.151). Using the monotonicity of $\boldsymbol{w}_1$ as well as $\min(\xi^-,y^0-\rho)\leqslant\xi^-$, we find:

$$\begin{aligned}(2.151)\ &\leqslant\ 1_{(x-y)^2<0,\xi^->0}\,\boldsymbol{w}_1(\xi^-)\int_0^{y^0} ds\ \boldsymbol{w}(s)\\ &=\ 1_{(x-y)^2<0,\xi^->0}\,\boldsymbol{w}_1(\xi^-)\boldsymbol{w}_1(y^0).\end{aligned} \tag{2.155}$$

For the constraints given by the indicator function, we have $\xi^-<x^0$. Thus:

$$(2.151)\leqslant 1_{(x-y)^2<0,\xi^->0}\,\boldsymbol{w}_1(x^0)\boldsymbol{w}_1(y^0). \tag{2.156}$$

2.2.4.1.1.7 Further estimate of (2.152). We have:

$$\begin{aligned}(2.152) &= 1_{(x-y)^2<0,\xi^->0}\,\frac{\boldsymbol{w}_2(x^0)-\boldsymbol{w}_2(\xi^-)}{|\vec{x}-\vec{y}|}\int_{\frac{y^0-x^0+|\vec{x}-\vec{y}|}{2}}^{y_0} d\rho\,\boldsymbol{w}(y^0-\rho)\\ &= 1_{(x-y)^2<0,\xi^->0}\,\frac{\boldsymbol{w}_2(x^0)-\boldsymbol{w}_2(\xi^-)}{|\vec{x}-\vec{y}|}\int_0^{\xi^-} ds\,\boldsymbol{w}(s)\\ &= 1_{(x-y)^2<0,\xi^->0}\,\frac{\boldsymbol{w}_2(x^0)-\boldsymbol{w}_2(\xi^-)}{|\vec{x}-\vec{y}|}\big[\boldsymbol{w}_1(\xi^-)-\underbrace{\boldsymbol{w}_1(0)}_{=0}\big].\end{aligned} \tag{2.157}$$

Applying the mean value theorem to $\boldsymbol{w}_2$ in the interval $[\xi^-, x^0]$ (note that here $\xi^- < x^0$), we obtain that:

$$(2.152) \leqslant 1_{(x-y)^2<0,\xi^->0}\,\frac{x^0-\xi^-}{|\vec{x}-\vec{y}|}\,\boldsymbol{w}_1(x^0)\boldsymbol{w}_1(\xi^-). \tag{2.158}$$

Next, we use that $\frac{x^0-\xi^-}{|\vec{x}-\vec{y}|} = \frac{x^0-y^0+|\vec{x}-\vec{y}|}{2|\vec{x}-\vec{y}|} \leqslant 1$ as $|x^0-y^0| < |\vec{x}-\vec{y}|$. Thus:

$$(2.152) \leqslant 1_{(x-y)^2<0,\xi^->0}\,\boldsymbol{w}_1(x^0)\boldsymbol{w}_1(\xi^-). \tag{2.159}$$

Using also that for the given constrains $\xi^- < y^0$, we finally obtain:

$$(2.152) \leqslant 1_{(x-y)^2<0,\xi^->0}\,\boldsymbol{w}_1(x^0)\boldsymbol{w}_1(y^0). \tag{2.160}$$

2.2.4.1.1.8 Further estimate of (2.153). Here, we can directly carry out the remaining integral using the definition of $\boldsymbol{w}_1$ as the integral of g:

$$(2.153) = 1_{(x-y)^2\geqslant 0,x^0>y^0}\,\frac{\boldsymbol{w}_2(\xi^+)-\boldsymbol{w}_2(\xi^-)}{|\vec{x}-\vec{y}|}\boldsymbol{w}_1(y^0). \tag{2.161}$$

Next, we apply the mean value theorem to $\boldsymbol{w}_2$ in the interval $[\xi^-, \xi^+]$ noting that $\xi^+ - \xi^- = |\vec{x}-\vec{y}|$. This implies:

$$(2.153) \leqslant 1_{(x-y)^2>0,x^0>y^0}\,\boldsymbol{w}_1(\xi^+)\boldsymbol{w}_1(y^0). \tag{2.162}$$

Next, we note that $(x-y)^2 \geqslant 0 \Leftrightarrow |x^0 - y^0| \geqslant |\vec{x} - \vec{y}|$. Together with $x^0 > y^0$, we obtain $x^0 \geqslant y^0 + |\vec{x} - \vec{y}|$ and therefore:

$$\xi^+ = \frac{x^0 + y^0 + |\vec{x} - \vec{y}|}{2} \leqslant x^0. \tag{2.163}$$

Thus, we obtain:

$$(2.153) \leqslant 1_{(x-y)^2 \geqslant 0, x^0 > y^0}\, \boldsymbol{w}_1(x^0)\boldsymbol{w}_1(y^0). \tag{2.164}$$

2.2.4.1.1.9 Further estimate of (2.154). Here, we carry out the remaining integral as well.

$$\begin{aligned}(2.154) &\leqslant 1_{(x-y)^2 > 0, y^0 > x^0}\, \boldsymbol{w}_1(x^0)[\boldsymbol{w}_1(\xi^+) - \boldsymbol{w}_1((y^0 - x^0 - |\vec{x} - \vec{y}|)/2)] \\ &\leqslant 1_{(x-y)^2 > 0, y^0 > x^0}\, \boldsymbol{w}_1(x^0)\boldsymbol{w}_1(y^0). \end{aligned} \tag{2.165}$$

as $\xi^+ \leqslant y^0$.

2.2.4.1.1.10 Summary of the result. Gathering the terms (2.156), (2.160), (2.164) and (2.165) yields:

$$\begin{aligned}&\frac{16\pi}{\lambda \|\psi\|_{\boldsymbol{w}}} |A_0 \psi|(x,y) \\ &\leqslant \boldsymbol{w}_1(x^0)\boldsymbol{w}_1(y^0) \left(2 \times 1_{(x-y)^2 < 0, \xi^- > 0} + 1_{(x-y)^2 \geqslant 0, x^0 > y^0} + 1_{(x-y)^2 \geqslant 0, y^0 > x^0}\right). \end{aligned} \tag{2.166}$$

Considering that the conditions in different indicator functions are mutually exclusive, we finally obtain:

$$\frac{16\pi}{\lambda \|\psi\|_{\boldsymbol{w}}} |A_0 \psi|(x,y) \leqslant 2\boldsymbol{w}_1(x^0)\boldsymbol{w}_1(y^0). \tag{2.167}$$

Dividing by $\boldsymbol{w}(x^0)\boldsymbol{w}(y^0)$, taking the supremum over $x, y \in \mathbb{M}_0^+$ and factorizing into one-dimensional suprema finally yields the claim (2.89).

2.2.4.1.2 Estimate of the mixed terms (2.90) and (2.91). We focus on A_2 first, starting from its definition (2.61). We take the absolute value and make use of $|\psi(x,y)| \leqslant \boldsymbol{w}(x^0)\boldsymbol{w}(y^0)\,\|\psi\|_{\boldsymbol{w}}$. Moreover, we use:

$$|J_1(t)/t| \leqslant \frac{1}{2}. \tag{2.168}$$

This yields:

$$\begin{aligned}
|A_2\psi|(x,y) \leqslant{}& \frac{\lambda\, m_2^2\,\|\psi\|_{\boldsymbol{w}}}{4(4\pi)^3} \int d^3\vec{x}' \int d^3\vec{y}'\; \frac{H(x^0-|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}\,\frac{\boldsymbol{w}(x^0-|\vec{x}-\vec{x}'|)}{|\vec{x}'-\vec{y}'|}\\
&\times \big[\boldsymbol{w}(x^0-|\vec{x}-\vec{x}'|+|\vec{x}'-\vec{y}'|)H(x^0-|\vec{x}-\vec{x}'|+|\vec{x}'-\vec{y}'|)\\
&\quad\times H(y^0-x^0+|\vec{x}-\vec{x}'|-|\vec{x}'-\vec{y}'|-|\vec{y}-\vec{y}'|)\\
&+\boldsymbol{w}(x^0-|\vec{x}-\vec{x}'|-|\vec{x}'-\vec{y}'|)H(x^0-|\vec{x}-\vec{x}'|-|\vec{x}'-\vec{y}'|)\\
&\quad\times H(y^0-x^0+|\vec{x}-\vec{x}'|+|\vec{x}'-\vec{y}'|-|\vec{y}-\vec{y}'|)\big]\,.
\end{aligned} \tag{2.169}$$

As the remaining singularities are independent of each other for a suitable choice of integration variables (see below), we are left with an integrable function on a finite domain.

The next task is to bring the expressions into a simpler form. One possibility to do this is to use

$$\begin{aligned}
&H(y^0-x^0+|\vec{x}-\vec{x}'|+|\vec{x}'-\vec{y}'|-|\vec{y}-\vec{y}'|)\\
&\qquad\leqslant H(y^0-x^0+|\vec{x}-\vec{x}'|+|\vec{x}'-\vec{y}'|)
\end{aligned} \tag{2.170}$$

for the second Heaviside function in the second summand. The first Heaviside function in the first summand equals 1 anyway, as $|\vec{x}-\vec{x}'| < x^0$. We furthermore use

$$\begin{aligned}
&H(y^0-x^0+|\vec{x}-\vec{x}'|-|\vec{x}'-\vec{y}'|-|\vec{y}-\vec{y}'|)\\
&\qquad\leqslant H(y^0-x^0+|\vec{x}-\vec{x}'|-|\vec{x}'-\vec{y}'|),
\end{aligned} \tag{2.171}$$

as it simplifies the domain of integration. Overall, the domain of integration remains bounded. Introducing $\vec{z}_1 = \vec{x}-\vec{x}'$, $\vec{z}_2 = \vec{x}'-\vec{y}'$ (with

Jacobi determinant of modulus 1) and using spherical coordinates for $\vec{z}_2$, this leads to:

$$
\begin{aligned}
&|A_2\psi|(x,y)\frac{4(4\pi)^3}{\lambda\, m_2^2\,\|\psi\|_{\boldsymbol{w}}} \\
&\leqslant \int_{B_{x^0}(0)} d^3\vec{z}_1 4\pi \int_0^{\max(0,y^0-x^0+|\vec{z}_1|)} d^3\vec{z}_2 \frac{\boldsymbol{w}(x^0-|\vec{z}_1|)\boldsymbol{w}(x^0-|\vec{z}_1|+|\vec{z}_2|)}{|\vec{z}_1||\vec{z}_2|}|\vec{z}_2|^2 \\
&+\int_{B_{x^0}(0)} d^3\vec{z}_1 4\pi \int_{\max(0,x^0-y^0-|\vec{z}_1|)}^{x^0-|\vec{z}_1|} d|\vec{z}_2| \frac{\boldsymbol{w}(x^0-|\vec{z}_1|)\boldsymbol{w}(x^0-|\vec{z}_1|-|\vec{z}_2|)}{|\vec{z}_1||\vec{z}_2|}|\vec{z}_2|^2.
\end{aligned}
\tag{2.172}
$$

Using spherical coordinates also for $\vec{z}_1$, this can be further simplified to:

$$
|A_2\psi|(x,y)\frac{16\pi}{\lambda m_2^2\,\|\psi\|_{\boldsymbol{w}}}
$$

$$
\leqslant \int_0^{x^0} dr_1 \int_0^{\max(0,y^0-x^0+r_1)} dr_2\; r_1 r_2\, \boldsymbol{w}(x^0-r_1)\boldsymbol{w}(x^0-r_1+r_2) \tag{2.173}
$$

$$
+ \int_0^{x^0} dr_1 \int_{\max(0,x^0-r_1-y^0)}^{x^0-r_1} dr_2\; r_1 r_2\; \boldsymbol{w}(x^0-r_1)\boldsymbol{w}(x^0-r_1-r_2). \tag{2.174}
$$

Our next task is to simplify the remaining integrals. We begin with making the change of variables $\rho = x^0 - r_1$:

$$
\begin{aligned}
&|A_2\psi|(x,y)\frac{16\pi}{\lambda m_2^2\,\|\psi\|_{\boldsymbol{w}}} \\
&\leqslant \int_0^{x^0} d\rho\; (x^0-\rho)\boldsymbol{w}(\rho) \int_0^{\max(0,y^0-\rho)} dr_2\; r_2\, \boldsymbol{w}(\rho+r_2) \\
&+ \int_0^{x^0} d\rho\; (x^0-\rho)\boldsymbol{w}(\rho) \int_{\max(0,\rho-y^0)}^{\rho} dr_2\; r_2\, \boldsymbol{w}(\rho-r_2).
\end{aligned}
\tag{2.175}
$$

Now we consider the r_2-integral in both terms and integrate by parts. This yields:

$$\int_0^{\max(0,y^0-\rho)} dr_2\ r_2\, \boldsymbol{w}(\rho+r_2) = \max(0,y^0-\rho)\boldsymbol{w}_1(y^0) \\ -\boldsymbol{w}_2(\max(\rho,y^0))+\boldsymbol{w}_2(\rho), \quad (2.176)$$

$$\int_{\max(0,\rho-y^0)}^{\rho} dr_2\ r_2\, \boldsymbol{w}(\rho-r_2) = \max(0,\rho-y^0)\boldsymbol{w}_1(y^0) \\ +\boldsymbol{w}_2(\min(\rho,y^0)). \quad (2.177)$$

We now use $-\boldsymbol{w}_2(\max(\rho,y^0))+\boldsymbol{w}_2(\rho) \leqslant 0$ in the first term and then re-insert the resulting estimate into (2.175). Considering also $\max(0,y^0-\rho)+\max(0,\rho-y^0) = |y^0-\rho|$, this yields:

$$|A_2\psi|(x,y)\,\frac{16\pi}{\lambda\, m_2^2\, \|\psi\|_{\boldsymbol{w}}} \\ \leqslant \int_0^{x^0} d\rho\ (x^0-\rho)\boldsymbol{w}(\rho)\left[|y^0-\rho|\boldsymbol{w}_1(y^0)+\boldsymbol{w}_2(\min(\rho,y^0))\right] \quad (2.178)$$

The first summand of (2.178) can be treated as follows. First we focus on whether $x^0 > y^0$ or $x^0 \leqslant y^0$. In the first case, we then differentiate between the cases $\rho < y^0$ and $\rho \geqslant y^0$ and split up the integrals accordingly. This yields:

$$\int_0^{x^0} d\rho\ (x^0-\rho)\boldsymbol{w}(\rho)|y^0-\rho|\boldsymbol{w}_1(y^0)$$

$$= \boldsymbol{w}_1(y^0)\,1_{x^0>y^0}\int_0^{y^0} d\rho\ (x^0-\rho)(y^0-\rho)\boldsymbol{w}(\rho) \quad (2.179)$$

$$-\boldsymbol{w}_1(y^0)\,1_{x^0>y^0}\int_{y^0}^{x^0} d\rho\ (x^0-\rho)(y^0-\rho)\boldsymbol{w}(\rho) \quad (2.180)$$

$$+\boldsymbol{w}_1(y^0)\,1_{x^0\leqslant y^0}\int_0^{x^0} d\rho\ (x^0-\rho)(y^0-\rho)\boldsymbol{w}(\rho). \quad (2.181)$$

We now calculate these terms separately using integration by parts. The first term yields:

$$(2.179) \;=\; \boldsymbol{w}_1(y^0)1_{x^0>y^0}\left[(x^0-y^0)\boldsymbol{w}_2(y^0)+2\boldsymbol{w}_3(y^0)\right]. \qquad (2.182)$$

We turn to (2.180):

$$(2.180) = -\boldsymbol{w}_1(y^0)1_{x^0>y^0}\big[(y^0-x^0)(\boldsymbol{w}_2(x^0) + \boldsymbol{w}_2(y^0))+2\boldsymbol{w}_3(x^0)-2\boldsymbol{w}_3(y^0)\big]. \qquad (2.183)$$

The result of (2.181) is:

$$(2.181) \;=\; \boldsymbol{w}_1(y^0)1_{x^0\leqslant y^0}\left[(y^0-x^0)\boldsymbol{w}_2(x^0)+2\boldsymbol{w}_3(x^0)\right]. \qquad (2.184)$$

Gathering the terms (2.182), (2.183) and (2.184) yields:

$$\begin{aligned}
|A_2\psi|(x,y)\frac{16\pi}{\lambda\, m_2^2\,\|\psi\|_{\boldsymbol{w}}}
&\leqslant \boldsymbol{w}_1(y^0)1_{x^0>y^0}\left[2(x^0-y^0)\boldsymbol{w}_2(y^0)+4\boldsymbol{w}_3(y^0)-2\boldsymbol{w}_3(x^0)\right]\\
&\quad+\boldsymbol{w}_1(y^0)|x^0-y^0|\boldsymbol{w}_2(x^0)+2\boldsymbol{w}_1(y^0)1_{x^0\leqslant y^0}\boldsymbol{w}_3(x^0)\\
&\leqslant 2\boldsymbol{w}_1(y^0)|x^0-y^0|\boldsymbol{w}_2(x^0)+2\boldsymbol{w}_1(y^0)\boldsymbol{w}_3(x^0)1_{x^0<y^0}\\
&\quad+\boldsymbol{w}_1(y^0)|x^0-y^0|\boldsymbol{w}_2(x^0)+2\boldsymbol{w}_1(y^0)\boldsymbol{w}_3(x^0)1_{x^0\leqslant y^0}\\
&= 3\boldsymbol{w}_1(y^0)|x^0-y^0|\boldsymbol{w}_2(x^0)+2\boldsymbol{w}_1(y^0)\boldsymbol{w}_3(x^0)\\
&\leqslant 3(x^0+y^0)\boldsymbol{w}_1(y^0)\boldsymbol{w}_2(x^0)+2\boldsymbol{w}_1(y^0)\boldsymbol{w}_3(x^0). \qquad (2.185)
\end{aligned}$$

In order to obtain $\|A_2\psi\|_{\boldsymbol{w}}$, we divide by $\boldsymbol{w}(x^0)\boldsymbol{w}(y^0)$ and take the supremum over $x,y\in\mathbb{M}_0^+$. This results in:

$$\begin{aligned}
&\sup_{\psi\in\mathcal{S}((\mathbb{M}_0^+)^2)}\frac{\|A_2\psi\|_{\boldsymbol{w}}}{\|\psi\|_{\boldsymbol{w}}} \qquad (2.186)\\
&\leqslant \frac{\lambda\, m_2^2}{16\pi}\left(3\sup_{x^0,y^0\geqslant 0}\frac{(x^0+y^0)\boldsymbol{w}_2(x^0)\,\boldsymbol{w}_1(y^0)}{\boldsymbol{w}(x^0)\boldsymbol{w}(y^0)}+2\sup_{x^0,y^0\geqslant 0}\frac{\boldsymbol{w}_3(x^0)\boldsymbol{w}_1(y^0)}{\boldsymbol{w}(x^0)\boldsymbol{w}(y^0)}\right).
\end{aligned}$$

After factorizing the two-dimensional suprema into one-dimensional ones, this exactly yields the claim, (2.91).
For the operator A_1, we find analogously:

$$\sup_{\psi\in\mathcal{S}((\mathbb{M}_0^+)^2)} \frac{\|A_1\psi\|_{\boldsymbol{w}}}{\|\psi\|_{\boldsymbol{w}}} \leqslant \frac{\lambda m_1^2}{16\pi}\left(3\sup_{x^0,y^0\geqslant 0}\frac{(x^0+y^0)\boldsymbol{w}_1(x^0)\boldsymbol{w}_2(y^0)}{\boldsymbol{w}(x^0)\boldsymbol{w}(y^0)} + 2\sup_{x^0,y^0\geqslant 0}\frac{\boldsymbol{w}_1(x^0)\boldsymbol{w}_3(y^0)}{\boldsymbol{w}(x^0)\boldsymbol{w}(y^0)}\right). \tag{2.187}$$

which, after factorization into one-dimensional suprema, yields the claim (2.90).

2.2.4.1.3 Estimate of the mass-mass term (2.92). We begin with (2.64). Taking the absolute value and using $|\psi(x,y)| \leqslant \|\psi\|_{\boldsymbol{w}}\, \boldsymbol{w}(x^0)\boldsymbol{w}(y^0)$ as well as $|J_1(t)/t| \leqslant \frac{1}{2}$ yields:

$$\begin{aligned} &|A_{12}\psi|(x,y) \\ &\leqslant \frac{\lambda\, m_1 m_2\, \|\psi\|_{\boldsymbol{w}}}{8(4\pi)^3}\int_0^\infty dx'^0\int d^3\vec{x}'\int_0^\infty dy'^0\int_0^{2\pi}d\varphi\int_0^\pi d\vartheta\,\sin(\vartheta)|x'^0-y'^0| \\ &\qquad\times H(x^0-x'^0-|\vec{x}-\vec{x}'|)H(y^0-y'^0-|\vec{y}-\vec{x}'+\vec{z}|) \\ &\qquad\qquad\times \boldsymbol{w}(x'^0)\boldsymbol{w}(y'^0)\Big|_{|\vec{z}|=|x^{0\prime}-y^{0\prime}|}, \end{aligned} \tag{2.188}$$

where, we recall, $\vec{z}$ is the variable for which the spherical coordinates are used.
Next, we consider the ranges of integration which the Heaviside functions imply. $H(x^0-x'^0-|\vec{x}-\vec{x}'|)$ restricts the range of integration of $\vec{x}'$ to the ball $B_{x^0-x'^0}(\vec{x})$ and the range of the x'^0-integration to $(0,x^0)$. The range implied by the second Heaviside function is more complicated. We therefore use the estimate

$$H(y^0-y'^0-|\vec{y}-\vec{x}'+\vec{z}|) \leqslant H(y^0-y'^0). \tag{2.189}$$

Then $y'^0 \in (0, y^0)$ and there is no further restriction for the angular variables. We obtain:

$$|A_{12}\psi|(x,y) \leqslant \frac{\lambda\, m_1 m_2\, \|\psi\|_{\boldsymbol{w}}}{8(4\pi)^3} \int_0^{x^0} dx'^0 \int_{B_{x^0-x'^0}(\vec{x})} d^3\vec{x}' \int_0^{y^0} dy'^0 \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \\ \times \sin(\vartheta)|x'^0 - y'^0|\, \boldsymbol{w}(x'^0)\boldsymbol{w}(y'^0). \quad (2.190)$$

Performing the $\vec{x}'$-integration, as well as the angular integrals yields:

$$|A_{12}\psi|(x,y) \\ \leqslant \frac{\lambda\, m_1 m_2\, \|\psi\|_{\boldsymbol{w}}}{96\pi} \int_0^{x^0} dx'^0 |x^0 - x'^0|^3 \boldsymbol{w}(x'^0) \int_0^{y^0} dy'^0 |x'^0 - y'^0|\, \boldsymbol{w}(y'^0). \quad (2.191)$$

Our next task is to estimate the term explicitly in terms of the functions $\boldsymbol{w}_n$ only. To do so, we use

$$|x'^0 - y'^0| \leqslant x'^0 + y'^0. \quad (2.192)$$

This yields:

$$|A_{12}\psi|(x,y) \\ \leqslant \frac{\lambda m_1 m_2 \|\psi\|_{\boldsymbol{w}}}{96\pi} \int_0^{x^0} dx'^0 |x^0 - x'^0|^3 \boldsymbol{w}(x'^0) \int_0^{y^0} dy'^0 (x'^0 + y'^0) \boldsymbol{w}(y'^0). \quad (2.193)$$

Let

$$I(x^0, y^0) = \int_0^{x^0} dx'^0 |x^0 - x'^0|^3 \boldsymbol{w}(x'^0) \int_0^{y^0} dy'^0\, (x'^0 + y'^0) \boldsymbol{w}(y'^0) \quad (2.194)$$

and

$$L(x'^0, y^0) = \int_0^{y^0} dy'^0\, (x'^0 + y'^0) \boldsymbol{w}(y'^0). \quad (2.195)$$

Integration by parts yields:

$$\begin{aligned} L(x'^0, y^0) =& x'^0 \boldsymbol{w}_1(y^0) + y^0 \boldsymbol{w}_1(y^0) - \boldsymbol{w}_2(y^0) \\ \leqslant& x'^0 \boldsymbol{w}_1(y^0) + y^0 \boldsymbol{w}_1(y^0). \end{aligned} \tag{2.196}$$

Next, let

$$\begin{aligned} I_a(x^0) &= \int_0^{x^0} dx'^0 \; |x^0 - x'^0|^3 \boldsymbol{w}(x'^0), \\ I_b(x^0) &= \int_0^{x^0} dx'^0 \; x'^0 |x^0 - x'^0|^3 \boldsymbol{w}(x'^0). \end{aligned} \tag{2.197}$$

Then:

$$I(x^0, y^0) \leqslant I_a(x^0)\, y^0 \boldsymbol{w}_1(y^0) + I_b(x^0)\, \boldsymbol{w}_1(y^0). \tag{2.198}$$

We consider I_a first, using $(x^0 - x'^0)^2 \leqslant (x^0)^2$ and integrating by parts:

$$I_a(x^0) \leqslant (x^0)^2 \int_0^{x^0} dx'^0 \; (x^0 - x'^0) \boldsymbol{w}(x'^0) \tag{2.199}$$

$$= (x^0)^2 \left(\underbrace{(x^0 - x'^0)\boldsymbol{w}_1(x'^0)|_{x'^0=0}^{x^0}}_{=0} + \boldsymbol{w}_2(x^0) \right) = (x^0)^2 \boldsymbol{w}_2(x^0).$$

We turn to I_b, using $x'^0(x^0 - x'^0) \leqslant \frac{1}{4}(x^0)^2$ and integrating by parts twice. This results in:

$$I_b(x^0) \;\leqslant\; \frac{(x^0)^2}{4} \int_0^{x^0} dx'^0 \; (x^0 - x'^0)^2 \boldsymbol{w}(x'^0) \;=\; \frac{(x^0)^2}{2} \boldsymbol{w}_3(x^0). \tag{2.200}$$

Considering (2.198), we therefore obtain:

$$I(x^0, y^0) \;\leqslant\; (x^0)^2 \boldsymbol{w}_2(x^0)\, y^0 \boldsymbol{w}_1(y^0) + \frac{(x^0)^2}{2} \boldsymbol{w}_3(x^0)\, \boldsymbol{w}_1(y^0). \tag{2.201}$$

Returning to (2.193), we divide by $\boldsymbol{w}(x^0)\boldsymbol{w}(y^0)$ and take the supremum, with the result:

$$\sup_{\psi\in\mathcal{S}((\mathbb{M}_0^+)^2)} \frac{\|A_{12}\psi\|_{\boldsymbol{w}}}{\|\psi\|_{\boldsymbol{w}}} \leqslant \frac{\lambda\, m_1 m_2\, \|\psi\|_{\boldsymbol{w}}}{96\pi}\left[\sup_{x^0,y^0\geqslant 0} \frac{(x^0)^2\boldsymbol{w}_2(x^0)\, y^0\boldsymbol{w}_1(y^0)}{\boldsymbol{w}(x^0)\boldsymbol{w}(y^0)} + \frac{1}{2}\sup_{x^0,y^0\geqslant 0}\frac{(x^0)^2\boldsymbol{w}_3(x^0)\,\boldsymbol{w}_1(y^0)}{\boldsymbol{w}(x^0)\boldsymbol{w}(y^0)}\right]. \tag{2.202}$$

Factorizing the two-dimensional suprema into one-dimensional ones yields the claim, (2.92).

2.2.4.2 Proof of Theorem 14

Let $\psi \in \mathcal{S}$. It only remains to calculate the supremum in (2.89) for $\boldsymbol{w}(t) = e^{\gamma t}$. We have:

$$\boldsymbol{w}_1(t) = \frac{1}{\gamma}\left(e^{\gamma t} - 1\right) \tag{2.203}$$

and hence

$$\sup_{\psi\in\mathcal{S}((\mathbb{M}_0^+)^2)} \frac{\|A_0\psi\|_{\boldsymbol{w}}}{\|\psi\|_{\boldsymbol{w}}} \leqslant \frac{\lambda}{8\pi}\left(\sup_{t\geqslant 0}\frac{\boldsymbol{w}_1(t)}{\boldsymbol{w}(t)}\right)^2 = \frac{\lambda}{4\pi}\left(\sup_{t\geqslant 0}\frac{1}{\gamma}(1-e^{-\gamma t})\right)^2 = \frac{\lambda}{8\pi\gamma^2}. \tag{2.204}$$

This shows that A_0 can be linearly extended to a bounded operator on $\mathcal{B}_{\boldsymbol{w}}$ which satisfies the same estimate, (2.77). Moreover, for $\gamma > \sqrt{\frac{\lambda}{4\pi}}$, A_0 is a contraction and Banach's fixed point theorem implies the existence of a unique solution $\psi \in \mathcal{B}_{\boldsymbol{w}}$ of the equation $\psi = \psi^{\text{free}} + A_0\psi$ for every $\psi^{\text{free}} \in \mathcal{B}_{\boldsymbol{w}}$.

2.2.4.3 Proof of Theorem 15

Let again $\psi \in \mathcal{S}$. We need to calculate the suprema in (2.89) to (2.92) for $\boldsymbol{w}(t) = (1+\alpha t^2)e^{\alpha t^2/2}$. We first note:

$$\begin{aligned}
\boldsymbol{w}_1(t) &= te^{\alpha t^2/2}, \\
\boldsymbol{w}_2(t) &= \frac{1}{\alpha}\left(e^{\alpha t^2/2} - 1\right), \\
\boldsymbol{w}_3(t) &= \frac{1}{\alpha}\left[\sqrt{\frac{\pi}{2\alpha}}\,\mathrm{erfi}(\sqrt{\alpha/2}t) - t\right].
\end{aligned} \tag{2.205}$$

We can see that with each successive integration, the functions $\boldsymbol{w}_n$ grow slower than $t \to \infty$. Furthermore, the leading terms in $\boldsymbol{w}_n$ are inversely proportional to increasing powers of α. These two properties (and of course the fact that $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ can be written down in terms of elementary functions) make this particular function $\boldsymbol{w}(t)$ a suitable choice for the proof.

As we need to estimate the behaviour of quotients like $\boldsymbol{w}_3(t)/\boldsymbol{w}(t)$ for $t \to \infty$, we look for a simpler estimate of $\boldsymbol{w}_3$ in terms of exponential functions. We note:

$$\begin{aligned}
\boldsymbol{w}_3(t) &= \int_0^t dt'\,\frac{1}{\alpha}\left(e^{\alpha t'^2/2} - 1\right) \\
&\leqslant \frac{e^{\alpha t^2/2}}{\alpha} e^{-\alpha t^2/2}\sqrt{2/\alpha}\int_0^{\sqrt{\alpha/2}t} d\tau\, e^{\tau^2} \\
&= \frac{\sqrt{2}}{\alpha^{3/2}}\, e^{\alpha t^2/2}\, D(\sqrt{\alpha/2}\,t),
\end{aligned} \tag{2.206}$$

where $D(t) = e^{-t^2}\int_0^t d\tau\, e^{\tau^2}$ denotes the Dawson function. Using the property $|tD(t)| < \frac{2}{3}$, we obtain:

$$t\boldsymbol{w}_3(t) \leqslant \frac{4}{3}\frac{e^{\alpha t^2/2}}{\alpha^2}. \tag{2.207}$$

We are now well-equipped to calculate the suprema occurring in (2.89) to (2.92). Using

$$\sup_{t\geqslant 0} \frac{t^\beta}{1+t^2} = \begin{cases} 1 & \text{for } \beta = 0 \\ \frac{1}{2} & \text{for } \beta = 1 \\ 1 & \text{for } \beta = 2 \end{cases} \tag{2.208}$$

we obtain:

$$\sup_{t\geqslant 0} \frac{\boldsymbol{w}_1(t)}{\boldsymbol{w}(t)} = \sup_{t\geqslant 0} \frac{t}{1+\alpha t^2} = \frac{1}{2}\frac{1}{\sqrt{\alpha}}, \tag{2.209}$$

$$\sup_{t\geqslant 0} \frac{t\boldsymbol{w}_1(t)}{\boldsymbol{w}(t)} = \sup_{t\geqslant 0} \frac{t^2}{1+\alpha t^2} = \frac{1}{\alpha}, \tag{2.210}$$

$$\sup_{t\geqslant 0} \frac{\boldsymbol{w}_2(t)}{\boldsymbol{w}(t)} \leqslant \sup_{t\geqslant 0} \frac{1}{\alpha}\frac{1}{1+\alpha t^2} = \frac{1}{\alpha}, \tag{2.211}$$

$$\sup_{t\geqslant 0} \frac{t\boldsymbol{w}_2(t)}{\boldsymbol{w}(t)} \leqslant \sup_{t\geqslant 0} \frac{1}{\alpha}\frac{t}{1+\alpha t^2} = \frac{1}{2}\frac{1}{\alpha^{3/2}}, \tag{2.212}$$

$$\sup_{t\geqslant 0} \frac{t^2\boldsymbol{w}_2(t)}{\boldsymbol{w}(t)} \leqslant \sup_{t\geqslant 0} \frac{1}{\alpha}\frac{t^2}{1+\alpha t^2} = \frac{1}{\alpha^2}. \tag{2.213}$$

Using, in addition, the property $|D(t)| < \frac{3}{5}$, we find:

$$\sup_{t\geqslant 0} \frac{\boldsymbol{w}_3(t)}{\boldsymbol{w}(t)} \leqslant \sup_{t\geqslant 0} \frac{\sqrt{2}}{\alpha^{3/2}}\frac{D(\sqrt{\alpha/2}t)}{1+\alpha t^2} = \frac{3\sqrt{2}}{5}\frac{1}{\alpha^{3/2}} < \frac{1}{\alpha^{3/2}}, \tag{2.214}$$

$$\sup_{t\geqslant 0} \frac{t^2\boldsymbol{w}_3(t)}{\boldsymbol{w}(t)} \leqslant \sup_{t\geqslant 0} \frac{4}{3}\frac{1}{\alpha^2}\frac{t}{1+\alpha t^2} = \frac{2}{3}\frac{1}{\alpha^{5/2}}. \tag{2.215}$$

In the last line, we have made use of (2.207).
With these results, we find for A_0:

$$(2.89) \leqslant \frac{\lambda}{8\pi}\left(\frac{1}{2}\frac{1}{\sqrt{\alpha}}\right)^2 = \frac{\lambda}{32\pi}\frac{1}{\alpha}. \tag{2.216}$$

This yields (2.79).
We continue with A_1.

$$(2.90) \leqslant \frac{\lambda m_1^2}{16\pi}\left[3\frac{1}{\alpha}\frac{1}{\alpha} + 3\frac{1}{2}\frac{1}{\sqrt{\alpha}}\frac{1}{2}\frac{1}{\alpha^{3/2}} + 2\frac{1}{2}\frac{1}{\sqrt{\alpha}}\frac{1}{\alpha^{3/2}}\right]$$
$$= \frac{\lambda m_1^2}{16\pi}\frac{19}{4}\frac{1}{\alpha^2} < \frac{\lambda m_1^2}{16\pi}\frac{5}{\alpha^2}. \qquad (2.217)$$

This yields (2.80). Analogously, we obtain the estimate (2.81) for A_2.
Finally, for A_{12}, we have

$$(2.92) \leqslant \frac{\lambda m_1^2 m_2^2}{96\pi}\left[\frac{1}{\alpha^2}\frac{1}{\alpha} + \frac{1}{2}\frac{2}{3}\frac{1}{\alpha^{5/2}}\frac{1}{2}\frac{1}{\sqrt{\alpha}}\right]$$
$$= \frac{\lambda m_1^2 m_2^2}{96\pi}\frac{7}{6}\frac{1}{\alpha^3} < \frac{\lambda m_1^2 m_2^2}{80\pi}\frac{1}{\alpha^3}, \qquad (2.218)$$

which yields (2.82).
Now, the estimates (2.79) to (2.82) show that the operators A_0, A_1, A_2 and A_{12} are bounded on test functions. Thus, they can be linearly extended to bounded operators on $\mathcal{B}_{\boldsymbol{w}}$ with the same bounds.
The operator $A = A_0 + A_1 + A_2 + A_{12}$ then also defines a bounded linear operator on $\mathcal{B}_{\boldsymbol{w}}$ with norm

$$\|A\| \leqslant \|A_0\| + \|A_1\| + \|A_2\| + \|A_{12}\|. \qquad (2.219)$$

Using the previous results (2.79)- (2.82), we obtain:

$$\|A\| \leqslant \frac{\lambda}{8\pi\alpha}\left(\frac{1}{4} + \frac{5(m_1^2+m_2^2)}{2}\frac{1}{\alpha} + \frac{m_1^2 m_2^2}{10}\frac{1}{\alpha^2}\right). \qquad (2.220)$$

If α is chosen such that this expression is strictly smaller than unity, A becomes a contraction and the existence and uniqueness of solutions of the equation $\psi = \psi^{\text{free}} + A\psi$ follows. This yields condition (2.83) and ends the proof.

2.2.4.4 Proof of Theorem 18

The proof can be reduced to the one for $\mathbb{M}_0^+$. To do so, we take the absolute value of (2.75) and use $|\psi|(\eta_1, \vec{x}, \eta_2, \vec{y}) \leqslant \boldsymbol{w}(\eta_1)\boldsymbol{w}(\eta_2)\|\psi\|_{\boldsymbol{w}}$. With

$$G(\eta) = a(\eta) \exp\left(\gamma \int_0^\eta d\eta' \, a(\eta')\right) \tag{2.221}$$

$$G_1(\eta) = \int_0^\eta d\eta' \, G(\eta) \tag{2.222}$$

we obtain the estimate

$$\begin{aligned}
&|\widetilde{A}_0\psi|(x,y) \\
&\leqslant \frac{\lambda\|\psi\|_{\boldsymbol{w}}}{4(4\pi)^3}\int_{B_{\eta_2}(\vec{y})} d^3\vec{y}'\int_0^{2\pi} d\varphi \int_{-1}^1 d\cos\vartheta \, \frac{|b^2|}{(b^0+|\vec{b}|\cos\vartheta)^2|\vec{y}'|} G(\eta_2-|\vec{y}'|) \\
&\times G\left(\eta_1 - \frac{1}{2}\frac{b^2}{b^2+|\vec{b}|\cos\vartheta}\right) \\
&\times \left(1_{b^2>0}1_{b^0>0}1_{\cos\vartheta>\frac{b^2}{2\eta_1^0|\vec{b}|}-\frac{b^0}{|\vec{b}|}} + 1_{b^2<0}1_{\cos\vartheta<\frac{b^2}{2\eta_1|\vec{b}|}-\frac{b^0}{|\vec{b}|}}\right).
\end{aligned} \tag{2.223}$$

This estimate is identical to (2.94) with the only difference that the function g is exchanged with G in the integral (but not in $\|\cdot\|_{\boldsymbol{w}}$). Thus, going through the same steps as in Secs. 2.2.4.1, 2.2.4.3, we obtain:

$$\sup_{\psi\in\mathcal{S}(([0,\infty)\times\mathbb{R}^3)^2)} \frac{\|\widetilde{A}_0\psi\|_{\boldsymbol{w}}}{\|\psi\|_{\boldsymbol{w}}} \leqslant \frac{\lambda}{8\pi}\left(\sup_{t\geqslant 0}\frac{G_1(t)}{\boldsymbol{w}(t)}\right)^2. \tag{2.224}$$

Now, recalling $\boldsymbol{w}(t) = \exp\left(\gamma\int_0^t d\tau \, a(\tau)\right)$ we have

$$G_1(t) = \frac{1}{\gamma}\boldsymbol{w}(t) \tag{2.225}$$

and it follows that

$$\sup_{\psi \in \mathcal{S}(([0,\infty)\times\mathbb{R}^3)^2)} \frac{\|\widetilde{A}_0\psi\|_{\boldsymbol{w}}}{\|\psi\|_{\boldsymbol{w}}} \leqslant \frac{\lambda}{8\pi\gamma^2}, \tag{2.226}$$

which yields (2.87). The rest of the claim follows as before.

2.3 Directly Interacting Dirac Particles

This section is based on the scientific publication [67], which is the result of joint work with Matthias Lienert. In it, we prove the existence and uniqueness of solutions of the spin-1/2 delay-equation (2.20) for a class of kernels K subject to the modifications (A) and (B). Similar to the results of the last section an analogous result is proven on FLRW spacetime. Furthermore, we show that the solutions are determined by Cauchy data at the initial time; however, no Cauchy problem is admissible at other times.

2.3.1 Introduction

In order to take a closer look at equation (2.20) and its constituents. The solution to the inhomogeneous equation 2.21 represented by $A^K\psi$ in (2.20) will be constructed by $\overline{D} := -i\gamma^\mu\partial_\mu - m$ acting on a solution to the inhomogeneous Klein-Gordon equation. The operator $\overline{D}$ will be referred to as the adjoint Dirac operator. Since $D\overline{D} = \square + m^2$ holds, this is equivalent to the convolution with a Green's function. The solution to the inhomogeneous Klein-Gordon equation will be constructed by Duhamel's principle [33] using known the propagator of the Klein-Gordon equation [10, 11, 12] as outlined in subsection 2.1.2 in simplifying assumption (A). Consequently, one has to define the operator A^K in (2.20) on a function space where one can take certain weak derivatives. In contrast to most of non-relativistic physics, this also concerns the time derivatives here. The choice of function space can be a tricky issue, as the fixed point scheme requires the integral operator to preserve the regularity, so that the regularity needs to be in harmony with the structure of the integral equation (see Sec. 2.3.2.2). This section is structured as follows. In subsection 2.3.2, we specify the integral equation (2.20) in detail. The difficulties with understanding the distributional derivatives are discussed and a suitable function space is identified. Subsection 2.3.3 contains the main results of this section. In subsection 2.3.3.1, we formulate an existence and uniqueness theorem (theorem 25) for equation (2.20) on $\mathbb{M}_0^+$. It is shown that the relevant initial data are equivalent to Cauchy data at $t = 0$. In subsection 2.3.3.2, we provide a physical justification for the cut-off at $t = 0$ by extending the results to a FLRW spacetime. In the massless case, we show that an existence and uniqueness theorem can be obtained from the one for $\mathbb{M}_0^+$ via conformal invariance. The result, theorem 27, covers a fully relativistic interacting dynamics in 1+3 spacetime dimensions. The proofs are carried out in subsection 2.3.4.

2.3.2 Setting of the problem

2.3.2.1 Definition of the integral operator on test functions

In this subsection, we show how the integral operator in (2.20) can be defined rigorously on test functions. We consider the integral equation (2.20) on the Minkowski half space $\mathbb{M}_0^+$ as we did in the last section. In order to define the meaning of the Green's functions as distributions, we introduce a suitable space of test functions:

$$\mathcal{S} = \mathcal{S}\big((\mathbb{M}_0^+)^2, \mathbb{C}^{16}\big), \tag{2.227}$$

the space of 16-component Schwarz functions on $(\mathbb{M}_0^+)^2$. For an interaction kernel $K \in C^3(\mathbb{R}^8, \mathbb{C})$ and a test function $\psi \in \mathcal{S}$, the equation (2.20) takes the form

$$\psi(x_1, x_2) = \psi^{\text{free}}(x_1, x_2) + (A\psi)(x_1, x_2) \tag{2.228}$$

$$:= \psi^{\text{free}}(x_1, x_2) + \overline{D}_1 \overline{D}_2 \int_0^{x_1^0} dx_1'^0 \int_0^{x_2^0} dx_2'^0 \int_{\mathbb{R}^3} d^3\vec{x}_1' \int_{\mathbb{R}^3} d^3\vec{x}_2'$$
$$\times\, G_1^{\text{ret}}(x_1 - x_1') G_2^{\text{ret}}(x_2 - x_2')[K\psi](x_1', x_2') \tag{2.229}$$

where we used Duhamel's principle to construct the solution to the inhomogeneous Klein-Gordon equation as outlined in subsection 2.1.2 in simplifying assumption (A).

Remark 20. *Recall the expression (2.45) for G^{ret}. Note that this includes terms of the form*

$$\int_{\mathbb{R}^3} d^3\vec{x}' \frac{\delta(t - |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} f(t, \vec{x}, \vec{x}'), \tag{2.230}$$

for continuous functions f, which will be interpreted as

$$:= \int_{\partial B_t(0)} d\vec{x}' \frac{f(t, \vec{x}, \vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{2.231}$$

Expression (2.229) fulfills our requirements:

Lemma 21. *For $K \in C^3(\mathbb{R}^8, \mathbb{C})$ and a test function $\psi \in \mathcal{S}$ we have*

$$D_2 D_1 A\psi = D_1 D_2 A\psi = K\psi. \tag{2.232}$$

where $D_k = (i\gamma_k^\mu \partial_{x_k^\mu} - m_k)$, $k = 1, 2$. *Furthermore, we have*

$$A\psi = \prod_{j=1,2} \Big(A_j^{(1)}(m_j)\overline{D}_j + A_j^{(2)}(m_j)\overline{D}_j + A_j^{(3)}(m_j) + A_j^{(4)}(m_j) \Big) K\psi, \tag{2.233}$$

$$D_1 A\psi = \Big(A_2^{(1)}(m_2)\overline{D}_2 + A_2^{(2)}(m_2)\overline{D}_2 + A_2^{(3)}(m_2) + A_2^{(4)}(m_2) \Big) K\psi, \tag{2.234}$$

$$D_2 A\psi = \Big(A_1^{(1)}(m_1)\overline{D}_1 + A_1^{(2)}(m_1)\overline{D}_1 + A_1^{(3)}(m_1) + A_1^{(4)}(m_1) \Big) K\psi, \tag{2.235}$$

where for $j = 1, 2$, $k = 1, 2, 3, 4$ *the operator* $A_j^{(k)}(m) : C^3(\mathbb{M}_0^+ \times \mathbb{M}_0^+, \mathbb{C}^{16}) \to C^3\big(\mathbb{M}_0^+ \times \mathbb{M}_0^+, \mathbb{C}^{16}\big)$ *is defined by letting the respective operator* $A^{(k)}(m)$, *given below, act on the* j*-th spacetime-variable and spin index of* $\psi(x_1, x_2)$, $\psi \in \mathcal{S}$. [1]

$$\big(A^{(1)}(m)\,\phi\big)(x) = \frac{1}{4\pi} \int_{B_{x^0}(0)} d^3\vec{y}\ \frac{1}{|\vec{y}|} \phi(x+y)|_{y^0 = -|\vec{y}|}, \tag{2.236}$$

$$\big(A^{(2)}(m)\,\phi\big)(x) = -\frac{m}{4\pi} \int_{-x^0}^{0} dy^0 \int_{B_{|y^0|}(0)} d^3\vec{y}\ \frac{J_1(m\sqrt{y^2})}{\sqrt{y^2}} \phi(x+y), \tag{2.237}$$

$$\big(A^{(3)}(m)\,\phi\big)(x) = -\frac{i\gamma^0}{4\pi} \int_{\partial B_{x^0}(0)} d\sigma(\vec{y})\ \frac{\phi(0, \vec{x} + \vec{y})}{x^0}, \tag{2.238}$$

$$\big(A^{(4)}(m)\,\phi\big)(x) = i\gamma^0 \frac{m}{4\pi} \int_{B_{x^0}(0)} d^3\vec{y}\ \frac{J_1(m\sqrt{(x^0)^2 - \vec{y}^2})}{\sqrt{(x^0)^2 - \vec{y}^2}} \times \phi(0, \vec{x} + \vec{y}), \tag{2.239}$$

[1] We deliberately avoid using tensor products here, as the completion of an algebraic tensor product of Banach spaces depends sensitively on which completion is taken.

here $\phi \in \mathcal{S}(\mathbb{M}_0^+, \mathbb{C}^4)$ *and the dependence of* $A_j^{(1)}$ *and* $A_j^{(3)}$ *on* m *is only for notational convenience.*

The proof can be found in subsection 2.3.4. The first part of this lemma will enable us to work in a Banach space adapted to equation (2.228), while the second part is useful for working out the bounds on $A\psi$. For the second part we gave the convolution with delta distributions explicitly in terms of integrals over manifolds of one dimension lower.
We now turn to the question of a suitable Banach space for equation (2.228).

2.3.2.2 Choice of Banach space

As we did in the last section, we would like to apply a contraction mapping argument, for that we need a proper Banach space $\mathcal{B}$ as the domain of A. The results reviewed in subsection 2.1.2 as well as the results of subsection 2.2 suggest choosing

$$\mathcal{B}_0 = L^\infty\left([0,\infty)^2_{(x_1^0,x_2^0)},\, L^2(\mathbb{R}^6,\mathbb{C}^{16})_{(\vec{x}_1,\vec{x}_2)}\right) \tag{2.240}$$

with norm

$$\|\psi\|_{\mathcal{B}_0} = \operatorname*{ess\,sup}_{x_1^0,x_2^0>0} \|\psi(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}. \tag{2.241}$$

However, as (2.233) involves the Dirac operators D_1 and D_2, $\mathcal{B}_0$ is not sufficient for our problem. An appropriate Banach space $\mathcal{B}$ must allow us to take at least weak derivatives of ψ. The choice of $\mathcal{B}$ is a delicate matter. One can easily go wrong with demanding too much regularity, as we shall see next.

2.3.2.2.1 Possible problems with the choice of space. The problem can best be illustrated with an example which is structurally related to (2.20) but otherwise simpler. Consider the equation

$$f(t,x) = f^{\text{free}}(t,z) + \int_0^t dz'\, K(z,z')\partial_t f(t,z'), \tag{2.242}$$

where $f^{\text{free}}, f, K : \mathbb{R}^2 \to \mathbb{C}$ and f^{free} is given. Equation (2.242) is inspired by the term $A_1\overline{D}_1$ in (2.233).
We would like to set up an iteration scheme for (2.242). As we cannot integrate by parts to shift the t-derivative to K, we must demand at least weak differentiability of f with respect to t. This suggests using a Sobolev space such as $\mathcal{B} = H^1(\mathbb{R}^2)$. To prove that the integral operator in (2.242) maps $\mathcal{B}$ to $\mathcal{B}$, we then have to estimate the L^2-norm of

$$\partial_t \int_0^t dz'\, K(z,z')\partial_t f(t,z') = K(t,t)(\partial_t f)(t,t) + \int_0^t dz'\, K(z,z')\partial_t^2 f(t,z'). \tag{2.243}$$

This expression, however, contains $\partial_t^2 f$. For this to make sense, we must be allowed to take the second weak time derivative of f. This, in turn, requires to choose a different Sobolev space, such as $H^2(\mathbb{R}^2)$, and to estimate the L^2-norm of the second time derivative of the integral operator acting on f which involves $\partial_t^3 f$, and so on. One is thus led to a Sobolev space where all weak n-th time derivatives have to exist. Such infinite-order Sobolev spaces have, in fact, been investigated in [30]. However, it does not seem realistic to get an iteration to converge on these spaces. We therefore take a different approach.

2.3.2.2.2 A Banach space adapted to our integral equation. Considering the form of the integral operator A (2.233) and the fact that $\overline{D} = -2m - D$ holds, one can see that it is sufficient that the derivatives $D_1\psi$, $D_2\psi$ and $D_1D_2\psi$ exist in a weak sense. As we want to prove later that A maps the Banach space to itself, we have to estimate, among other things, a suitable norm of $D_1(A\psi)$. If $\psi \in \mathcal{S}$ is a test function and K is smooth, we have

$$D_1(A\psi) = \left(A_2^{(1)}(m_2)\overline{D}_2 + A_2^{(2)}(m_2)\overline{D}_2 + A_2^{(3)}(m_2) + A_2^{(4)}(m_2)\right) K\psi, \tag{2.244}$$

according to lemma 21. The crucial point now is that (2.244) does not contain higher-order derivatives such as $D_1^2\psi$. The same holds true also for $D_2(A\psi)$ and $D_1D_2(A\psi)$. Thus, the problem of the toy example (2.242) is avoided.

Together with the previous considerations about $\mathcal{B}_0$ (2.240), we are led to define the Banach space $\mathcal{B}_{\boldsymbol{w}}$ as the completion of $\mathcal{S}$ with respect to the following Sobolev-type norm:

$$\|\psi\|_{\boldsymbol{w}}^2 = \operatorname*{ess\,sup}_{x_1^0, x_2^0 > 0} \frac{1}{\boldsymbol{w}(x_1^0)\boldsymbol{w}(x_2^0)} [\psi]^2(x_1^0, x_2^0) \tag{2.245}$$

where $\boldsymbol{w} : [0,\infty[\to [0,\infty[$ is a monotonically increasing function which is such that the function $1/\boldsymbol{w}$ is bounded. We admit such a weight factor with hindsight. As we shall see, a suitable choice of $\boldsymbol{w}$ will make a contraction mapping argument possible.

In (2.245) we use the notation

$$[\psi]^2(x_1^0, x_2^0) \; = \sum_{k=0}^{3} \|(\mathcal{D}_k\psi)(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2(\mathbb{R}^6, \mathbb{C}^{16})}^2 \tag{2.246}$$

with

$$\mathcal{D}_k = \begin{cases} 1, & k = 0 \\ D_1, & k = 1 \\ D_2, & k = 2 \\ D_1D_2, & k = 3. \end{cases} \tag{2.247}$$

Remark 22. *One can see the purpose of equation (2.228) in determining an interacting correction to a solution ψ^{free} of the free multi-time Dirac equations $D_i\psi^{\text{free}} = 0,\ i = 1, 2$. Therefore, it is important to check that sufficiently many solutions of these free equations lie in $\mathcal{B}_{\boldsymbol{w}}$. This is ensured by the following Lemma (see Sec. 2.3.4 for a proof).*

Lemma 23. *Let ψ^{free} be a solution of the free multi-time Dirac equations $D_i\psi^{\text{free}} = 0,\ i = 1, 2$ with initial data $\psi^{\text{free}}(0, \cdot, 0, \cdot) = \psi_0 \in C_c^\infty(\mathbb{R}^6, \mathbb{C}^{16})$. Furthermore, let $\boldsymbol{w} : [0, \infty[\to]0, \infty[$ be a monotonically increasing function with $\boldsymbol{w}(t) \to \infty$ for $t \to \infty$. Then ψ^{free} lies in $\mathcal{B}_{\boldsymbol{w}}$.*

Given the definition of A on $\mathcal{S}$ as in Sec. 2.3.2.1, we shall now proceed with showing that A is bounded on this space. Furthermore, we show that for a suitable choice of the weight factor $\boldsymbol{w}$ in $\mathcal{B}_{\boldsymbol{w}}$, we can achieve $\|A\| < 1$ on $\mathcal{S}$. This allows to extend A to a contraction on $\mathcal{B}_{\boldsymbol{w}}$ so that the Neumann series $\psi = \sum_{k=0}^\infty A^k\psi^{\text{free}}$ yields the unique solution of $\psi = \psi^{\text{free}} + A\psi$.

2.3.3 Results

2.3.3.1 Results for a Minkowski half space

The core of our results is the following Lemma which allows us to control the growth of the spatial norm of ψ with the two time variables.

Lemma 24. *Let $\psi \in \mathcal{S}$, $\not{\partial}_k = \gamma_k^\mu \partial_{k,\mu},\ k = 1, 2$ and let $K \in C^3(\mathbb{R}^8, \mathbb{C})$ with*

$$\|K\| := \sup_{x_1, x_2 \in \mathbb{M}_0^+} \max\Big\{|K(x_1, x_2)|, |\not{\partial}_1 K(x_1, x_2)|, \tag{2.248}$$

$$|\not{\partial}_2 K(x_1, x_2)|, |\not{\partial}_1\not{\partial}_2 K(x_1, x_2)|\Big\} < \infty. \tag{2.249}$$

Then we have:

$$[A\psi]^2(x_1^0, x_2^0) \;\leqslant\; \|K\|^2 \prod_{j=1,2} \big(\mathbb{1} + 8(1+2m_j)^2 \mathcal{A}_j(m_j)\big)\, [\psi]^2(x_1^0, x_2^0), \tag{2.250}$$

where $\mathcal{A}_j(m) = \sum_{k=1}^4 \mathcal{A}_j^{(k)}(m)$ *with* $\mathcal{A}_j^{(k)}$ *as defined in* (2.327)*. The expression* $[\psi]^2(x_1^0, x_2^0)$ *is understood as a function in* $C^\infty\big((\mathbb{M}_0^+)^2, \mathbb{R}_0^+\big)$ *to which the operators in front of it are applied.*

The proof can be found in Sec. 2.3.4.1.
Lemma 24 can now be used to identify (with some trial and error) a suitable weight factor $\boldsymbol{w}$ which allows us to extend A to a contraction on $\mathcal{B}_{\boldsymbol{w}}$. Our main result is:

Theorem 25 (Existence and uniqueness of dynamics on a Minkowski half space.)**.** *Let* $0 < \|K\| < 1$, $\mu = \max\{m_1, m_2\}$ *and*

$$\boldsymbol{w}(t) \;=\; \sqrt{1 + bt^8}\, \exp(bt^8/16), \tag{2.251}$$

$$b \;=\; \frac{\|K\|^4}{(1-\|K\|)^4} \left(16 + \mu^4\right)^4 (1+2\mu)^8. \tag{2.252}$$

Then for every $\psi^{\text{free}} \in \mathcal{B}_{\boldsymbol{w}}$*, the equation* $\psi = \psi^{\text{free}} + A\psi$ *possesses a unique solution* $\psi \in \mathcal{B}_{\boldsymbol{w}}$*.*

The proof is given in Sec. 2.3.4.2.

Remark 26. *1. Note that theorem 25 establishes the existence and uniqueness of a global-in-time solution. The non-Markovian nature of the dynamics makes it necessary to prove such a result directly instead of concatenating short-time solutions. The key step in our proof which makes the global-in-time result possible is the suitable choice of the weight factor* $\boldsymbol{w}$*.*

2. *The main condition in theorem 25 is $\|K\| < 1$. This may be interpreted as the interaction must not being too strong. A condition of that kind is to be expected solely because of the contribution $\|(D_1 D_2(A\psi))(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2} = \|K\psi(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}$ to $[A\psi](x_1^0, x_2^0)$. Taking our strategy for setting up the Banach space for granted, we therefore think that one cannot avoid a condition on the interaction strength. Note that conditions on the interaction strength also occur at other places in quantum theory (albeit in a different sense). For example, the Dirac Hamiltonian plus a Coulomb potential is only self-adjoint if the prefactor of the latter is smaller than a certain value.*

3. *Cauchy problem. Theorem 25 shows that ψ^{free} uniquely determines the solution ψ. However, specifying a whole function in $\mathcal{B}_{\boldsymbol{w}}$ amounts to a lot of data. In case ψ^{free} is a solution of the free multi-time Dirac equations $D_1\psi^{\text{free}} = 0 = D_2\psi^{\text{free}}$ much less data are needed. ψ^{free} is then determined uniquely by Cauchy data, and hence ψ is as well. Furthermore, if ψ^{free} is differentiable,* (2.20) *yields*

$$\psi(0, \vec{x}_1, 0, \vec{x}_2) \;=\; \psi^{\text{free}}(0, \vec{x}_1, 0, \vec{x}_2). \tag{2.253}$$

Thus, Cauchy data for ψ^{free} at $x_1^0 = x_2^0 = 0$ are also Cauchy data for ψ. The procedure works for arbitrary Cauchy data which are appropriate for the free multi-time Dirac equations. Note, however, that a Cauchy problem for ψ for times $x_1^0 = t_0 = x_2^0$ with $t_0 > 0$ is not possible. The reason is that $\psi(t_0, \vec{x}_1, t_0, \vec{x}_2) \neq \psi^{\text{free}}(t_0, \vec{x}_1, t_0, \vec{x}_2)$ in general (and contrary to (2.253) *the pointwise evaluation may not make sense for ψ).*

2.3.3.2 Results for a FLRW universe with a Big Bang singularity

This subsection is analogous to subsection 2.2.3.3, we show that a Big Bang singularity provides a natural and covariant justification for the cut-off at $t = 0$. As this justification is our main goal, we make the point at the example of a particular class of FLRW spacetimes and do not strive to treat more general spacetimes here. The reason for studying these FLRW spacetimes is that they are conformally equivalent to $\frac{1}{2}\mathbb{M}_0^+$ [56]. Together with the conformal invariance of the massless Dirac operator this allows for an efficient method of calculating the Green's functions which occur in the curved spacetime analogue of the integral equation (2.17). By doing this, we show that the existence and uniqueness result on these spacetimes can be reduced to theorem 25. We start by giving heuristic arguments for our interpretation of equation (2.17) on this spacetime and then show that this interpretation has a unique solution in a sense specified below.
As shown in [70], equation (2.17) possesses a natural generalization to curved spacetimes $\mathcal{M}$,

$$\psi(x_1, x_2) = \psi^{\text{free}}(x_1, x_2) + \int dV(x_1') \int dV(x_2')\, \mathcal{S}_1(x_1, x_1')\mathcal{S}_2(x_2, x_2') \times \tilde{K}(x_1', x_2')\psi(x_1', x_2'). \tag{2.254}$$

Here, $dV(x)$ is the spacetime volume element, $\mathcal{S}_i$ are (retarded) Green's functions of the respective free Dirac equation, i.e.

$$D\mathcal{S}(x, x') = [-g(x)]^{-1/2}\, \delta^{(4)}(x, x'), \tag{2.255}$$

where $g(x)$ is the metric determinant, D the covariant Dirac operator on $\mathcal{M}$, and ψ a section of the tensor spinor bundle over $\mathcal{M} \times \mathcal{M}$.
In order to explicitly formulate (2.254), we need to know the detailed form of $\mathcal{S}$. Note that results for general classes of spacetimes showing

that $\mathcal{S}$ is a bounded operator on a suitable function space are not sufficient to obtain a strong (global in time) existence and uniqueness result. We therefore focus on the case of a flat FLRW universe where it is easy to determine the Green's functions explicitly. In that case, the metric is given by

$$ds^2 = a^2(\eta)[d\eta^2 - d\vec{x}^2] \tag{2.256}$$

where, as before, η is conformal time and $a(\eta)$ denotes the *scale function*. The coordinate ranges are given by $\eta \in [0, \infty[$ and $\vec{x} \in \mathbb{R}^3$. For a FLRW universe with a Big Bang singularity, $a(\eta)$ is a continuous, monotonically increasing function of η with $a(\eta) = 0$, corresponding to the Big Bang singularity. The spacetime volume element reads

$$dV(x) = a^4(\eta)\, d\eta\, d^3\vec{x}. \tag{2.257}$$

The crucial point now is that according to (2.256) the spacetime is globally conformally equivalent to $\mathbb{M}_0^+$, with conformal factor

$$\Omega(x) = a(\eta). \tag{2.258}$$

In addition, for $m = 0$, the Dirac equation is known to be conformally invariant (see e.g. [98, 80]). More accurately, consider two spacetimes $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ with metrics

$$\widetilde{g}_{ab} = \Omega^2\, g_{ab}. \tag{2.259}$$

Then the massless Dirac operator D on $\mathcal{M}$ is related to the massless Dirac operator $\widetilde{D}$ on $\widetilde{\mathcal{M}}$ by (see [98, 34]):

$$\widetilde{D} = \Omega^{-5/2}\, D\, \Omega^{3/2}. \tag{2.260}$$

This implies the following transformation behaviour of the Green's functions:

$$\widetilde{G}(x, x') = \Omega^{-3/2}(x)\, \Omega^{-3/2}(x')\, G(x, x'). \tag{2.261}$$

One can verify this using (2.260) and the definition of Green's functions on curved spacetimes (2.255).
Denoting the Green's functions of the Dirac operator on Minkowski spacetime by $G(x, x') = S(x - x')$ and using coordinates $\eta, \vec{x}$ we thus obtain the Green's functions $\widetilde{G}$ on flat FLRW spacetimes as:

$$\widetilde{G}(\eta, \vec{x}; \eta', \vec{x}\,') = a^{-3/2}(\eta) a^{-3/2}(\eta')\, \mathcal{S}(\eta - \eta', \vec{x} - \vec{x}\,'). \tag{2.262}$$

With this result, we can write out in detail the multi-time integral equation (2.254) for massless Dirac particles on flat FLRW spacetimes (using retarded Green's functions):

$$\begin{aligned}\psi(\eta_1,\vec{x}_1, \eta_2, \vec{x}_2) &= \psi^{\text{free}}(\eta_1, \vec{x}_1, \eta_2, \vec{x}_2) + a^{-3/2}(\eta_1) a^{-3/2}(\eta_2) \\ &\times \int_0^\infty d\eta_1' \int d^3\vec{x}_1' \int_0^\infty d\eta_2' \int d^3\vec{x}_2' a^{5/2}(\eta_1') a^{5/2}(\eta_2') \\ &\times \mathcal{S}_1^{\text{ret}}(\eta_1 - \eta_1', \vec{x}_1 - \vec{x}_1')\mathcal{S}_2^{\text{ret}}(\eta_2 - \eta_2', \vec{x}_2 - \vec{x}_2')\, (\tilde{K}\psi)(\eta_1', \vec{x}_1', \eta_2', \vec{x}_2').\end{aligned} \tag{2.263}$$

Note that we can regard ψ as a map $\psi : (\mathbb{M}_0^+)^2 \to \mathbb{C}^{16}$ as the coordinates $\eta, \vec{x}$ cover the flat FLRW spacetime manifold globally.
It seems reasonable to allow for a singularity of the interaction kernel, i.e.

$$\tilde{K}(\eta_1, \vec{x}_1, \eta_2, \vec{x}_2) = a^{-\alpha}(\eta_1) a^{-\alpha}(\eta_1)\, K(\eta_1, \vec{x}_1, \eta_2, \vec{x}_2). \tag{2.264}$$

Here, $\alpha \geqslant 0$. The singular behaviour is motivated by that of the Green's functions of the conformal wave equation, see subsection 2.2.3.3. Recall from section 2.1 that the most natural interaction kernel on $\mathbb{M}_0^+$ would be $\tilde{K}(x_1, x_2) \propto \delta((x_1 - x_2)_\mu (x_1 - x_2)^\mu)$ which is a Green's function of the wave equation – a concept that can be generalized to curved spacetimes using the conformal wave equation. Now, under conformal transformations, Green's functions of that equation transform as (2.40)

$$\widetilde{G}(x, x') = \Omega^{-1}(x)\, \Omega^{-1}(x')\, G(x, x'), \tag{2.265}$$

which corresponds to $\alpha = 1$ in (2.264).
Considering (2.264), our integral equation becomes:

$$\psi(\eta_1, \vec{x}_1, \eta_2, \vec{x}_2) = \psi^{\text{free}}(\eta_1, \vec{x}_1, \eta_2, \vec{x}_2) + a^{-3/2}(\eta_1) a^{-3/2}(\eta_2) \int_0^\infty d\eta_1' \times \int d^3\vec{x}_1' \int_0^\infty d\eta_2' \int d^3\vec{x}_2' \times a^{5/2-\alpha}(\eta_1') a^{5/2-\alpha}(\eta_2')\, \mathcal{S}_1^{\text{ret}}(\eta_1 - \eta_1', \vec{x}_1 - \vec{x}_1') \times \mathcal{S}_2^{\text{ret}}(\eta_2 - \eta_2', \vec{x}_2 - \vec{x}_2')\, (K\psi)(\eta_1', \vec{x}_1', \eta_2', \vec{x}_2'). \quad (2.266)$$

Apart from the scale factors which produce a certain singularity of ψ for $\eta_1, \eta_2 \to 0$, this integral equation has the form of (2.17) on $\mathbb{M}_0^+$. Indeed, we can use the transformation

$$\chi(\eta_1, \vec{x}_1, \eta_2, \vec{x}_2) = a^{3/2}(\eta_1) a^{3/2}(\eta_2)\, \psi(\eta_1, \vec{x}_1, \eta_2, \vec{x}_2) \quad (2.267)$$

to transform the two equations into each other. Hence, we interpret equation (2.266) to mean

$$a \otimes a(3/2)\psi = a \otimes a(3/2)\psi^{\text{free}} + A(a \otimes a(5/2 - \alpha)\psi), \quad (2.268)$$

where the operator $a \otimes a(\beta) : C((\mathbb{R}_0^+ \times \mathbb{R}^3)^2, \mathbb{C}^{16}) \to C((\mathbb{R}_0^+ \times \mathbb{R}^3)^2, \mathbb{C}^{16})$ multiplies pointwise with a in both coordinates:

$$a \otimes a(\beta) f(x_1, x_2) = a(x_1^0) a(x_2^0) f(x_1, x_2) \quad (2.269)$$

and A is the operator of equation (2.228).

Theorem 27 (Existence and uniqueness of dynamics on a flat FLRW universe)**.** *Let,* $0 \leqslant \alpha \leqslant 1$ *and let* $a : [0, \infty) \to [0, \infty)$ *be a differentiable function with* $a(0) = 0$ *and* $a(\eta) > 0$ *for* $\eta > 0$*. Moreover, assume that* $K \in C^3\left(([0, \infty) \times \mathbb{R}^3)^2, \mathbb{C}\right)$ *with*

$$\|a^{1-\alpha}(\eta_1) a^{1-\alpha}(\eta_2)\, K\| < 1. \quad (2.270)$$

Then for every ψ^{free} with $a^{3/2}(\eta_1)a^{3/2}(\eta_2)\psi^{\text{free}} \in \mathcal{B}_{\boldsymbol{w}}$, (2.268) has a unique solution ψ with $a^{3/2}(\eta_1)a^{3/2}(\eta_2)\psi \in \mathcal{B}_{\boldsymbol{w}}$ (and with $\boldsymbol{w}$ as in theorem 25).

Proof. Using the transformation (2.268) on equation (2.268) yields

$$\chi = \chi^{\text{free}} + A(a \otimes a(1-\alpha)\chi). \tag{2.271}$$

This equation has the form of (2.228) on $\mathbb{M}_0^+$ with K replaced by $a^{1-\alpha}(\eta_1')a^{1-\alpha}(\eta_2')K$. Thus, theorem 25 yields the claim. □

Remark 28. *1. Both ψ^{free} and ψ have a singularity proportional to $a^{-3/2}(\eta_1)a^{-3/2}(\eta_2)$ for $\eta_1, \eta_2 \to 0$.*

2. For $\alpha < 1$, K has to compensate the singularities caused by $a^{-3/2}(\eta_1)a^{-3/2}(\eta_2)$ in order for (2.270) to hold. In the most natural case $\alpha = 1$, however, K only needs to satisfy $\|K\| < 1$, i.e. the same condition as for $\widetilde{K}$ in theorem 25.

3. Let $\chi^{\text{free}} = a^{3/2}(\eta_1)a^{3/2}(\eta_2)\psi^{\text{free}}$ be differentiable and let χ be the unique solution of (2.271). Then, by (2.271), we have:

$$\chi^{\text{free}}(0, \vec{x}_1, 0, \vec{x}_2) = \chi(0, \vec{x}_1, 0, \vec{x}_2), \tag{2.272}$$

i.e. χ satisfies a Cauchy problem "at the Big Bang".

4. Remarkably, theorem 27 covers a class of manifestly covariant, interacting integral equations in 1+3 dimensions. Then the interaction kernel K has to be covariant as well. A class of examples (see also [70]) is given by $\alpha = 1$ and

$$K(x_1, x_2) = \begin{cases} f(d(x_1, x_2)) & \text{if } x_1, x_2 \text{ are time-like related} \\ 0 & \text{else,} \end{cases} \tag{2.273}$$

where $d(x_1, x_2)$ denotes the geodesic distance of time-like separated the events $x_1 = (\eta_1, \vec{x}_1)$ and $x_2 = (\eta_2, \vec{x}_2)$, and f is an arbitrary smooth function which leads to $\|K\| < 1$.

2.3.4 Proofs

We begin this subsection with a set of technical lemmas.

Lemma 29. *For any* $f \in C^3(\mathbb{M}_0^+ \times \mathbb{M}_0^+, \mathbb{C})$ *the functions* $f_1', f_2' \in C^3(\mathbb{M}_0^+ \times \mathbb{M}_0^+, \mathbb{C})$ *defined by*

$$f_1'(x,z) := \int_0^{x_1^0} dx'^0 \int_{\mathbb{R}^3} d^3x' G^{\mathrm{ret}}(x-x') f(x',z) \tag{2.274}$$

$$f_2'(x,z) := \int_0^{x_1^0} dx'^0 \int_{\mathbb{R}^3} d^3x' G^{\mathrm{ret}}(x-x') f(z,x') \tag{2.275}$$

fulfil for $k \in \{x^l \mid l \in \{0,1,2,3\}\} \cup \{z^l \mid l \in \{0,1,2,3\}\}$

$$\begin{aligned}\operatorname{grad} f_1'(x,z) &= e_0 \int_{\mathbb{R}^3} d^3x' G^{\mathrm{ret}}(x-x') f(x',z)|_{x'^0=x^0} \\ &\quad + \int_0^{x_1^0} dx'^0 \int_{\mathbb{R}^3} d^3x' G^{\mathrm{ret}}(x-x') \operatorname{grad} f(x',z)\end{aligned} \tag{2.276}$$

$$\begin{aligned}\operatorname{grad} f_2'(x,z) &= e_0 \int_{\mathbb{R}^3} d^3x' G^{\mathrm{ret}}(x-x') f(z,x')|_{x'^0=x^0} \\ &\quad + \int_0^{x_1^0} dx'^0 \int_{\mathbb{R}^3} d^3x' G^{\mathrm{ret}}(x-x') \operatorname{grad} f(z,x'),\end{aligned} \tag{2.277}$$

where e_0 *is the unit vector in the zeroth direction and*

$$(\Box_x + m^2) f_1'(x,z) = f(x,z) \tag{2.278}$$

$$(\Box_x + m^2) f_2'(z,x) = f(x,z). \tag{2.279}$$

Proof. Pick $f \in C^3(\mathbb{M}_0^+ \times \mathbb{M}_0^+, \mathbb{C})$, we will only check the first of each of the three sets of equalities in the lemma, the proof of the second case is identical. We abbreviate f_1' as f', write it more explicitly and check the stated regularity. We have

$$f'(x,z) = \int_0^{x_1^0} dx'^0 \int_{\mathbb{R}^3} d^3x' G^{\text{ret}}(x-x') f(x',z) \tag{2.280}$$

$$=: \int_0^{x_1^0} dy^0 h(x,y^0,z). \tag{2.281}$$

Remark 20 and the form of G^{ret}

$$G^{\text{ret}}(x) = \frac{1}{4\pi|\vec{x}|}\delta(x^0-|\vec{x}|) - \frac{m}{4\pi}H(x^0-|\vec{x}|)\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} \tag{2.282}$$

together result for any $y^0 \in [0, x^0[$ in:

$$h(x,y^0,z) = \int_{\partial B_{x^0}(0)} d^2S(y)\frac{1}{4\pi|\vec{y}|} f(x+y,y^0,z)|_{y^0=-|\vec{y}|} \tag{2.283}$$

$$- \frac{m^2}{4\pi}\int_{B_{|y^0|}(0)} d^3y \frac{J_1(m\sqrt{y^2})}{m\sqrt{y^2}} f(x+y,y^0,z) \tag{2.284}$$

$$= \int_{\partial B_1(0)} d^2S(y)\frac{1}{4\pi|\vec{y}|} f(x+x^0y,y^0,z)|_{y^0=-|\vec{y}|}x^0 \tag{2.285}$$

$$- \frac{m^2}{4\pi}\int_{B_1(0)} d^3y (y^0)^3 \frac{J_1(my^0\sqrt{y^2})}{my^0\sqrt{y^2}} f(x+w,y^0,z)|_{w^0=y^0,\vec{w}=y^0\vec{y}}. \tag{2.286}$$

Now, each of the two summands is in C^3. We begin with the first summand for $h \in C^l(\{(x,y^0,z) \in \mathbb{R}^9 \mid x^0 \geqslant y^0 \geqslant 0, z^0 \geqslant 0\}, \mathbb{C})$ with $l \geqslant 1$ we define

$$h'(x,y^0,z) := \int_{\partial B_1(0)} d^2S(y)\frac{1}{4\pi|\vec{y}|} h(x+x^0y,y^0,z)|_{y^0=-|\vec{y}|}x^0. \tag{2.287}$$

Now, pick a direction of differentiation k. The factor $\frac{1}{4\pi|\vec{y}|}$ is absolutely integrable, while the remaining integrand $h(x+x^0y,y^0,z)|_{y^0=-|\vec{y}|}(x^0)^2$

is in C^l and the integration region is compact. Therefore, the difference quotient of $h(x + x^0y, y^0, z)|_{y^0=-|\vec{y}|}(x^0)^2$ in the direction $k \in \{x_l \mid l \in \{0,1,2,3\}\} \cup \{y^0\} \cup \{z_l \mid l \in \{0,1,2,3\}\}$ is continuous and hence bounded on any small neighbourhood of any point (x, y^0, z) which means that by Lebesgue dominated convergence the first summand is differentiable at least once and the resulting derivative equals

$$\int_{\partial B_1(0)} d^2S(y) \frac{1}{4\pi|\vec{y}|} \partial_k \left(h(x + x^0y, y^0, z)|_{y^0=-|\vec{y}|} x^0 \right). \tag{2.288}$$

However, this is just the expression we started with where $h(x + x^0y, y^0, z)|_{y^0=-|\vec{y}|}x^0$ is replaced by its derivative. Hence, by induction it follows that the first summand in (2.285) is three times differentiable in any set of directions k_1, k_2, k_3 with derivative

$$\int_{\partial B_1(0)} d^2S(y) \frac{1}{4\pi|\vec{y}|} \partial_{k_1}\partial_{k_2}\partial_{k_3} \left(h(x + x^0y, y^0, z)|_{y^0=-|\vec{y}|} x^0 \right). \tag{2.289}$$

This function is continuous, since Lebesgue dominated convergence can be applied once more for the same reasons to the limit $(x, y^0, z) \to (x', y'^0, z')$. For the term (2.286) the argument is similar. Recall the power series representation of J_1 [28, 10.2.2]:

$$J_1(z) = \sum_{k\in\mathbb{N}_0} \frac{(-1)^k}{k!(k+1)!} \left(\frac{z}{2}\right)^{2k+1}, \tag{2.290}$$

from this it follows that $u \mapsto J_1(u)/u$ is a smooth function. Hence, the term (2.286) is an integral of a function in C^3 over a compact domain, and therefore we can pull three derivatives inside and still have a continuous function. So overall, f' is an integral over a function in C^3 and hence is itself in C^3.

Equation (2.276) follows directly from the above.

Next, we show that f' is a solution to the inhomogeneous Klein-Gordon equation. For this we use that for the positive values of x^0 we have

$$f'(x,z) = \int_0^{x_1^0} dy^0 \int_{\mathbb{R}^3} d^3y G^{\text{ret}}(x-y)f(y,z) \tag{2.291}$$

$$= \int_0^{x_1^0} dy^0 \int_{\mathbb{R}^3} d^3y (G^{\text{ret}} - G^{\text{adv}})(x-y)f(y,z), \tag{2.292}$$

according to the support properties of G^{adv}. Now, because the inner integral can be interpreted as a convolution against the propagator of the Klein-Gordon equation [10, 11, 12] it follows that

$$g(x,y^0,z) := \int_{\mathbb{R}^3} d^3y (G^{\text{ret}} - G^{\text{adv}})(x-y)f(y,z) \tag{2.293}$$

fulfils

$$(\Box_x + m^2)g(x,y^0,z) = 0 \quad \text{on } x^0 \geqslant y^0, \vec{x} \in \mathbb{R}^3, z \in \mathbb{M}_0^+ \tag{2.294}$$

$$g(x,y^0,z) = 0 \quad \text{on } x^0 = y^0, x \in \mathbb{R}^3, z \in \mathbb{M}_0^+ \tag{2.295}$$

$$\partial_{x^0} g(x,y^0,z) = f(x,z) \quad \text{on } x^0 = y^0, x \in \mathbb{R}^3, z \in \mathbb{M}_0^+ \tag{2.296}$$

$$\tag{2.297}$$

holds. Therefore, we can directly calculate [33]:

$$(\Box_x + m^2) \int_0^{x^0} dy^0 g(x,y^0,z) \tag{2.298}$$

$$= (\partial_{x^0}^2 - \Delta + m^2) \int_0^{x^0} dy^0 g(x,y^0,z) \tag{2.299}$$

$$= \partial_{x^0} \left(\overbrace{g(x,x^0,z)}^{=0} + \int_0^{x^0} dy^0 \partial_{x^0} g(x,y^0,z) \right) \tag{2.300}$$

$$+ \int_0^{x^0} dy^0 (-\Delta + m^2) g(x,y^0,z) \tag{2.301}$$

$$= \overbrace{\partial_{x^0} g(x, x^0, z)}^{=f(x)} + \int_0^{x^0} dy^0 \overbrace{(\Box_x + m^2) g(x, y^0, z)}^{=0} . \tag{2.302}$$

□

Lemma 30. *For any $f \in C^3(\mathbb{M}_0^+ \times \mathbb{M}_0^+, \mathbb{C}^{16})$ define the function $f' \in C^3(\mathbb{M}_0^+ \times \mathbb{M}_0^+, \mathbb{C}^{16})$ by*

$$f'(x_1, x_2) := \int_0^{x_1^0} dy_1^0 \int_0^{x_2^0} dy_2^0 \int_{\mathbb{R}^3} d^3\vec{y}_1 \int_{\mathbb{R}^3} d^3\vec{y}_2 \, G_1^{\mathrm{ret}}(x_1 - y_1) \times G_2^{\mathrm{ret}}(x_2 - y_2) f(y_1, y_2). \tag{2.303}$$

Moreover, we have

$$f'(x_1, x_2) = \int_0^{x_1^0} dy_1^0 \int_{\mathbb{R}^3} d^3\vec{y}_1 \, G_1^{\mathrm{ret}}(x_1 - y_1) \int_0^{x_2^0} dy_2^0 \int_{\mathbb{R}^3} d^3\vec{y}_2 \times G_2^{\mathrm{ret}}(x_2 - y_2) f'(y_1, y_2) \tag{2.304}$$

$$= \int_0^{x_2^0} dy_2^0 \int_{\mathbb{R}^3} d^3\vec{y}_2 G_2^{\mathrm{ret}}(x_2 - y_2) \int_0^{x_1^0} dy_1^0 \int_{\mathbb{R}^3} d^3\vec{y}_1 \times G_1^{\mathrm{ret}}(x_1 - y_1) f'(y_1, y_2). \tag{2.305}$$

Proof. Plugging in the form of G^{ret} and using remark 20 for equation (2.303) one obtains four terms. The proof that all of them are absolutely integrable is analogous, so we will only do one term. Consider the term

$$f'(x_1, x_2) := \frac{m_2^2}{16\pi^2} \int_{B_{x_1^0}(0)} d^3y_1 \frac{1}{|\vec{y}_1|} \int_{-x_2^0}^{0} dy_2^0 \int_{B_{|y_2^0|}(0)} d^3y_2 \frac{J_1(m_2\sqrt{y_2^2})}{m_2\sqrt{y_2^2}} f(x_1 + y_1, x_2 + y_2)|_{y_1^0 = -|\vec{y}_1|}, \tag{2.306}$$

these integrals are absolutely integrable, because the domain of integration is bounded and the function $\vec{y_1} \mapsto \frac{1}{|\vec{y_1}|}$ is absolutely integrable on bounded domains in three dimensions and the remaining integrand is continuous and so bounded on the integral domain. Therefore, one may interchange the order of integration for f. Since the analogous argument holds for all summands we can exchange the order of integration in (2.303). The claimed regularity for f' holds, because f' in the form of equation (2.305) is generated by applying lemma 29 once to each variable x_1 and x_2. □

Proof of lemma 21. Pick $\psi \in \mathcal{S}, K \in C^3(\mathbb{R}^8, \mathbb{C})$ as in the lemma. We will only check $D_2 D_1 A\psi = K\psi$ and (2.234), the other cases are analogous. First we rewrite the integral expression in (2.229) using remark 20 and the form of G^{ret} the same way we did in the proofs of the last two lemmas. This results in the form

$$\int_0^{x_1^0} dy_1^0 \int_0^{x_2^0} dy_2^0 \int_{\mathbb{R}^3} d^3\vec{y}_1 \int_{\mathbb{R}^3} d^3\vec{y}_2\, G_1^{\text{ret}}(x_1 - y_1) G_2^{\text{ret}}(x_2 - y_2) \times [K\psi](y_1, y_2) \tag{2.307}$$

$$= \prod_{j=1,2} \left(A_j^{(1)}(m_j) + A_j^{(2)}(m_j) \right) K\psi(x_1, x_2), \tag{2.308}$$

where the operators $A_j^{(l)}$ are defined in the statement of the lemma. Using lemma 2.3.4 we see that this expression is three times continuously differentiable. Reordering the integrals in $A\psi$ we get

$$A\psi(x_1, x_2) = \overline{D}_1 \overline{D}_2 \int_0^{x_1^0} dy_1^0 \int_{\mathbb{R}^3} d^3\vec{y}_1 G_1^{\text{ret}}(x_1 - y_1) \tag{2.309}$$

$$\int_0^{x_2^0} dy_2^0 \int_{\mathbb{R}^3} d^3\vec{y}_2 G_2^{\text{ret}}(x_2 - y_2)[K\psi](y_1, y_2) \tag{2.310}$$

Next, we apply D_1 to it. We compute

$$D_1 A\psi(x_1,x_2) = (\Box_{x_1} + m_1^2)\overline{D}_2 \int_0^{x_1^0} dy_1^0 \int_{\mathbb{R}^3} d^3\vec{y}_1 G_1^{\text{ret}}(x_1-y_1) \quad (2.311)$$

$$\int_0^{x_2^0} dy_2^0 \int_{\mathbb{R}^3} d^3\vec{y}_2 G_2^{\text{ret}}(x_2-y_2)[K\psi](y_1,y_2), \quad (2.312)$$

now we exchange differentiation, which is justified due to the lemma provided by lemma 29. Then, we apply this lemma once more to obtain:

$$D_1 A\psi(x_1,x_2) = \overline{D}_2(\Box_{x_1} + m_1^2) \int_0^{x_1^0} dy_1^0 \int_{\mathbb{R}^3} d^3\vec{y}_1 G_1^{\text{ret}}(x_1-y_1) \quad (2.313)$$

$$\int_0^{x_2^0} dy_2^0 \int_{\mathbb{R}^3} d^3\vec{y}_2 G_2^{\text{ret}}(x_2-y_2)[K\psi](y_1,y_2) \quad (2.314)$$

$$= \overline{D}_2 \int_0^{x_2^0} dy_2^0 \int_{\mathbb{R}^3} d^3\vec{y}_2 G_2^{\text{ret}}(x_2-y_2)[K\psi](x_1,y_2). \quad (2.315)$$

At this point we realize that once plugging in the form of G_2^{ret} and using the regularity proven in lemma 29 the last expression evaluates to (2.234). The resulting expression is regular enough to apply D_2, which yields

$$D_2 D_1 A\psi(x_1,x_2) \quad (2.316)$$

$$= D_2\overline{D}_2 \int_0^{x_2^0} dy_2^0 \int_{\mathbb{R}^3} d^3\vec{y}_2 G_2^{\text{ret}}(x_2-y_2)[K\psi](x_1,y_2) \quad (2.317)$$

$$= (\Box_{x_2} + m_2^2) \int_0^{x_2^0} dy_2^0 \int_{\mathbb{R}^3} d^3\vec{y}_2 G_2^{\text{ret}}(x_2-y_2)[K\psi](x_1,y_2) \quad (2.318)$$

$$= K(x_1,x_2)\psi(x_1,x_2), \quad (2.319)$$

proving the claim. The representation (2.233) then follows by exchanging Dirac operators and integrals and differentiating the change of integration domain with the time dimensions, which is justified according to lemma 29. □

Proof of lemma 23. Consider a solution ψ of $D_i\psi^{\text{free}} = 0,\ i = 1,2$ for compactly supported initial data at $x_1^0 = 0 = x_2^0$. As the Dirac equation has finite propagation speed, ψ^{free} is spatially compactly supported for all times. Without loss of generality we may assume $\|\psi^{\text{free}}(t_1,\cdot,t_2,\cdot)\|_{L^2(\mathbb{R}^6)} = 1$ for all times t_1, t_2, so it follows that also $[\psi^{\text{free}}](t_1,t_2) = 1$. In the following we will construct a sequence of test functions $(\psi_m)_{m\in\mathbb{N}}$ satisfying $\psi_m \xrightarrow[\|\cdot\|_{\boldsymbol{w}}]{m\to\infty} \psi^{\text{free}}$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be zero for arguments less than 0, be 1 for arguments greater than 1 and in between given by (see also Fig. 2.1)

$$\eta(t) = \exp\left(-\frac{1}{t}\exp\left(\frac{1}{t-1}\right)\right). \tag{2.320}$$

Note that η is smooth and monotonically increasing. Next, we define for every $m \in \mathbb{N}$

$$\psi_m^{\text{free}}(t_1,\vec{x}_1,t_2,\vec{x}_2) := e^{-(t_1-m)\eta(t_1-m)}e^{-(t_2-m)\eta(t_2-m)}\psi^{\text{free}}(t_1,\vec{x}_1,t_2,\vec{x}_2). \tag{2.321}$$

This function is smooth and decreases rapidly in all variables and thus lies in $\mathcal{S}$. Now we estimate $\|\psi^{\text{free}} - \psi_m\|_{\boldsymbol{w}}$. Pick $m \in \mathbb{N}$. First consider $\|\psi^{\text{free}} - \psi_m\|_{L^2(\mathbb{R}^6)}(t_1,t_2)$. This function is identically zero for all $t_1 < m$ and $t_2 < m$, so we obtain the estimate

$$\begin{aligned}
&\sup_{t_1,t_2>0} \frac{1}{\boldsymbol{w}(t_1)^2\boldsymbol{w}(t_2)^2}\|\psi^{\text{free}} - \psi_m\|^2_{L^2(\mathbb{R}^6)} \\
&= \sup_{t_1,t_2>0} \frac{1}{\boldsymbol{w}(t_1)^2\boldsymbol{w}(t_2)^2}\left|1 - e^{-\eta(t_1-m)(t_1-m)}e^{-\eta(t_2-m)(t_2-m)}\right| \qquad (2.322)\\
&\leqslant \frac{1}{\boldsymbol{w}(0)^2\boldsymbol{w}(m)^2}. \qquad (2.323)
\end{aligned}$$

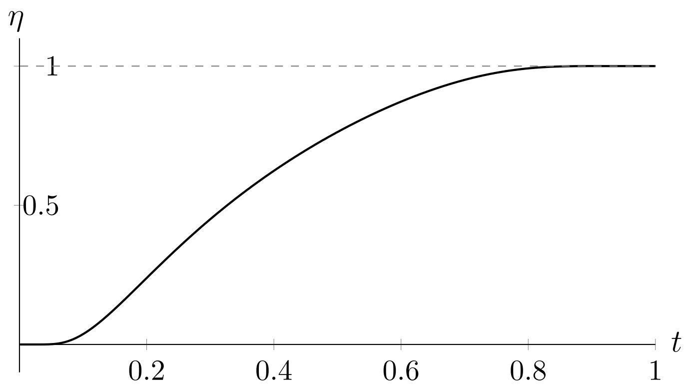

Figure 2.1: The function $\eta(t)$.

For the other terms we use that ψ^{free} solves the free Dirac equation in each variable and that $\sup_{t>0} \partial_t e^{-\eta(t)t} =: \alpha < \infty$ is realized for some positive value of t. So we find for $i \in \{0,1\}$:

$$\begin{aligned}
&\sup_{t_1,t_2>0} \frac{1}{\boldsymbol{w}(t_1)^2\boldsymbol{w}(t_2)^2} \|D_i(\psi^{\text{free}} - \psi_m)\|^2_{L^2(\mathbb{R}^6)}(t_1,t_2) \\
&= \sup_{t_1,t_2>0} \frac{1}{\boldsymbol{w}(t_1)^2\boldsymbol{w}(t_2)^2} && (2.324)\\
&\quad \times \|\gamma_i^0 \psi^{\text{free}}(t_1,\cdot,t_2,\cdot) e^{-\eta(t_{3-i}-n)(t_{3-i}-n)} \partial_{t_i} e^{-\eta(t_i-n)(t_i-m)}\|^2_{L^2(\mathbb{R}^6)} && (2.325)\\
&\leqslant \frac{\alpha}{\boldsymbol{w}(0)^2\boldsymbol{w}(m)^2}. && (2.326)
\end{aligned}$$

For the inequality it has been used that the factor with a derivative vanishes for $t_i < m$.
An analogous estimate repeated for the D_1D_2-term yields

$$\sup_{t_1,t_2>0} \frac{1}{\boldsymbol{w}(t_1)^2\boldsymbol{w}(t_2)^2} \|D_1D_2(\psi^{\text{free}} - \psi_m)\|^2_{L^2(\mathbb{R}^6)}(t_1,t_2) \leqslant \frac{\alpha^2}{\boldsymbol{w}(m)^4}$$

$$\leqslant \frac{\alpha^2}{\boldsymbol{w}(0)^2 \boldsymbol{w}(m)^2}.$$

All in all, adding the estimates and taking the square root we find $\|\psi^{\text{free}} - \psi_n\|_{\boldsymbol{w}} \leqslant \frac{1+\alpha}{\boldsymbol{w}(0)\boldsymbol{w}(n)}$, which together with the asymptotic behaviour of $\boldsymbol{w}$ implies convergence. It follows that the free solution ψ^{free} can be approximated by Cauchy sequences in $\mathcal{S}$ and hence is contained in $\mathcal{B}_{\boldsymbol{w}}$ which, we recall, has been defined as the completion of $\mathcal{S}$ with respect to $\| \cdot \|_{\boldsymbol{w}}$.

□

2.3.4.1 Proof of Lemma 24

Throughout the following subsections, let $\psi \in \mathcal{S}$ and $K \in C^3(\mathbb{R}^8, \mathbb{C})$. Furthermore, define $\delta := 1 - \|K\|^2 > 0, \mu = \max\{m_1, m_2\}$ and let $\boldsymbol{w}$ be as in the statement of theorem 25.
We begin with some lemmas which are useful for estimating $[A\psi]^2(x_1^0, x_2^0)$.

Lemma 31. *Let the following operators be defined on $C([0,\infty))$:*

$$\begin{aligned}
\big(\mathcal{A}^{(1)}(m) f\big)(t) &= t \int_0^t d\rho \; (t-\rho)^2 \, f(\rho), \\
\big(\mathcal{A}^{(2)}(m) f\big)(t) &= \frac{m^4 t^4}{2^4\, 3^2} \int_0^t d\rho \; (t-\rho)^3 \, f(\rho), \\
\big(\mathcal{A}^{(3)}(m) f\big)(t) &= t^2 \, f(0), \\
\big(\mathcal{A}^{(4)}(m) f\big)(t) &= \frac{m^4 t^6}{2^2\, 3^2} \, f(0).
\end{aligned} \tag{2.327}$$

Then, for $j = 1, 2$ and $k = 1, 2, 3, 4$, we define the operator $\mathcal{A}_j^{(k)}(m)$ acting on functions $\phi \in C([0,\infty)^2)$ by letting $\mathcal{A}^{(k)}(m)$ act on the j-th *variable of $\phi(t_1, t_2)$. Then we have for all $\psi \in \mathcal{S}$, all $m_1, m_2 \geqslant 0$ and all $k, l = 1, 2, 3, 4$:*

$$\left\| A_1^{(k)}(m_1) A_2^{(l)}(m_2) \psi(t_1, \cdot, t_2, \cdot) \right\|_{L^2}^2 \leqslant \mathcal{A}_j^{(k)}(m_1) \mathcal{A}_j^{(l)}(m_2) \left\| \psi(t_1, \cdot, t_2, \cdot) \right\|_{L^2}^2. \tag{2.328}$$

Here, it is understood that the operators $\mathcal{A}_j^{(k)}$ are applied to the functions defined by the norms which follow them, e.g.
$\mathcal{A}_1^{(4)}(m_1) \left\| \psi(t_1, \cdot, t_2, \cdot) \right\|_{L^2}^2 = \frac{m_1^4 t_1^6}{2^2\, 3^2} \left\| \psi(0, \cdot, t_2, \cdot) \right\|_{L^2}^2.$

Proof. We prove (2.328) for $k = 1$, $l = 2$ and $k = 3$, $l = 4$. The remaining cases can be treated in the same way. We begin with $k = 1$, $l = 2$, using $|J_1(x)/x| \leqslant \frac{1}{2}$:

$$\begin{aligned}
\| A_1^{(1)}(m_1) A_2^{(2)}(m_2)\, \psi(x_1^0, \cdot, x_2^0, \cdot) \|_{L^2}^2 \;&= \frac{m_2^2}{(4\pi)^4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3\vec{x}_1\, d^3\vec{x}_2 \qquad (2.329)\\
\times \left| \int_{B_{x_1^0}(0)} d^3\vec{y}_1 \int_{-x_2^0}^0 dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\vec{y}_2\; \frac{1}{|\vec{y}_1|} \frac{J_1(m_2\sqrt{y_2^2})}{\sqrt{y_2^2}} \psi(x_1 + y_1, x_2 + y_2)|_{y_1^0 = -|\vec{y}_1|} \right|^2 \\
&\leqslant \frac{m_2^2}{(4\pi)^4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3\vec{x}_1\, d^3\vec{x}_2 \\
&\times \left(\int_{B_{x_1^0}(0)} d^3\vec{y}_1 \int_{-x_2^0}^0 dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\vec{y}_2\; \frac{1}{|\vec{y}_1|^2} \left| \frac{J_1(m_2\sqrt{y_2^2})}{\sqrt{y_2^2}} \right|^2 \right) \\
&\times \left(\int_{B_{x_1^0}(0)} d^3\vec{y}_1 \int_{-x_2^0}^0 dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\vec{y}_2\; |\psi|^2(x_1 + y_1, x_2 + y_2)|_{y_1^0 = -|\vec{y}_1|} \right) \\
&\leqslant \frac{m_2^2}{(4\pi)^4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3\vec{x}_1\, d^3\vec{x}_2\; 4\pi x_1^0 \left(\frac{\pi m_2^2 (x_2^0)^4}{12} \right) \\
&\times \left(\int_{B_{x_1^0}(0)} d^3\vec{y}_1 \int_{-x_2^0}^0 dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\vec{y}_2\; |\psi|^2(x_1 + y_1, x_2 + y_2)|_{y_1^0 = -|\vec{y}_1|} \right)
\end{aligned}$$

$$\leqslant \frac{m_2^4\, x_1^0\, (x_2^0)^4}{3\pi^2\, 2^8} \int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\vec{x}_1\, d^3\vec{x}_2 \int_{B_{x_1^0}(0)} d^3\vec{y}_1 \int_{-x_2^0}^0 dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\vec{y}_2$$
$$\times\, |\psi|^2(x_1^0 - |\vec{y}_1|, \vec{x}_1 + \vec{y}_1, x_2^0 + y_2^0, \vec{x}_2 + \vec{y}_2). \tag{2.330}$$

Exchanging the x and y integrals yields:

$$\begin{aligned}
(2.330) \leqslant\ & \frac{m_2^4\, x_1^0\, (x_2^0)^4}{3\pi^2\, 2^8} \int_{B_{x_1^0}(0)} d^3\vec{y}_1 \int_{-x_2^0}^0 dy_2^0 \int_{B_{|y_2^0|}(0)} d^3\vec{y}_2 \\
& \times \|\psi(x_1^0 - |\vec{y}_1|, \cdot, x_2^0 + y_2^0, \cdot)\|_{L^2} \\
\leqslant\ & \frac{m_2^4\, x_1^0\, (x_2^0)^4}{3\pi^2\, 2^8}\, 4\pi \int_0^{x_1^0} dr_1\ r_1^2 \int_{-x_2^0}^0 dy_2^0\ \frac{4\pi}{3} |y_2^0|^3 \\
& \times \|\psi(x_1^0 - r_1, \cdot, x_2^0 + y_2^0, \cdot)\|_{L^2} \\
\leqslant\ & \frac{m_2^4\, x_1^0\, (x_2^0)^4}{2^4\, 3^2} \int_0^{x_1^0} d\rho_1\ (x_1^0 - \rho_1)^2 \int_0^{x_2^0} d\rho_2\ (x_2^0 - \rho_2)^3\, \|\psi(\rho_1, \cdot, \rho_2, \cdot)\|_{L^2} \\
=\ & \mathcal{A}_1^{(1)}(m_1) \mathcal{A}_2^{(2)}(m_2)\, \|\psi(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2.
\end{aligned} \tag{2.331}$$

Next, we turn to the case $k = 3, l = 4$. Using that the modulus of the largest eigenvalue of γ^0 is 1, we obtain:

$$\|A_1^{(3)}(m_1) A_2^{(4)}(m_2)\, \psi(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2\ \leqslant\ \frac{m_2^2}{(4\pi)^4 (x_1^0)^2} \int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\vec{x}_1\, d^3\vec{x}_1$$
$$\times \left| \int_{\partial B_{x_1^0}(0)} d\sigma(\vec{y}_1) \int_{B_{x_2^0}(0)} d^3\vec{y}_2\ \frac{J_1\left(m_2\sqrt{(x_2^0)^2 - \vec{y}_2^2}\right)}{\sqrt{(x_2^0)^2 - \vec{y}_2^2}} |\psi|(0, \vec{x}_1 + \vec{y}_2, 0, \vec{x}_2 + \vec{y}_2) \right|^2$$
$$\leqslant \frac{m_2^4}{(4\pi)^4 (x_1^0)^2} \int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\vec{x}_1\, d^3\vec{x}_1$$
$$\times \left(\int_{\partial B_{x_1^0}(0)} d\sigma(\vec{y}_1) \int_{B_{x_2^0}(0)} d^3\vec{y}_2 \left| \frac{J_1\left(m_2\sqrt{(x_2^0)^2 - \vec{y}_2^2}\right)}{m_2\sqrt{(x_2^0)^2 - \vec{y}_2^2}} \right|^2 \right)$$

$$
\begin{aligned}
&\times \left(\int_{\partial B_{x_1^0}(0)} d\sigma(\vec{y}_1) \int_{\partial B_{x_2^0}(0)} d\sigma(\vec{y}_2)\ |\psi|^2(0, \vec{x}_1 + \vec{y}_2, 0, \vec{x}_2 + \vec{y}_2) \right) \\
&= \frac{m_2^4}{(4\pi)^4 (x_1^0)^2} 4\pi (x_1^0)^2 \frac{\pi (x_2^0)^3}{3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3\vec{x}_1\, d^3\vec{x}_1 \int_{\partial B_{x_1^0}(0)} d\sigma(\vec{y}_1) \\
&\quad \times \int_{B_{x_2^0}(0)} d^3\vec{y}_2\ |\psi|^2(0, \vec{x}_1 + \vec{y}_2, 0, \vec{x}_2 + \vec{y}_2). \qquad (2.332)
\end{aligned}
$$

Exchanging the order of the x and y integrals yields:

$$
\begin{aligned}
(2.332) &= \frac{m_2^4}{3(4\pi)^3} \pi (x_2^0)^3 \int_{\partial B_{x_1^0}(0)} d\sigma(\vec{y}_1) \int_{B_{x_2^0}(0)} d^3\vec{y}_2\ \|\psi(0,\cdot,0,\cdot)\|_{L^2}^2 \\
&= \frac{m_2^4\, (x_1^0)^2\, (x_2^0)^6}{2^2\, 3^2} \|\psi(0,\cdot,0,\cdot)\|_{L^2}^2 \\
&= \mathcal{A}_1^{(3)}(m_1) \mathcal{A}_2^{(4)}(m_2)\, \|\psi(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2. \qquad (2.333)
\end{aligned}
$$

□

Lemma 32. *For $j = 1, 2$ let $\mathcal{A}_j(m) = \sum_{k=1}^4 \mathcal{A}_j^{(k)}(m)$. Then the following estimates hold:*

$$
\begin{aligned}
\|(A\psi)(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 &\leqslant 64\, \|K\|^2 (1+2m_1)^2 (1+2m_2)^2 \qquad (2.334) \\
&\quad \times \mathcal{A}_1(m_1) \mathcal{A}_2(m_2)\, [\psi]^2(x_1^0, x_2^0), \qquad (2.335) \\
\|(D_1(A\psi))(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 &\leqslant 8\, \|K\|^2 (1+2m_2)^2 \qquad (2.336) \\
&\quad \times \mathcal{A}_2(m_2)\, [\psi](x_1^0, x_2^0), \qquad (2.337) \\
\|(D_2(A\psi))(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 &\leqslant 8\, \|K\|^2 (1+2m_1)^2 \qquad (2.338) \\
&\quad \times \mathcal{A}_1(m_1)\, [\psi](x_1^0, x_2^0), \qquad (2.339) \\
\|(D_1 D_2(A\psi))(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 &\leqslant \|K\|^2\, [\psi]^2(x_1^0, x_2^0), \qquad (2.340)
\end{aligned}
$$

where $[\psi]^2(x_1^0, x_2^0)$ is regarded as a function of x_1^0, x_2^0 to which the operators in front of it are applied.

Proof. We start with (2.335). Recalling (2.233), and using $\overline{D} = -2m - D$ the expression $A\psi$ contains terms such as $D_1 D_2(K\psi)$ and $D_i(K\psi)$, $i = 1,2$. Recalling also the definition of $\mathcal{D}_k$ (equation (2.247)), we have:

$$D_1 D_2(K\psi) = \sum_{k=0}^{3} (\boldsymbol{\nabla}_{3-k} K)(\mathcal{D}_k \psi) \tag{2.341}$$

with

$$\boldsymbol{\nabla}_k := \begin{cases} 1, & k = 0 \\ i\not{\partial}_1, & k = 1 \\ i\not{\partial}_2, & k = 2 \\ -\not{\partial}_1 \not{\partial}_2, & k = 3. \end{cases} \tag{2.342}$$

Hence, noting (2.249):

$$|D_1 D_2 \psi| \leqslant \|K\| \sum_{k=0}^{3} |\mathcal{D}_k \psi|. \tag{2.343}$$

Similarly, we find:

$$D_i(K\psi) \leqslant \|K\| \sum_{k=0}^{3} |\mathcal{D}_k \psi|, \quad i = 1,2. \tag{2.344}$$

Considering the definition of $A_j^{(k)}(m)$, $j = 1,2$, $k = 1,2,3,4$ it follows that

$$\begin{aligned} |A\psi| \;\leqslant\; & \|K\|(1+2m_1)(1+2m_2) \sum_{k=0}^{3} \prod_{j=1,2} \big[A_j(m_j)^{(1)} + \\ & A_j^{(2)}(m_j) + A_j^{(3)}(m_j) + A_j^{(4)}(m_j)\big] |\mathcal{D}_k \psi|. \end{aligned} \tag{2.345}$$

In slight abuse of notation, we here use the same symbols for the operators $A_j^{(k)}(m)$ acting on functions with and without spin components.

The idea now is to make use of lemma 31. In order to be able to apply the lemma, we first note that by Young's inequality for $a_1, ..., a_N \in \mathbb{R}$, we have $\left(\sum_{i=1}^N a_i\right)^2 \leqslant N \sum_{i=1}^N a_i^2$ and thus:

$$|A\psi(x_1,x_2)|^2 \leqslant 64\,\|K\|^2(1+2m_1)^2(1+2m_2)^2 \times \sum_{i,j=1}^4 \sum_{k=0}^3 \left|A_1^{(i)}(m_1)A_2^{(j)}(m_2)\,|\mathcal{D}_k\psi|\right|^2. \quad (2.346)$$

Integrating over this expression and using lemma 31, we obtain:

$$\|(A\psi)(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 \leqslant 64\,\|K\|^2(1+2m_1)^2(1+2m_2)^2 \times \sum_{i,j=1}^4 \sum_{k=0}^3 \mathcal{A}_1^{(i)}(m_1)\mathcal{A}_2^{(j)}(m_2)\|(\mathcal{D}_k\psi)(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2. \quad (2.347)$$

Recalling the definition of $[\psi]^2(x_1^0,x_2^0)$, equation (2.246) yields (2.335). Next, we turn to (2.337). Here, we use lemma 21 and $\overline{D} = -2m - D$ to obtain

$$D_1(A\psi) = \Big(A_2^{(1)}(m_2)\,(-2m_2-D_2) + A_2^{(2)}(m_2)(-2m_2-D_2) + A_2^{(3)}(m_2) + A_2^{(4)}(m_2)\Big)(K\psi). \quad (2.348)$$

Considering the form of $A_j^{(k)}(m_j)$ this implies:

$$|D_1(A\psi)| \leqslant \|K\|(1+2m_2)\sum_{i=1}^4 \sum_{k\in\{0,2\}} A_2^{(i)}(m_2)\,|\mathcal{D}_k\psi|. \quad (2.349)$$

We now square and use Young's inequality, finding:

$$|D_1(A\psi)|^2 \leqslant 8\,\|K\|^2(1+2m_2)^2 \sum_{i=1}^4 \sum_{k\in\{0,2\}} A_2^{(i)}(m_2)\,|\mathcal{D}_k\psi|^2. \quad (2.350)$$

Integrating and using lemma 31 yields:

$$\begin{aligned}\|D_1(A\psi)(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2 \;&\leqslant\; 8\,\|K\|^2(1+2m_2)^2\\ &\times\sum_{i=1}^{4}\sum_{k\in\{0,2\}}\mathcal{A}_2^{(i)}(m_2)\,\|(\mathcal{D}_k\psi)(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2. \end{aligned} \tag{2.351}$$

Adding the terms with $k=1,3$ and using the definition of $[\psi]^2(x_1^0,x_2^0)$ gives us (2.337).
The estimate (2.339) follows in an analogous way.
Finally, for (2.340) we also use lemma 21 to get

$$D_1D_2(A\psi) = K\psi. \tag{2.352}$$

Squaring and integrating gives us:

$$\begin{aligned}\|D_1D_2(A\psi)(x_1^0,\cdot,x_2^0,\cdot)\|^2 \;&\leqslant\; \|K\|^2\,\|\psi(x_1^0,\cdot,x_2^0,\cdot)\|_{L^2}^2\\ &\leqslant\; \|K\|^2\,[\psi]^2(x_1^0,x_2^0),\end{aligned} \tag{2.353}$$

which yields (2.340). □

These estimates are the core of:

Proof of Lemma 24: We use lemma 32 together with the definition of $[\psi]^2(x_1^0,x_2^0)$ to obtain:

$$[A\psi]^2(x_1^0,x_2^0) \;\leqslant\; (2.335) + (2.337) + (2.339) + (2.340). \tag{2.354}$$

Summarizing the operators into a product yields (2.250).

□

2.3.4.2 Proof of Theorem 25

In order to prove theorem 25, we combine the previous estimates to show that $\|A\| < 1$, first on test functions $\psi \in \mathcal{S}$ and by linear extension also on the whole of $\mathcal{B}_{\boldsymbol{w}}$. We start with equation (2.250) of lemma 24 using the definition of $\mathcal{A}_j$ for $j = 1, 2$, as well as the following estimate, valid for all $\psi \in \mathcal{S}, t_1, t_2 > 0$:

$$[\psi](t_1, t_2) \;=\; [\psi](t_1, t_2)\,\frac{\boldsymbol{w}(t_1)\boldsymbol{w}(t_2)}{\boldsymbol{w}(t_1)\boldsymbol{w}(t_2)} \;\leqslant\; \|\psi\|_{\boldsymbol{w}}\,\boldsymbol{w}(t_1)\boldsymbol{w}(t_2). \tag{2.355}$$

Using this in (2.250) yields:

$$\begin{aligned}
\|A\psi\|^2_{\boldsymbol{w}} \leqslant \; & \sup_{x_1^0, x_2^0 > 0} \frac{1}{(\boldsymbol{w}(x_1^0)\boldsymbol{w}(x_2^0))^2}\,\|K\|^2 && (2.356)\\
& \times \prod_{j=1,2} \big(\mathbb{1} + 8(1+2m_j)^2\mathcal{A}_j(m_j)\big)\,[\psi]^2(x_1^0, x_2^0), && (2.357)\\
\leqslant \; & \sup_{x_1^0, x_2^0 > 0} \frac{\|\psi\|^2}{(\boldsymbol{w}(x_1^0)\boldsymbol{w}(x_2^0))^2}\,\|K\|^2 \\
& \times \prod_{j=1,2} \big(\mathbb{1} + 8(1+2m_j)^2\mathcal{A}_j(m_j)\big)\,(\boldsymbol{w}^2 \otimes \boldsymbol{w}^2)(x_1^0, x_2^0), && (2.358)\\
\leqslant \; & \|K\|^2\,\|\psi\|^2_{\boldsymbol{w}} \Big(\sup_{t>0} \frac{1}{\boldsymbol{w}(t)^2} && (2.359)\\
& \times \big(\mathbb{1} + 8(1+2m_j)^2\mathcal{A}(\mu)\big)\,\boldsymbol{w}^2(t)\Big)^2, && (2.360)
\end{aligned}$$

where $\mu = \max\{m_1, m_2\}$ and $\mathcal{A}(\mu) = \sum_{k=1}^4 \mathcal{A}^{(k)}(\mu)$ with $\mathcal{A}^{(k)}(\mu)$ as in (2.327).

Next, we shall estimate the term in the big round bracket. To this end, we first note some special properties of $\boldsymbol{w}^2$, which motivated choosing $\boldsymbol{w}$ as in (2.251).

Lemma 33. *For all $t > 0$, we have*

$$\int_0^t d\tau\, \boldsymbol{w}^2(\tau) = \frac{t}{1+bt^8}\, \boldsymbol{w}^2(t). \tag{2.361}$$

Proof: Differentiating the right side of the equation and using the concrete function $\boldsymbol{w}^2$ as in (2.251) shows that it is, indeed, the anti-derivative of $\boldsymbol{w}^2$. Since this function vanishes at $t = 0$, the claim follows. □

Lemma 34. *For $c < 8$ we have*

$$\sup_{t>0} \frac{t^c}{1+bt^8} \;=\; \frac{c}{8}\, b^{-c/8} \left(\frac{8}{c} - 1\right)^{1-c/8}, \tag{2.362}$$

and furthermore for $c = 8$:

$$\sup_{t>0} \frac{t^8}{1+bt^8} \;=\; \frac{1}{b}. \tag{2.363}$$

Proof. To prove (2.362), considering the shape of the function $h(t) = t^c/(1+bt^8)$ we find that the supremum is in fact a maximum which is located at $t = b^{-1/8}\,(8/c-1)^{-1/8}$. Inserting this back into the function $h(t)$ yields (2.362). Equation (2.363) follows from $\frac{t^8}{1+bt^8} = \frac{1}{b}\frac{1}{1/(bt^8)+1} \leqslant \frac{1}{b}$. □

Proof of theorem 25: Applying Lemma 33 to $\mathcal{A}(\mu)\, \boldsymbol{w}^2$ yields:

$$\begin{aligned}
\big(\mathcal{A}^{(1)}(\mu)\,\boldsymbol{w}^2\big)(t) &= t\int_0^t d\rho\,(t-\rho)^2\,\boldsymbol{w}^2(\rho) \;\leqslant\; t^3\int_0^t d\rho\,\boldsymbol{w}^2(\rho)\\
&\qquad\qquad\qquad = \frac{t^4}{1+bt^8}\,\boldsymbol{w}^2(t),\\
\big(\mathcal{A}^{(2)}(\mu)\,\boldsymbol{w}^2\big)(t) &= \frac{\mu^4\,t^4}{2^4\,3^2}\int_0^t d\rho\,(t-\rho)^3\,\boldsymbol{w}^2(\rho) \;\leqslant\; \frac{\mu^4\,t^8}{2^4\,3^2}\frac{\boldsymbol{w}^2(t)}{1+bt^8},\\
\big(\mathcal{A}^{(3)}(\mu)\,\boldsymbol{w}^2\big)(t) &= t^2,\\
\big(\mathcal{A}^{(4)}(\mu)\,\boldsymbol{w}^2\big)(t) &= \frac{\mu^4\,t^6}{2^2\,3^2}. \qquad (2.364)
\end{aligned}$$

Multiplying with $1/\boldsymbol{w}^2(t)$ and using Lemma 34 as well as $1/\boldsymbol{w}(t)^2 \leqslant (1+bt^8)^{-1}$, we find:

$$\begin{aligned}
\boldsymbol{w}^{-2}(t)\,\big(\mathcal{A}^{(1)}(\mu)\,\boldsymbol{w}^2\big)(t) &\leqslant \sqrt{2}b^{-\frac{1}{2}},\\
\boldsymbol{w}^{-2}(t)\,\big(\mathcal{A}^{(2)}(\mu)\,\boldsymbol{w}^2\big)(t) &\leqslant \frac{\mu^4}{2^4\,3^2\,b},\\
\boldsymbol{w}^{-2}(t)\,\big(\mathcal{A}^{(3)}(\mu)\,\boldsymbol{w}^2\big)(t) &\leqslant \frac{3^{3/4}}{2^2\,b^{1/4}},\\
\boldsymbol{w}^{-2}(t)\,\big(\mathcal{A}^{(4)}(\mu)\,\boldsymbol{w}^2\big)(t) &\leqslant \frac{\mu^4}{2^4\,3^{5/4}}\,b^{-3/4}. \qquad (2.365)
\end{aligned}$$

Using (2.360), we can employ these inequalities (whose right-hand sides are inversely proportional to powers of b) to estimate the norm of A. According to (2.360), we have, first on $\mathcal{S}$ and by linear extension also on the whole of $\mathcal{B}_{\boldsymbol{w}}$:

$$\|A\| \;\leqslant\; \|K\|\,\sup_{t>0}\,\boldsymbol{w}^{-2}(t)\Big(\big(\mathbb{1}+8(1+2\mu)^2\mathcal{A}(\mu)\big)\,\boldsymbol{w}^2\Big)(t). \qquad (2.366)$$

Now we use (2.365) for the various contributions $A^{(k)}(\mu)$ to $\mathcal{A}(\mu) =$

$\sum_{k=1}^{4} A^{(k)}(\mu)$, finding:

$$\begin{aligned}\|A\| &\leqslant \|K\| + \Big[\frac{2^{3.5}\|K\|}{b^{1/2}} + \frac{\mu^4\|K\|}{18b} + \frac{3^{3/4}2\|K\|}{b^{1/4}} + \frac{\mu^4\|K\|}{2(3^5b^3)^{1/4}}\Big](1+2\mu)^2 \\ &\overset{b\geqslant 1}{\leqslant} \|K\| + \frac{\|K\|}{b^{1/4}}\left(2^{3.5} + \mu^4/18 + 3^{3/4}2 + \mu^4/(2\cdot 3^{5/4})\right)(1+2\mu)^2 \end{aligned} \tag{2.367}$$

$$< \|K\| + \frac{\|K\|}{b^{1/4}}(16+\mu^4)(1+2\mu)^2. \tag{2.368}$$

Recalling that $b = \frac{\|K\|^4}{(1-\|K\|)^4}(16+\mu^4)^4(1+2\mu)^8$ (see (2.252)), we finally obtain that:

$$\|A\| < \|K\| + \frac{\|K\|}{b^{1/4}}(16+\mu^4)(1+2\mu)^2 = \|K\| + 1 - \|K\| = 1. \tag{2.369}$$

We have thus shown that A defines (by linear extension) a contraction on $\mathcal{B}_{\boldsymbol{w}}$. Thus, the Neumann series $\psi = \sum_{k=0}^{\infty} A^k \psi^{\text{free}}$ yields the unique (global-in-time) solution of the equation $\psi = \psi^{\text{free}} + A\psi$. □

2.4 Summary and Conclusions

In this chapter we have extended the analysis of spin-0 and spin-1/2 delayed-equations. The resulting degree of understanding is quite different in the two cases.

Extending previous work for Klein-Gordon particles [72, 70] to the Dirac case, we have established the existence of dynamics for a class of integral equations (2.20) which express direct interactions with time delay at the quantum level. To obtain this result, we have used both simplifying assumptions (A) of a cut-off of the spacetime before $t = 0$, and (B) of a smoother interaction kernel than the choice (2.26). While we have tried to justify assumption (A) by considering the equation on

the FLRW universe that features a Big Bang, no physical justification has been given for assumption (B).
In fact, assumption (B) consists of two parts here:
Firstly, we have assumed that K is complex-valued while it could be matrix-valued in the most general case. The reason for this assumption is that our proof requires the integral operator A to be a map from a certain Sobolev space onto itself in which weak derivatives with respect to the Dirac operators of the two particles can be taken. If K were matrix-valued, it would not commute with these Dirac operators in general. Then $A\psi$ would contain new types of weak derivatives which cannot be taken in the initial Sobolev space. As illustrated in Sec. 2.3.2.2, this creates a situation where more and more derivatives have to be controlled, possibly up to infinite order where the success of an iteration scheme seems unlikely. At present, we do not know how to deal with this issue. Improving on this point, however, defines an important task for future research, as e.g. electromagnetic interactions involve interaction kernels proportional to $\gamma_1^\mu \gamma_{2\mu}$ (see [64]).
Secondly, the physically most natural interaction kernel is given by a delta distribution along the light cone, $K(x_1, x_2) \propto \delta((x_1 - x_2)_\mu (x_1 - x_2)^\mu)$. In the Dirac case, the distributional derivatives make generalizations of results about more singular interaction kernels obtained in the KG case such as [72, 70] as well as section 2.2 difficult, and we have not attempted it here. Another interesting question is whether the smallness condition on K can be alleviated such that arbitrarily peaked functions are admitted. This could allow taking a limit where K approaches the delta distribution along the light cone.
Improving upon any of these two points would be very desirable.
In the case of scalar particles, we have proved the existence and uniqueness of solutions of the fully singular scalar integral equation (2.22) and its N-particle generalization (2.68). Following previous works and the Dirac case, we have depended upon assumption (A), i.e. a cut-off in time; however, in contrast to those cases considering a more regular

interaction kernel than what is present in (2.22), i.e. assumption (B) was not necessary. We have given the same justification for assumption (A) as in the Dirac case and in [70] by extending the main part of our result to the FLRW spacetime.
We have worked with a weighted L^∞ norm both for time and space variables in the case of scalar particles, while it would be more natural to use a weighted $L^\infty L^2$ norm instead. It would then be a challenging task to find the right inequalities to obtain similar estimates as we did. Moreover, one could also try to prove higher regularity not only in the sense of integrability but also differentiability. An interesting question, for example, is whether one can apply the Klein-Gordon operators $(\square_k + m_k^2)$ to the solutions of (2.22) in a weak sense.
This work provides a rigorous proof of the existence of interacting relativistic quantum dynamics in 1+3 spacetime dimensions; in particular, this model does not suffer from ultraviolet divergences which are typically encountered in quantum field theoretic models. Of course, the model does not describe particle creation and annihilation and is therefore a toy model rather than an alternative to QFT. Nevertheless, one might find the fact that direct interactions, even singular ones along the light cone, can be made mathematically rigorous, remarkable. One might wonder whether in the long run the mechanism of interaction through multi-time integral equations and direct interactions could contribute to a rigorous formulation of Quantum Field Theory.

Chapter 3

Quantum Field Theoretic Approach to Interactions

3.1 Introduction

3.1.1 Motivation

In this chapter we will turn our attention to the widely used dispersion relation for relativistic quantum systems obtained by filling up the negative energy states; see more details below. It is the result of joint work with my supervisors. For the fermionic parts of a Quantum Field Theory this approach is the standard one. The motivation and introduction closely follows [16]. While the rigorous quantum field theoretic formulation of free relativistic fermions is well-established [20] the introduction of interaction faces difficulties. In fact, introducing an external electromagnetic field acting on the fermions, while neglecting all interactions between the fermions, is already a non-trivial

matter. The completely satisfactory formulation of such an *external field quantum electrodynamics* (QED) is, to the best of my knowledge, still to be formulated, despite the famous works of Dirac [22], Feynman [35], Schwinger [95], Ruisenaars [86, 87], Langmann and Mickelsson [59], Dereziński and Gérard [20] and Thaller [99]. Not to mention a full QED including an interaction with a photon field. In recent years, there have been several novel attempts to the problem among them are what is called the causal fermion systems [37], which has a wider scope. There has been progress on the rigorous treatment of the search for minimizers in the static problem where a formulation in terms of Fock spaces has been emitted in favour of one in terms of projectors onto polarizations [50, 52, 49, 48, 43, 42], see [51] for a summary and [53] for a dynamic result in this formulation including Coulomb interaction but not magnetic fields and [44] for a derivation of the Euler-Heisenberg energy. Finally, there is the approach to construct the geometric phase of external field QED [90, 75, 13, 15, 14], which succeeded in constructing a time evolution operator but failed to identify a unique canonical choice. This subsection continues with a heuristic introduction into the latter approach.
When Dirac found the equation now bearing his name he recognized that the range of kinetic energies accessible to the particles is $]-\infty, -m] \cup [m, \infty[$ [24]. So he was worried that particles coupled to an electromagnetic field might radiate and lower their kinetic energy without bound, also cf. [46, Example 12.1]. Since particles capable of such behaviour would not form stable matter, he devised a way to introduce a stable ground state into the system. Instead of applying the Dirac equation

$$0 = (i\partial\!\!\!/ - m)\psi, \tag{3.1}$$

for $\psi \in L^2(\mathbb{R}^3, \mathbb{C}^4) =: \mathcal{H}$ to a fixed finite number of electrons, he sought to apply its evolution to an infinite antisymmetric product of the form

$$\Omega = \varphi_1 \wedge \varphi_2 \wedge \dots, \tag{3.2}$$

where $(\varphi_k)_{k\in\mathbb{N}}$ forms an ONB of the negative spectral subspace $\mathcal{H}^-$ of $\mathcal{H}$ with respect to the free Hamiltonian corresponding to the equation (3.1). Thus, making use of Pauli's exclusion principle for fermions no particle is any longer able to lower its kinetic energy. This object Ω resulting from filling up all the negative energy states is called the *Dirac sea* and constitutes the ground state introduced to this system. Let $U : \mathcal{H} \circlearrowleft$ be a one-particle evolution such as the time evolution subject to an external field $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4$ to act upon the particles, changing (3.1) to

$$0 = (i\partial\!\!\!/ - A\!\!\!/ - m)\psi, \tag{3.3}$$

where we have set the electric charge of the electron to one. Then Ω might be evolved according to

$$\mathcal{L}_U\Omega = U\varphi_1 \wedge U\varphi_2 \wedge \ldots. \tag{3.4}$$

The first step towards a theory including interactions is to allow for such an external field. We might now imagine a field A that acts only during a brief period of time. Such a field could pull a particle of the Dirac Sea Ω, say φ_1, out of $\mathcal{H}^-$ and above the surface $\mathcal{H}^+ \ni \xi = U^A\varphi_1$, where $\mathcal{H}^+$ is the positive spectral subspace of the Hamiltonian corresponding to (3.1). Such a field might not disturb the other wave functions much and the Dirac Sea after the action of the field might be represented by

$$\Psi = \xi \wedge \varphi_2 \wedge \varphi_3 \wedge \ldots. \tag{3.5}$$

Now, according to Dirac, since $\xi \in \mathcal{H}^+$, the corresponding particle behaves qualitatively differently compared to the remaining part of the Dirac sea which consists of wave functions taken from $\mathcal{H}^-$. This particle appears above the surface where it leaves behind a hole, the missing wave function $\varphi_1 \in \mathcal{H}^-$. These holes are also called positrons.

Following Dirac's argument, whenever the wave functions of very negative energy are left relatively unperturbed by the action of the process we can switch to a leaner description. If Ω remains unchanged below a certain level, such as in our heuristic example, it suffices to follow the generated particles e.g. ξ and the created holes e.g. φ_1. If on the other hand all wave functions are affected one has to keep track of a net evolution of Ω as well. As in this description the particle number is not a constant, creation operators are introduced. These act as

$$a^*(\xi)\varphi_2 \wedge \varphi_3 \wedge \cdots = \xi \wedge \varphi_2 \wedge \varphi_3 \wedge \ldots . \tag{3.6}$$

The adjoint of the creation operator is called annihilation operator and is denoted by a. Using these equation (3.5) can be condensed to

$$\Psi = a^*(\xi)a(\varphi_1)\Omega. \tag{3.7}$$

Given a one-particle time evolution operator U^A, its *lift* $\tilde{U}^A$ acting on objects like the wedge product (3.2) needs to fulfil

$$\tilde{U}^A a^*(\psi) = a^*(U^A\psi)\tilde{U}^A. \tag{3.8}$$

Requirement (3.8) is enough to fix $\tilde{U}^A$ up to a phase. Now, still $a^*(\chi)\Omega$ behaves differently for $\chi \in \mathcal{H}^+$ compared to $\chi \in \mathcal{H}^-$, so in order to completely forget about the Dirac sea in the notation one performs the splitting

$$a^*(f) = b^*(f) + c^*(f) \quad , b^*(f) = a^*(P^+f), \quad c^*(f) = a^*(P^-f), \tag{3.9}$$

exploiting linearity of a^*, where the orthogonal projectors onto the negative and positive energy subspaces of $\mathcal{H}$ are denoted by $P^- : \mathcal{H} \to \mathcal{H}^-$ and $P^+ := 1 - P^-$ respectively. Now, the space generated by elements of the form $b^*(f_1)\, b^*(f_2) \ldots b^*(f_n)\Omega$ is called electron Fock space $\mathcal{F}_e$ while the space generated by $c(f_1)\; c(f_2)\; \ldots c(f_n)\Omega$ is called hole Fock space $\mathcal{F}_h$. Here, the hole Fock space is generated by the annihilation operators of negative energy acting on the vacuum, which is another remnant of the fact that Ω is an infinite product state. This can be hidden by one more change in notation

$$d^*(f) = c(f). \tag{3.10}$$

Since c is antilinear, but d^* is supposed to be linear, one replaces $\mathcal{H}^-$ by $\overline{\mathcal{H}^-}$ as the domain of definition of d^*. By $\overline{\mathcal{H}^-}$ we denote the space that is identical to $\mathcal{H}^-$ as a set, but multiplication is defined by $(\mathbb{C} \times \overline{\mathcal{H}^+}) \ni (\lambda, f) \mapsto \lambda^* f$. Resulting in a corresponding replacement of $\mathcal{F}_h$ by $\overline{\mathcal{F}}_h$. Turning the full space into

$$\mathcal{F} = \mathcal{F}_e \otimes \overline{\mathcal{F}}_h. \tag{3.11}$$

Using this we can represent Ω by $|0\rangle = 1 \otimes 1$ and Ψ by $b^*(\xi) d^*(\varphi_1)|0\rangle$. Forgetting about the structure of Ω leads to problems, as we will see now. Using the techniques recalled in the following sections one can express the probability of creation of at least one pair due to time evolution from t_0 to t_1 by a lift fulfilling (3.8) subject to an external potential A as a Fredhom determinant, $2(1 - \sqrt{\det_{\mathcal{H}^-}(1 - |U^A_{+-}|^2)})$. Picking an ONB $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathcal{H}^+$ and $(\varphi_{-n})_{n\in\mathbb{N}}$ of $\mathcal{H}^-$ this yields to leading order

$$\sum_{k,n\in\mathbb{N}} |\langle \varphi_k, U^A(t_1, t_0)\varphi_{-n}\rangle|^2 = \|U^A_{+-}(t_1, t_0)\|^2_{I_2}, \tag{3.12}$$

where $U_{\pm\mp} := P^\pm U P^\mp$ for any operator U and $\|\cdot\|_{I_2}$ is the Hilbert-Schmidt norm. The space of operators of type $\mathcal{H} \circlearrowleft$ induced by this norm is denoted by $I_2(\mathcal{H})$. As a probability the expression (3.12) needs to be bounded by one; however, this is not always the case.

Theorem 35 (Ruijsenaars [86, 87]). *For times $t_0, t_1 \in \mathbb{R}$ the right-hand side of* (3.12) $< \infty$ *if and only if* $\vec{A}(t_0) = 0 = \vec{A}(t_1)$.

There is one more classical theorem to take note of in this context.

Theorem 36 (Shale-Stinespring [97]). *The one-particle operator U has a lift $\tilde{U} : \mathcal{F} \circlearrowleft$ satisfying (3.8) if and only if $U_{\pm}, U_{\mp} \in I_2(\mathcal{H})$.*

Combined these theorems imply that unless the condition $\vec{A} = 0$, which is highly artificial in the light of Lorentz and gauge invariance, is fulfilled there is no representation of the time evolution operator subject to the field A into Fock space.
Stated in this way the last statement might nudge one into concluding that the Fock space representation is a dead end. In fact, the situation is more subtle. To get a heuristic idea, recall that positive and negative energy states differ in the direction of their spinors. Meaning that multiplying a negative energy state with e.g. γ matrices will in general result in a mixture of positive and negative energy states. Incidentally, this is exactly what happens in the Hamiltonian

$$H^A = \gamma^0(-i\vec{\gamma} \cdot \text{grad} + m) + A_0 - \gamma^0\vec{\gamma} \cdot \vec{A}. \tag{3.13}$$

Since there are infinitely many particles in the wedge product of Ω and there is no mechanism of suppression for states of large momentum, the sum in (3.12) does not converge for $\vec{A} \neq 0$. However, as it turns out there is a $\tilde{U}^A(t_1, t_0) : \mathcal{F} \circlearrowleft$ whenever $\vec{A}(t_1) = 0 = \vec{A}(t_0)$. This implies that once the vector parts $\vec{A}$ vanishes, only finitely many pairs remain, justifying the term “virtual pairs” for the infinity of pairs that appear and vanish together with $\vec{A}$.
As we can read off of our construction of Fock space, this space consists of infinite wedge products that are sufficiently close to the initial state Ω. We just found out that, this space is not large enough to contain the state also at later times when the external field is non-zero, but that does not mean that we cannot find a mapping from the initial state to the later ones. It would be enough to adapt the choice of space at later times to the external field present at that time. These spaces

will in general not give a physically meaningful distinction between electrons and parts of the Dirac sea. Such a distinction may have to wait for a full interacting theory of QED, where it may be given in terms of ground states or states homogeneous and isotropic enough such that excitations above it behave effectively free. However, such a distinction is not necessary to answer physical questions such as which currents are induced by strong external fields or how Maxwell's equations are modified by those currents.
In the last decade progress has been made to construct the evolution operator of external field QED mapping states of one Fock space to another and to identify the remaining freedom of picking Fock spaces at each hypersurface or point in time. The results generalize theorems 35 and 36 in a way that exposes the gauge and relativistic invariance inherent to the problem, which is not apparent in the original versions. In order to state those theorems we need some mathematical notation which we introduce first.

3.1.2 Overview of Previous Results

This subsection will first introduce some notation of [13, 15, 14, 16] in order to state the results of [14] and prepare for the sections to come. While doing so we will closely follow [16]. Throughout the whole chapter the class of four potentials we are interested in is

$$\mathcal{V} := C_c^\infty(\mathbb{R}^4, \mathbb{R}^4). \tag{3.14}$$

All the results could be extended with a reasonable amount of additional work to slightly more general four-potentials, but not to physically realistic ones such as the Coulomb potential.
Any notion of time evolution in a relativistic setting needs to generalize the notion of simultaneity. For this reason we introduce Cauchy surfaces.

Definition 37 (Cauchy surface, definition 2.1 of [16])**.** *A Cauchy surface $\Sigma \subset \mathbb{R}^4$ is a smooth, 3-dimensional submanifold of $\mathbb{R}^4$ that fulfils the following three conditions:*

a) Every inextensible, two-sided, time- or light-like, continuous path in $\mathbb{R}^4$ intersects Σ in a unique point.

b) For every $x \in \Sigma$, the tangential space $T_x\Sigma$ is space-like.

c) The tangential spaces to Σ are bounded away from light-like directions in the following sense: The only light-like accumulation point of $\bigcup_{x\in\Sigma} T_x\Sigma$ is zero.

Definition 38 ($\mathcal{H}_\Sigma$, definition 2.2 of [16])**.** *For every Cauchy surface Σ there is a parametrization*

$$\Sigma = \{\pi_\Sigma(\vec{x}) := (t_\Sigma(\vec{x}), \vec{x}) \mid \vec{x} \in \mathbb{R}^3\}, \tag{3.15}$$

with a smooth function $t_\Sigma : \mathbb{R}^3 \to \mathbb{R}$. Agreeing with standard notation, $d^4x = dx^0dx^1dx^2dx^3$ denotes the standard volume form over $\mathbb{R}^4$, where the product of forms is the wedge product. By d^3x we denote the form $d^3x = dx^1dx^2dx^3$ both on $\mathbb{R}^4$ and on $\mathbb{R}^3$. When contracting a form ω with a vector v we will be denoting this by $i_v(\omega)$. We will keep writing $i_v(\omega)$ also for the spinor matrix valued vector $\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = \gamma^\mu e_\mu$:

$$i_\gamma(d^4x) = \gamma^\mu i_{e_\mu}(d^4x). \tag{3.16}$$

For any $x \in \Sigma$ restricting the 3-form $i_\gamma(d^4x)$ to the tangent space $T_x\Sigma$ results in

$$i_\gamma(d^4x) = \not{n}(x) i_n(d^4x) = \left(\gamma^0 - \sum_{\mu=1}^{3} \gamma^\mu \frac{\partial t_\Sigma(\vec{x})}{\partial x^\mu}\right) d^3x \tag{3.17}$$

Being able to write a Poincaré covariant measure on Cauchy surfaces we may introduce the scalar product

$$\phi, \psi \mapsto \int_\Sigma \overline{\phi}(x) i_\gamma(d^4\gamma)\psi(x) =: \langle \phi, \psi \rangle \tag{3.18}$$

and $\overline{\phi} = \phi^\dagger \gamma^0$*. With respect to this scalar product we define* $\mathcal{H}_\Sigma = L^2(\Sigma, \mathbb{C}^4)$*.*

As is well known (e.g. [3, 15]) the Dirac equation coupled to an external potential, equation (3.3), has a one-particle evolution operator for each pair of Cauchy surfaces Σ, Σ'

$$U_{\Sigma',\Sigma} : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}, \tag{3.19}$$

Using this covariant replacement of the standard Hilbert-space we repeat the heuristic Fock space construction of subsection 3.1.1 in a slightly more general and detailed fashion.

Definition 39 (Fock space of generalized polarization, definition 2.4 of [16])**.** *Let* $Pol(\mathcal{H}_\Sigma)$ *denote the set of all closed, linear subspaces* $V \subset \mathcal{H}_\Sigma$ *such that both* V *and* $V^\perp$ *are infinite dimensional. Any* $V \in Pol(\mathcal{H}_\Sigma)$ *is called* polarization *of* $\mathcal{H}_\Sigma$*. For* $V \in Pol(\mathcal{H}_\Sigma)$*, let* $P_\Sigma^V : \mathcal{H}_\Sigma \to V$ *denote the orthogonal projection of* $\mathcal{H}_\Sigma$ *onto* V*. The Fock space corresponding to* V *on the Cauchy surface* Σ *is defined to be*

$$\mathcal{F}(V, \mathcal{H}_\Sigma) := \bigoplus_{c \in \mathbb{Z}} \mathcal{F}_c(V, \mathcal{H}_\Sigma), \quad \mathcal{F}_c(V, \mathcal{H}_\Sigma) := \bigoplus_{\substack{n,m \in \mathbb{N}_0 \\ c = m-n}} (V^\perp)^{\wedge n} \otimes \overline{V}^{\wedge m}, \tag{3.20}$$

where $\bigoplus$ *is the Hilbert space direct sum,* $\wedge$ *the antisymmetric tensor product of Hilbert spaces and* $\mathcal{H}^{\wedge m}$ *is the* n*-fold antisymmetric product of the Hilbert space* $\mathcal{H}$ *and* $\overline{V}$ *is the conjugate complex vector space of* V*, which is identical to* V *as a set, but multiplication is defined by*

$(\mathbb{C} \times \overline{V}) \ni (\lambda, f) \mapsto \lambda^* f$. *Furthermore, for the special case* $\mathcal{H} := \mathcal{H}_{\Sigma_0}$ *with* Σ_0 *earlier than the support of any four-potential considered in this chapter we introduce the abbreviations*

$$P^- := P^{\mathcal{H}^-}_{\Sigma_0} \tag{3.21}$$

$$P^+ = 1 - P^-, \tag{3.22}$$

where $\mathcal{H}^-$ *is the negative spectral subspace of the Hamiltonian* (3.13) *for* $A = 0$. *We also introduce the abbreviation*

$$\mathcal{F} := \mathcal{F}(\mathcal{H}^-, \mathcal{H}), \tag{3.23}$$

as well as $\Omega \in \mathcal{F}$ *for a fixed element with* $m, n = 0$ *in equation* (3.20) *and* $\|\Omega\| = 1$.

Pick Cauchy surfaces Σ, Σ' and polarizations $V \in \mathrm{Pol}(\mathcal{H}_\Sigma)$, $V' \in \mathrm{Pol}(\mathcal{H}_{\Sigma'})$ then we can give the analogue of the lift condition (3.8) in this setting: for all $\psi \in \mathcal{H}_\Sigma$

$$\tilde{U}^A_{V',\Sigma';V,\Sigma} a^*_\Sigma(\psi) = a^*_{\Sigma'}(U^A_{\Sigma',\Sigma}\psi)\tilde{U}^A_{V',\Sigma';V,\Sigma}, \qquad \text{(lift condition)}$$

holds, where $a^*_{\Sigma'}$ and a^*_Σ are the creation operator associated to $\mathcal{F}(V, \mathcal{H}_\Sigma)$ and $\mathcal{F}(V', \mathcal{H}_{\Sigma'})$ respectively.
The rephrasing of theorem 36 adapted to the more general notation we have developed now is

Theorem 40 ([14], also cor. 2.5 of [16])**.** *Let* Σ, Σ' *be Cauchy surfaces,* $V \in$ *Pol*$(\mathcal{H}_\Sigma)$*, and* $V' \in$ *Pol*$(\mathcal{H}_{\Sigma'})$ *be polarizations. Then the following two statements are equivalent:*

a) There is a unitary operator $\tilde{U}^A_{V',\Sigma';V,\Sigma} : \mathcal{F}(V, \Sigma) \to \mathcal{F}(V', \Sigma')$ *that satisfies the* (lift condition)

b) The operators $P^{V'\perp}_{\Sigma'} U^A_{\Sigma'.\Sigma} P^V_\Sigma$ *and* $P^{V'}_{\Sigma'} U^A_{\Sigma'.\Sigma} P^{V\perp}_\Sigma$ *are Hilbert-Schmidt operators.*

So the question given an initial state in an initial Fock space $\mathcal{F}(V,\Sigma)$, which Fock space we may pick at a final Cauchy surface Σ' such that there is a lift fulfilling the (lift condition) now becomes a question of polarizations. We know a priori that $U^A_{\Sigma',\Sigma}$ has a lift from $\mathcal{F}(V,\Sigma)$ to $\mathcal{F}(U^A_{\Sigma',\Sigma}V,\Sigma')$. Furthermore, we have a distinguished polarization for very early times, namely the negative energy states with respect to the free Hamiltonian. Thus, we may characterize all relevant polarization classes into the following equivalence classes

Definition 41 (polarization classes, def. 2.6 of [16])**.** *For a Cauchy surface Σ and a potential $A \in \mathcal{V}$ we define*

$$C_\Sigma(A) := \left\{ W \in Pol(\mathcal{H}_\Sigma) \mid W \approx U^A_{\Sigma,\Sigma_{\mathrm{in}}} \mathcal{H}^-_{\Sigma_{\mathrm{in}}} \right\}, \tag{3.24}$$

where Σ_{in} is a Cauchy surface earlier than $\operatorname{supp} A$ *and $\mathcal{H}^-_{\Sigma_{in}}$ is the subspace spanned by the wave functions in the negative spectrum of the free Dirac Hamiltonian. Furthermore, for $V, V' \in Pol(\mathcal{H}_\Sigma)$ we write $V \approx V'$ whenever $P^V_\Sigma - P^{V'}_\Sigma$ is a Hilbert-Schmidt operator.*

Using this we immediately find

Corollary 42 (polarization classes and lifts, cor 2.7 of [16])**.** *Let Σ, Σ' be Cauchy surfaces and $V \in C_\Sigma(A), V' \in Pol(\mathcal{H}_{\Sigma'})$ be polarizations. Then there is a unitary operator $\tilde{U}^A_{V',\Sigma';V,\Sigma} : \mathcal{F}(V,\Sigma) \to \mathcal{F}(V',\Sigma')$ satisfying the* (lift condition) *if and only if $V' \in C_{\Sigma'}(A)$.*

The definition 41 suggests a dependence of $C_\Sigma(A)$ on all of A as a function of time. As indicated in the last subsection, this is not the case.

Theorem 43 ($C_\Sigma(A)$ depends on $A|_{T\Sigma}$, thm. 1.5 of [14])**.** *Pick a Cauchy surface Σ and $A, A' \in \mathcal{V}$. Then we have*

$$C_\Sigma(A) = C_\Sigma(A') \iff A|_{T\Sigma} = A'|_{T\Sigma}, \tag{3.25}$$

where $A|_{T\Sigma} = A'|_{T\Sigma}$ *means that for all* $x \in \Sigma$ *and* $y \in T_x\Sigma$ *the relation* $A_\mu(x)y^\mu = A'_\mu(x)y^\mu$ *holds.*

This is a version of theorem 35 for general Cauchy surfaces. The next theorem shows how the polarization classes change with the gauge and Lorentz transforms.

Theorem 44 (transformation of polarization classes, thm. 1.6 of [14])**.** *Let* $A \in \mathcal{V}$ *be a four-potential and* Σ *a Cauchy surface.*

a) Let $(\mathfrak{s}, \Lambda) \in \mathbb{C}^{4\times 4} \times \mathrm{SO}^\uparrow(1,3)$ *be an orthochronous Lorentz transform, i.e. the tuple fulfils* $\Lambda^\mu_\sigma g_{\mu\nu} \Lambda^\nu_\tau = g_{\sigma,\tau}$ *and* $\Lambda^\mu_\nu \gamma^\nu = \mathfrak{s}^{-1}\gamma^\mu \mathfrak{s}$ *and acts on wave functions as* $\psi \mapsto \mathfrak{s}\psi(\Lambda^{-1}\cdot)$ *(see sec. 2.3 of [15]). Then we have*

$$V \in C_\Sigma(A) \iff (\mathfrak{s}, \Lambda)V \in C_{\Lambda\Sigma}(\Lambda A(\Lambda^{-1}\cdot)). \tag{3.26}$$

b) Let $A' = A + d\zeta$ *be the gauge transformed potential, for some* $\zeta \in C_c^\infty(\mathbb{R}^4, \mathbb{R})$*. Then the gauge transformation acts on wave functions as* $e^{-i\zeta} : \mathcal{H}_\Sigma \circlearrowleft, \psi \mapsto e^{-i\zeta}\psi$ *and one obtains*

$$V \in C_\Sigma(A) \iff e^{-i\zeta}V \in C_\Sigma(A + d\zeta). \tag{3.27}$$

Theorems 44, 43 and corollary 42 make clear in which way the original plan to work in a single Fock space was misguided and how it may be adapted to make it work.
When trying to construct an evolution operator from a Cauchy surface Σ to a second one Σ' subject to an external field A, one has to choose an initial polarization $V \in C_\Sigma(A)$ and a final polarization $V' \in C_{\Sigma'}(A)$. Then there is an evolution operator $\tilde{U}^A_{V',\Sigma';V,\Sigma}$, unique up to a phase. Picking polarizations is akin to picking a patch of coordinates on a non-trivial manifold, in the sense that there may not

be a canonical choice and the choice is going to influence the representation of all the relevant objects. Nevertheless, one can obtain valuable information from calculations done with respect to one such choice and always transform the results to the representation induced by any other choice. Carrying out such a procedure will for every $\Phi \in \mathcal{F}(V, \Sigma)$ and $\Psi \in \mathcal{F}(V', \Sigma')$ result in finite transition probabilities $|\langle \Psi, \tilde{U}^A_{V',\Sigma',V,\Sigma}\Phi\rangle|^2$ without the need for renormalization.
To make the discussion more concrete, we are going to introduce a particular representation of the Fock spaces and evolution operators discussed so far. This representation is heavily inspired by Dirac's original idea discussed in subsection 3.1.1 and is usually referred to as infinite wedge space. For further details please have a look at section 2 of [13].
The basic idea behind this representation is a generalization of the following point of view of the scalar product of finitely many fermions but works for any Hilbert space $\mathcal{H}$. Pick $N \in \mathbb{N}$ and two states $\Lambda\Psi, \Lambda\Phi \in \mathcal{H}^{\wedge N}$ that can be written as the wedge product of N wave functions

$$\Lambda\Psi = \psi_1 \wedge \cdots \wedge \psi_N \tag{3.28}$$
$$\Lambda\Phi = \phi_1 \wedge \cdots \wedge \phi_N, \tag{3.29}$$

then the standard scalar product $\langle \Psi, \Phi\rangle$ in $\mathcal{H}^{\wedge N}$ can be written as a determinant

$$\det_{\mathbb{R}^N}(\Psi^*\Phi), \tag{3.30}$$

where Φ and Ψ are interpreted to be linear maps of type $\mathbb{R}^N \to \mathcal{H}$

$$\forall k : \Psi : e_k \mapsto \psi_k \tag{3.31}$$
$$\forall k : \Phi : e_k \mapsto \phi_k \tag{3.32}$$

where $(e_k)_{k\in\{1,\dots,N\}}$ is an ONB of $\mathbb{R}^N$ and the star denotes the adjoint. Non product states first have to be decomposed into a sum of product states, then the determinant is continued to be linear in the left and right factor. Replacing $\mathbb{R}^N$ by an index space ℓ, e.g. $l^2(\mathbb{N})$ the space of square summable sequences, and using the Fredholm determinant this particular representation of the scalar product of $\mathcal{H}^N$ can be directly generalized to an infinite number of particles. In this representation, product states Φ can be thought of as maps

$$\Phi : \ell \to \mathcal{H}, e_k \mapsto \varphi_k. \tag{3.33}$$

Such maps will be called *Dirac seas*. The corresponding wedge product can be thought of as

$$\Lambda\Phi = \varphi_1 \wedge \varphi_2 \dots . \tag{3.34}$$

However, as operators on an infinite dimensional Vector space ℓ only have a Fredholm determinant if they are in the set $1 + I_1(\ell)$ (here I_1 denotes the space of operators with finite trace-norm), only Dirac seas $\Phi, \Psi : \ell \to \mathcal{H}$ satisfying

$$\Psi^*\Phi, \Psi^*\Psi, \Phi^*\Phi \in 1 + I_1(\ell) \tag{3.35}$$

have a scalar product. Starting with one particular infinite wedge product $\Lambda\Phi$ and collecting all infinite wedge products $\Lambda\Psi$ such that (3.35) is fulfilled and formal linear combinations thereof and taking the completion with respect to the pairing (3.30) results in what is referred to as infinite wedge space $\mathcal{F}_{\Lambda\Phi}$. For a rigorous construction please have a look at [13, section 2.1].
It is worth noticing here, that if one starts from some other infinite wedge product $\Lambda\Psi$ such that $\Phi^*\Psi \in 1+I_1(\ell)$ to construct $\mathcal{F}_{\Lambda\Psi}$ one finds $\mathcal{F}_{\Lambda\Psi} = \mathcal{F}_{\Lambda\Phi}$, so there is no unique vacuum state in the infinite wedge space. Pick a second Hilbert space $\mathcal{H}'$ and some unitary operator $U : \mathcal{H} \to \mathcal{H}'$, which can be thought of as $U^A_{\Sigma',\Sigma}$. Next we define the operation form the left of U by

$$\mathcal{L}_U : \mathcal{F}_{\Lambda\Phi} \to \mathcal{F}_{\Lambda U\Phi}, \tag{3.36}$$

$$\mathcal{L}_U\Lambda\Psi = \Lambda U\Psi = (U\psi_1) \wedge (U\psi_2) \wedge \dots, \tag{3.37}$$

where $\Psi : \ell \to \mathcal{H}$ satisfies $\Psi^*\Phi \in 1 + I_1(\ell)$. The operator $\mathcal{L}_U$ is a lift of U in the sense of the (lift condition) and maps one Fock space to another. The resulting target space $\mathcal{F}_{\Lambda U\Phi}$ is quite implicit and for $\mathcal{H} = \mathcal{H}_\Sigma$, $\mathcal{H}' = \mathcal{H}_{\Sigma'}$ and $U = U^A_{\Sigma',\Sigma}$ in general not identical to $\mathcal{F}_{\Lambda\Phi'}$ for some $\Phi' : \ell \to \mathcal{H}'$ even if range$(\Phi) \in C_\Sigma(A)$ and range$(\Phi') \in C_{\Sigma'}(A)$ hold. This shortcoming of the construction can be overcome by adding an additional operation from the right. Let $\Psi : \ell' \to \mathcal{H}_{\Sigma'}$ such that range(Ψ) = range(Φ') holds, then there is a unitary $R : \ell' \to \ell$ such that $\Phi' = \Psi R$. Analogously to the action from the left we define one from the right:

$$\mathcal{R}_R : \mathcal{F}_{\Lambda\Psi} \to \mathcal{F}_{\Lambda\Phi'}, \tag{3.38}$$

$$\mathcal{R}_R \Lambda\Phi' = \Lambda(\Phi' R), \tag{3.39}$$

also here the Φ' generate the infinite wedge space $\mathcal{F}_{\Lambda\Psi}$. The spaces $\mathcal{F}_{\Lambda\Psi}$ and $\mathcal{F}_{\Lambda\Psi R}$ only coincides if $\ell' = \ell$ and R has a determinant, i.e. $R \in 1 + I_1(\ell)$. The next theorem helps us to decide in which cases there is a unitary $R : \ell' \to \ell$ such that $\mathcal{F}_{\Lambda U\Phi R} = \mathcal{F}_{\Lambda\Phi'}$ holds.

Theorem 45 (thm. 36 for $\mathcal{L}$ and $\mathcal{R}$, [13, thm. 2.26], [16, thm 3.1])**.** *Let $\mathcal{H}, \ell, \mathcal{H}', \ell'$ be Hilbert spaces, $V \in$ Pol$(\mathcal{H})$ and $V' \in$ Pol$(\mathcal{H}')$ polarizations, $\Phi : \ell \to \mathcal{H}$ and $\Phi' : \ell' \to \mathcal{H}'$ be Dirac seas such that* range$(\Phi) = V$ *and* range$(\Phi') = V'$ *Then the following two statements are equivalent*

a) The off diagonal operators $P^{V'^\perp} U P^V$ and $P^{V'} U P^{V^\perp}$ are Hilbert-Schmidt operators.

b) There is a unitary $R : \ell \to \ell'$ such that $\mathcal{F}_{\Lambda\Phi'} = \mathcal{F}_{\Lambda U\Phi R}$.

So returning to $\Phi : \ell \to \mathcal{H}_\Sigma$, $\Phi' : \ell \to \mathcal{H}_{\Sigma'}$, with range$(\Phi) \in C_\Sigma(A)$, range$(\Phi') \in C_{\Sigma'}(A)$ for some Cauchy surfaces Σ, Σ' and some $A \in \mathcal{V}$, condition a) of the last theorem is satisfied so the existence of $R : \ell \to \ell$ such that the evolution operator

$$\tilde{U}^A_{V',\Sigma',V,\Sigma} : \mathcal{F}_{\wedge\Phi} \to \mathcal{F}_{\wedge\Phi'}, \quad \tilde{U}^A_{V',\Sigma',V,\Sigma} = \mathcal{L}_{U^A_{\Sigma',\Sigma}} \circ \mathcal{R}_R \tag{3.40}$$

is well-defined and unitary is ensured. The operators $\mathcal{R}_R$ are unique up to a phase, see [13, cor. 2.28], as might have been expected since the (lift condition) allows for exactly this much freedom. A simple choice of ℓ, ℓ' that makes it possible to guess a choice for R is $\ell = \mathrm{range}(\Phi) \subseteq \mathcal{H}_\Sigma$, $\ell' = \mathrm{range}(\Phi') \subseteq \mathcal{H}_{\Sigma'}$. As discussed in subsection 3.1.1 one might hope that the motion of the electrons of very negative energy is irrelevant for understanding excitations at the surface. However, as we saw the motion of the electrons at great depth made it impossible to directly compare the time evolved states with the original ones. We can use R to revert this motion. So the idea is to pick $R' = (P^{V'}UP^V)^{-1}$ whenever $P^{V'}UP^V$ is invertible, but this choice is not unitary. If $P^{V'}UP^V$ is not invertible one can perform the construction outlined next in several steps and assemble a lift of the total operator U. This is possible, because of for two unitary operators U_1, U_2 and corresponding lifts $\tilde{U}_1, \tilde{U}_2$,

$$\tilde{U}_1\tilde{U}_2 \tag{3.41}$$

is a lift of U_1U_2. By virtue of the scalar product of two infinite wedge products being a determinant and the equation

$$\det((AR)^*BR) = \det(R^*A^*BR) = \det(RR^*A^*B) \tag{3.42}$$

$$= \det(RR^*)\det(A^*B) = \det(R^*R)\det(A^*B), \tag{3.43}$$

is also true for bounded operators A, B, R of appropriate type whenever R is invertible and R^*R, A^*B each have a determinant. So the operation from the right $\mathcal{R}_{R'}$ may still be defined and a posteriori be corrected by a factor of $\sqrt{\det(R'^*R')^{-1}}$ to turn it into a unitary operator. Using

$$1 = U^{A*}U^A = (P^V + P^{V^\perp})U^{A*}(P^{V'} + P^{V'^\perp})U^A(P^V + P^{V^\perp}), \quad (3.44)$$

and splitting the equation up according to the different initial and target spaces, this implies

$$P^V U^{A*} P^{V'} U^A P^V = 1 - P^V U^{A*} P^{V'^\perp} U^A P^V. \quad (3.45)$$

Polarizations V, V' belonging to the appropriate polarization classes $V \in C_\Sigma(A), V' \in C_{\Sigma'}(A)$ satisfy condition a) of theorem 45 and because the product of Hilbert-Schmidt operators is trace class $R'^*R' \in 1 + I_1(V)$, i.e. has a determinant. Hence, we may define

$$\tilde{U}^A_{V',\Sigma',V,\Sigma} : \mathcal{F}_{\Lambda\Phi} \to \mathcal{F}_{\Lambda\Phi'}, \quad (3.46)$$

$$\tilde{U}^A_{V',\Sigma',V,\Sigma} = \det|(P^{V'}U^A_{\Sigma',\Sigma}P^V)|\mathcal{R}_{(P^{V'}U^A_{\Sigma',\Sigma}P^V)} \circ \mathcal{L}_{U^A_{\Sigma',\Sigma}}. \quad (3.47)$$

Well-definedness can be checked directly, the operator

$$\Phi'^* U^A_{\Sigma',\Sigma}\Phi R' = P^{V'}U^A_{\Sigma',\Sigma}P^V R' = 1_{V'} \quad (3.48)$$

clearly has a determinant on V'. By similar calculations as (3.42) it follows that also all operators $\tilde{\Phi}' U^A_{\Sigma',\Sigma}\tilde{\Phi}R'$ with $\Lambda\tilde{\Phi}' \in \mathcal{F}_{\Lambda\Phi'}$, $\Lambda\tilde{\Phi} \in \mathcal{F}_{\Lambda\Phi}$ have a determinant.
So the construction of an evolution operator in external field QED was successful.

3.2 Geometric Construction of the Phase

In this section we perform a geometric construction of the phase of the evolution operator based on an object c^+ whose existence is conjectured due to physical intuition. This construction is heavily inspired

by [96]. This object c^+ has not itself been constructed yet, but the author and his collaborators hope to construct it in the near future. We restrict ourselves to the scattering regime, because it reduces the available freedom. In this case we identify the initial and final Hilbert-space both with $\mathcal{H}$ and use the polarization introduced in definition 39.
Because our notions of the argument of a complex number and the argument of an invertible bounded operator is non-standard, we introduce it next.

Definition 46 (polar decomposition, logarithmic derivative and derivative with respect to four-potential)**.** *For $X : \mathcal{H} \to \mathcal{H}$ bounded and invertible we introduce*

$$\mathrm{AG}(X) := X|X|^{-1}. \tag{3.49}$$

Furthermore, we define for any complex number $z \in \mathbb{C}\backslash\{0\}$

$$\mathrm{ag}(z) := \frac{z}{|z|}. \tag{3.50}$$

In abuse of notation we define the expression

$$\partial_t \ln f(t) := \frac{\partial_t f(t)}{f(t)}, \tag{3.51}$$

for any differentiable $f : \mathbb{R} \to \mathbb{C}\backslash\{0\}$, even if the expression $\ln f(t)$ cannot be interpreted as the principal branch of the logarithm.
We also introduce $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. We will denote the space of bounded linear functions from one normed vector space V into itself by $\mathcal{B}(V)$.
Lastly we introduce the partial derivative in the direction of any four-potential F of an operator valued function $F : \mathcal{V} \to \mathcal{B}(\mathcal{F})$ by

$$\partial_F T(F) := \partial_\varepsilon T(\varepsilon F)|_{\varepsilon=0}, \tag{3.52}$$

where the limit is taken with respect to the operator norm topology.

Definition 47 (scattering operator and phases)**.** *We define for all* $A, B \in \mathcal{V}$

$$S_{A,B} := U^{A}_{\Sigma_{\text{in}},\Sigma_{\text{out}}} U^{B}_{\Sigma_{\text{out}},\Sigma_{\text{in}}}, \tag{3.53}$$

where Σ_{out} *and* Σ_{int} *are Cauchy surfaces of Minkowski spacetime such that*

$$\forall (x,y) \in \operatorname{supp} A \cup \operatorname{supp} B \times \Sigma_{\text{in}} : (x-y)^2 \geqslant 0 \Rightarrow x^0 > y^0, \tag{3.54}$$
$$\forall (x,y) \in \operatorname{supp} A \cup \operatorname{supp} B \times \Sigma_{\text{out}} : (x-y)^2 \geqslant 0 \Rightarrow x^0 < y^0 \tag{3.55}$$

holds. For the special case $A = 0$ *we define the shorthand* $S^B = S_{0,B}$. *Define for any* a, b *elements of the same real or complex vector space the line segment connecting them*

$$\overline{a\ b} := \{sa + (1-s)b \mid s \in [0,1]\}. \tag{3.56}$$

Let

$$\text{dm} := \{(A,B) \in \mathcal{V}^2 \mid P^- S_{A,B} P^- \ \textit{and} \tag{3.57}$$
$$P^- S_{B,A} P^- : \mathcal{H}^- \circlearrowleft \ \textit{are invertible}\}, \tag{3.58}$$

we define

$$\operatorname{dom} \overline{S} := \{(A,B) \in \text{dm} \mid \overline{A\ B} \times \overline{A\ B} \subseteq \text{dm}\}. \tag{3.59}$$

Furthermore, we choose for all $A, B \in \operatorname{dom} \overline{S}$ *the lift discussed at the end of the last section*

$$\overline{S}_{A,B} = \mathcal{R}_{\text{AG}((P^- S_{A,B} P^-)^{-1})} \mathcal{L}_{S_{A,B}}. \tag{3.60}$$

For $(A,B), (B,C), (C,A) \in \operatorname{dom} \overline{S}$, *we define the complex numbers*

$$\gamma_{A,B,C} := \det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-), \tag{3.61}$$
$$\Gamma_{A,B,C} := \text{ag}(\gamma_{A,B,C}). \tag{3.62}$$

We will see in lemma 53 that $\gamma_{A,B,C} \neq 0$ *and* $P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} \in 1 + I_1(\mathcal{H}^-)$*, so that* $\Gamma_{A,B,C}$ *is well-defined. Next, we introduce for* $A, B, C \in \mathcal{V}$ *the function*

$$c_A(F, G) := -i\partial_F \partial_G \Im \operatorname{tr}[P^- S_{A,A+F} P^+ S_{A,A+G} P^-]. \tag{3.63}$$

Finally, let a, b *be two subsets of Minkowski spacetime, we say* $a \prec b$ *(in words: "a is causally prior to b") if and only if for all* $(x, y) \in a \times b$

$$\big((x-y)^2 \geqslant 0 \wedge x \neq y\big) \Rightarrow x^0 < y^0 \tag{3.64}$$

holds. For $A, B \in \mathcal{V}$ *in the expressions* $a \prec A, A \prec a, A \prec B$ *the four-potentials* A, B *are to be interpreted as* $\operatorname{supp} A, \operatorname{supp} B$ *respectively.*

Lemma 48 (properties of $\operatorname{dom} \overline{S}$)**.** *The set* $\operatorname{dom} \overline{S}$ *has the following properties:*

1. *contains the diagonal:* $\{(A, A) \mid A \in \mathcal{V}\} \subseteq \operatorname{dom} \overline{S}$*.*
2. *openness:* $\forall n \in \mathbb{N} : \left\{s \in \mathbb{R}^{2n} \mid \left(\sum_{k=1}^{n} s_k A_k, \sum_{k=n+1}^{2n} s_k A_k\right) \in \operatorname{dom} \overline{S}\right\}$ *is an open subset of* $\mathbb{R}^{2n}$ *for all* $A_1, \ldots A_{2n} \in \mathcal{V}$*.*
3. *symmetry:* $(A, A') \in \operatorname{dom} \overline{S} \iff (A', A) \in \operatorname{dom} \overline{S}$
4. *star-shaped:* $(A, tA) \in \operatorname{dom} \overline{S} \Rightarrow \forall s \in \overline{1\ t} : (A, sA) \in \operatorname{dom} \overline{S}$
5. *well-definedness of* $\overline{S}$*:* $\operatorname{dom} \overline{S} \subseteq \{A, B \in \mathcal{V} \mid P^- S_{A,B} P^- : \mathcal{H}^- \circlearrowleft$ *is invertible*$\} \to \mathcal{B}(\mathcal{F})$*.*

Proof. We will only prove openness, as the other properties follow directly from the definition (3.59). So pick $n \in \mathbb{N}$, $A_i \in \mathcal{V}$ for $i \in \mathbb{N}, i \leqslant 2n$ and $s \in \mathbb{R}^{2n}$ such that $\left(\sum_{k=1}^{n} s_k A_k, \sum_{k=n+1}^{2n} s_k A_k\right) \in \operatorname{dom} \overline{S}$. We have to find a neighbourhood $U \subseteq \mathbb{R}^{2n}$ of s such that $\{\left(\sum_{k=1}^{n} s'_k A_k, \sum_{k=n+1}^{2n} s'_k A_k\right) \mid s' \in U\} \subseteq \operatorname{dom} \overline{S}$ holds. In doing so we have to ensure that the square

$$\overline{\sum_{k=1}^{n} s'_k A_k \qquad \sum_{k=n+1}^{2n} s'_k A_k}^{\,2} \tag{3.65}$$

stays a subset of dm for all $s' \in U$. Now pick a metric d on $\mathbb{R}^{2n}$ and define

$$r := \inf \left\{ d(s, s') \mid \overline{\sum_{k=1}^{n} s'_k A_k \qquad \sum_{k=n+1}^{2n} s'_k A_k}^{\,2} \cap \mathrm{dm}^c \neq \varnothing \right\}.$$

It cannot be the case that $r = 0$, because the metric is continuous, the square compact in $\mathbb{R}^{2n}$ and the set of invertible bounded operators (defining dm) is open in the topology generated by the operator norm. If $r = \infty$ then $U = \mathbb{R}^{2n}$ will suffice. If $r \in \mathbb{R}^+$ then $U = B_r(s, t)$ the open ball of radius r around s works. □

3.2.1 Main Result of Construction

Since the phase of a lift of the one-particle scattering operator relative to any other lift is fixed by a single matrix element, we may use vacuum expectation values to characterize the phase of a lift. The function c captures the dependence of this object on variations of the external field. As it turns out, this function also has a very close connection to the derivative of the current as given by Bogolyubov's formula, cf. definition 50 and the main result theorem 51. However, the support properties of c do not align nicely with the causality condition we require, motivated in remark 52. So the need of a splitting $c = c^+ - c^-$ arises so that c^+ has only support in the causal past and c^- only in the causal future in the sense specified below.

Definition 49 (causal splitting)**.** *We define a causal splitting as a function*

$$c^+ : \mathcal{V}^3 \to \mathbb{C}, \tag{3.66}$$
$$(A, F, G) \mapsto c^+_A(F, G), \tag{3.67}$$

such that c^+ restricted to any finite dimensional subspace is smooth in the first argument and linear in the second and third argument. Furthermore, c^+ should satisfy

$$c_A(F, G) = c^+_A(F, G) - c^+_A(G, F), \tag{3.68}$$
$$\partial_H c^+_{A+H}(F, G) = \partial_G c^+_{A+G}(F, H), \tag{3.69}$$
$$\forall F \prec G : c^+_A(F, G) = 0. \tag{3.70}$$

Definition 50 (current)**.** *Given a lift $\hat{S}_{A,B}$ of the one-particle scattering operator $S_{A,B}$ for which the derivative in the following expression exists, we define the associated current by Bogolyubov's formula:*

$$j^{\hat{S}}_A(F) := i\partial_F \left\langle \Omega, \hat{S}_{A,A+F}\Omega \right\rangle. \tag{3.71}$$

Theorem 51 (existence of causal lift)**.** *Given a causal splitting c^+, there is a second quantized scattering operator $\tilde{S}$, lift of the one-particle scattering operator S with the following properties*

$$\forall A, B, C \in \mathcal{V} : \tilde{S}_{A,B}\tilde{S}_{B,C} = \tilde{S}_{A,C} \tag{3.72}$$
$$\forall F \prec G : \tilde{S}_{A,A+F} = \tilde{S}_{A+G,A+F+G} \tag{3.73}$$

and the associated current satisfies

$$\partial_G j^{\tilde{S}}_{A+G}(F) = \begin{cases} -2ic_A(F, G) & \text{for } G \prec F \\ 0 & \text{for } F \prec G \end{cases} \tag{3.74}$$

Remark 52. *One may wonder why we construct a lift with the properties* (3.72) *and* (3.73)*. The project of finding a rigorous formulation of external field QED could be considered a success once a lift* $\tilde{U}^A_{\Sigma',\Sigma}$ *of the evolution from one Cauchy surface to another* $U_{\Sigma',\Sigma}$ *has been constructed from which a current can be calculated that agrees with experiments to the degree that the approximations inherent to the model are applicable. In the light of this goal properties* (3.72) *and* (3.73) *should be judged. Property* (3.72) *is a basic requirement, any phase that does not fulfil it will not be directly generalizable to the evolution between different Cauchy surfaces. In the proof of the theorem we will see that in order to satisfy it, it suffices to construct any global section, i.e. a lift of* $S_{0,A}$ *for any* $A \in \mathcal{V}$*. In order to see why our formulation of Bogolyubov's causality condition [5], i.e. equation* (3.73)*, is a reasonable second requirement we may quickly go through its proof in the one-particle situation. Let* $A, F, G \in \mathcal{V}$ *such that* $F \prec G$*. We may then pick a Cauchy surface* Σ' *such that* $\operatorname{supp} F \prec \Sigma'$ *and* $\Sigma' \prec \operatorname{supp} G$*. This implies*

$$S_{A+G,A+G+F} = U^{A+G}_{\Sigma_{in},\Sigma_{out}} U^{A+G+F}_{\Sigma_{out},\Sigma_{in}} \tag{3.75}$$

$$= U^{A+G}_{\Sigma_{in},\Sigma'} U^{A+G}_{\Sigma',\Sigma_{out}} U^{A+G+F}_{\Sigma_{out},\Sigma'} U^{A+G+F}_{\Sigma',\Sigma_{in}} \tag{3.76}$$

$$\overset{*}{=} U^{A}_{\Sigma_{in},\Sigma'} U^{A+G}_{\Sigma',\Sigma_{out}} U^{A+G}_{\Sigma_{out},\Sigma'} U^{A+F}_{\Sigma',\Sigma_{in}} = U^{A}_{\Sigma_{in},\Sigma'} U^{A+F}_{\Sigma',\Sigma_{in}} \tag{3.77}$$

$$= U^{A}_{\Sigma_{in},\Sigma'} U^{A}_{\Sigma',\Sigma_{out}} U^{A}_{\Sigma_{out},\Sigma'} U^{A+F}_{\Sigma',\Sigma_{in}} = U^{A}_{\Sigma_{in},\Sigma_{out}} U^{A+F}_{\Sigma_{out},\Sigma_{in}} = S_{A,A+F}, \tag{3.78}$$

where for the marked equality we used the support properties of F *and* G *relative to* Σ' *and that* $S^A_{\Sigma',\Sigma}$ *only depends on values of* A *in the volume delimited by* Σ' *and* Σ*. For a lift of* $U^A_{\Sigma',\Sigma}$ *we would expect the last calculation to hold in the second quantized language as well. So property* (3.73) *is a way of incorporating attributes of the lift of the time evolution between different hypersurfaces without mentioning those hypersurfaces directly.*

3.2.2 Proof of Theorem 51

The connection between vacuum expectation values and c becomes clearer with the next lemma.

Lemma 53 (properties of Γ)**.** *The function Γ has the following properties for all $A, B, C, D \in \mathcal{V}$ such that the expressions occurring in each equation are well-defined:*

$$\gamma_{A,B,C} \neq 0 \tag{3.79}$$
$$\Gamma_{A,B,C} = \operatorname{ag}(\det_{\mathcal{H}^-}(P^- - P^- S_{A,C} P^+ S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-)) \tag{3.80}$$
$$\Gamma^{-1}_{A,B,C} = \operatorname{ag}(\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle) \tag{3.81}$$
$$\Gamma_{A,B,C} = \Gamma_{B,C,A} = \frac{1}{\Gamma_{B,A,C}} \tag{3.82}$$
$$\Gamma_{A,A,B} = 1 \tag{3.83}$$
$$\Gamma_{A,B,C}\Gamma_{B,A,D}\Gamma_{A,C,D}\Gamma_{C,B,D} = 1 \tag{3.84}$$
$$\Gamma_{A,B,C} = \Gamma_{D,B,C}\Gamma_{A,D,C}\Gamma_{A,B,D} \tag{3.85}$$
$$\overline{S}_{A,C} = \Gamma_{A,B,C} \overline{S}_{A,B} \overline{S}_{B,C} \tag{3.86}$$
$$c_A(B, C) = \partial_B \partial_C \ln \Gamma_{A,A+B,A+C} \tag{3.87}$$
$$\langle \Omega, \overline{S}_{X,Y} \Omega \rangle = \det_{\mathcal{H}^-} |P^- S_{A,B} P^-|. \tag{3.88}$$

Proof. Pick $A, B, C \in \mathcal{V}$ such that $(X, Y) \in \operatorname{dom} \overline{S}$ for $X, Y \in \{A, B, C\}$. By definition γ is

$$\gamma_{A,B,C} = \det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-). \tag{3.89}$$

The operator whose determinant we take in the last line is a product

$$P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^- = P^- S_{A,B} P^- \quad P^- S_{B,C} P^- \quad P^- S_{C,A} P^-. \tag{3.90}$$

The three factors appearing in this product are all invertible due to the definition of $\operatorname{dom} \overline{S}$, therefore if the determinant exists we have $\gamma_{A,B,C} \neq 0$. To see that it does exist, we reformulate

$$\gamma_{A,B,C} = \det_{\mathcal{H}^-}(P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-) \tag{3.91}$$

$$= \det_{\mathcal{H}^-}(P^- S_{A,C} P^- S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-) \tag{3.92}$$

$$= \det_{\mathcal{H}^-}(P^- - P^- S_{A,C} P^+ S_{C,A} P^- - P^- S_{A,B} P^+ S_{B,C} P^- S_{C,A} P^-), \tag{3.93}$$

now we know by theorem 35 of Ruijsenaars that $P^+ S_{X,Y} P^-$ is a Hilbert-Schmidt operator for our setting, hence γ and also Γ are well-defined.

Equation (3.93) also proves (3.80). Next we show (3.81). In the notation of the last section we have $\Omega = e^{i\varphi} \bigwedge \Phi$ for some $\varphi \in \mathbb{R}$ with the injection $\Phi : \mathcal{H}^- \hookrightarrow \mathcal{H}$. We begin by reformulating the right-hand side of (3.81)

$$\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle \tag{3.94}$$

$$= \langle e^{i\varphi} \bigwedge \Phi, e^{i\varphi} \bigwedge \big(S_{A,B} S_{B,C} S_{C,A} \Phi \operatorname{AG}(P^- S_{C,A} P^-)^{-1} \operatorname{AG}(P^- S_{B,C} P^-)^{-1} \operatorname{AG}(P^- S_{A,B} P^-)^{-1} \big) \rangle$$

$$= \langle \bigwedge \Phi, \bigwedge \big(\Phi \operatorname{AG}(P^- S_{C,A} P^-)^{-1} \times \operatorname{AG}(P^- S_{B,C} P^-)^{-1} \operatorname{AG}(P^- S_{A,B} P^-)^{-1} \big) \rangle \tag{3.95}$$

$$= \det_{\mathcal{H}^-} \big((\Phi)^* \big[\Phi \operatorname{AG}(P^- S_{C,A} P^-)^{-1} \operatorname{AG}(P^- S_{B,C} P^-)^{-1} \times \operatorname{AG}(P^- S_{A,B} P^-)^{-1} \big] \big) \tag{3.96}$$

$$= \det_{\mathcal{H}^-} \big(\operatorname{AG}(P^- S_{C,A} P^-)^{-1} \operatorname{AG}(P^- S_{B,C} P^-)^{-1} \times \operatorname{AG}(P^- S_{A,B} P^-)^{-1} \big) \tag{3.97}$$

$$= \frac{1}{\det_{\mathcal{H}^-} \operatorname{AG}(P^- S_{A,B} P^-) \operatorname{AG}(P^- S_{B,C} P^-) \operatorname{AG}(P^- S_{C,A} P^-)}. \tag{3.98}$$

We first note that $\det_{\mathcal{H}^-} |P^- S_{X,Y} P^-| \in \mathbb{R}^+$ for $X, Y \in \{A, B, C\}$. This is well-defined because

$$\langle \Omega, \overline{S}_{X,Y} \Omega \rangle = \langle e^{i\varphi} \bigwedge \Phi, e^{i\varphi} \bigwedge (S_{X,Y} \Phi \operatorname{AG}(P^- S_{X,Y} P^-)^{-1}) \rangle \tag{3.99}$$

$$= \det_{\mathcal{H}^-} \left(\Phi^* S_{X,Y} \Phi \operatorname{AG}(P^- S_{X,Y} P^-)^{-1} \right) \tag{3.100}$$

$$= \det_{\mathcal{H}^-} \left(P^- S_{X,Y} P^- \operatorname{AG}(P^- S_{X,Y} P^-)^{-1} \right) \tag{3.101}$$

$$= \det_{\mathcal{H}^-} \left(\operatorname{AG}(P^- S_{X,Y} P^-)^{-1} P^- S_{X,Y} P^- \right) \tag{3.102}$$

$$= \det_{\mathcal{H}^-} \left(\operatorname{AG}(P^- S_{X,Y} P^-)^{-1} \operatorname{AG}(P^- S_{X,Y} P^-) |P^- S_{X,Y} P^-| \right) \tag{3.103}$$

$$= \det_{\mathcal{H}^-} |P^- S_{X,Y} P^-| \tag{3.104}$$

holds, proving equation (3.88). Moreover, this determinant does not vanish, since $P^- S_{X,Y} P^-$ is invertible. Also, clearly the eigenvalues are positive since $|P^- S_{X,Y} P^-|$ is an absolute value. We continue with the result of (3.98). Thus, we find

$$\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega \rangle^{-1} \tag{3.105}$$

$$= \det_{\mathcal{H}^-} \left(\operatorname{AG}(P^- S_{A,B} P^-) \operatorname{AG}(P^- S_{B,C} P^-) \operatorname{AG}(P^- S_{C,A} P^-) \right) \tag{3.106}$$

$$= \det_{\mathcal{H}^-} \Big(\operatorname{AG}(P^- S_{A,B} P^-) \operatorname{AG}(P^- S_{B,C} P^-) P^- S_{C,A} P^- \times |P^- S_{C,A} P^-|^{-1} \Big) \tag{3.107}$$

$$= \det_{\mathcal{H}^-} \left(\operatorname{AG}(P^- S_{A,B} P^-) \operatorname{AG}(P^- S_{B,C} P^-) P^- S_{C,A} P^- \right) \times \det_{\mathcal{H}^-} |P^- S_{C,A} P^-|^{-1} \tag{3.108}$$

$$= \det_{\mathcal{H}^-} \left(P^- S_{C,A} P^- \operatorname{AG}(P^- S_{A,B} P^-) \operatorname{AG}(P^- S_{B,C} P^-) \right) \times \det_{\mathcal{H}^-} |P^- S_{C,A} P^-|^{-1} \tag{3.109}$$

$$= \frac{\det_{\mathcal{H}^-} \left(P^- S_{A,B} P^- P^- S_{B,C} P^- P^- S_{C,A} P^- \right)}{\det_{\mathcal{H}^-} |P^- S_{A,B} P^-| \cdot \det_{\mathcal{H}^-} |P^- S_{B,C} P^-| \cdot \det_{\mathcal{H}^-} |P^- S_{C,A} P^-|}. \tag{3.110}$$

Now since the denominator of this fraction is real we can use (3.62) to identity

$$\mathrm{ag}(\langle\Omega, \overline{S}_{A,B}\overline{S}_{B,C}\overline{S}_{C,A}\Omega\rangle) = \Gamma^{-1}_{A,B,C}, \tag{3.111}$$

which proves (3.81).

For the first equality in (3.82) we use $\det(X(1+Y)X^{-1}) = \det(1+Y)$ for any Y trace-class and X bounded and invertible. So we can cyclically permute the factors $P^-S_{X,Y}P^-$ in the determinant and find

$$\begin{aligned}\Gamma_{A,B,C} &= \mathrm{ag}(\det_{\mathcal{H}^-} P^-S_{A,B}P^-S_{B,C}P^-S_{C,A}P^-)\\ &= \mathrm{ag}(\det_{\mathcal{H}^-} P^-S_{C,A}P^-S_{A,B}P^-S_{B,C}P^-) = \Gamma_{C,A,B}.\end{aligned}$$

For the second equality of (3.82) we use (3.62) to represent both $\Gamma_{A,B,C}$ and $\Gamma_{B,A,C}$. Using this and the manipulations of the determinant we already employed, we arrive at

$$\Gamma_{A,B,C}\Gamma_{B,A,C} \tag{3.112}$$
$$= \mathrm{ag}(\det_{\mathcal{H}^-}(P^-S_{A,B}P^-S_{B,C}P^-S_{C,A}P^-)) \tag{3.113}$$
$$\times\ \mathrm{ag}(\det_{\mathcal{H}^-}(P^-S_{B,A}P^-S_{A,C}P^-S_{C,B}P^-)) \tag{3.114}$$
$$= \mathrm{ag}(\det_{\mathcal{H}^-}(P^-S_{A,B}P^-S_{B,C}P^-S_{C,A}P^-)) \tag{3.115}$$
$$\times\ (\mathrm{ag}(\det_{\mathcal{H}^-}(P^-S_{B,C}P^-S_{C,A}P^-S_{A,B}P^-)))^* \tag{3.116}$$
$$= \mathrm{ag}(\det_{\mathcal{H}^-}(P^-S_{A,B}P^-S_{B,C}P^-S_{C,A}P^-)) \tag{3.117}$$
$$\times\ (\mathrm{ag}(\det_{\mathcal{H}^-}(P^-S_{A,B}P^-S_{B,C}P^-S_{C,A}P^-)))^* \tag{3.118}$$
$$= |\,\mathrm{ag}(\det_{\mathcal{H}^-}(P^-S_{A,B}P^-S_{B,C}P^-S_{C,A}P^-))|^2 = 1, \tag{3.119}$$

which proves (3.82).

Next, using (3.61) inserting twice the same argument yields

$$\gamma_{A,A,C} = \det_{\mathcal{H}^-} P^-S_{A,C}P^-S_{C,A}P^- = \det_{\mathcal{H}^-}(P^-S_{C,A}P^-)^*P^-S_{C,A}P^- \in \mathbb{R}^+, \tag{3.120}$$

hence (3.83) follows.
For proving (3.84) we will use the definition of Γ directly and repeatedly use that we can cyclically permute operator groups of the form $P^- S_{X,Y} P^-$ for $X, Y \in \{A, B, C, D\}$ in the determinant, i.e.

$$\det P^- S_{X,Y} P^- O = \det O P^- S_{X,Y} P^-, \qquad (\circlearrowleft)$$

whenever O has a determinant. This is possible, because $P^- S_{X,Y} P^-$ is bounded and invertible. Furthermore, we will use that

$$\det O_1 O_2 = \det O_1 \det O_2 \qquad (\leftrightarrow)$$

holds whenever both O_1 and O_2 have a determinant. Moreover, for any X, Y the expression $(P^- S_{X,Y} P^-)^* P^- S_{X,Y} P^-$ is the modulus squared of an invertible operator and hence its determinant is positive which means that

$$\operatorname{ag} \det (P^- S_{X,Y} P^-)^* P^- S_{X,Y} P^- = 1. \qquad (\operatorname{ag}|\,|)$$

These three rules will be repeatedly used. We calculate

$$\Gamma_{A,B,C}\Gamma_{B,A,D}\Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.121}$$
$$= \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^-$$
$$\times \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{B,A} P^- S_{A,D} P^- S_{D,B} P^- \;\; \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.122}$$
$$\overset{(\circlearrowleft)}{=} \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{A,D} P^- S_{D,B} P^- S_{B,A} P^-$$
$$\times \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{A,B} P^- S_{B,C} P^- S_{C,A} P^- \;\; \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.123}$$
$$\overset{(\leftrightarrow)}{=} \operatorname{ag} \det_{\mathcal{H}^-} \big(P^- S_{A,D} P^- S_{D,B} \big[P^- S_{B,A} P^- S_{A,B} P^-\big]$$
$$\times S_{B,C} P^- S_{C,A} P^-\big) \; \Gamma_{A,C,D}\Gamma_{C,B,D} \tag{3.124}$$
$$\overset{(\circlearrowleft)}{=} \operatorname{ag} \det_{\mathcal{H}^-} P^- S_{B,C} P^- S_{C,A} P^- S_{A,D} P^- S_{D,B} \big[P^- S_{B,A} P^- S_{A,B} P^-\big]$$

$$\times \Gamma_{A,C,D}\Gamma_{C,B,D} \quad (3.125)$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{B,C} P^- S_{C,A} P^- S_{A,D} P^- S_{D,B} P^- \times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{B,A} P^- S_{A,B} P^- \quad \Gamma_{A,C,D}\Gamma_{C,B,D} \quad (3.126)$$

$$\overset{(\mathrm{ag}|\,|)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{B,C} P^- S_{C,A} P^- S_{A,D} P^- S_{D,B} P^- \quad \Gamma_{A,C,D}\Gamma_{C,B,D} \quad (3.127)$$

$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{A,D} P^- S_{D,B} P^- S_{B,C} P^- S_{C,A} P^- \quad \Gamma_{A,C,D}\Gamma_{C,B,D} \quad (3.128)$$

$$= \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{A,C} P^- S_{C,D} P^- S_{D,A} P^- \times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{A,D} P^- S_{D,B} P^- S_{B,C} P^- S_{C,A} P^- \quad \Gamma_{C,B,D} \quad (3.129)$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} \big(P^- S_{A,C} P^- S_{C,D} P^- \big[P^- S_{D,A} P^- P^- S_{A,D} P^-\big] \times S_{D,B} P^- S_{B,C} P^- S_{C,A} P^-\big) \quad \Gamma_{C,B,D} \quad (3.130)$$

$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} \big(P^- S_{D,B} P^- S_{B,C} P^- S_{C,A} P^- S_{A,C} P^- S_{C,D} P^- \times \big[P^- S_{D,A} P^- P^- S_{A,D} P^-\big]\big) \quad \Gamma_{C,B,D} \quad (3.131)$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{D,B} P^- S_{B,C} P^- S_{C,A} P^- S_{A,C} P^- S_{C,D} P^- \times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{D,A} P^- P^- S_{A,D} P^- \quad \Gamma_{C,B,D} \quad (3.132)$$

$$\overset{(\mathrm{ag}|\,|)}{=} \operatorname{ag}\det_{\mathcal{H}^-} \big(P^- S_{D,B} P^- S_{B,C} P^- \big[P^- S_{C,A} P^- S_{A,C} P^-\big] \times P^- S_{C,D} P^-\big)\Gamma_{C,B,D} \quad (3.133)$$

$$\overset{(\circlearrowleft)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D} P^- S_{D,B} P^- S_{B,C} P^- \big[P^- S_{C,A} P^- S_{A,C} P^-\big] \times \Gamma_{C,B,D} \quad (3.134)$$

$$\overset{(\leftrightarrow)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D} P^- S_{D,B} P^- S_{B,C} P^-$$
$$\times \operatorname{ag}\det P^- S_{C,A} P^- S_{A,C} P^- \quad \Gamma_{C,B,D} \tag{3.135}$$
$$\overset{(\operatorname{ag}|\,|)}{=} \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D} P^- S_{D,B} P^- S_{B,C} P^- \quad \Gamma_{C,B,D} \tag{3.136}$$
$$= \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D} P^- S_{D,B} P^- S_{B,C} P^-$$
$$\times \operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,B} P^- S_{B,D} P^- S_{D,C} P^- \tag{3.137}$$
$$= |\operatorname{ag}\det_{\mathcal{H}^-} P^- S_{C,D} P^- S_{D,B} P^- S_{B,C} P^-|^2 = 1. \tag{3.138}$$

Equation (3.85) is a direct consequence of (3.84) and (3.82).
For (3.86) we realize that according to [13] two lifts of the same one-particle operator can only differ by a phase, that is

$$\overline{S}_{A,C} = \xi\ \overline{S}_{A,B} \overline{S}_{B,C} \tag{3.139}$$

for some $\xi \in \mathbb{C}$, $|\xi| = 1$.
In order to identify ξ we recognize that $\overline{S}_{X,Y} = \overline{S}^{-1}_{Y,X}$ for four potentials X, Y and find

$$1\xi^{-1} = \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A}. \tag{3.140}$$

Now we take the vacuum expectation value on both sides of this equation and use (3.81) to find

$$\xi^{-1} = \operatorname{ag}\langle \Omega, \overline{S}_{A,B} \overline{S}_{B,C} \overline{S}_{C,A} \Omega\rangle = \Gamma^{-1}_{A,B,C}. \tag{3.141}$$

Finally, we prove (3.87). We start from the right-hand side of this equation and work our way towards the left-hand side of it. In the following calculation we will repeatedly make use of the fact that $(P^- S_{A,A+B} P^- S_{A+B,A} P^-)$ is the absolute value squared of an invertible operator and has a determinant, which is therefore positive. For the marked equality we will use that for a differentiable function $z : \mathbb{R} \to \mathbb{C}$

at points t where $z(t) \in \mathbb{R}^+$ holds, we have

$$(z/|z|)'(t) = \frac{z'}{|z|}(t) + \frac{-z}{|z|^2}\frac{z'z^* + z^{*\prime}z}{2|z|}(t) = \frac{z'}{2|z|}(t) - \frac{z^2 z^{*\prime}}{2|z|^3}(t)$$
$$= i(\Im(z'))/z(t). \quad (3.142)$$

Furthermore, we will use the following expressions for the derivative of the determinant which holds for all operator valued functions on the reals $M : \mathbb{R} \to 1 + I_1(\mathcal{H})$ such that M is invertible for at 0

$$\partial_\varepsilon \det M(\varepsilon)|_{\varepsilon=0} = \det M(0) \operatorname{tr}(M^{-1}(0)\partial_\varepsilon M(\varepsilon)|_{\varepsilon=0}), \quad (3.143)$$

likewise we need the following expression for the derivative of M^{-1} for $M : \mathbb{R} \to (\mathcal{H} \to \mathcal{H})$ such that $M(t)$ is invertible and bounded for every $t \in \mathbb{R}$

$$\partial_\varepsilon M^{-1}(\varepsilon)|_{\varepsilon=0} = -M^{-1}(0)\partial_\varepsilon M(\varepsilon)|_{\varepsilon=0} M^{-1}(0). \quad (3.144)$$

The handling of derivatives of operators in the following calculation is justified by the section on regularity 4.1. We compute

$$\partial_B \partial_C \ln \Gamma_{A,A+B,A+C} \quad (3.145)$$
$$\overset{(3.62)}{=} \partial_B \partial_C \ln \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-)) \quad (3.146)$$
$$= \partial_B \frac{\partial_C \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-))}{\operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-))} \quad (3.147)$$
$$= \partial_B \partial_C \operatorname{ag}(\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-)) \quad (3.148)$$
$$\overset{*}{=} i\partial_B \frac{\Im \partial_C \det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-)}{\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-)} \quad (3.149)$$
$$= i\partial_B \Big[\frac{\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-)}{\det_{\mathcal{H}^-}(P^- S_{A,A+B} P^- S_{A+B,A} P^-)}$$
$$\times \Im \operatorname{tr}((P^- S_{A,A+B} P^- S_{A+B,A} P^-)^{-1}$$
$$\times \partial_C P^- S_{A,A+B} P^- S_{A+B,A+C} P^- S_{A+C,A} P^-) \Big] \quad (3.150)$$

The fraction in front of the trace equals 1. As a next step we replace the second but last projector $P^- = 1 - P^+$, the resulting first summand vanishes, because the dependence on C cancels. This results in

$$(3.150) = -i\partial_B \Im \operatorname{tr}((P^- S_{A,A+B} P^- S_{A+B,A} P^-)^{-1} \times \partial_C P^- S_{A,A+B} P^- S_{A+B,A+C} P^+ S_{A+C,A} P^-). \quad (3.151)$$

Now, because $P^+ P^- = 0$ only one summand of the product rule survives:

$$(3.151) = -i\partial_B \Im \operatorname{tr}((P^- S_{A,A+B} P^- S_{A+B,A} P^-)^{-1} \times \partial_C P^- S_{A,A+B} P^- S_{A+B,A} P^+ S_{A+C,A} P^-). \quad (3.152)$$

Next we use $(MN)^{-1} = N^{-1} M^{-1}$ for invertible operators M and N for the first factor in the trace and cancel as much as possible of the second factor:

$$\begin{aligned}
(3.152) &= -i\partial_B \Im \operatorname{tr}((P^- S_{A+B,A} P^-)^{-1} P^- S_{A+B,A} \\
&\qquad \times P^+ \partial_C S_{A+C,A} P^-) && (3.153)\\
&= -i \Im \operatorname{tr}(\partial_B [(P^- S_{A+B,A} P^-)^{-1} P^- S_{A+B,A} \\
&\qquad \times P^+ \partial_C S_{A+C,A} P^-]) && (3.154)\\
&= -i \Im \operatorname{tr}(\partial_B P^- S_{A+B,A} P^+ \partial_C S_{A+C,A} P^-) && (3.155)\\
&= -i \Im \operatorname{tr}(\partial_B P^- S_{A,A+B} P^+ \partial_C S_{A,A+C} P^-) && (3.156)\\
&= -i \partial_B \partial_C \Im \operatorname{tr}(P^- S_{A,A+B} P^+ S_{A,A+C} P^-) && (3.157)
\end{aligned}$$

which proves the claim.

□

In order to construct the lift announced in theorem 51, we first construct a reference lift $\hat{S}$, that is well-defined on all of $\mathcal{V}$. Afterwards we will study the dependence of the relative phase between this global lift $\hat{S}_{0,A}$ and a local lift given by $\hat{S}_{0,B} \overline{S}_{B,A}$ for $B - A$ small. By exploiting

properties of this phase and the causal splitting c^+ we will construct a global lift that has the desired properties.
Since $\mathcal{V}$ is star shaped, we have $\overline{0\ A} \subset \mathcal{V}$.

Definition 54 (ratio of lifts)**.** *For any $A, B \in \mathcal{V}$ and any two lifts $S'_{A,B}, S''_{A,B}$ of the one-particle scattering operator $S_{A,B}$ we define the ratio*

$$\frac{S'_{A,B}}{S''_{A,B}} \in S^1 \tag{3.158}$$

to be the unique complex number $z \in S^1$ such that

$$z\ S''_{A,B} = S'_{A,B} \tag{3.159}$$

holds.

Theorem 55 (existence of global lift)**.** *There is a unique map $\hat{S}_{0,\cdot} : \mathcal{V} \to U(\mathcal{F})$ which maps $A \in \mathcal{V}$ to a lift of $S_{0,A}$ and solves the parallel transport differential equation*

$$A, B \in \mathcal{V} \textit{ linearly dependent} \Rightarrow \partial_B \frac{\hat{S}_{0,A+B}}{\hat{S}_{0,A} \overline{S}_{A,A+B}} = 0, \tag{3.160}$$

subject to the initial condition $\hat{S}_{0,0} = 1$.

The proof of theorem 55 is divided into two lemmas due to its length. We will introduce the integral flow ϕ_A associated with the differential equation (3.160) for some $A \in \mathcal{V}$. We will then study the properties of ϕ_A in the two lemmas and finally construct $\hat{S}_{0,A} := \phi_A(0, 1)$. In the first lemma we will establish the existence of a local solution. The solution will be constructed along the line $\overline{0\ \ A}$. In the second lemma we patch local solutions together to a global one.

Lemma 56 (ϕ local existence and uniqueness). *There is a unique* $\phi_A : \{(t,s) \in \mathbb{R}^2 \mid (tA, sA) \in \operatorname{dom} \overline{S}\} := \operatorname{dom} \phi_A \to U(\mathcal{F})$ *for every* $A \in \mathcal{V}$ *satisfying*

$$\forall (t,s) \in \operatorname{dom} \phi_A : \phi_A(t,s) \text{ is a lift of } S_{tA,sA} \tag{3.161}$$

$$\forall (t,s),(s,l),(l,t) \in \operatorname{dom} \phi_A : \phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.162}$$

$$\forall t \in \mathbb{R} : \phi_A(t,t) = 1 \tag{3.163}$$

$$\forall s \in \mathbb{R} : \partial_t \left. \frac{\phi_A(s,t)}{\overline{S}_{sA,tA}} \right|_{t=s} = 0. \tag{3.164}$$

Proof. We first define the phase

$$z : \{(A,B) \in \operatorname{dom} \overline{S} \mid A, B \text{ linearly dependent}\} \to S^1 \tag{3.165}$$

by the differential equation

$$\frac{d}{dx} \ln z(tA, xA) = - \left. \left(\frac{d}{dy} \ln \Gamma_{tA,xA,yA} \right) \right|_{y=x} \tag{3.166}$$

and the initial condition

$$z(A,A) = 1 \tag{3.167}$$

for any $A \in \mathcal{V}$. The phase z takes the form

$$z(tA, xA) = \exp\left(- \int_t^x dx' \left. \left(\frac{d}{dx'} \ln \Gamma_{tA,yA,x'A} \right) \right|_{y=x'} \right). \tag{3.168}$$

Please note that both differential equation and initial condition are invariant under rescaling of the potential A, so z is well-defined. We will now construct a local solution to (3.160) and define ϕ_A using this solution. Pick $A \in \mathcal{V}$ the expression

$$\hat{S}_{0,sA} = \hat{S}_{0,A} \overline{S}_{A,sA} z(A, sA) \tag{3.169}$$

solves (3.160) locally. Local here means that s is close enough to 1 such that $(A, sA) \in \operatorname{dom} \overline{S}$. Calculating the argument of the derivative of (3.160) we find:

$$\frac{\hat{S}_{0,(s+\varepsilon)A}}{\hat{S}_{0,sA}\overline{S}_{sA,(s+\varepsilon)A}} = \frac{\hat{S}_{0,A}\overline{S}_{A,(s+\varepsilon)A}z(A,(s+\varepsilon)A)}{\hat{S}_{0,A}\overline{S}_{A,sA}\overline{S}_{sA,(s+\varepsilon)}z(A,sA)} \tag{3.170}$$

$$\overset{(3.86)}{=} \frac{\hat{S}_{0,A}\overline{S}_{A,sA}\overline{S}_{sA,(s+\varepsilon)}\Gamma_{A,sA,(s+\varepsilon)A}z(A,(s+\varepsilon)A)}{\hat{S}_{0,A}\overline{S}_{A,sA}\overline{S}_{sA,(s+\varepsilon)}z(A,sA)} \tag{3.171}$$

$$= \frac{\Gamma_{A,sA,(s+\varepsilon)A}z(A,(s+\varepsilon)A)}{z(A,sA)} \tag{3.172}$$

Now we take the derivative with respect to ε at $\varepsilon = 0$, cancel the factor that does not depend on ε and relabel $s = x$ to obtain

$$0 = \left(\frac{d}{dy}(\Gamma_{A,xA,yA}\; z(A,yA))\right)\bigg|_{y=x} \tag{3.173}$$

$$\iff \frac{d}{dx}\ln z(tA,xA) = \left(-\frac{d}{dy}\ln \Gamma_{tA,xA,yA}\right)\bigg|_{y=x}, \tag{3.174}$$

where we rescaled A so that the expression matches exactly the defining differential equation of z. The initial condition of z equation (3.167) is necessary to match the initial condition in (3.169) for $s = 1$. The connection to ϕ from the statement of the lemma can now be made. We define

$$\phi_A(t,s) := z(tA,sA)\overline{S}_{tA,sA}, \tag{3.175}$$

for $(tA, sA) \in \operatorname{dom} \overline{S}$. Since $\overline{S}$ is a lift of S, we see that (3.161) holds. Equation (3.163) follows from (3.167) and $\overline{S}_{tA,tA} = 1$ for general $t \in \mathbb{R}$. Equation (3.164) follows by plugging in (3.175) and using the defining differential equation for z (3.166) as well as its initial condition (3.167) and the property (3.82) of Γ:

$$\partial_s \left.\frac{\phi_A(t,s)}{\overline{S}_{tA,sA}}\right|_{s=t} = \partial_s \left.\frac{z(tA,sA)\overline{S}_{tA,sA}}{\overline{S}_{tA,sA}}\right|_{s=t} \tag{3.176}$$

$$= \partial_s\, z(tA,sA)|_{t=s} = -z(tA,tA)\left(\frac{d}{ds}\ln\Gamma_{tA,tA,sA}\right)\Bigg|_{s=t} = 0. \tag{3.177}$$

It remains to see that (3.162), i.e. that

$$\phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.178}$$

holds for $(tA,sA),(sA,lA),(tA,lA) \in \operatorname{dom}\overline{S}$. In order to do so we plug in the definition (3.175) of ϕ_A and obtain

$$\phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.179}$$

$$\iff z(tA,sA)z(sA,lA)\overline{S}_{tA,sA}\overline{S}_{sA,lA} = z(tA,lA)\overline{S}_{tA,lA} \tag{3.180}$$

$$\iff z(tA,sA)z(sA,lA)\overline{S}_{tA,sA}\overline{S}_{sA,lA} \tag{3.181}$$

$$= z(tA,lA)\overline{S}_{tA,sA}\overline{S}_{sA,lA}\Gamma_{tA,sA,lA} \tag{3.182}$$

$$\iff z(tA,sA)z(sA,lA)z(tA,lA)^{-1} = \Gamma_{tA,sA,lA}. \tag{3.183}$$

In order to check the validity of the last equality we plug in the integral formula (3.168) for z, we also abbreviate $\frac{d}{dx} = \partial_x$

$$z(tA,sA)z(sA,lA)z(tA,lA)^{-1} \tag{3.184}$$

$$= e^{-\int_t^s dx'\,(\partial_{x'}\ln\Gamma_{tA,yA,x'A})\big|_{y=x'} - \int_s^l dx'\,(\partial_{x'}\ln\Gamma_{sA,yA,x'A})\big|_{y=x'}} \tag{3.185}$$

$$\times e^{+\int_t^l dx'\,(\partial_{x'}\ln\Gamma_{tA,yA,x'A})\big|_{y=x'}} \tag{3.186}$$

$$= e^{-\int_l^s dx'\,(\partial_{x'}\ln\Gamma_{tA,yA,x'A})\big|_{y=x'} - \int_s^l dx'\,(\partial_{x'}\ln\Gamma_{sA,yA,x'A})\big|_{y=x'}} \tag{3.187}$$

$$\overset{(3.85)}{=} e^{-\int_l^s dx'\,(\partial_{x'}\ln\Gamma_{sA,yA,x'A})\big|_{y=x'} - \int_l^s dx'\,(\partial_{x'}\ln\Gamma_{tA,sA,x'A})\big|_{y=x'}} \tag{3.188}$$

$$\times e^{-\int_l^s dx'\,(\partial_{x'}\ln\Gamma_{tA,yA,sA})\big|_{y=x'}-\int_s^l dx'\,(\partial_{x'}\ln\Gamma_{sA,yA,x'A})\big|_{y=x'}} \tag{3.189}$$

$$= e^{-\int_l^s dx'\,(\partial_{x'}\ln\Gamma_{tA,sA,x'A})\big|_{y=x'}} \tag{3.190}$$

$$= e^{-\int_l^s dx'\partial_{x'}\ln\Gamma_{tA,sA,x'A}} \tag{3.191}$$

$$\overset{(3.83)}{=} \Gamma_{tA,sA,lA}, \tag{3.192}$$

which proves the validity of the consistency relation (3.178).
In order to prove uniqueness we pick $A \in \mathcal{V}$ and assume there is ϕ' also defined on $\operatorname{dom}\phi_A$ and satisfies (3.161) to (3.164). Then we may use (3.161) to conclude that for any $(t,s) \in \operatorname{dom}\phi_A$ there is $\gamma(t,s) \in S^1$ such that

$$\phi_A(t,s) = \phi'(t,s)\gamma(t,s) \tag{3.193}$$

holds true. Picking l such that $(t,s),(s,l),(t,l) \in \operatorname{dom}\phi_A$ and using (3.162) we find

$$\phi'(t,s)\gamma(t,s) = \phi_A(t,s) = \phi_A(t,l)\phi_A(l,s) \tag{3.194}$$

$$= \gamma(t,l)\phi'(t,l)\gamma(l,s)\phi'(l,s) = \gamma(t,l)\gamma(l,s)\phi'(t,s), \tag{3.195}$$

hence we have

$$\gamma(t,s) = \gamma(t,l)\gamma(l,s). \tag{3.196}$$

From property (3.163) we find

$$\gamma(t,t) = 1, \tag{3.197}$$

for any t. Using equation (3.164) we conclude that

$$0 = \partial_t \left.\frac{\phi'(s,t)}{\overline{S}_{sA,tA}}\right|_{t=s} = \partial_t \left.\frac{\phi_A(s,t)\gamma(s,t)}{\overline{S}_{sA,tA}}\right|_{t=s} \tag{3.198}$$

$$= \partial_t\, \gamma(s,t)\left.\frac{\phi_A(s,t)}{\overline{S}_{sA,tA}}\right|_{t=s} = \partial_t\, \gamma(s,t)\big|_{t=s} + \partial_t \left.\frac{\phi_A(s,t)}{\overline{S}_{sA,tA}}\right|_{t=s} \tag{3.199}$$

$$= \partial_t\, \gamma(s,t)\big|_{t=s}. \tag{3.200}$$

Finally, we find for general $(s,t) \in \operatorname{dom}\phi_A$:

$$\partial_x\gamma(s,x)|_{x=t} = \partial_x(\gamma(s,t)\gamma(t,x))|_{x=t} = \gamma(s,t)\partial_x\gamma(t,x)|_{x=t} = 0. \tag{3.201}$$

So $\gamma(t,s) = 1$ everywhere. We conclude $\phi_A = \phi'$.

□

Lemma 57 (ϕ global existence and uniqueness)**.** *For any $A \in \mathcal{V}$ the map ϕ_A constructed in lemma 56 can be uniquely extended to all of $\mathbb{R}^2$ keeping its defining properties*

$$\forall (t,s) \in \mathbb{R}^2 : \phi_A(t,s) \text{ is a lift of } S_{tA,sA} \tag{3.202}$$

$$\forall (t,s),(s,l),(l,t) \in \mathbb{R}^2 : \phi_A(t,s)\phi_A(s,l) = \phi_A(t,l) \tag{3.203}$$

$$\forall t \in \mathbb{R} : \phi_A(t,t) = 1 \tag{3.204}$$

$$\forall s \in \mathbb{R} : \partial_t \left.\frac{\phi_A(s,t)}{\overline{S}_{sA,tA}}\right|_{t=s} = 0. \tag{3.205}$$

Proof. Pick $A \in \mathcal{V}$. For $x \in \mathbb{R}$ we define the set

$$U_x := \{y \in \mathbb{R} \mid (xA, yA) \in \operatorname{dom}\overline{S}\}, \tag{3.206}$$

which according to properties 2 and 4 of lemma 48 is an open interval and fulfils that $\bigcup_{x\in\mathbb{R}} U_x \times U_x$ is an open neighbourhood of the diagonal $\{(x,x) \mid x \in \mathbb{R}\}$. Therefore, ϕ_A is defined for arguments that are close enough to each other. Since properties (3.205) and (3.204) only concern the behaviour of ϕ_A at the diagonal any extension fulfils them. We pick a sequence $(x_k)_{k\in\mathbb{N}_0} \subset \mathbb{R}$ such that

$$\bigcup_{k\in\mathbb{N}_0} U_{x_k} = \mathbb{R} \tag{3.207}$$

holds and

$$\forall n \in \mathbb{N}_0 : \bigcup_{k=0}^{n} U_{x_k} =: \operatorname{dom}_n \tag{3.208}$$

is an open interval. Please note that such a sequence always exists. We are going to prove that for any $n \in \mathbb{N}_0$ There is a function $\psi_n : \mathrm{dom}_n \times \mathrm{dom}_n \to U(\mathcal{F})$, which satisfies the conditions

$$\forall (t,s) \in \mathrm{dom}_n \times \mathrm{dom}_n : \psi_n(t,s) \text{ is a lift of } S_{tA,sA} \tag{3.209}$$

$$\forall s,k,l \in \mathrm{dom}_n : \psi_n(k,s)\psi_n(s,l) = \psi_n(k,l) \tag{3.210}$$

$$\forall x,y \in \mathrm{dom}_n : (xA,yA) \in \mathrm{dom}\,\overline{S} \Rightarrow \psi_n(x,y) = \phi_A(x,y) \tag{3.211}$$

and is the unique function to do so, i.e. any other function $\tilde{\psi}_n$ fulfilling properties (3.209)-(3.211) possibly being defined on a larger domain coincides with ψ_n on $\mathrm{dom}_n \times \mathrm{dom}_n$.
We start with $\psi_0 = \phi_A$ restricted to $U_{x_0} \times U_{x_0}$. This function is a restriction of ϕ_A and because of lemma 56 it fulfils all of the required properties directly.
For the induction step we pick $t \in \mathrm{dom}_n \cap U_{x_{n+1}}$ and define ψ_{n+1} on the domain $\mathrm{dom}_{n+1} \times \mathrm{dom}_{n+1}$ by

$$\psi_{n+1}(x,y) := \begin{cases} \psi_n(x,y) & \text{for } x,y \in \mathrm{dom}_n \\ \phi_A(x,y) & \text{for } x,y \in U_{x_{n+1}} \\ \psi_n(x,t)\phi_A(t,y) & \text{for } x \in \mathrm{dom}_n, y \in U_{x_{n+1}} \\ \phi_A(x,t)\psi_n(t,y) & \text{for } y \in \mathrm{dom}_n, x \in U_{x_{n+1}}. \end{cases} \tag{3.212}$$

In order to complete the induction step we have to show that ψ_{n+1} is well-defined and fulfils properties (3.209)-(3.211) with n replaced by $n+1$ and is the unique function to do so.
To see that ψ_{n+1} is well-defined we have to check that the cases in the definition agree when they overlap.

1. If we have $x, y \in \mathrm{dom}_n \cap U_{x_{n+1}}$ all four cases overlap; however, the alternative definitions all equal $\phi_A(x,y)$:

$$\psi_n(x,y) \overset{(3.211)}{=} \phi_A(x,y) \overset{(3.162)}{=} \phi_A(x,t)\phi_A(t,y) \overset{(3.211)}{=} \begin{cases} \psi_n(x,t)\phi_n(t,y) \\ \phi_A(x,t)\psi_n(t,y). \end{cases} \tag{3.213}$$

2. Furthermore, if we have $x \in \mathrm{dom}_n$, $y \in \mathrm{dom}_n \cap U_{x_{n+1}}$ cases one and three overlap. Here both alternatives are equal to $\psi_n(x,y)$, since $x, y \in \mathrm{dom}_n$ and we obtain:

$$\psi_n(x,y) \overset{(3.210)}{=} \psi_n(x,t)\psi_n(t,y) \overset{(3.211)}{=} \psi_n(x,t)\phi_A(t,y). \quad (3.214)$$

3. Additionally, if $y \in \mathrm{dom}_n$, $x \in \mathrm{dom}_n \cap U_{x_{n+1}}$ cases one and four overlap. Here they are equal to $\psi_n(x,y)$, since $x, y \in \mathrm{dom}_n$ a quick calculation yields:

$$\psi_n(x,y) \overset{(3.210)}{=} \psi_n(x,t)\psi_n(t,y) \overset{(3.211)}{=} \phi_A(x,t)\psi_n(t,y). \quad (3.215)$$

4. Moreover, if we have $y \in U_{x_{n+1}}$, $x \in \mathrm{dom}_n \cap U_{x_{n+1}}$ cases two and three overlap. Here both candidate definitions are equal to $\phi_A(x,y)$, since $x, t \in U_{x_{n+1}}$ we arrive at:

$$\phi_A(x,y) \overset{(3.162)}{=} \phi_A(x,t)\phi_A(t,y) \overset{(3.211)}{=} \psi_n(x,t)\phi_A(t,y). \quad (3.216)$$

5. Also, if we have $x \in U_{x_{n+1}}$, $y \in \mathrm{dom}_n \cap U_{x_{n+1}}$ cases two and four overlap. In this case both alternatives are equal to $\phi_A(x,y)$, since $y, t \in U_{x_{n+1}}$ we get:

$$\phi_A(x,y) \overset{(3.162)}{=} \phi_A(x,t)\phi_A(t,y) \overset{(3.211)}{=} \phi_A(x,t)\psi_n(t,y). \quad (3.217)$$

We proceed to show the induction claim, starting with $(3.209)_{n+1}$. By the induction hypothesis we know that $\psi_n(x,y)$ as well as $\phi_A(x,y)$ are lifts of $S_{xA,yA}$ for any (x,y) in their domain of definition. Therefore, we have for $x, y \in \mathrm{dom}_n \cup U_{x_{n+1}}$

$$\psi_{n+1}(x,y) = \begin{cases} \psi_n(x,y) & \text{for } x, y \in \mathrm{dom}_n, \\ \phi_A(x,y) & \text{for } x, y \in U_{x_{n+1}}, \\ \psi_n(x,t)\phi_A(t,y) & \text{for } x \in \mathrm{dom}_n, y \in U_{x_{n+1}}, \\ \phi_A(x,t)\psi_n(t,y) & \text{for } y \in \mathrm{dom}_n, x \in U_{x_{n+1}}, \end{cases} \quad (3.212)$$

where each of the lines is a lift of $S_{xA,yA}$ whenever the expression is defined.
Equation $(3.210)_{n+1}$ we will again show in a case by case manner depending on the s, k and l:

1. $s, k, l \in \mathrm{dom}_n$: $(3.210)_{n+1}$ follows directly from the induction hypothesis;

2. $s, k \in \mathrm{dom}_n$ and $l \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \psi_n(s,k)\psi_n(k,t)\phi_A(t,l) \\ &\overset{(3.210)}{=} \psi_n(s,t)\phi_A(t,l) = \psi_{n+1}(s,l),\end{aligned} \tag{3.218}$$

3. $s, l \in \mathrm{dom}_n$ and $k \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \psi_n(s,t)\phi_A(t,k)\phi_A(t,k)\psi_n(t,l) \\ \overset{(3.163),\,(3.162)}{=} \psi_n(s,t)\psi_n(t,l) &\overset{(3.210)}{=} \psi_n(s,l) = \psi_{n+1}(s,l),\end{aligned}$$

4. $s \in \mathrm{dom}_n$ and $k, l \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \psi_n(s,t)\phi_A(t,k)\phi_A(k,l) \\ &\overset{(3.162)}{=} \psi_n(s,t)\phi_A(t,l) = \psi_{n+1}(s,l),\end{aligned}$$

5. $k, l \in \mathrm{dom}_n$ and $s \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,t)\psi_n(t,k)\psi_n(k,l) \\ &\overset{(3.210)}{=} \phi_A(s,t)\psi_n(t,l) = \psi_{n+1}(s,l),\end{aligned}$$

6. $k \in \mathrm{dom}_n$ and $s, l \in U_{x_{n+1}}$:
$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,t)\psi_n(t,k)\psi_n(k,t)\phi_A(t,l) \\ \overset{(3.210)}{=} \phi_A(s,t)\psi(t,t)\phi_A(t,l) &\overset{(3.211),\,(3.163)}{=} \phi_A(s,t)\phi_A(t,l) \\ &\overset{(3.162)}{=} \phi_A(s,l) = \psi_{n+1}(s,l),\end{aligned}$$

7. $l \in \mathrm{dom}_n$ and $s, k \in U_{x_{n+1}}$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,k)\phi_A(k,t)\psi_n(t,l) \\ &\overset{(3.162)}{=} \phi_A(s,t)\psi_n(t,l) = \psi_{n+1}(s,l),\end{aligned}$$

8. and if $s, k, l \in U_z$:

$$\begin{aligned}\psi_{n+1}(s,k)\psi_{n+1}(k,l) &= \phi_A(s,k)\phi_A(k,l) \\ &\overset{(3.162)}{=} \phi_A(s,l) = \psi_{n+1}(s,l).\end{aligned}$$

To see $(3.211)_{n+1}$, i.e. that ψ_{n+1} coincides with ϕ_A where both functions are defined pick $x, y \in \mathrm{dom}_{n+1}$ such that $(xA, yA) \in \mathrm{dom}\,\overline{S})$. Recall the definition of ψ_{n+1}

$$\psi_{n+1}(x,y) = \begin{cases} \psi_n(x,y) & \text{for } x, y \in \mathrm{dom}_n, \\ \phi_A(x,y) & \text{for } x, y \in U_{x_{n+1}}, \\ \psi_n(x,t)\phi_A(t,y) & \text{for } x \in \mathrm{dom}_n, y \in U_{x_{n+1}}, \\ \phi_A(x,t)\psi_n(t,y) & \text{for } y \in \mathrm{dom}_n, x \in U_{x_{n+1}}. \end{cases} \tag{3.212}$$

Therefore, if $x, y \in \mathrm{dom}_n$ we may use the induction hypothesis directly and if $x, y \in U_{x_{n+1}}$ we also arrived at the claim we want to prove. Excluding these cases, we are left with rows number three and four of this definition with the restriction

3. $x \in \mathrm{dom}_n \setminus U_{x_{n+1}}, y \in U_{x_{n+1}} \setminus \mathrm{dom}_n$ or

4. $y \in \mathrm{dom}_n \setminus U_{x_{n+1}}, x \in U_{x_{n+1}} \setminus \mathrm{dom}_n$,

respectively. Because t satisfies $t \in \mathrm{dom}_n \cap U_{x_{n+1}}$, we have in both cases $t \in \overline{x\,y}$. By using property 4 of lemma 48 we infer from $(xA, yA) \in \mathrm{dom}\,\overline{S}$ that in both cases $(xA, tA), (tA, yA) \in \mathrm{dom}\,\overline{S}$ also holds. Hence, we may apply the induction hypothesis $(3.211)_n$.

It remains to show uniqueness. So let $\tilde{\psi}_{n+1}$ be defined on $\mathrm{dom}_{n+1}\times\mathrm{dom}_{n+1}$ fulfil

$$\forall(t,s)\in\mathrm{dom}_{n+1}\times\mathrm{dom}_{n+1}:\tilde{\psi}(t,s)\text{ is a lift of }S_{tA,sA},\quad(3.209_{\tilde{\psi}})$$

$$\forall s,k,l\in\mathbb{R}:\tilde{\psi}(k,s)\tilde{\psi}(s,l)=\tilde{\psi}(k,l),\quad(3.210_{\tilde{\psi}})$$

$$\forall(x,y)\in\mathrm{dom}_{n+1}:(xA,yA)\in\mathrm{dom}\,\overline{S}\Rightarrow\tilde{\psi}(x,y)=\phi_A(x,y).\quad(3.211_{\tilde{\psi}})$$

Now pick $x,y\in\mathrm{dom}_{n+1}$. We proceed in a case by case manner

1. If $x,y\in\mathrm{dom}_n$ holds, then $\psi_{n+1}(x,y)=\tilde{\psi}_{n+1}(x,y)$ follows directly from the induction hypothesis.

2. Similarly, if $x,y\in U_{x_{n+1}}$ holds, we have

$$\psi_{n+1}(x,y)=\phi_A(x,y)=\tilde{\psi}_{n+1}(x,y).\tag{3.219}$$

3. Additionally, if $x\in\mathrm{dom}_n, y\in U_{x_{n+1}}$ holds, then

$$\psi_{n+1}(x,y)\overset{(3.212)}{=}\psi_n(x,t)\phi_A(t,y)\tag{3.220}$$

$$\overset{t\in\mathrm{dom}_n\cap U_{x_{n+1}}}{=}\tilde{\psi}_{n+1}(x,t)\tilde{\psi}_{n+1}(t,y)\overset{(3.210_{\tilde{\psi}})}{=}\tilde{\psi}_{n+1}(x,y)\tag{3.221}$$

   is satisfied.

4. Conversely, if $y\in\mathrm{dom}_n, x\in U_{x_{n+1}}$ holds, we may use the same calculation to obtain

$$\psi_{n+1}(x,y)\overset{(3.212)}{=}\phi_A(x,t)\psi_n(t,y)\tag{3.222}$$

$$\overset{t\in\mathrm{dom}_n\cap U_{x_{n+1}}}{=}\tilde{\psi}_{n+1}(x,t)\tilde{\psi}_{n+1}(t,y)\overset{(3.210_{\tilde{\psi}})}{=}\tilde{\psi}_{n+1}(x,y).\tag{3.223}$$

Now we have established a unique extension ψ_n of ϕ_A fulfilling properties (3.209)-(3.211).

We know that for each $n \in \mathbb{N}$ the function $\psi_{n+1} : \mathrm{dom}_{n+1}^2 \to U(\mathcal{F})$ is an extension of $\psi_n : \mathrm{dom}_n^2 \to U(\mathcal{F})$. Furthermore, the sets dom_n cover $\mathbb{R}$ according to equation (3.207). Consequently, there is a unique common extension, by small abuse of notation again called $\phi_A : \mathbb{R}^2 \to U(\mathcal{F})$, of all ψ_n. This function fulfils the claim (3.202)-(3.205), because any $t, l, s \in \mathbb{R}$ are contained in some dom_n.

□

Lemma 57 enables us to define a global lift.

Definition 58 (global lift)**.** *For any $A \in \mathcal{V}$ we define*

$$\hat{S}_{0,A} := \phi_A(0,1). \tag{3.224}$$

Using lemma 57 we are now in a position to prove theorem 55.

proof of theorem 55. The operator $\hat{S}$ fulfils the claimed differential equation (3.160) due to the global multiplication property (3.203) and the differential equation (3.205). Its uniqueness is inherited from the uniqueness of ϕ_A for $A \in \mathcal{V}$ from lemma 57. □

Definition 59 (relative phase)**.** *Let $(A, B) \in \mathrm{dom}\,\overline{S}$, we define $z(A, B) \in S^1$ by*

$$z(A,B) := \frac{\hat{S}_{0,B}}{\hat{S}_{0,A}\overline{S}_{A,B}}. \tag{3.225}$$

Please note that for such A, B the lift $\bar{S}_{A,B}$ is well-defined. This means that the product in the denominator is a lift of $S_{0,B}$ and according to definition 58 the ratio is well-defined.

Remark 60. *The function z defined here is an extension of the function z appearing locally in the proof of lemma 56, cf. formula (3.165). Please note that z is smooth when restricted to $\mathcal{W}^2 \cap \mathrm{dom}\,\overline{S}$ for any finite dimensional subspace $\mathcal{W} \subseteq \mathcal{V}$, since $\hat{S}$ is smooth as a solution to* *a differential equation with smooth initial conditions. The parameter $\overline{S}$ appearing in the defining differential equation of $\hat{S}$ is smooth since it is directly constructed in terms of the one-particle scattering operator which is smooth due to section 4.1 in the appendix.*

Lemma 61 (properties of the relative phase)**.** *For all $(A, F), (F, G), (G, A) \in \operatorname{dom} \overline{S}$, as well as or all $H, K \in \mathcal{V}$, we have*

$$z(A, F) = z(F, A)^{-1} \tag{3.226}$$

$$z(F, A)z(A, G)z(G, F) = \Gamma_{F,A,G} \tag{3.227}$$

$$\partial_H \partial_K \ln z(A + H, A + K) = c_A(H, K). \tag{3.228}$$

Proof. Pick $A, F, G \in \mathcal{V}$ as in the lemma. We start off by analysing

$$\hat{S}_{0,F}\overline{S}_{F,G} \overset{(3.225)}{=} z(A, F)\hat{S}_{0,A}\overline{S}_{A,F}\overline{S}_{F,G} \tag{3.229}$$

$$\overset{(3.86)}{=} z(A, F)\Gamma^{-1}_{A,F,G}\hat{S}_{0,A}\overline{S}_{A,G}. \tag{3.230}$$

Exchanging A and F in this equation yields

$$\hat{S}_{0,A}\overline{S}_{A,G} = z(F, A)\Gamma^{-1}_{F,A,G}\hat{S}_{0,F}\overline{S}_{F,G}. \tag{3.231}$$

This is equivalent to

$$\hat{S}_{0,F}\overline{S}_{F,G} = z(F, A)^{-1}\Gamma_{F,A,G}\hat{S}_{0,A}\overline{S}_{A,G}\ . \tag{3.232}$$

Comparing the last equation with formula (3.230) and taking the permutation properties (3.82) of Γ into account this implies that

$$z(A, F) = z(F, A)^{-1} \tag{3.233}$$

holds true. Equation (3.230) solved for $\hat{S}_{0,A}\overline{S}_{A,G}$ also gives us

$$\hat{S}_{0,G} \overset{(3.225)}{=} z(A, G)\hat{S}_{0,A}\overline{S}_{A,G} \tag{3.234}$$

$$\overset{(3.230)}{=} z(A, G)z(A, F)^{-1}\Gamma_{A,F,G}\hat{S}_{0,F}\overline{S}_{F,G}. \tag{3.235}$$

The latter equation compared with

$$\hat{S}_{0,G} \stackrel{(3.225)}{=} z(F,G)\hat{S}_{0,F}\overline{S}_{F,G}, \tag{3.236}$$

yields a direct connection between Γ and z:

$$\frac{z(A,G)}{z(A,F)}\Gamma_{A,F,G} = z(F,G), \tag{3.237}$$

which we rewrite using the antisymmetry (3.226) of z as

$$\Gamma_{A,F,G} = z(F,G)z(A,F)z(G,A). \tag{3.238}$$

Finally, in this equation, we substitute $F = A + \varepsilon_1 H$ as well as $G = A + \varepsilon_2 K$, where $\varepsilon_1, \varepsilon_2$ is small enough so that z and Γ are still well-defined. Then we take the second logarithmic derivative to find

$$\begin{aligned}\partial_{\varepsilon_1}\partial_{\varepsilon_2}\ln z(A+\varepsilon_1 H, A+\varepsilon_2 K) &= \partial_{\varepsilon_1}\partial_{\varepsilon_2}\ln\Gamma_{A,A+\varepsilon_1 H,A+\varepsilon_2 K}\\ &\stackrel{(3.87)}{=} c_A(H,K).\end{aligned} \tag{3.239}$$

□

So we find that c_A is the second mixed logarithmic derivative of z. In the following we will characterize z more thoroughly by c and c^+.

Definition 62 (p-forms of four potentials, phase integral)**.** *For $p \in \mathbb{N}$, we introduce the set Ω^p of p-forms to consist of all maps $\omega : \mathcal{V}\times\mathcal{V}^p \to \mathbb{C}$ such that ω is linear and antisymmetric in its p last arguments and smooth in its first argument when restricted to any finite dimensional subspace of $\mathcal{V}$.*
Additionally, we define the 1-form $\chi \in \Omega^1$ by

$$\chi_A(B) := \partial_B \ln z(A, A+B) \tag{3.240}$$

for all $A, B \in \mathcal{V}$*. Furthermore, for* $p \in \mathbb{N}$ *and any differential form* $\omega \in \Omega^p$*, we define its exterior derivative,* $d\omega \in \Omega^{p+1}$ *by*

$$(d\omega)_A(B_1, \ldots, B_{p+1}) := \sum_{k=1}^{p+1} (-1)^{k+1} \partial_{B_k} \omega_{A+B_k}(B_1, \ldots, \widehat{B_k}, \ldots, B_{p+1}), \tag{3.241}$$

for $A, B_1, \ldots, B_{p+1} \in \mathcal{V}$*, where the notation* $\widehat{B_k}$ *denotes that* B_k *is dropped as an argument.*

Lemma 63 (connection between c and the relative phase)**.** *The differential form* χ *fulfils*

$$(d\chi)_A(F, G) = 2c_A(F, G) \tag{3.242}$$

for all $A, F, G \in \mathcal{V}$*.*

Proof. Pick $A, F, G \in \mathcal{V}$, we calculate

$$\begin{aligned}
(d\chi)_A(F, G) &= \partial_F \partial_G \ln z(A + F, A + F + G) \\
&\qquad - \partial_F \partial_G \ln z(A + G, A + F + G) && (3.243)\\
&= \partial_F \partial_G (\ln z(A, A + F + G) + \ln z(A + F, A + G)) && (3.244)\\
&\qquad - \partial_F \partial_G (\ln z(A, A + F + G) + \ln z(A + G, A + F)) && (3.245)\\
&\overset{(3.226)}{=} 2\partial_F \partial_G \ln z(A + F, A + G) \overset{(3.228)}{=} 2c_A(F, G). && (3.246)
\end{aligned}$$

□

Now since $dc = 0$, we might use Poincaré's lemma as a method independent of z to construct a differential form ω such that $d\omega = c$.

Lemma 64 (Poincaré)**.** *Let* $\omega \in \Omega^p$ *for* $p \in \mathbb{N}$ *be closed, i.e.* $d\omega = 0$*. Then* ω *is also exact, more precisely we have*

$$\omega = d \int_0^1 \iota_t^* i_X f^* \omega dt, \tag{3.247}$$

where X, ι_t for $t \in \mathbb{R}$ and f are given by

$$X : \mathbb{R} \times \mathcal{V} \to \mathbb{R} \times \mathcal{V}, \tag{3.248}$$
$$(t, B) \mapsto (1, 0) \tag{3.249}$$
$$\forall t \in \mathbb{R} : \iota_t : \mathcal{V} \to \mathbb{R} \times \mathcal{V}, \tag{3.250}$$
$$B \mapsto (t, B) \tag{3.251}$$
$$f : \mathbb{R} \times \mathcal{V} \mapsto \mathcal{V}, \tag{3.252}$$
$$(t, B) \mapsto tB \tag{3.253}$$
$$i_W : \Omega^p \to \Omega^{p-1}, \tag{3.254}$$
$$\omega \mapsto ((A; Y_1, \ldots, Y_{p-1}) \mapsto \omega_A(W, Y_1, \ldots, Y_{p-1})) \tag{3.255}$$

For a proof see section 4.2 of the appendix. This lemma gives the next definition meaning.

Definition 65 (antiderivative of a closed p form)**.** *For a closed exterior form $\omega \in \Omega^p$ we define the form $\Pi[\omega]$*

$$\Omega^{p-1} \ni \Pi[\omega] := \int_0^1 \iota_t^* i_X f^* \omega dt. \tag{3.256}$$

For $A, B_1, \ldots, B_{p-1} \in \mathcal{V}$ it takes the form

$$\Pi[\omega]_A(B_1, \ldots, B_{p-1}) = \int_0^1 t^{p-1} \omega_{tA}(A, B_1, \ldots, B_{p-1}) dt. \tag{3.257}$$

By lemma 64 we know $d\Pi[\omega] = \omega$ if $d\omega = 0$.

Now we found two one forms each produces c when the exterior derivative is taken. The next lemma informs us about their relationship.

Lemma 66 (inversion of lemma 63)**.** *The following equality holds*

$$\chi = 2\Pi[c]. \tag{3.258}$$

Proof. By definition 65 of Π and lemma 63 we have $d(\chi - 2\Pi[c]) = 0$. Hence, by the Poincaré lemma 64, we know that there is $v : \mathcal{V} \to \mathbb{R}$ such that

$$dv = \chi - 2\Pi[c] \tag{3.259}$$

holds. Using the definition 59 of z, the parallel transport equation (3.160) translates into the following ODE for z:

$$\partial_B \ln z(0, B) = 0, \quad \partial_\varepsilon \ln z(A, (1 + \varepsilon)A)|_{\varepsilon=0} = 0 \tag{3.260}$$

for all $A, B \in \mathcal{V}$. Therefore, we have

$$\chi_0(B) = 0 = \Pi[c]_0(B), \quad \chi_A(A) = 0 = \Pi[c]_A(A), \tag{3.261}$$

which implies

$$\partial_\varepsilon v_{\varepsilon A}|_{\varepsilon=0} = 0, \quad \partial_\varepsilon v_{A+\varepsilon A}|_{\varepsilon=0} = 0. \tag{3.262}$$

In conclusion, v is constant. □

From this point on we will assume the existence of a function c^+ fulfilling (3.68), (3.69) and (3.70). Recall property (3.69):

$$\forall A, F, G, H : \partial_H c^+_{A+H}(F, G) = \partial_G c^+_{A+G}(F, H). \tag{3.263}$$

For a fixed $F \in \mathcal{V}$, this condition can be read as $d(c^+_\cdot(F, \cdot)) = 0$. As a consequence we can apply Poincaré's lemma 64 to define a one form.

Definition 67 (integral of the causal splitting)**.** *For any $A, F \in \mathcal{V}$, we define*

$$\beta_A(F) := 2\Pi[c^+_\cdot(F, \cdot)]_A. \tag{3.264}$$

Lemma 68 (relation between the integral of the causal splitting and the phase integral)**.** *The following two equations hold:*

$$d\beta = -2c, \tag{3.265}$$

$$d(\beta + \chi) = 0. \tag{3.266}$$

Proof. We start with the exterior derivative of β. Pick $A, F, G \in \mathcal{V}$:

$$d\beta_A(F,G) = \partial_F \beta_{A+F}(G) - \partial_G \beta_{A+G}(F) \tag{3.267}$$
$$= d\Big(2\Pi[c_\cdot^+(G,\cdot)]\Big)_A(F) - d\Big(2\Pi[c_\cdot^+(F,\cdot)]\Big)_A(G) \tag{3.268}$$
$$= 2c_A^+(G,F) - 2c_A^+(F,G) \overset{(3.68)}{=} -2c_A(F,G). \tag{3.269}$$

This proves the first equality. The second equality follows directly by $d\chi = 2c$. □

Definition 69 (corrected lift)**.** *Since $\beta + \chi$ is closed, we may use lemma 64 again to define the phase*

$$\alpha := \Pi[\beta + \chi]. \tag{3.270}$$

Furthermore, for all $A, B \in \mathcal{V}$ we define the corrected second quantized scattering operator

$$\tilde{S}_{0,A} := e^{-\alpha_A} \hat{S}_{0,A}, \tag{3.271}$$
$$\tilde{S}_{A,B} := \tilde{S}_{0,A}^{-1} \tilde{S}_{0,B}. \tag{3.272}$$

Using this definition one immediately gets:

Corollary 70 (group structure of the corrected lift)**.** *We have $\tilde{S}_{A,B}\tilde{S}_{B,C} = \tilde{S}_{A,C}$ for all $A, B, C \in \mathcal{V}$.*

Theorem 71 (causality of the corrected lift)**.** *The corrected second quantized scattering operator fulfils the following causality condition for all $A, F, G \in \mathcal{V}$ such that $F \prec G$:*

$$\tilde{S}_{A,A+F} = \tilde{S}_{A+G,A+G+F}. \tag{3.273}$$

Proof. Let $A, F, G \in \mathcal{V}$ such that $F \prec G$. For the first quantized scattering operator we have

$$S_{A+G,A+G+F} = S_{A,A+F}, \tag{3.274}$$

which we proved in remark 52. So that by definition of $\overline{S}$ we obtain

$$\overline{S}_{A+G,A+G+F} = \overline{S}_{A,A+F}. \tag{3.275}$$

Therefore, any lift this equality is true up to a phase, meaning that

$$f(A,F,G) := \frac{\tilde{S}_{A+G,A+G+F}}{\tilde{S}_{A,A+F}} \tag{3.276}$$

is well-defined. We see immediately

$$f(A,0,G) = 1 = f(A,F,0). \tag{3.277}$$

Pick $F_1, F_2 \prec G_1, G_2$. We abbreviate $F = F_1 + F_2, G = G_1 + G_2$ and we calculate

$$\begin{aligned} f(A,F,G) &= \frac{\tilde{S}_{A+G,A+F+G}}{\tilde{S}_{A,A+F}} && (3.278)\\ &= \frac{\tilde{S}_{A+G,A+F+G}}{\tilde{S}_{A+G_1,A+G_1+F}} \frac{\tilde{S}_{A+G_1,A+G_1+F}}{\tilde{S}_{A,A+F}} && (3.279)\\ &= \frac{\tilde{S}_{A+G,A+G+F_1}\tilde{S}_{A+G+F_1,A+F+G}}{\tilde{S}_{A+G_1,A+F_1+G_1}\tilde{S}_{A+G_1+F_1,A+G_1+F}} \frac{\tilde{S}_{A+G_1,A+G_1+F}}{\tilde{S}_{A,A+F}} && (3.280)\\ &= \frac{\tilde{S}_{A+G,A+G+F_1}}{\tilde{S}_{A+G_1,A+F_1+G_1}} \frac{\tilde{S}_{A+G+F_1,A+F+G}}{\tilde{S}_{A+G_1+F_1,A+G_1+F}} f(A,G_1,F_1+F_2) && (3.281)\\ &= f(A+G_1,F_1,G_2) f(A+G_1+F_1,F_2,G_2) f(A,G_1,F_1+F_2). && (3.282) \end{aligned}$$

Taking the mixed logarithmic derivative we find:

$$\partial_{F_2}\partial_{G_2} \ln f(A, F_1+F_2, G_1+G_2) = \partial_{F_2}\partial_{G_2} \ln f(A+F_1+G_1, F_2, G_2). \tag{3.283}$$

Next we pick $F_2 = \alpha_1 F_1$ and $G_2 = \alpha_2 G_1$ for $\alpha_1, \alpha_2 \in \mathbb{R}^+$ small enough so that $(A + (1+\alpha_1)F_1 + (1+\alpha_2)G_1, A + F_1 + G_1), (A + (1+\alpha_1)F_1 +$

$(1+\alpha_2)G_1, A+F_1+(1+\alpha_2)G_1), (A+(1+\alpha_1)F_1+(1+\alpha_2)G_1, A+(1+\alpha_1)F_1+G_1) \in \operatorname{dom}\overline{S}$ holds. We abbreviate $A' = A+G_1+F_1$, use the definition of z (3.225) and compute

$$\begin{aligned}
& f(A', F_2, G_2) \\
& \overset{(3.271)}{=} \exp(-\alpha_{A'+F_2+G_2} + \alpha_{A'+G_2} + \alpha_{A'+F_2} - \alpha_{A'}) \\
& \qquad \times \frac{\hat{S}^{-1}_{0,A'+G_2}\hat{S}_{0,A'+G_2+F_2}}{\hat{S}^{-1}_{0,A'}\hat{S}_{0,A'+F_2}} && (3.284) \\
& \overset{(3.225)}{=} \exp(-\alpha_{A'+F_2+G_2} + \alpha_{A'+G_2} + \alpha_{A'+F_2} - \alpha_{A'}) \\
& \qquad \times \frac{z(A'+G_2, A'+G_2+F_2)}{z(A', A'+F_2)} \frac{\overline{S}_{A'+G_2,A'+G_2+F_2}}{\overline{S}_{A',A'+F_2}} && (3.285) \\
& \overset{F_2 \lesssim G_2}{=} \exp(-\alpha_{A'+F_2+G_2} + \alpha_{A'+G_2} + \alpha_{A'+F_2} - \alpha_{A'}) \\
& \qquad \times \frac{z(A'+G_2, A'+G_2+F_2)}{z(A', A'+F_2)} && (3.286)
\end{aligned}$$

Most of the factors do not depend on both F_2 and G_2, so taking the mixed logarithmic derivative things simplify:

$$\begin{aligned}
& \partial_{G_2}\partial_{F_2} \ln f(A', F_2, G_2) = \\
& \partial_{G_2}\partial_{F_2}(-\alpha_{A'+F_2+G_2} + \ln z(A'+G_2, A'+G_2+F_2)) && (3.287) \\
& \overset{(3.270),\ (3.240)}{=} \partial_{G_2}(-\beta_{A'+G_2}(F_2) - \chi_{A'+G_2}(F_2) + \chi_{A'+G_2}(F_2)) && (3.288) \\
& \overset{(3.264)}{=} -2c^{+}_{A'}(F_2, G_2) \overset{F_2 < G_2,\ (3.70)}{=} 0. && (3.289)
\end{aligned}$$

So by (3.283) we also have

$$\begin{aligned}
\partial_{F_2}\partial_{G_2} \ln f(A, F_1+F_2, G_1+G_2) &= 0 && (3.290) \\
= \partial_{\alpha_1}\partial_{\alpha_2} \ln f(A, F_1(1+\alpha_1), G_1(1+\alpha_2)). & && (3.291)
\end{aligned}$$

Using this then we can integrate and obtain

$$0 = \int_{-1}^{0} d\alpha_1 \int_{-1}^{0} d\alpha_2 \partial_{\alpha_1}\partial_{\alpha_2} \ln f(A, F_1(1+\alpha_1), G_1(1+\alpha_2)) \tag{3.292}$$

$$= \ln f(A, F_1, G_1) - \ln f(A, 0, G_1) - \ln f(A, F_1, 0) \tag{3.293}$$
$$+ \ln f(A, 0, 0)$$

$$\overset{(3.277)}{=} \ln f(A, F_1, G_1). \tag{3.294}$$

Recalling equation (3.276), the definition of f, this ends our proof. □

Next, we investigate the current associated with $\tilde{S}$.

Theorem 72 (evaluation of the current of the corrected lift)**.** *For general $A, F \in \mathcal{V}$ we have*

$$j_A^{\tilde{S}}(F) = -i\beta_A(F). \tag{3.295}$$

So in particular for $G \in \mathcal{V}$

$$\partial_G j_{A+G}^{\tilde{S}}(F) = -2ic_A^{+}(F, G). \tag{3.296}$$

holds.

Proof. Pick $A, F \in \mathcal{V}$ as in the theorem. We calculate

$$i\partial_F \ln \left\langle \Omega, \tilde{S}_{A,A+F}\Omega \right\rangle \tag{3.297}$$

$$\overset{(3.271)}{=} i\partial_F \left(-\alpha_{A+F} - \alpha_A + \ln \left\langle \Omega, \hat{S}_{0,A}^{-1}\hat{S}_{0,A+F}\Omega \right\rangle \right) \tag{3.298}$$

$$\overset{(3.225)}{=} i\partial_F \left(-\alpha_{A+F} + \ln z(A, A+F) + \ln \langle \Omega, \overline{S}_{A,A+F}\Omega \rangle \right) \tag{3.299}$$

The last summand vanishes, as can be seen by the following calculation

$$\partial_F \ln \left\langle \Omega, \overline{S}_{A,A+F}\Omega \right\rangle \tag{3.300}$$

$$\overset{(3.88)}{=} i\partial_F \ln \det_{\mathcal{H}^-} |P^- S_{A,A+F} P^-| \tag{3.301}$$

$$= \frac{i}{2}\partial_F \ln \det_{\mathcal{H}^-}((P^- S_{A,A+F} P^-)^* P^- S_{A,A+F} P^-) \tag{3.302}$$

$$= \frac{i}{2}\partial_F \det_{\mathcal{H}^-}(P^- S_{A+F,A} P^- S_{A,A+F} P^-) \tag{3.303}$$

$$= \frac{i}{2}\operatorname{tr}(\partial_F P^- S_{A+F,A} P^- S_{A,A+F} P^-) \tag{3.304}$$

$$= \frac{i}{2}\operatorname{tr}(\partial_F P^- S_{A,A+F} P^- + \partial_F P^- S_{A+F,A} P^-) = 0 \tag{3.305}$$

where we made use of (3.143). Theorem 90 serves to justify the necessary regularity. So we are left with

$$j_A(F) = i\partial_F(-\alpha_{A+F} + \ln z(A, A+F)) \tag{3.306}$$

$$= i(-\beta_A(F) - \chi_A(F) + \chi_A(F)) = -i\beta_A(F). \tag{3.307}$$

Finally, by taking the derivative with respect to $G \in \mathcal{V}$ and using the definition of β we find

$$\partial_G j_{A+G}(F) = -2ic_A^+(F, G). \tag{3.308}$$

□

proof of theorem 51. The operator $\tilde{S}$ constructed in this subsection fulfils properties (3.72) and (3.73) by corollary 70 and theorem 71. The characterization of the current (3.74) follows form theorem 72 and the properties of c^+, (3.68) to (3.70). □

3.3 Analyticity of the Scattering Operator

In this section we will present a relatively simple formula for the second quantized scattering operator in terms of the one-particle scattering operator. This formula is valid for small external fields, where "small" will be made precise later. The formula has implications for the analyticity of the second quantized scattering operator for general external fields, which we will also present. Also, this section is concerned with the scattering regime. The notation introduced in the last sections still applies. The shorthand defined next will turn out to be useful. Let $B : \mathcal{H} \to \mathcal{H}$ be linear and bounded, then we introduce

$$B_{\#,\tilde{\#}} = P^{\#} B P^{\tilde{\#}}, \tag{3.309}$$

where $\#, \tilde{\#} \in \{+,-\}$ holds. Recall the definition of Fock space in this setting

$$\mathcal{F} := \bigoplus_{m,p=0}^{\infty} \left(\mathcal{H}^+\right)^{\wedge m} \otimes \left(\overline{\mathcal{H}^-}\right)^{\wedge p}. \tag{3.310}$$

We denote the sectors of Fock space of fixed particle numbers by $\mathcal{F}_{m,p}$. A fixed element of $\mathcal{F}_{0,0}$ of norm 1 will be denoted by Ω.
The annihilation operator a acts on an arbitrary sector of Fock space $\mathcal{F}_{m,p}$, for any $m, p \in \mathbb{N}_0$ with the operator type

$$a : \overline{\mathcal{H}} \times \mathcal{F}_{m,p} \to \mathcal{F}_{m-1,p} \oplus \mathcal{F}_{m,p+1}, \tag{3.311}$$

where the second argument is usually not included in the parenthesis, as is common practice for bounded operators on a Hilbert space. We start out by defining a on elements of $\{\bigwedge_{l=1}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c \mid \forall c : \varphi_c \in \mathcal{H}^+, \phi_c \in \mathcal{H}^-\}$ which spans a dense subset of $\mathcal{F}_{m,p}$, then one continues this operator uniquely by linearity and finally by the bounded linear extension theorem to all of $\mathcal{F}_{m,p}$ and then again by linearity to all of $\overline{\mathcal{H}} \otimes \mathcal{F}_{m,p}$. On the dense set we define $a(\phi)$ by

$$a(\phi) \bigwedge_{l=1}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c \tag{3.312}$$

$$= \sum_{k=1}^{m} (-1)^{1+k} \langle P^+\phi, \varphi_k\rangle \bigwedge_{\substack{l=1 \\ l\neq k}}^{m} \varphi_l \otimes \bigwedge_{c=1}^{p} \phi_c + \bigwedge_{l=1}^{m} \varphi_l \otimes P^-\phi \wedge \bigwedge_{c=1}^{p} \phi_c \tag{3.313}$$

where $\langle , \rangle$ denotes that the scalar product of $\mathcal{H}$. The first summand on the right-hand side is taken to vanish for $m = 0$. The operator norm of a is given by

$$\|a(\phi)\| = \|\phi\|. \tag{3.314}$$

The operators a and its adjoint a^* fulfil the canonical anticommutation relations:

$$\forall \phi, \psi : \{a(\phi), a^*(\psi)\} = a(\phi)a^*(\psi) + a^*(\psi)a(\phi) = \langle \phi, \psi\rangle \tag{3.315}$$

$$\forall \phi, \psi : \{a^*(\phi), a^*(\psi)\} = \{a(\phi), a(\psi)\} = 0. \tag{3.316}$$

Now for the construction of the second quantized S-matrix please recall the lift condition

$$\forall \phi \in \mathcal{H} : \quad \tilde{S}^A \circ a(\phi) = a\left(S^A\phi\right) \circ \tilde{S}^A, \qquad \text{(lift condition)}$$

which is to be satisfied by any lift $\tilde{S}^A$ of the one-particle scattering matrix S^A.

In the appendix we carry out an explicit, albeit heuristic, construction of a power series expression of a lift of S^A in section 4.3 that culminates in the formula which will be directly verified in this section.

In order to state this formula, we have to introduce some more notation.

3.3.1 Differential second quantization

Let $B \in \mathcal{B}(\mathcal{H})$ be skew adjoint, i.e. iB is self adjoint such that B_{+-} is a Hilbert-Schmidt operator. We would like to construct a version $\mathrm{d}\Gamma(B)$ of B that acts on Fock space and also is skew adjoint. The proof of the skew adjointness of $\mathrm{d}\Gamma(B)$ is a bit lengthy, as is typical of such proofs. Even though one might speed it up a little by using the tools available e.g. in [20] the author is of the opinion that it is instructive to give a direct proof. In this subsection we will associate with every set $C \subset \mathbb{N}$ such that $|C| < \infty$ the sequence $(C_k)_{1\leqslant k\leqslant |C|}$ such that

$$\forall 1 \leqslant k \leqslant |C| : C_k \in C \tag{3.317}$$

$$\forall 1 \leqslant k < l \leqslant |C| : C_k < C_l \tag{3.318}$$

hold. This notation is confined to within the proofs of this subsection. The strategy of this subsection is to construct an operator in two steps that is essentially self adjoint on the dense domain of the next

Definition 73 (Fock space of finitely many particles)**.** *We introduce*

$$\mathcal{F}' := \bigoplus_{m,p=0}^{\infty} \mathcal{F}_{m,p}, \tag{3.319}$$

where $\bigoplus$ *refers to the algebraic direct sum. Furthermore, we define*

$$\mathcal{F}^0 \subset \mathcal{F}' \tag{3.320}$$

such that for each element $\alpha \in \mathcal{F}^0$ *there is a basis an ONB* $(\tilde{\varphi}_k)_{k\in\mathbb{N}}$ *of* $\mathcal{H}^+$ *and an ONB* $(\tilde{\varphi}_{-k})_{k\in\mathbb{N}}$ *of* $\mathcal{H}^-$ *such that*

$$\alpha \in \operatorname{span}\Bigg\{ \prod_{k=1}^{m} a^*(\tilde{\varphi}_{L_k}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega \tag{3.321}$$

$$\Bigg| \; m,p \in \mathbb{N}, (L_k)_k, (C_c)_c \subset \mathbb{N}, |L| = m, |C| = p \Bigg\} \tag{3.322}$$

holds. Here and elsewhere, the product is defined inductively:

$$\prod_{k=1}^{M} f_k = \left(\prod_{k=1}^{M-1} f_k\right) f_M, \tag{3.323}$$

for all $M \in \mathbb{N}$ and factors f_k, i.e. factors increase in index from left to right.

Constructing $\mathrm{d}\Gamma$ piecewise turns out to be advantageous.

Definition 74. *We define the following operators of type $\mathcal{F}^0 \to \mathcal{F}$*

$$\mathrm{d}\Gamma(B_{++}) := \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)a(\varphi_n) \tag{3.324}$$

$$\mathrm{d}\Gamma(B_{--}) := -\sum_{n\in\mathbb{N}} a(\varphi_{-n})a^*(B_{--}\varphi_{-n}) \tag{3.325}$$

$$\mathrm{d}\Gamma(B_{-+}) := \sum_{n\in\mathbb{N}} a^*(B_{-+}\varphi_n)a(\varphi_n) \tag{3.326}$$

where the sum converges in the strong operator topology and $(\varphi_n)_n$, $(\varphi_{-n})_n$ are arbitrary ONBs of $\mathcal{H}^+$ and $\mathcal{H}^-$.

Lemma 75. *The operators $\mathrm{d}\Gamma(B_{++}), \mathrm{d}\Gamma(B_{--})$ and $\mathrm{d}\Gamma(B_{-+})$ restricted to $\mathcal{F}^0_{m,p}$ have the following type*

$$\mathrm{d}\Gamma(B_{++})|_{\mathcal{F}^0_{m,p}} : \quad \mathcal{F}^0_{m,p} \to \mathcal{F}_{m,p} \tag{3.327}$$

$$\mathrm{d}\Gamma(B_{--})|_{\mathcal{F}^0_{m,p}} : \quad \mathcal{F}^0_{m,p} \to \mathcal{F}_{m,p} \tag{3.328}$$

$$\mathrm{d}\Gamma(B_{-+})|_{\mathcal{F}^0_{m,p}} : \quad \mathcal{F}^0_{m,p} \to \mathcal{F}_{m-1,p-1} \tag{3.329}$$

and fulfil the following bounds for all m, p

$$\|\mathrm{d}\Gamma(B_{++})|_{\mathcal{F}^0_{m,p}}\| \leqslant (m+4)\|B_{++}\| \tag{3.330}$$

$$\|\mathrm{d}\Gamma(B_{--})|_{\mathcal{F}^0_{m,p}}\| \leqslant (p+4)\|B_{--}\| \tag{3.331}$$

$$\|\mathrm{d}\Gamma(B_{-+})|_{\mathcal{F}^0_{m,p}}\| \leqslant 2\|B_{-+}\|_{I_2}, \tag{3.332}$$

Moreover, the operator $\mathrm{d}\Gamma(B_{-+})|_{\mathcal{F}^0_{m,p}}$ *also assumes the following form*

$$\mathrm{d}\Gamma(B_{-+})|_{\mathcal{F}^0_{m,p}} = -\sum_{k\in\mathbb{N}} a^*(\varphi_{-k})a(B_{+-}\varphi_{-k})|_{\mathcal{F}^0_{m,p}}. \tag{3.333}$$

The equality of operators can then be continued to all of $\mathcal{F}$.

Proof. Pick $\alpha \in \mathcal{F}^0_{m,p}$ for $m, p \in \mathbb{N}_0$, α can be expressed in terms of some ONB $(\tilde{\varphi}_k)_{k\in\mathbb{N}}$ of $\mathcal{H}^+$ and $(\tilde{\varphi}_{-k})_{k\in\mathbb{N}}$ of $\mathcal{H}^-$

$$\alpha = \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m,|C|=p}} \alpha_{L,C} \prod_{l=1}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega. \tag{3.334}$$

In this expansion only finitely many coefficients $\alpha_{\cdot,\cdot}$ are non-zero. Our operators all map the vacuum onto the zero vector, so commuting them through the products of creation and annihilation operators in the expansion of α we can make the action of them more explicit:

$$\mathrm{d}\Gamma(B_{++})\alpha = \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m} \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l}) \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)\langle\varphi_n, \tilde{\varphi}_{L_b}\rangle \times \prod_{l=b+1}^{m} a^*(\tilde{\varphi}_l) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega \tag{3.335}$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m} \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l}) a^*(B_{++}\tilde{\varphi}_{L_b}) \prod_{l=b+1}^{m} a^*(\tilde{\varphi}_l) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega. \tag{3.336}$$

We notice, that $\mathrm{d}\Gamma(B_{++})\alpha \in \mathcal{F}_{m,p}$ holds. What is left to show for the first operator is therefore its norm. For estimating this we see that

B_{++} in the last line can be replaced by

$$B^L_{L_b} := \left(1 - \sum_{\substack{l=1\\ l\neq b}}^{m} |\tilde{\varphi}_{L_l}\rangle\langle\tilde{\varphi}_{L_l}|\right) B_{++}, \tag{3.337}$$

due to the antisymmetry of fermions. Expanding

$$\begin{aligned}
&\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 = \langle \mathrm{d}\Gamma(B_{++})\alpha, \mathrm{d}\Gamma(B_{++})\alpha\rangle \\
&= \sum_{\substack{L,C,L',C'\subset\mathbb{N}\\ |L'|=|L|=m,|C'|=|C|=p}} \overline{\alpha_{L,C}}\alpha_{L',C'} \sum_{b,b'=1}^{m} \Bigg\langle \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l}) \;\; a^*(B^L_{L_b}\tilde{\varphi}_{L_b}) \\
&\prod_{l=b+1}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega, \prod_{l=1}^{b'-1} a^*(\tilde{\varphi}_{L'_l}) \;\; a^*(B^{L'}_{L'_{b'}}\tilde{\varphi}_{L'_b}) \\
&\prod_{l=b'+1}^{m} a^*(\tilde{\varphi}_{L'_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C'_c})\Omega \Bigg\rangle
\end{aligned} \tag{3.338}$$

we see that in fact C and C' need to agree, because we can just commute the corresponding annihilation operators from one end of the scalar product to the other. Furthermore, only a single wave function on each side of the scalar product is modified, this implies that in order for the scalar product not to vanish $|L \cap L'| \geqslant m-2$ has to hold. For the case $L \neq L'$ we split up the sum over sets into the sum over a new L such that $|L| = m-2$ holds and an additional sum over four indices $n_1 < n_2, p_1 < p_2$. The double sum over b, b' only has contributions where $b = n_1$ or $b = n_2$ and $b' = p_1$ or $b' = p_2$ are selected. Because each factor in the first half is orthogonal to each other factor in this half and analogously for the second half, this will result in a sum of eight terms. In the case $L' = L$ the full sum contributes, yielding

$$\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 =$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}} \sum_{\substack{n_1<n_2,p_1<p_2\in\mathbb{N}\setminus L\\ \{n_1,n_2\}\neq\{p_1,p_2\}}} \overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C} \begin{bmatrix} n_1\\ n_2\end{bmatrix}\begin{bmatrix} p_1\\ p_2\end{bmatrix}$$

$$\times \Bigg(\langle\tilde{\varphi}_{n_1},\tilde{\varphi}_{p_1}\rangle\langle B_{n_2}^{L\cup\{n_1,n_2\}}\tilde{\varphi}_{n_2}, B_{p_2}^{L\cup\{p_1,p_2\}}\tilde{\varphi}_{p_2}\rangle \tag{3.339}$$

$$- \langle\tilde{\varphi}_{n_1}, B_{p_2}^{L\cup\{p_1,p_2\}}\tilde{\varphi}_{p_2}\rangle\langle B_{n_2}^{L\cup\{n_1,n_2\}}\tilde{\varphi}_{n_2},\tilde{\varphi}_{p_1}\rangle \tag{3.340}$$

$$+ \langle B_{n_1}^{L\cup\{n_1,n_2\}}\tilde{\varphi}_{n_1},\tilde{\varphi}_{p_1}\rangle\langle\tilde{\varphi}_{n_2}, B_{p_2}^{L\cup\{p_1,p_2\}}\tilde{\varphi}_{p_2}\rangle \tag{3.341}$$

$$- \langle B_{n_1}^{L\cup\{n_1,n_2\}}\tilde{\varphi}_{n_1}, B_{p_2}^{L\cup\{p_1,p_2\}}\tilde{\varphi}_{p_2}\rangle\langle\tilde{\varphi}_{n_2},\tilde{\varphi}_{p_1}\rangle \tag{3.342}$$

$$+ \langle\tilde{\varphi}_{n_1}, B_{p_1}^{L\cup\{p_1,p_2\}}\tilde{\varphi}_{p_1}\rangle\langle B_{n_2}^{L\cup\{n_1,n_2\}}\tilde{\varphi}_{n_2},\tilde{\varphi}_{p_2}\rangle \tag{3.343}$$

$$- \langle\tilde{\varphi}_{n_1},\tilde{\varphi}_{p_2}\rangle\langle B_{n_2}^{L\cup\{n_1,n_2\}}\tilde{\varphi}_{n_2}, B_{p_1}^{L\cup\{p_1,p_2\}}\tilde{\varphi}_{p_1}\rangle \tag{3.344}$$

$$+ \langle B_{n_1}^{L\cup\{n_1,n_2\}}\tilde{\varphi}_{n_1}, B_{p_1}^{L\cup\{p_1,p_2\}}\tilde{\varphi}_{p_1}\rangle\langle\tilde{\varphi}_{n_2},\tilde{\varphi}_{p_2}\rangle \tag{3.345}$$

$$- \langle B_{n_1}^{L\cup\{n_1,n_1\}}\tilde{\varphi}_{n_1},\tilde{\varphi}_{p_2}\rangle\langle\tilde{\varphi}_{n_2}, B_{p_1}^{L\cup\{p_1,p_2\}}\tilde{\varphi}_{p_1}\rangle\Bigg) \tag{3.346}$$

$$+ \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} |\alpha_{L,C}|^2 \sum_{b,b'=1}^{m} \Bigg\langle \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l})\ \ a^*(B_{L_b}^{L}\tilde{\varphi}_{L_b}) \tag{3.347}$$

$$\prod_{l=b+1}^{m} a^*(\tilde{\varphi}_{L_l})\Omega, \prod_{l=1}^{b'-1} a^*(\tilde{\varphi}_{L_l})\ \ a^*(B_{L_{b'}}^{L}\tilde{\varphi}_{L_b}) \prod_{l=b'+1}^{m} a^*(\tilde{\varphi}_{L_l}))\Omega\Bigg\rangle,$$

where

$$\begin{bmatrix} n_1\\ n_2\end{bmatrix} := (-1)^{|\{l\in L|l<n_1\}|+|\{l\in L|l<n_2\}|} \tag{3.348}$$

keeps track of the number of anti commutations. This is non-standard notation, but it is meant to keep the notation as compact as possible and its use is contained to this subsection.

Due to the antisymmetry each summand containing a factor without an occurrence of the B operator are only non-zero if $n_1 = p_1$ or $n_2 = p_2$. Each factor containing exactly one occurrence of B obtains a similar restriction. So we can split the block of terms into one corresponding to the two cases just mentioned.

$$
\begin{aligned}
&\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 = \\
&= \sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{\substack{n_1<n_2,p_1<p_2\in\mathbb{N}\setminus L\\ \{n_1,n_2\}\neq\{p_1,p_2\}}} \overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C} \begin{bmatrix} n_1\\ n_2\end{bmatrix}\begin{bmatrix} p_1\\ p_2\end{bmatrix} \\
&\times \Big(-\langle\tilde\varphi_{n_1}, B_{p_2}^{L\cup\{p_1,p_2\}}\tilde\varphi_{p_2}\rangle\langle B_{n_2}^{L\cup\{n_1,n_2\}}\tilde\varphi_{n_2},\tilde\varphi_{p_1}\rangle 1_{n_1\neq p_1} \qquad (3.349)\\
&+ \langle B_{n_1}^{L\cup\{n_1,n_2\}}\tilde\varphi_{n_1},\tilde\varphi_{p_1}\rangle\langle\tilde\varphi_{n_2}, B_{p_2}^{L\cup\{p_1,p_2\}}\tilde\varphi_{p_2}\rangle 1_{n_2\neq p_1}\\
&+ \langle\tilde\varphi_{n_1}, B_{p_1}^{L\cup\{p_1,p_2\}}\tilde\varphi_{p_1}\rangle\langle B_{n_2}^{L\cup\{n_1,n_2\}}\tilde\varphi_{n_2},\tilde\varphi_{p_2}\rangle 1_{n_1\neq p_2}\\
&- \langle B_{n_1}^{L\cup\{n_1,n_2\}}\tilde\varphi_{n_1},\tilde\varphi_{p_2}\rangle\langle\tilde\varphi_{n_2}, B_{p_1}^{L\cup\{p_1,p_2\}}\tilde\varphi_{p_1}\rangle 1_{n_2\neq p_2}\Big)\\
&+ \sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-1}} \sum_{n\neq p\in\mathbb{N}\setminus L} \overline{\alpha_{L\cup\{n\},C}}\alpha_{L\cup\{p\},C}\begin{bmatrix} n\\ p\end{bmatrix}\langle B_n^{L\cup\{n\}}\tilde\varphi_n, B_p^{L\cup\{p\}}\tilde\varphi_p\rangle \qquad (3.350)\\
&+ \sum_{\substack{L,C\subset\mathbb{N}\\ |L|=m,|C|=p}} |\alpha_{L,C}|^2 \sum_{b,b'=1}^{m} \Big\langle \prod_{l=1}^{b-1} a^*(\tilde\varphi_{L_l}) \;\; a^*(B_{L_b}^{L}\tilde\varphi_{L_b}) \qquad (3.351)\\
&\prod_{l=b+1}^{m} a^*(\tilde\varphi_{L_l})\Omega, \prod_{l=1}^{b'-1} a^*(\tilde\varphi_{L_l}) \;\; a^*(B_{L_{b'}}^{L}\tilde\varphi_{L_b}) \prod_{l=b'+1}^{m} a^*(\tilde\varphi_{L_l}))\Omega\Big\rangle,
\end{aligned}
$$

where we also summarized the terms of the second block. The restrictions $n_1 < n_2$ and $p_1 < p_2$ have the effect that the negative terms sum

up to just one term without restrictions, while the positive terms add up to two such terms.
For the term (3.350) we add and subtract the term where $n = p$. The enlarged sum can then be reformulated

$$\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1\\|C|=p}}\sum_{n,n'\in\mathbb{N}\setminus L}\overline{\alpha_{L\cup\{n\},C}}\alpha_{L\cup\{n'\},C}\langle B_n^{L\cup\{n\}}\tilde{\varphi}_n, B_{n'}^{L\cup\{n'\}}\tilde{\varphi}_{n'}\rangle\begin{bmatrix}n\\n'\end{bmatrix}$$

$$=\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1,|C|=p}}\left\|\sum_{n\in\mathbb{N}\setminus L}\alpha_{L\cup\{n\},C}B_n^{L\cup\{n\}}\tilde{\varphi}_n\begin{bmatrix}n\\0\end{bmatrix}\right\|^2$$

$$=\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1\\|C|=p}}\left\|\left(1-\sum_{l\in L}|\tilde{\varphi}_l\rangle\langle\tilde{\varphi}_l|\right)B_{++}\sum_{n\in\mathbb{N}\setminus L}\alpha_{L\cup\{n\},C}\tilde{\varphi}_n\begin{bmatrix}n\\0\end{bmatrix}\right\|^2 \tag{3.352}$$

Now the operator product inside the norm has operator norm $\|B_{++}\|$ and so we can estimate the whole object by

$$(3.352)\leqslant\|\alpha\|^2\|B_{++}\|^2. \tag{3.353}$$

We need to estimate the term we added to complete the norm square in (3.350), this is done as follows

$$\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m-1,|C|=p}}\sum_{n\in\mathbb{N}\setminus L}|\alpha_{L\cup\{n\},C}|^2\|B_n^{L\cup\{n\}}\tilde{\varphi}_n\|^2$$

$$\leqslant\sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}}\|B_{++}\|^2|\alpha_{L,C}|^2=\|\alpha\|^2\|B_{++}\|^2. \tag{3.354}$$

For (3.349) and the following 3 lines we notice that we may replace all one-particle operators with B_{++}, since the projector acts as the identity in these cases. Subsequently, the two terms of equal sign are identical except for the extra condition on the sum, resulting in

$$
\begin{aligned}
&(3.349) = \\
&\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}} \sum_{\substack{n_1<n_2,p_1<p_2\in\mathbb{N}\setminus L\\\{n_1,n_2\}\neq\{p_1,p_2\}}} \overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C} \begin{bmatrix} n_1\\ n_2\end{bmatrix}\begin{bmatrix} p_1\\ p_2\end{bmatrix} && (3.355)\\
&\times\Big(\langle B_{++}\tilde\varphi_{n_1},\tilde\varphi_{p_1}\rangle\langle\tilde\varphi_{n_2},B_{++}\tilde\varphi_{p_2}\rangle(1_{n_1\neq p_2}+1_{n_2\neq p_1}) && (3.356)\\
&-\langle\tilde\varphi_{n_1},B_{++}\tilde\varphi_{p_2}\rangle\langle B_{++}\tilde\varphi_{n_2},\tilde\varphi_{p_1}\rangle(1_{n_1\neq p_1}+1_{n_2\neq p_2})\Big). && (3.357)
\end{aligned}
$$

Next, we are going to repeatedly add and subtract terms, such that we may factorize the n and the p sums. In order to do so we impose the condition $\{n_1, n_2\} \neq \{p_1, p_2\}$ in the sum by the factor $1 - \delta_{n_1,p_1}\delta_{n_2,p_2}$. Similarly, we rewrite the other conditions in the following way

$$1_{n_1\neq p_2} + 1_{n_2\neq p_1} = 2 - \delta_{n_1,p_2} - \delta_{n_2,p_1}, \quad (3.358)$$

$$1_{n_1\neq p_1} + 1_{n_2\neq p_2} = 2 - \delta_{n_1,p_1} - \delta_{n_2,p_2}. \quad (3.359)$$

These are to be multiplied by $1 - \delta_{n_1,p_1}\delta_{n_2,p_2}$ resulting in the two expressions

$$2 - \delta_{n_1,p_2} - \delta_{n_2,p_1} - 2\delta_{n_1,p_1}\delta_{n_2,p_2} \quad (3.360)$$

$$2 - \delta_{n_1,p_1} - \delta_{n_2,p_2}, \quad (3.361)$$

where the upper expression yields the restrictions on the sum of over $\langle B_{++}\tilde\varphi_{n_1},\tilde\varphi_{p_1}\rangle\langle\tilde\varphi_{n_2},B_{++}\tilde\varphi_{p_2}\rangle$ and the lower expression analogously for $\langle\tilde\varphi_{n_1},B_{++}\tilde\varphi_{p_2}\rangle\langle B_{++}\tilde\varphi_{n_2},\tilde\varphi_{p_1}\rangle$. For the term without further restrictions we may add the sum of the terms (3.356) and (3.357), the rest is treated separately.

The terms are all estimated after rewriting the scalar products as a single sum of two scalar products in $\mathcal{H}^+ \otimes \mathcal{H}^+$.

2 :

$$2\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\sum_{\substack{n_1<n_2\in\mathbb{N}\setminus L\\p_1<p_2\in\mathbb{N}\setminus L}}\overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C}\begin{bmatrix}n_1\\n_2\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix} \tag{3.362}$$

$$\times\Big(\langle B_{++}\tilde\varphi_{n_1},\tilde\varphi_{p_1}\rangle\langle\tilde\varphi_{n_2},B_{++}\tilde\varphi_{p_2}\rangle-\langle\tilde\varphi_{n_1},B_{++}\tilde\varphi_{p_2}\rangle\langle B_{++}\tilde\varphi_{n_2},\tilde\varphi_{p_1}\rangle\Big) \tag{3.363}$$

$$=2\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\Big\langle\sum_{n_1<n_2\in\mathbb{N}\setminus L}\begin{bmatrix}n_1\\n_2\end{bmatrix}\alpha_{L\cup\{n_1,n_2\},C}B_{++}\tilde\varphi_{n_1}\otimes\tilde\varphi_{n_2}, \tag{3.364}$$

$$\sum_{p_1<p_2\in\mathbb{N}\setminus L}\begin{bmatrix}p_1\\p_2\end{bmatrix}\alpha_{L\cup\{p_1,p_2\},C}\tilde\varphi_{p_1}\otimes B_{++}\tilde\varphi_{p_2}\Big\rangle \tag{3.365}$$

$$-2\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\Big\langle\sum_{n_1<n_2\in\mathbb{N}\setminus L}\begin{bmatrix}n_1\\n_2\end{bmatrix}\alpha_{L\cup\{n_1,n_2\},C}\tilde\varphi_{n_1}\otimes B_{++}\tilde\varphi_{n_2}, \tag{3.366}$$

$$\sum_{p_1<p_2\in\mathbb{N}\setminus L}\begin{bmatrix}p_1\\p_2\end{bmatrix}\alpha_{L\cup\{p_1,p_2\},C}B_{++}\tilde\varphi_{p_2}\otimes\tilde\varphi_{p_1}\Big\rangle \tag{3.367}$$

$$\leqslant 4\|B_{++}\|^2\|\alpha\|^2 \tag{3.368}$$

$\boldsymbol{\delta_{n_1,p_1}}$ **:**

$$\sum_{\substack{L,C\subset\mathbb{N}\\|C|=p\\|L|=m-2}}\sum_{\substack{n_1<n_2\in\mathbb{N}\setminus L\\p_1<p_2\in\mathbb{N}\setminus L}}\overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C}\begin{bmatrix}n_1\\n_2\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix}\delta_{n_1,p_1} \tag{3.369}$$

$$\times\langle\tilde\varphi_{n_1},B_{++}\tilde\varphi_{p_2}\rangle\langle B_{++}\tilde\varphi_{n_2},\tilde\varphi_{p_1}\rangle \tag{3.370}$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{l<n,p\in\mathbb{N}\setminus L} \overline{\alpha_{L\cup\{l,n\},C}}\alpha_{L\cup\{l,p\},C} \begin{bmatrix}0\\ n\end{bmatrix}\begin{bmatrix}0\\ p\end{bmatrix} \tag{3.371}$$

$$\times \langle\tilde{\varphi}_l, B_{++}\tilde{\varphi}_p\rangle\langle B_{++}\tilde{\varphi}_n, \tilde{\varphi}_l\rangle \tag{3.372}$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{l\in\mathbb{N}\setminus L} \left|\left\langle \tilde{\varphi}_l, B_{++} \sum_{l<p\in\mathbb{N}\setminus L} \alpha_{L\cup\{l,p\},C}\begin{bmatrix}0\\ p\end{bmatrix}\tilde{\varphi}_p\right\rangle\right|^2 \tag{3.373}$$

$$\leqslant \|B_{++}\|^2 \sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{l<p\in\mathbb{N}\setminus L} |\alpha_{L\cup\{l,p\},C}|^2 \leqslant \|B_{++}\|^2\|\alpha\|^2 \tag{3.374}$$

$\boldsymbol{-\delta_{n_1,p_2} - \delta_{n_2,p_1}}$ **:**

$$-\sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{\substack{n_1<n_2\in\mathbb{N}\setminus L\\ p_1<p_2\in\mathbb{N}\setminus L}} \overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C} \begin{bmatrix}n_1\\ n_2\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}(\delta_{n_1,p_2}+\delta_{n_2,p_1}) \tag{3.375}$$

$$\times \langle B_{++}\tilde{\varphi}_{n_1}, \tilde{\varphi}_{p_1}\rangle\langle\tilde{\varphi}_{n_2}, B_{++}\tilde{\varphi}_{p_2}\rangle \tag{3.376}$$

$$= -\sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{n,p,l\in\mathbb{N}\setminus L} \overline{\alpha_{L\cup\{l,n\},C}}\alpha_{L\cup\{l,p\},C} \begin{bmatrix}0\\ n\end{bmatrix}\begin{bmatrix}0\\ p\end{bmatrix} \tag{3.377}$$

$$\times \Big(1_{p<l<n}\langle B_{++}\tilde{\varphi}_l, \tilde{\varphi}_p\rangle\langle\tilde{\varphi}_n, B_{++}\tilde{\varphi}_l\rangle \tag{3.378}$$

$$+ 1_{n<l<p}\langle B_{++}\tilde{\varphi}_n, \tilde{\varphi}_l\rangle\langle\tilde{\varphi}_l, B_{++}\tilde{\varphi}_p\rangle\Big) \tag{3.379}$$

$$= -\sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{l\in\mathbb{N}\setminus L} \tag{3.380}$$

$$\Bigg[\Bigg\langle B_{++}\tilde{\varphi}_l, \sum_{\substack{p\in\mathbb{N}\setminus L\\ p<l}} \begin{bmatrix}0\\p\end{bmatrix} \alpha_{L\cup\{l,p\},C}\tilde{\varphi}_p\Bigg\rangle \Bigg\langle \sum_{\substack{n\in\mathbb{N}\setminus L\\ l<n}} \begin{bmatrix}0\\n\end{bmatrix} \alpha_{L\cup\{l,n\},C}\tilde{\varphi}_n, B_{++}\tilde{\varphi}_l\Bigg\rangle \tag{3.381}$$

$$+\Bigg\langle B_{++}\tilde{\varphi}_l, \sum_{\substack{p\in\mathbb{N}\setminus L\\ p>l}} \begin{bmatrix}0\\p\end{bmatrix} \alpha_{L\cup\{l,p\},C}\tilde{\varphi}_p\Bigg\rangle \Bigg\langle \sum_{\substack{n\in\mathbb{N}\setminus L\\ l>n}} \begin{bmatrix}0\\n\end{bmatrix} \alpha_{L\cup\{l,n\},C}\tilde{\varphi}_n, B_{++}\tilde{\varphi}_l\Bigg\rangle\Bigg] \tag{3.382}$$

$$\leqslant 2\|B_{++}\|^2\|\alpha\|^2 \tag{3.383}$$

$\boldsymbol{\delta_{n_2,p_2}}$ **:** completely analogous to δ_{n_1,p_1} one arrives at

$$\leqslant \|B_{++}\|^2\|\alpha\|^2. \tag{3.384}$$

$\boldsymbol{\delta_{n_1,p_1}\delta_{n_2,p_2}}$ **:**

$$-2\sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{\substack{n_1<n_2\in\mathbb{N}\setminus L\\ p_1<p_2\in\mathbb{N}\setminus L}} \overline{\alpha_{L\cup\{n_1,n_2\},C}}\alpha_{L\cup\{p_1,p_2\},C} \begin{bmatrix}n_1\\n_2\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix}\delta_{n_1,p_1}\delta_{n_2,p_2} \tag{3.385}$$

$$\times\langle B_{++}\tilde{\varphi}_{n_1}, \tilde{\varphi}_{p_1}\rangle\langle\tilde{\varphi}_{n_2}, B_{++}\tilde{\varphi}_{p_2}\rangle \tag{3.386}$$

$$= -2\sum_{\substack{L,C\subset\mathbb{N}\\ |C|=p\\ |L|=m-2}} \sum_{n<p\in\mathbb{N}\setminus L} |\alpha_{L\cup\{n,p\},C}|^2\langle B_{++}\tilde{\varphi}_n, \tilde{\varphi}_n\rangle\langle\tilde{\varphi}_p, B_{++}\tilde{\varphi}_p\rangle \tag{3.387}$$

$$\leqslant 2\|B_{++}\|^2\|\alpha\|^2 \tag{3.388}$$

So all together we have

$$(3.349) \leqslant 10\|B_{++}\|^2\|\alpha\|^2. \tag{3.389}$$

What remains is term (3.351), for this term there are two cases. If $b = b'$ then the scalar product is equal to $\langle B^L_{L_b}\tilde{\varphi}_b, B^L_{L_b}\tilde{\varphi}_b\rangle$. If $b \neq b'$ the scalar product is, up to a sign, equal to $\langle B^L_{L_b}\tilde{\varphi}_b, \tilde{\varphi}_b\rangle\langle\tilde{\varphi}_{b'}, B^L_{L_{b'}}\tilde{\varphi}_{b'}\rangle$.

However, both of these terms can be estimated by $\|B_{++}\|^2$. So all m^2 summands of this sum contribute $\|B_{++}\|^2$. Overall this estimate yields

$$\begin{aligned}\|\mathrm{d}\Gamma(B_{++})\alpha\|^2 &\leqslant (3.353) + (3.354) + (3.389) + \|\alpha\|^2 m^2 \|B_{++}\|^2 \\ &= \|\alpha\|^2(12+m^2)\|B_{++}\|^2.\end{aligned}$$

For convenience of notation the estimate can be weakened to

$$\|\mathrm{d}\Gamma(B_{++})\alpha\| \leqslant (m+4)\|B_{++}\|\|\alpha\|. \tag{3.390}$$

A completely analogous argument works for $\mathrm{d}\Gamma(B_{--})$. So lets move on to $\mathrm{d}\Gamma(B_{-+})$. Applying it to the same $\alpha \in \mathcal{F}^0_{m,p}$ again we permute all the operators to the right, where they annihilate the vacuum. The remaining terms are

$$\begin{aligned}&\sum_{n\in\mathbb{N}} a^*(B_{-+}\varphi_n)a(\varphi_n) \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \prod_{l=1}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega \\ &= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \sum_{n\in\mathbb{N}} \alpha_{L,C} \sum_{b=1}^{m}\sum_{d=1}^{p} (-1)^{m-1+b+d} \langle \varphi_n, \tilde{\varphi}_{L_b}\rangle\langle \tilde{\varphi}_{-C_d}, B_{-+}\varphi_n\rangle \\ &\qquad\qquad \times \prod_{\substack{l=1\\l\neq b}}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{\substack{c=1\\c\neq d}}^{p} a(\tilde{\varphi}_{-C_c})\Omega.\end{aligned} \tag{3.391}$$

From here we can eliminate the sum over n, and reintroduce a sum over the ONB of $\mathcal{H}^-$ to arrive at the expression for $d\Gamma(B_{-+})$:

$$= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m}\sum_{d=1}^{p}(-1)^{m+b+d}\langle B_{+-}\tilde{\varphi}_{-C_d},\tilde{\varphi}_{L_b}\rangle \times \prod_{\substack{l=1\\l\neq b}}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{\substack{c=1\\c\neq d}}^{p} a(\tilde{\varphi}_{-C_c})\Omega \quad (3.392)$$

$$= \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \sum_{b=1}^{m}\sum_{d=1}^{p}\sum_{k\in\mathbb{N}}(-1)^{m+b+d}\langle\tilde{\varphi}_{-C_d},\varphi_{-k}\rangle\langle B_{+-}\varphi_{-k},\tilde{\varphi}_{L_b}\rangle \times \prod_{\substack{l=1\\l\neq b}}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{\substack{c=1\\c\neq d}}^{p} a(\tilde{\varphi}_{-C_c})\Omega \quad (3.393)$$

$$= -\sum_{k\in\mathbb{N}} a^*(\varphi_{-k})a(B_{+-}\varphi_{-k}) \sum_{\substack{L,C\subset\mathbb{N}\\|L|=m,|C|=p}} \alpha_{L,C} \prod_{l=1}^{m} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{p} a(\tilde{\varphi}_{-C_c})\Omega. \quad (3.394)$$

For the estimate of the operator norm we continue with expression (3.392). By counting the remaining creation and annihilation operators we immediately see that $d\Gamma(B_{-+})\alpha \in \mathcal{F}_{m-1,p-1}$. We take the norm squared of the expression and notice that the scalar product is only not zero in cases where $|L\backslash L'| \leqslant 1$ and $|C\backslash C'| \leqslant 1$. Furthermore, whenever $L = L'$ holds, the two sums of $1 \leqslant b, b' \leqslant m$ collapses to a single sum over this range and analogously for $C = C'$ and d, d'. In case $L \neq L'$ no sum over b or b' remains for the same reason. Hence, we arrive at

$$\|d\Gamma(B_{-+})\alpha\|^2 \leqslant \sum_{\substack{C,L\subset\mathbb{N}\\|L|=m-1\\|C|=p-1}} \sum_{\substack{n_1,n_2\in\mathbb{N}\backslash L\\l_1,l_2\in\mathbb{N}\backslash C}} |\alpha_{L\cup\{n_2\},C\cup\{l_2\}}||\alpha_{L\cup\{n_1\},C\cup\{l_1\}}| \quad (3.395)$$

$$\Bigg[\delta_{n_1,n_2}\delta_{l_1,l_2}\sum_{b\in L\cup\{n_1\}}\sum_{d\in C\cup\{l_1\}}|\langle B_{+-}\tilde{\varphi}_{-d},\tilde{\varphi}_b\rangle|^2 \tag{3.396}$$

$$+(1-\delta_{n_1,n_2})\delta_{l_1,l_2}\sum_{d\in C\cup\{l_1\}}|\langle B_{+-}\tilde{\varphi}_{-d},\tilde{\varphi}_{n_1}\rangle||\langle\tilde{\varphi}_{n_2},B_{+-}\tilde{\varphi}_{-d}\rangle| \tag{3.397}$$

$$+\delta_{n_1,n_2}(1-\delta_{l_1,l_2})\sum_{b\in L\cup\{n_1\}}|\langle B_{+-}\tilde{\varphi}_{-l_1},\tilde{\varphi}_b\rangle||\langle\tilde{\varphi}_b,B_{+-}\tilde{\varphi}_{-l_2}\rangle| \tag{3.398}$$

$$+(1-\delta_{n_1,n_2})(1-\delta_{l_1,l_2})|\langle B_{+-}\tilde{\varphi}_{-C_{l_1}},\tilde{\varphi}_{n_1}\rangle||\langle\tilde{\varphi}_{n_2},B_{+-}\tilde{\varphi}_{-C_{l_2}}\rangle|\Bigg]. \tag{3.399}$$

In the next step we split the sum into the four already indicated, estimate the terms $(1-\delta)\leqslant 1$ and eliminate sums with the remaining Kronecker deltas. For the first term, we subsequently enlarge the sum over part of the Basis $\tilde{\varphi}$ in the scalar product to the sum over all basis elements, yielding

$$\|\mathrm{d}\Gamma(B_{-+})\alpha\|^2\leqslant\|\alpha\|^2\sum_{b\in\mathbb{N}}\sum_{c\in\mathbb{N}}|\langle B_{+-}\tilde{\varphi}_{-c},\tilde{\varphi}_b\rangle|^2 \tag{3.400}$$

$$+\sum_{\substack{C,L\subset\mathbb{N}\\|L|=m-1\\|C|=p}}\sum_{\substack{n_1,n_2\in\mathbb{N}\setminus L\\d\in C}}|\alpha_{L\cup\{n_1\},C}||\langle B_{+-}\tilde{\varphi}_{-d},\tilde{\varphi}_{n_1}\rangle| \tag{3.401}$$

$$\times|\alpha_{L\cup\{n_2\},C}||\langle\tilde{\varphi}_{n_2},B_{+-}\tilde{\varphi}_{-d}\rangle| \tag{3.402}$$

$$+\sum_{\substack{C,L\subset\mathbb{N}\\|L|=m\\|C|=p-1}}\sum_{l_1,l_2\in\mathbb{N}\setminus C}|\alpha_{L,C\cup\{l_2\}}||\alpha_{L,C\cup\{l_1\}}| \tag{3.403}$$

$$\times\sum_{b\in L}|\langle B_{+-}\tilde{\varphi}_{-l_1},\tilde{\varphi}_b\rangle||\langle\tilde{\varphi}_b,B_{+-}\tilde{\varphi}_{-l_2}\rangle| \tag{3.404}$$

$$+\sum_{\substack{C,L\subset\mathbb{N}\\|L|=m-1\\|C|=p-1}}\sum_{\substack{n_1,n_2\in\mathbb{N}\setminus L\\l_1,l_2\in\mathbb{N}\setminus C}}|\alpha_{L\cup\{n_2\},C\cup\{l_2\}}||\alpha_{L\cup\{n_1\},C\cup\{l_1\}}| \tag{3.405}$$

$$\times|\langle B_{+-}\tilde{\varphi}_{-l_1},\tilde{\varphi}_{n_1}\rangle||\langle\tilde{\varphi}_{n_2},B_{+-}\tilde{\varphi}_{-l_2}\rangle|. \tag{3.406}$$

Now we identify the sums over l_1, l_2 and n_1, n_2 as scalar products between factors of $|\alpha|$ and factors involving B_{+-} in tensor products of $\ell^2(\mathbb{N})$ and apply the Cauchy-Schwarz inequality. Additionally, we identify $\sum_{b\in\mathbb{N}}\sum_{c\in\mathbb{N}} |\langle B_{+-}\tilde{\varphi}_{-c}, \tilde{\varphi}_b\rangle|^2 = \|B_{+-}\|^2_{I_2}$. This results in

$$\|\mathrm{d}\Gamma(B_{-+})\alpha\|^2 \leqslant \|\alpha\|^2 \|B_{-+}\|^2_{I_2} \tag{3.407}$$

$$+ \sum_{\substack{C,L\subset\mathbb{N}\\ |L|=m-1\\ |C|=p}} \sum_{n\in\mathbb{N}\backslash L} |\alpha_{L\cup\{n\},C}|^2 \sum_{d\in C}\sum_{u\in\mathbb{N}\backslash L} |\langle B_{+-}\tilde{\varphi}_{-d}, \tilde{\varphi}_u\rangle|^2 \tag{3.408}$$

$$+ \sum_{\substack{C,L\subset\mathbb{N}\\ |L|=m\\ |C|=p-1}} \sum_{l\in\mathbb{N}\backslash C} |\alpha_{L,C\cup\{l\}}|^2 \sum_{b\in L}\sum_{u\in\mathbb{N}\backslash C} |\langle B_{+-}\tilde{\varphi}_{-u}, \tilde{\varphi}_b\rangle|^2 \tag{3.409}$$

$$+ \sum_{\substack{C,L\subset\mathbb{N}\\ |L|=m-1\\ |C|=p-1}} \sum_{\substack{n\in\mathbb{N}\backslash L\\ l\in\mathbb{N}\backslash C}} |\alpha_{L\cup\{n\},C\cup\{l\}}|^2 \sum_{\substack{u\in\mathbb{N}\backslash L\\ d\in\mathbb{N}\backslash C}} |\langle B_{+-}\tilde{\varphi}_{-d}, \tilde{\varphi}_u\rangle|^2 \tag{3.410}$$

$$\leqslant 4\|\alpha\|^2 \|B_{-+}\|^2_{I_2}. \tag{3.411}$$

□

Corollary 76. *The operators* $\mathrm{d}\Gamma(B_{--})$ *and* $\mathrm{d}\Gamma(B_{++})$ *can be extended by continuity on* $\mathcal{F}^0_{m,p}$ *to unbounded operators on all of* $\mathcal{F}'$. *The operator* $\mathrm{d}\Gamma(B_{-+})$ *can be continuously extended to all of* $\mathcal{F}$.

Lemma 77. *The operator* $(\mathrm{d}\Gamma(B_{-+}))^*$ *acts on elements of* $\mathcal{F}^0$ *as*

$$-\sum_{n\in\mathbb{N}} a^*(B_{+-}\varphi_{-n})a(\varphi_{-n}) =: -\mathrm{d}\Gamma(B_{+-}). \tag{3.412}$$

So also $\mathrm{d}\Gamma(B_{+-}) : \mathcal{F}^0 \to \mathcal{F}$ *can be extended continuously to all of* $\mathcal{F}$. *Moreover,* $\mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{+-})$ *is skew-adjoint.*

Proof. Pick $\beta, \alpha \in \mathcal{F}^0$. We use the form (3.333) to obtain

$$
\begin{aligned}
\langle\beta, \mathrm{d}\Gamma(B_{-+})\alpha\rangle &= \left\langle \beta, -\sum_{n\in\mathbb{N}} a^*(\varphi_{-n})a(B_{+-}\varphi_{-n})\alpha \right\rangle \\
&= -\sum_{n\in\mathbb{N}}\langle\beta, a^*(\varphi_{-n})a(B_{+-}\varphi_{-n})\alpha\rangle = -\sum_{n\in\mathbb{N}}\langle a^*(B_{+-}\varphi_{-n})a(\varphi_{-n})\beta, \alpha\rangle \\
&= \left\langle -\sum_{n\in\mathbb{N}} a^*(B_{+-}\varphi_{-n})a(\varphi_{-n})\beta, \alpha \right\rangle
\end{aligned} \tag{3.413}
$$

So we see that $\mathrm{d}\Gamma(B_{+-})$ and $\mathrm{d}\Gamma(B_{-+})^*$ agree on $\mathcal{F}^0$ which is dense. So they are the same bounded and continuous operator on all of Fock space. □

Definition 78. *We define the set*

$$
\mathfrak{B} := \{B : \mathcal{H} \circlearrowleft |\text{linear}, \|B\| + \|B_{+-}\|_{I_2} + \|B_{-+}\|_{I_2} \in \mathbb{R}, B^* = -B\} \tag{3.414}
$$

and the operator

$$
\mathrm{d}\Gamma(B) := \mathrm{d}\Gamma(B_{++}) + \mathrm{d}\Gamma(B_{+-}) + \mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{--}). \tag{3.415}
$$

Furthermore, we endow $\mathfrak{B}$ with the topology induced by the norm $B \mapsto \|B\| + \|B_{+-}\|_{I_2} + \|B_{-+}\|_{I_2}$.

Lemma 79. *The operator $\mathrm{d}\Gamma(B)$ is skew symmetric and real linear in its argument $B \in \mathfrak{B}$. Moreover, for each $m, p \in \mathbb{N}$ the functional $\mathrm{d}\Gamma(\cdot)|_{\mathcal{F}'_{m,p}}$ is bounded and hence continuous as a map from $\mathfrak{B}$ to the set of bounded linear operators of type $\mathcal{F}'_{m,p} \to \mathcal{F}'_{m-1,p-1} \oplus \mathcal{F}'_{m,p} \oplus \mathcal{F}'_{m+1,p+1}$.*

Proof. Since the sum of skew symmetric operators is skew symmetric, it suffices to show skew symmetry of $\mathrm{d}\Gamma(B_{++})$ and $\mathrm{d}\Gamma(B_{--})$. Moreover, since both of these operators are extended versions of operators of type $\mathcal{F}^0 \to \mathcal{F}$ it suffices to show skew symmetry on this domain.

We will only do the calculation for $\mathrm{d}\Gamma(B_{++})$, the other calculation is analogous. Pick $\beta, \alpha \in \mathcal{F}^0$ and basis $\tilde{\varphi}$, φ' such that β, α are expressible with finite sums over elements of the generating sets with respect to their respective basis. We calculate

$$\begin{aligned}
\langle \beta, \mathrm{d}\Gamma(B_{++})\alpha \rangle = \sum_{L,L',C,C' \subset \mathbb{N}} \overline{\beta}_{L',C'} \alpha_{L,C} \Bigg\langle \prod_{l=1}^{|L'|} a^*(\tilde{\varphi}_{L'_l}) \prod_{c=1}^{|C'|} a(\tilde{\varphi}_{-C'_c})\Omega,\\
\sum_{n \in \mathbb{N}} a^*(B_{++}\varphi_n) a(\varphi_n) \prod_{l=1}^{|L|} a^*(\varphi'_{L_l}) \prod_{c=1}^{|C|} a(\varphi'_{-C_c})\Omega \Bigg\rangle \\
= \sum_{L,L',C,C' \subset \mathbb{N}} \overline{\beta}_{L',C'} \alpha_{L,C} \sum_{n \in \mathbb{N}} \Bigg\langle \prod_{l=1}^{|L'|} a^*(\tilde{\varphi}_{L'_l}) \prod_{c=1}^{|C'|} a(\tilde{\varphi}_{-C'_c})\Omega,\\
a^*(B_{++}\varphi_n) a(\varphi_n) \prod_{l=1}^{|L|} a^*(\varphi'_{L_l}) \prod_{c=1}^{|C|} a(\varphi'_{-C_c})\Omega \Bigg\rangle \\
= \sum_{L,L',C,C' \subset \mathbb{N}} \overline{\beta}_{L',C'} \alpha_{L,C} \sum_{n \in \mathbb{N}} \Bigg\langle a^*(\varphi_n) a(B_{++}\varphi_n) \prod_{l=1}^{|L'|} a^*(\tilde{\varphi}_{L'_l}) \prod_{c=1}^{|C'|} a(\tilde{\varphi}_{-C'_c})\Omega,\\
\prod_{l=1}^{|L|} a^*(\varphi'_{L_l}) \prod_{c=1}^{|C|} a(\varphi'_{-C_c})\Omega \Bigg\rangle .
\end{aligned}$$

In the next step we perform the standard anticommutations to move the operator B_{++} from the annihilation operator to the creation operator:

$$\sum_{L,C \subset \mathbb{N}} \beta_{L,C} \sum_{n \in \mathbb{N}} a^*(\varphi_n) a(B_{++}\varphi_n) \prod_{l=1}^{|L|} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{|C|} a(\tilde{\varphi}_{-C_c})\Omega \tag{3.416}$$

$$= \sum_{L,C\subset\mathbb{N}} \beta_{L,C} \sum_{b=1}^{|L|} \prod_{l=1}^{b-1} a^*(\tilde{\varphi}_{L_l})(-1)a^*(B_{++}\tilde{\varphi}_{L_b}) \tag{3.417}$$

$$\times \prod_{l=b+1}^{|L|} a^*(\tilde{\varphi}_{L_l}) \prod_{c=1}^{|C|} a(\tilde{\varphi}_{-C_c})\Omega \tag{3.418}$$

$$= \sum_{L,C\subset\mathbb{N}} \beta_{L,C}(-1) \sum_{k\in\mathbb{N}} a^*(B_{++}\varphi_k)a(\varphi_k) \prod_{l=1}^{|L|} a^*(\tilde{\varphi_{L_l}}) \prod_{c=1}^{|C|} a(\tilde{\varphi}_{-C_c})\Omega \tag{3.419}$$

$$= -\sum_{k\in\mathbb{N}} a^*(B_{++}\varphi_k)a(\varphi_k)\beta, \tag{3.420}$$

where we used $B_{++}^* = -B_{++}$. This yields

$$\langle \beta, \mathrm{d}\Gamma(B_{++})\alpha \rangle = -\langle \mathrm{d}\Gamma(B_{++})\beta, \alpha \rangle . \tag{3.421}$$

Real linearity follows directly from the definition of $\mathrm{d}\Gamma$ on $\mathcal{F}^0$ and hence by extension on all of $\mathcal{F}'$. Continuity of the restriction to any $\mathcal{F}'_{m,p}$ follows directly from the forms of the bounds of lemma 75. □

Now we would like to define $e^{\mathrm{d}\Gamma(B)}$, in order to do so, we will show that $\mathrm{d}\Gamma(B)$ is essentially skew-adjoint. One way of doing so is by Nelson's analytic vector theorem.

Theorem 80 (Nelson's analytic vector theorem)**.** *Let C be a symmetric operator on a Hilbert space $\mathcal{H}$. If* $\mathrm{dom}(C)$ *contains a total set $S \subset \bigcap_{n=1}^{\infty} \mathrm{dom}(C^n)$ of analytic vectors, then C is essentially self adjoint. A vector $\phi \in \bigcap_{n=1}^{\infty} \mathrm{dom}(C^n)$ is called analytic if there is $t > 0$ such that $\sum_{k=0}^{\infty} \frac{\|C^n\phi\|}{n!} t^n < \infty$ holds. A set S is said to be total if $\overline{span(S)} = \mathcal{H}$*

For a proof see e.g. [85].

Lemma 81. *For any* $\alpha \in \mathcal{F}', t > 0$ *and* $B \in \mathfrak{B}$ *the operator* $\mathrm{d}\Gamma(B) : \mathcal{F}' \to \mathcal{F}$ *satisfies*

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha\|}{k!} t^k < \infty. \tag{3.422}$$

Proof. By definition of $\mathcal{F}'$ there are $m, p \in \mathbb{N}$ such that $\alpha \in \bigoplus_{l=0}^{m} \bigoplus_{c=0}^{p} \mathcal{F}_{l,p}$. Fix $t > 0$. We dissect α into its parts of fixed particle numbers:

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha\|}{k!} t^k \leqslant \sum_{l=0}^{m} \sum_{c=0}^{p} \sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha_{l,c}\|}{k!} t^k. \tag{3.423}$$

Using the following abbreviations

$$\Gamma_{-1} := \mathrm{d}\Gamma(B_{-+}) \tag{3.424}$$

$$\Gamma_{0} := \mathrm{d}\Gamma(B_{++}) + \mathrm{d}\Gamma(B_{--}) \tag{3.425}$$

$$\Gamma_{+1} := \mathrm{d}\Gamma(B_{+-}) \tag{3.426}$$

$$\beta := \max\{\|B_{++}\| + \|B_{--}\|, \|B_{-+},\|_{I_2}\} \tag{3.427}$$

we estimate

$$\|\mathrm{d}\Gamma(B)^k \alpha_{l,c}\| \leqslant \sum_{x \in \{-1,0,+1\}^k} \left\| \prod_{b=1}^{k} \Gamma_{x_b} \alpha_{l,c} \right\|$$

$$\leqslant \sum_{x \in \{-1,0,+1\}^k} \prod_{b=1}^{k} \left\| \Gamma_{x_b}|_{\mathcal{F}_{l+\sum_{d=1}^{b-1} x_d, c+\sum_{d=1}^{b-1} x_d}} \right\| \|\alpha_{l,c}\| \tag{3.428}$$

$$\leqslant 3^k \|\alpha\| \max_{x \in \{-1,0,+1\}^k} \prod_{b=1}^{k} \left\| \Gamma_{x_b}|_{\mathcal{F}_{l+\sum_{d=1}^{b-1} x_d, c+\sum_{d=1}^{b-1} x_d}} \right\|. \tag{3.429}$$

At this point the factors only depend on the number of particles the Fock space vector attains as we act on it with the operators $\Gamma_{\#}$ for

$\# \in \{-1, 0, 1\}$. As these bounds increase with the particle number we can restrict the set $\{-1, 0, +1\}$ in the last line to $\{0, +1\}$. We notice that the bound in (3.429) will only increase if we exchange each pair $x_{i+1} = 1, x_i = 0$ by the pair $x_i = 1, x_{i+1} = 0$ so that the norm of the operator that acts like a particle number operator is taken after the particle number is increased. Therefore, we for each fixed $\sum_{b=1}^{k} x_b = d$ we can estimate the maximum by lemma 75 to be $\beta^k(c+l+8+2d)^{k-d}$, which we bound by $(2\beta)^k(c/2 + l/2 + 4 + d)^{k-d}$. The factor constant in d will be omitted for the maximization problem. For maximizing

$$(c/2 + l/2 + 4 + d)^{k-d} \tag{3.430}$$

we treat d as a continuous variable take the derivative and set it to zero. From the form of the function to be maximized it is clear that it is equal to 1 for $d = k$ and at $d = -c/2 - l/2 - 3$, it is bigger in between. We abbreviate $y = c/2 + l/2 + 4$. We calculate

$$0 = (y + d)^{k-d}(-\ln(y + d) + \frac{k - d}{y + d}) \tag{3.431}$$

$$\iff \frac{k - d}{y + d} = \ln(y + d) \tag{3.432}$$

$$\iff \frac{k + y}{y + d} - 1 = -1 + \ln(e(y + d)) \tag{3.433}$$

$$\iff e(k + y) = e(y + d)\ln(e(y + d)) \tag{3.434}$$

$$\iff e(k + y) = \ln(e(y + d))e^{\ln(e(y+d))} \tag{3.435}$$

$$\iff W_0(e(k + y)) = \ln(e(y + d)) \tag{3.436}$$

$$\iff e^{W_0(e(k+y))-1} - y = d, \tag{3.437}$$

where we made use of the Lambert W function, which is the inverse function of $x \mapsto xe^x$ and has multiple branches; however as $e(y+d) > 0$ W_0 is the only real branch which is applicable here, it corresponds to the inverse of $x \mapsto xe^x$ for $x > -1$. From the form of the maximizing value we see, that it is always bigger than $-y$. Plugging this back onto our function we find its maximum

$$\begin{aligned}
&\max_{d\in]-y,\infty[}(y+d)^{k-d} = e^{(W_0(e(k+y))-1)(k+y)-(W_0(e(k+y))-1)e^{W_0(e(k+y))-1}} \\
&= e^{-(k+y)+(k+y)W_0(e(k+y))+e^{W_0(e(k+y))-1}-((k+y)e)/e} \\
&= e^{-2(k+y)+(k+y)W_0((k+y)e)+\frac{e(k+y)}{eW_0((k+y)e)}} \\
&= e^{(k+y)(-2+W_0((k+y)e)+W_0((k+y)e)^{-1})},
\end{aligned} \tag{3.438}$$

where we repeatedly used $W_0(x)e^{W_0(x)} = x$. Putting things together we find

$$\|\Gamma(B)^k\alpha_{l,c}\| \leqslant (6\beta)^k\|\alpha\|e^{(k+y)(-2+W_0((k+y)e)+W_0((k+y)e)^{-1})}. \tag{3.439}$$

Dividing this by $k!$ and using the lower bound given by Sterling's formula we would like to prove that

$$\sum_{k=1}^{\infty}(6\beta t)^k e^{k(1-\ln(k))-\frac{1}{2}\ln(k)+(k+y)(-2+W_0((k+y)e)+W_0((k+y)e)^{-1})} < \infty \tag{3.440}$$

holds, where we neglected constant factors and the summand $k=0$ which do not matter for the task at hand. Next we are going to use an inequality about the growth of W_0 proven in [54]. For any $x \geqslant e$

$$W_0(x) \leqslant \ln(x) - \ln(\ln(x)) + \frac{e}{e-1}\frac{\ln(\ln(x))}{\ln(x)} \tag{3.441}$$

holds true. Plugging this into our sum the exponent is bounded from above by

$$
\begin{aligned}
&k(1-\ln(k)) - \frac{1}{2}\ln(k) + (k+y)\Big[-1+\ln(k+y)-\ln(1+\ln(k+y)) \\
&+\frac{e}{e-1}\frac{\ln(1+\ln(k+y))}{1+\ln(k+y)} + W_0((k+y)e)^{-1}\Big] \\
&= -y + k\ln\left(1+\frac{y}{k}\right) + y\ln(k+y) - \frac{1}{2}\ln(k) + \\
&(k+y)\left[\ln(1+\ln(k+y))\frac{1-(e-1)\ln(k+y)}{(e-1)(1+\ln(k+y))} + W_0((k+y)e)^{-1}\right] \\
&\leqslant y\ln(k+y) - \frac{1}{2}\ln(k) + (k+y)W_0((k+y)e)^{-1} + \\
&(k+y)\ln(1+\ln(k+y))\frac{1-(e-1)\ln(k+y)}{(e-1)(1+\ln(k+y))}.
\end{aligned}
\tag{3.442}
$$

Now it is important to notice that the only remaining term that grows faster than linearly in magnitude is the last summand. This term; however, is negative for large k, as the fraction converges to -1 for large k, while the double logarithm in front grows without bounds. So there is a k^* big enough such that for all $k > k^*$ (3.442) is smaller than $-k(\ln(6\beta t) + 1)$, proving that (3.440) in fact holds.

□

Theorem 82. *The operator* $\mathrm{d}\Gamma(B) : \mathcal{F}' \to \mathcal{F}$ *is essentially skew adjoint and hence by Stones theorem generates a strongly continuous unitary group* $\left(e^{t\,\widehat{\mathrm{d}\Gamma(B)}}\right)_t$, *where* $\widehat{\mathrm{d}\Gamma(B)}$ *is the closure of* $\mathrm{d}\Gamma(B)$.

Proof. In order to apply Nelson's analytic vector theorem we pick $S = \mathcal{F}'$. Pick $\alpha \in \mathcal{F}'$. We need to show that there is $t > 0$ such that

$$
\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\Gamma(B)^k \alpha\|}{k!} t^k < \infty \tag{3.443}
$$

holds. This is guaranteed by the last lemma. □

Lastly in this subsection, we will investigate the commutation properties of $\mathrm{d}\Gamma(B)$ with general creation and annihilation operators. These properties are the reason we are interested in this operator, they will prove to be very useful in the next subsection.

Theorem 83. *For $\psi \in \mathcal{H}$ and $\alpha \in \mathcal{F}'$ we have*

$$[\mathrm{d}\Gamma(B), a^{\#}(\psi)]\alpha = a^{\#}(B\psi)\alpha, \tag{3.444}$$

where $a^{\#}$ can be either a or a^*.

Proof. Because $\mathrm{d}\Gamma(B)$ is defined as the extension of an operator on $\mathcal{F}^0$ it suffices to show the desired identity on this space. We will do the case $a(\psi)$, the other case is completely analogous. As a first step we decompose $\mathrm{d}\Gamma(B)$ into its four parts

$$[\mathrm{d}\Gamma(B), a(\psi)] = [\mathrm{d}\Gamma(B_{++}) + \mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{-+}) + \mathrm{d}\Gamma(B_{--}), a(\psi)], \tag{3.445}$$

each of those parts is evaluated directly. We begin with the B_{++} part, this can be expressed as

$$[\mathrm{d}\Gamma(B_{++}), a(\psi)] \tag{3.446}$$

$$= \sum_{n\in\mathbb{N}} a^*(B_{++}\varphi_n)a(\varphi_n)a(\psi) - \sum_{n\in\mathbb{N}} a(\psi)a^*(B_{++}\varphi_n)a(\varphi_n) \tag{3.447}$$

$$= \sum_{n\in\mathbb{N}} [-\langle\psi, B_{++}\varphi_n\rangle a(\varphi_n) + a(\psi)a^*(B_{++}\varphi_n)a(\varphi_n)] \tag{3.448}$$

$$- \sum_{n\in\mathbb{N}} a(\psi)a^*(B_{++}\varphi_n)a(\varphi_n). \tag{3.449}$$

Let $\alpha \in \mathcal{F}^0$. Now applying the expression in the last two lines to α, considering each $\alpha_{m,p} \in \mathcal{F}_{m,p}$ separately and commuting the annihilation operators in (3.448) and (3.449) to the right, the sums over n will be absolutely convergent. Hence, we may split the firs sum into two and observe the cancellation between the last two terms. Continuing we find

$$[\mathrm{d}\Gamma(B_{++}), a(\psi)]\,\alpha = -\sum_{n\in\mathbb{N}} \langle\psi, B_{++}\varphi_n\rangle a(\varphi_n)\alpha \quad (3.450)$$

$$= -a\left(\sum_{n\in\mathbb{N}} \langle B_{++}\varphi_n, \psi\rangle\varphi_n\right)\alpha = a\left(\sum_{n\in\mathbb{N}} \langle\varphi_n, B_{++}\psi\rangle\varphi_n\right)\alpha \quad (3.451)$$

$$= a(B_{++}\psi)\alpha, \quad (3.452)$$

where we used $B^* = -B^*$. The final extension of this equation to all $\alpha \in \mathcal{F}'$ happens via the continuous linear extension theorem on $\mathcal{F}_{m,p}$ for each $m, p \in \mathbb{N}$. The proof in all seven other cases are completely analogous, except that the off diagonal terms switch. More precisely, from the excatly analogous calculation it follows that

$$\left[\mathrm{d}\Gamma(B_{-+}), a^{\#}(\psi)\right]\alpha = a^{\#}\left(B_{+-}\psi\right)\alpha \quad (3.453)$$

and

$$\left[\mathrm{d}\Gamma(B_{+-}), a^{\#}(\psi)\right]\alpha = a^{\#}\left(B_{-+}\psi\right)\alpha \quad (3.454)$$

hold. Putting things together again we obtain

$$[\mathrm{d}\Gamma(B_{++}), a(\psi)] + [\mathrm{d}\Gamma(B_{-+}), a(\psi)] \quad (3.455)$$

$$+ [\mathrm{d}\Gamma(B_{+-}), a(\psi)] + [\mathrm{d}\Gamma(B_{--}), a(\psi)] = \quad (3.456)$$

$$a(B_{++}\psi) + a(B_{+-}\psi) + a(B_{-+}\psi) + a(B_{--}\psi) \iff \quad (3.457)$$

$$[\mathrm{d}\Gamma(B), a(\psi)] = a(B\psi) \quad (3.458)$$

on all of $\mathcal{F}'$.

□

3.3.2 Presentation and Proof of the Formula

In this subsection we verify the formula for the S-matrix directly. For a heuristic derivation see section 4.3 of the appendix.

Theorem 84 (Analyticity for small A)**.** *Let $A \in \mathcal{V}$ be such that the one particle scattering operator S^A fulfils*

$$\|\mathbb{1} - S^A\| < 1. \tag{3.459}$$

We define $\ln(S^A)$ by the norm convergent Taylor series of the logarithm around the identity. Then the operator $d\Gamma(\ln(S^A))$ is unbounded, essentially skew-adjoint, which gives meaning to $\tilde{S}$:

$$\tilde{S}^A = e^{d\Gamma(\ln(S^A))}. \tag{3.460}$$

The unitary operator $\tilde{S}^A$ is a lift of S^A, i.e. it fulfils the (lift condition)*.*

Proof. In order to establish this theorem we need to verify that the expression given in equation (3.460) for the scattering operator is a well-defined object and fulfils the (lift condition).
Well-definedness is established, by theorem 82, because for unitary S^A with $\|1 - S^A\| < 1$ the power series of the logarithm converges and fulfils

$$\|\ln(S^A)\| = \|\ln(1 - (1 - S^A))\| = \left\| - \sum_{k=1}^{\infty} \frac{(1 - S^A)^k}{k} \right\| \tag{3.461}$$

$$\leqslant \sum_{k=1}^{\infty} \frac{\|1 - S^A\|^k}{k} = -\ln(1 - \|1 - S^A\|) \tag{3.462}$$

implying that the power series of the logarithm around the identity is a well-defined map from the one-particle operators of norm less than one to the bounded one-particle operators. Moreover, this operator fulfils $[\ln(S^A)]^* = \ln(S^A)^* = \ln(S^A)^{-1} = -\ln(S^A)$, so $\mathrm{d}\Gamma(\ln S^A)$ is a well-defined unbounded operator that is essentially skew adjoint on the finite particle sector of Fock space $\mathcal{F}'$. Finally, the off diagonal Hilbert-Schmidt norm can also be controlled by the same norm of S^A:

$$\|P^+ \ln(S^A) P^-\|_{I_2} = \|P^+ \ln(1-(1-S^A))P^-\|_{I_2} \tag{3.463}$$

$$= \left\| -P^+ \sum_{k=1}^{\infty} \frac{(1-S^A)^k}{k} P^- \right\|_{I_2} \tag{3.464}$$

$$\leqslant \|S^A_{+-}\|_{I_2} \sum_{k=1}^{\infty} \max\{\|1-S^A\|, \|P^- - S^A_{--}\|\}^{k-1} \tag{3.465}$$

$$= \frac{\|S^A_{+-}\|_{I_2}}{1-\|1-S^A\|}. \tag{3.466}$$

Let $\varphi \in \mathcal{H}$ and $\alpha \in \mathcal{F}'$, for any $k \in \mathbb{N}_0$ we see applying the commutation relation of $\mathrm{d}\Gamma$:

$$\begin{aligned}
&\mathrm{d}\Gamma(\ln S^A) \sum_{l=0}^{k} \binom{k}{l} a\left(\left(\ln S^A\right)^l \varphi\right) \left(\mathrm{d}\Gamma(\ln S^A)\right)^{k-l} \alpha \\
&\quad = \sum_{l=0}^{k} \binom{k}{l} a\left(\left(\ln S^A\right)^{l+1} \varphi\right) \left(\mathrm{d}\Gamma(\ln S^A)\right)^{k-l} \alpha \\
&\quad + \sum_{l=0}^{k} \binom{k}{l} a\left(\left(\ln S^A\right)^l \varphi\right) \left(\mathrm{d}\Gamma(\ln S^A)\right)^{k-l+1} \alpha \\
&= \sum_{b=0}^{k+1} \left(\binom{k}{b-1} + \binom{k}{b}\right) a\left((\ln S^A)^b \varphi\right) \left(\mathrm{d}\Gamma(\ln S^A)\right)^{k+1-b} \alpha \\
&\quad = \sum_{b=0}^{k+1} \binom{k+1}{b} a\left((\ln S^A)^b \varphi\right) \left(\mathrm{d}\Gamma(\ln S^A)\right)^{k+1-b} \alpha,
\end{aligned}$$

so we see that for $k \in \mathbb{N}_0$

$$\left(\mathrm{d}\Gamma(\ln S^A)\right)^k a(\varphi)\alpha = \sum_{b=0}^{k} \binom{k}{b} a\left((\ln S^A)^b \varphi\right) \left(\mathrm{d}\Gamma(\ln S^A)\right)^{k-b} \alpha \tag{3.467}$$

holds. Using what we just obtained, we conclude

$$\begin{aligned} e^{\mathrm{d}\Gamma(\ln S^A)} a(\varphi)\alpha &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\mathrm{d}\Gamma(\ln S^A)\right)^k a(\varphi)\alpha \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{b=0}^{k} \binom{k}{b} a\left((\ln S^A)^b \varphi\right) \left(\mathrm{d}\Gamma(\ln S^A)\right)^{k-b} \alpha \\ &\overset{*}{=} \sum_{c=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{c!l!} a\left((\ln S^A)^c \varphi\right) \left(\mathrm{d}\Gamma(\ln S^A)\right)^l \alpha \\ &= a\left(e^{\ln S^A} \varphi\right) e^{\mathrm{d}\Gamma(\ln S^A)}\alpha = a\left(S^A \varphi\right) e^{\mathrm{d}\Gamma(\ln S^A)}\alpha. \end{aligned}$$

For the marked equality changing order of summation is justified, because by the bounds $\|a((\ln S^A)^c\varphi)\| \leqslant \|\ln S^A\|^c$ and lemma 81 the sum obtained by changing the order of summands converges absolutely. Clearly multiplying the second quantized operator by an additional phase as in (84) does not influence this calculation. So (lift condition) holds when applied to any $\alpha \in \mathcal{F}'$ and can be continued to all of $\mathcal{F}$ by continuity of $\tilde{S}^A$. □

The last theorem can be restated as for A small enough there is a power series of operators on $\mathcal{F}$ that converges against a lift of S^A. Power series in A is used here in the sense that it is of the form $\sum_{k\in\mathbb{N}_0} T_k(A)$, where $T_k(A)$ is homogeneous in A of degree k. The next theorem establishes such a power series for all $A \in \mathcal{V}$.

Theorem 85 (Analyticity for all A)**.** *Let $A \in \mathcal{V}$ and S^A be the corresponding one-particle scattering operator. There is a lift $\tilde{S}^A$ of S^A that fulfils for all $\alpha \in \mathcal{F}'$*

$$\|\tilde{S}^A\alpha - \sum_{a=0}^{M} \tilde{T}_a(A)\alpha\| \xrightarrow{M\to\infty} 0, \tag{3.468}$$

where $\tilde{T}_a(A)$

$$\tilde{T}_a(A) = \sum_{k\in\mathbb{N}_0^N} \sum_{\substack{\forall l \leqslant N:\, b_l \in \mathbb{N}^{k_l}\\ \forall l \leqslant N:\, c_l \in \mathbb{N}_0^{b_l}\\ \sum_{l=1}^N |c_l| = a}} \prod_{l_1=1}^{N} \frac{(-1)^{|b_{l_1}|+k_{l_1}}}{k_{l_1}!c_{l_1}!} \tag{3.469}$$

$$\times \prod_{l_2=1}^{k_{l_1}} \left(\frac{1}{b_{l_1,l_2}} \mathrm{d}\Gamma \left(\prod_{v=1}^{b_{l_1,l_2}} Z^A_{l_1,c_{l_1,l_2,v}} \right)\right), \tag{3.470}$$

are unbounded operators defined on $\mathcal{F}'$ that are homogeneous of degree a in A. Here, $N \in \mathbb{N}$ is chosen such that

$$S^A = \prod_{k=1}^{N} S_k^A \tag{3.471}$$

and

$$\forall k \leqslant N : \|1 - S_k^A\| < 1 \tag{3.472}$$

holds true, where S_k^A is given by

$$S_k^A = U^{(k-1)A/N}_{\Sigma_{in},\Sigma_{out}} U^{Ak/N}_{\Sigma_{out},\Sigma_{in}}, \tag{3.473}$$

and

$$S_k^A = 1 + \sum_{d\in\mathbb{N}} Z^A_{k,d}, \tag{3.474}$$

is a norm converent series where $Z_{k,d}$ is homogeneous of degree d.

Proof. Pick $A \in \mathcal{V}$. Recall the definition 47 of the one-particle scattering operator S^A

$$S^A = U^0_{\Sigma_{\text{in}},\Sigma_{\text{out}}} U^A_{\Sigma_{\text{out}},\Sigma_{\text{in}}}. \tag{3.475}$$

Equation (3.471) then holds by virtue of equation (3.473). Please note that the existence of an N as given in the theorem can be inferred from the norm convergent Dyson series, i.e. equation (4.8) of the appendix. Then, by the theorem 84,

$$\tilde{S}^A_k = e^{\mathrm{d}\Gamma(\ln(S^A_k))} \tag{3.476}$$

is a lift of S^A_k for each k, so the product

$$\tilde{S}^A = \prod_{k=1}^{N} \tilde{S}^A_k \tag{3.477}$$

is a lift of S^A. Pick $\alpha \in \mathcal{F}'$. Note that the convergence of

$$\sum_{k_N \in \mathbb{N}_0} \frac{1}{k_N!} \Big(\mathrm{d}\Gamma(\ln S^A_N)\Big)^{k_N} \alpha \tag{3.478}$$

is guaranteed by lemma 81. We calculate

$$\tilde{S}^A \alpha = \prod_{k=1}^{N} \tilde{S}^A_k \alpha = \prod_{k=1}^{N-1} \tilde{S}^A_k \quad \sum_{k_N \in \mathbb{N}_0} \frac{1}{k_N!} \Big(\mathrm{d}\Gamma(\ln S^A_N)\Big)^{k_N} \alpha \tag{3.479}$$

$$= \prod_{k=1}^{N-2} \tilde{S}^A_k \quad \sum_{k_N \in \mathbb{N}_0} \tilde{S}^A_{N-1} \frac{1}{k_N!} \Big(\mathrm{d}\Gamma(\ln S^A_N)\Big)^{k_N} \alpha. \tag{3.480}$$

By unitary of $\tilde{S}^A_l$ we may pull them into the sum, and expand again since inside the sum its argument is again in $\mathcal{F}'$. We may continue this process by induction. Since all of these sums are absolutely convergent, we may forget about the order in which they are to be carried out in our notation:

$$\tilde{S}^A \alpha = \sum_{k_1,\ldots,k_N \in \mathbb{N}_0} \prod_{l=1}^{N} \frac{1}{k_l!} \left(\mathrm{d}\Gamma(\ln S_l^A) \right)^{k_l} \quad \alpha. \tag{3.481}$$

Since $\alpha \in \mathcal{F}'$, there is a maximal number of particles of each summand. This implies that $d\Gamma$ is continuous as a function of $\ln S_l^A$, which allows us to pull the Taylor series out of $d\Gamma$. The logarithm of S_l^A is given by a norm convergent series in A,

$$\ln S_l^A = -\sum_{k=1}^{\infty} \frac{1}{k}(1 - S_l^A)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\sum_{c \in \mathbb{N}_0} \frac{1}{c!} Z_{l,c}^A \right)^k \tag{3.482}$$

$$= \sum_{\substack{k \in \mathbb{N} \\ c \in \mathbb{N}_0^k}} \frac{(-1)^{k+1}}{k \; c!} \prod_{v=1}^{k} Z_{l,c_v}^A. \tag{3.483}$$

This may be plugged into the expression for $\tilde{S}^A$ and the sum may be pulled out of $\mathrm{d}\Gamma$ due to linearity and continuity

$$\tilde{S}^A \alpha = \sum_{k \in \mathbb{N}_0^N} \frac{1}{k!} \prod_{l=1}^{N} \left(\sum_{\substack{b \in \mathbb{N} \\ c \in \mathbb{N}_0^b}} \frac{(-1)^{b+1}}{b \; c!} \mathrm{d}\Gamma \left(\prod_{v=1}^{b} Z_{l,c_v}^A \right) \right)^{k_l} \alpha. \tag{3.484}$$

Since all of these sums are absolutely convergent, we may reorder the sum according to degree of homogeneity:

$$\tilde{S}^A \alpha = \sum_{a \in \mathbb{N}} \sum_{k \in \mathbb{N}_0^N} \sum_{\substack{\forall l \leqslant N :\, b_l \in \mathbb{N}^{k_l} \\ \forall l \leqslant N :\, c_l \in \mathbb{N}_0^{b_l} \\ \sum_{l=1}^{N} |c_l| = a}} \prod_{l_1=1}^{N} \frac{(-1)^{|b_{l_1}| + k_{l_1}}}{k_{l_1}! c_{l_1}!} \tag{3.485}$$

$$\times \prod_{l_2=1}^{k_{l_1}} \left(\frac{1}{b_{l_1,l_2}} \mathrm{d}\Gamma \left(\prod_{v=1}^{b_{l_1,l_2}} Z_{l_1, c_{l_1,l_2,v}}^A \right) \right) \cdot \alpha. \tag{3.486}$$

□

3.4 Summary and Conclusions

In the third chapter of this thesis we worked on a quantum field theoretic formulation of electromagnetic interactions. While this approach is more conventional than what was presented in chapter 2 that does not at all imply that the general theory has been worked out on a mathematically rigorous level. So much so that our work on external field QED, i.e. neglecting all interaction between particles, can be regarded as at the frontier of our present understanding. The chapter started with a short summary of the approach to construct a lift of the one-particle time evolution operator where we mentioned the shortcoming of the present method, not uniquely identifying the phase of this operator. Subsequently, we gave a geometric construction of said phase in the scattering regime from an object c^+ very closely related to the current induced by an external field. If such an object were identified the residual freedom might be reduced to an irrelevant constant phase and a single number related to the charge of the electron. Furthermore, we showed that there is a lift of the one-particle scattering operator that is analytic in the external field and gave a compact explicit formula for weak fields.
I hope that in the near future the construction of c^+ will succeed, so that one can further analyse a self-consistent model. In such a model one feeds the current generated by the fermion field induced by the action of an electrodynamic field into Maxwell's equations and acts with the resulting fields again on the fermion field. Such a model would incorporate a mean field interaction between the fermions and would thus be a further step in the direction of a fully interacting theory.

Chapter 4

Appendix

4.1 Regularity of the One-Particle Scattering Operator

In this section we analyse the construction of the one-particle scattering operator S_A carried out in [13] and answer whether operators like

$$P^+ \partial_B S_A^* S_{A+B} P^- \tag{4.1}$$

are Hilbert-Schmidt operators. This is important for the geometric construction carried out in section 3.2.

Since this section is heavily inspired by [13], we need to introduce some notation from this paper.

Definition 86. *Let $A \in \mathcal{V}$, we define the integral operator $Q^A : \mathcal{H} \circlearrowleft$*

by giving its integral kernel, which is also denoted by Q^A:

$$\mathbb{R}^3 \times \mathbb{R}^3 \ni (p,q) \mapsto Q^A(p,q) := \frac{Z^A_{+-}(p,q) - Z^A_{-+}(p,q)}{i(E(p)+E(q))} \tag{4.2}$$

$$\text{with } Z^A_{\pm\mp}(p,q) := P_\pm(p) Z^A(p-q) P_\mp(q), \tag{4.3}$$

$$Z^A = -ie\gamma^0\gamma^\alpha \hat{A}_\alpha, \tag{4.4}$$

$$\hat{A}_\mu := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} A_\mu(x) e^{-ipx} d^3x, \tag{4.5}$$

$$\text{and } E(p) := \sqrt{m^2 + |p|^2}. \tag{4.6}$$

Fact 87. *Please recall that for general $A, F \in \mathcal{V}$ and $t_0, t_1 \in \mathbb{R}$ we have the well known equations for the one-particle time evolution operators*

$$U^A(t_1,t_0) = U^0(t_1,t_0) + \int_{t_0}^{t_1} dt\; U^0(t_1,t) Z^A(t) U^A(t,t_0) \tag{4.7}$$

$$U^{A+F}(t_1,t_0) = U^A(t_1,t_0) + \int_{t_0}^{t_1} dt\; U^A(t_1,t) Z^F(t) U^{A+F}(t,t_0). \tag{4.8}$$

Definition 88. *For any $A \in \mathcal{V}$, we introduce the integral operator $Q'^A : \mathcal{H} \circlearrowleft$ by its kernel*

$$\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \ni (t,p,q) \mapsto Q'^A(t,p,q) = \partial_t Q^A(t,p,q), \tag{4.9}$$

where the time dependence is due to the time dependence of the four-potential A. The following notion of even and odd part of an arbitrary bounded linear operator $T : \mathcal{F} \circlearrowleft$ on Fock space will come in handy:

$$T_{\mathrm{odd}} := P^+TP^- + P^-TP^+ \tag{4.10}$$

$$T_{\mathrm{ev}} := P^+TP^+ + P^-TP^-. \tag{4.11}$$

Additionally, we define the norm

$$T : \mathcal{H} \circlearrowleft \; \|T\|_{\mathrm{op}+I_2} = \|T\| + \|T_{\mathrm{odd}}\|_{I_2}, \tag{4.12}$$

where $\|\cdot\|$ is the operator norm and $\|\cdot\|_{I_2}$ is the Hilbert-Schmidt norm and the space

$$I_2^{\mathrm{odd}} := \{T : \mathcal{F} \to \mathcal{F} \mid \|T\| < \infty, \|T_{\mathrm{odd}}\|_{I_2} < \infty\}. \tag{4.13}$$

Lemma 89. *The space I_2^{odd} equipped with the norm $\|\cdot\|_{\mathrm{op}+I_2}$ is a Banach space.*

Proof. Let $(T_n)_{n\in\mathbb{N}} \subset I_2^{\mathrm{odd}}$ be a Cauchy sequence with respect to $\|\cdot\|_{\mathrm{op}+I_2}$. Then it follows directly that $(T_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence with respect to $\|\cdot\|$ and $(T_{n,\mathrm{odd}})_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{I_2}$. Since the space of bounded operators equipped with $\|\cdot\|$ and the space of Hilbert-Schmidt operators equipped with $\|\cdot\|_{I_2}$ both are complete we have

$$T_n \xrightarrow[\|\cdot\|]{n\to\infty} T^1 \tag{4.14}$$

$$T_{n,\mathrm{odd}} \xrightarrow[\|\cdot\|_{I_2}]{n\to\infty} T^2 \tag{4.15}$$

for some bounded operator T^1 and some Hilbert-Schmidt operator T^2. Now because the Hilbert-Schmidt norm fulfils

$$\|T\| \leqslant \|T\|_{I_2}, \tag{4.16}$$

we obtain directly

$$T_{n,\mathrm{odd}} \xrightarrow[\|\cdot\|]{n\to\infty} T^2, \tag{4.17}$$

hence $T^1_{\mathrm{odd}} = T^2$. Therefore, $T^1 \in I_2^{\mathrm{odd}}$ holds. Finally, since $\|\cdot\|_{\mathrm{op}+I_2} = \|\cdot\| + \|\cdot_{\mathrm{odd}}\|_{I_2}$ is true, we find

$$T_n \xrightarrow[\|\cdot\|_{\mathrm{op}+I_2}]{n\to\infty} T^1, \tag{4.18}$$

proving completeness. □

For the following theorem and lemma we are going to make use of the following shorthand notation of [13]. For operator valued maps $T_1, T_2 : \mathbb{R}^2 \to \mathcal{B}(\mathcal{H})$ we define for $t_1, t_0 \in \mathbb{R}$

$$T_1 T_2 := \int_{t_0}^{t_1} dt\ T_1(t_1, t) T_2(t, t_0), \tag{4.19}$$

as a map of the same type as T_1 and T_2 whenever this is well-defined. Furthermore, for operator valued functions $W_1, W_2 : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ we define

$$T_1 W_1\ (t', t) := T_1(t', t) C_1(t) \tag{4.20}$$

$$W_1 T_1\ (t', t) := W(t_1) T_1(t', t) \tag{4.21}$$

$$W_1 W_2\ (t) := W_1(t) W_2(t), \tag{4.22}$$

as maps of the same type as T_1, T_1 and C_1 respectively.
Pick $k \in \mathbb{N}$, $A, H_b \in \mathcal{V}$ for $b \leqslant k$ and t_1, t_0 such that t_1 is later and t_0 is earlier than the support of A and all H_b. Whenever the shorthand (4.20) and (4.21) is used without specific arguments, by convention $t' = t_1, t = t_0$. We abbreviate

$$H := \sum_{b=1}^{k} H_b, \quad B := A + H. \tag{4.23}$$

We introduce

$$R^B(t', t) := (1 - Q^B) U^B (1 + Q^B)(t', t), \tag{4.24}$$

for general $t', t \in \mathbb{R}$. Because of the choice of t_1, t_0 we have

$$R^B(t_1, t_0) = (1 - Q^B) U^B (1 + Q^B) = U^B(t_1, t_0), \tag{4.25}$$

because $B = 0$ both at t_1 and t_0.

So it suffices to study the family of operators R^B. As shown in the proof of [13, lemma 3.5] R^B for $B \in \mathcal{V}$ is the limit in the sense of the operator norm of the sequence

$$R_0^B := 0, \quad R_{n+1}^B := U^0 \mathsf{F}^B R_n^B + U^0 + \mathsf{G}^B, \tag{4.26}$$

where F and G are given By

$$\mathsf{F}^B := (-Q'^B + Z_{\mathrm{ev}}^B - Q^B Z^B)(1 + Q^B), \tag{4.27}$$
$$\mathsf{G}^B := -U^0 Q^B Q^B \tag{4.28}$$
$$+ U^0(-Q'^B + Z_{\mathrm{ev}} - Q^B Z^B) Q^B Q^B U^B (1 + Q^B). \tag{4.29}$$

Finally, we introduce the auxiliary norms for operators T and W depending on one and two scalar variables respectively.

$$\|T\|_{\mathrm{op}+I_2,\gamma} := \sup_{t \in [t_1,t_0]} e^{-\gamma(t-t_0)} \|T(t)\| + \sup_{t \in [t_1,t_0]} e^{-\gamma(t-t_0)} \|T_{\mathrm{odd}}(t)\|_{I_2} \tag{4.30}$$
$$\|T\|_\gamma := \sup_{t \in [t_1,t_0]} e^{-\gamma(t-t_0)} \|T(t)\| \tag{4.31}$$
$$\|W\|_0 := \sup_{t,t' \in [t_1,t_0]} \|W(t,t')\| \tag{4.32}$$
$$\|T\|_{I_2,\gamma} := \sup_{t \in [t_1,t_0]} e^{-\gamma(t-t_0)} \|T(t)\|_{I_2}, \tag{4.33}$$
$$\|W\|_{I_2,0} := \sup_{t,t' \in [t_1,t_0]} \|W(t,t')\|_{I_2}, \tag{4.34}$$

for $\gamma \geqslant 0$.
Now we have collected enough tools to prove

Theorem 90 (Smoothness of S)**.** *Let $n \in \mathbb{N}$, $A, H_k \in \mathcal{V}$ for $k \leqslant n$, pick t_1 after* $\operatorname{supp} A \cup \bigcup_{k \leqslant n} \operatorname{supp} H_k$ *and t_0 before* $\operatorname{supp} A \cup \bigcup_{k \leqslant n} \operatorname{supp} H_k$ *then the derivative*

$$\partial_{H_1} \dots \partial_{H_k} U^{A + \sum_{b=1}^k H_b}(t_1, t_0) \tag{4.35}$$

exists with respect to the topology induced by the norm $\| \cdot \|_{\mathrm{op}+I_2}$*.*

Proof. We will follow the corresponding proof in [13]. In the proof of the Grönwall lemma in [13, equation (3.42)] we also have that the recursive equation

$$R_n^B = U^0 \mathsf{F}_{\mathrm{ev}}^B R_{n-1}^B + U^0 \mathsf{F}_{\mathrm{odd}}^B U^0 \mathsf{F}^B R_{n-2}^B \tag{4.36}$$

$$+ U^0 \mathsf{F}_{\mathrm{odd}}^B \mathsf{G}^B + U^0 F_{\mathrm{odd}}^B U^0 + U^0 + \mathsf{G}^B \tag{4.37}$$

is fulfilled by the same sequence of operators for $n \geqslant 2$. Furthermore, we introduce the notation

$$[k] = \{l \in \mathbb{N} \mid l \leqslant k\} \tag{4.38}$$

$$\forall u \subseteq [k] : \partial_u = \prod_{k \in u} \partial_{H_k}, \tag{4.39}$$

$$\Delta^n = R_{n+1}^B - R_n^B, \tag{4.40}$$

where the product of derivatives is to be understood as the mixed derivative with respect to all the factors and we use the derivative defined in (3.52).
Hence, we have for such n:

$$\Delta_n = U^0 \mathsf{F}_{\mathrm{ev}}^B \Delta_{n-1} + U^0 \mathsf{F}_{\mathrm{odd}}^B U^0 \mathsf{F}^B \Delta_{n-2}.$$

Abbreviating $U^0 \mathsf{F}_{\mathrm{ev}}^B =: a, \quad U^0 \mathsf{F}_{\mathrm{odd}}^B U^0 \mathsf{F}^B := b$, we obtain

$$\Delta_n = a \Delta_{n-1} + b \Delta_{n-2}. \tag{4.41}$$

we estimate for any set $u \subset [k]$:

$$\sup_{p\subseteq u} \|\partial_p \Delta^n_{\mathrm{odd}}(\cdot, t_0)\|_{I_2,\gamma} \tag{4.42}$$

$$\leqslant \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \left\| \sum_{w\subseteq p} (\partial_{p\setminus w} a \ \partial_w \Delta^{n-1}_{\mathrm{odd}})(t,t_0) \right\|_{I_2} \tag{4.43}$$

$$+ \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \left\| P^+ \sum_{w\subseteq p} (\partial_{p\setminus w} b \ \partial_w \Delta^{n-2})(t,t_0) P^- \right\|_{I_2} \tag{4.44}$$

$$+ \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \left\| P^- \sum_{w\subseteq p} (\partial_{p\setminus w} b \ \partial_w \Delta^{n-2})(t,t_0) P^+ \right\|_{I_2} \tag{4.45}$$

$$\leqslant \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq p} \|(\partial_{p\setminus w} a \ \partial_w \Delta^{n-1}_{\mathrm{odd}})(t,t_0)\|_{I_2} \tag{4.46}$$

$$+ 2 \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq p} \|(\partial_{p\setminus w} b \ \partial_w \Delta^{n-2})(t,t_0)\|_{I_2} \tag{4.47}$$

$$\leqslant \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq p} \int_{t_0}^{t} dt' \|\partial_{p\setminus w} a(t,t') \ \partial_w \Delta^{n-1}_{\mathrm{odd}}(t',t_0)\|_{I_2} \tag{4.48}$$

$$+ 2 \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq p} \int_{t_0}^{t} dt' \|\partial_{p\setminus w} b(t,t') \ \partial_w \Delta^{n-2}(t',t_0)\|_{I_2} \tag{4.49}$$

$$\leqslant \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq p} \int_{t_0}^{t} dt' \|\partial_{p\setminus w} a\|_0 \|\partial_w \Delta^{n-1}_{\mathrm{odd}}(t',t_0)\|_{I_2} \tag{4.50}$$

$$+ 2 \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma(t-t_0)} \sum_{w\subseteq p} \int_{t_0}^{t} dt' \|\partial_{p\setminus w} b(t,t')\|_{I_2} \|\partial_w \Delta^{n-2}(t',t_0)\| \tag{4.51}$$

$$\leqslant \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma t} \sum_{w\subseteq p} \int_{t_0}^{t} dt' e^{\gamma t'} \|\partial_{p\setminus w} a\|_0 \|\partial_w \Delta^{n-1}_{\mathrm{odd}}(\cdot,t_0)\|_{I_2,\gamma} \tag{4.52}$$

$$+ 2 \sup_{p\subseteq u} \sup_{t\in[t_0,t_1]} e^{-\gamma t} \sum_{w\subseteq p} \int_{t_0}^{t} dt' e^{\gamma t'} \|\partial_{p\setminus w} b\|_{I_2,0} \|\partial_w \Delta^{n-2}(\cdot,t_0)\|_{\gamma} \tag{4.53}$$

$$\leqslant \frac{1}{\gamma} \sup_{p \subseteq u} \sum_{w \subseteq p} \|\partial_{p \setminus w} a\|_0 \|\partial_w \Delta^{n-1}_{\text{odd}}(\cdot, t_0)\|_{I_2,\gamma} \tag{4.54}$$

$$+ \frac{2}{\gamma} \sup_{p \subseteq u} \sum_{w \subseteq p} \|\partial_{u \setminus w} b\|_{I_2,0} \|\partial_w \Delta^{n-2}(\cdot, t_0)\|_{\gamma} \tag{4.55}$$

$$\leqslant \frac{2^{|u|}}{\gamma} \sup_{u' \subseteq u} \|\partial_{u'} a\|_0 \sup_{p \subseteq u} \|\partial_p \Delta^{n-1}_{\text{odd}}(\cdot, t_0)\|_{I_2,\gamma} \tag{4.56}$$

$$+ \frac{2^{|u|+1}}{\gamma} \sup_{u' \subseteq u} \|\partial_{u'} b\|_{I_2,0} \sup_{p \subseteq u} \|\partial_p \Delta^{n-2}(\cdot, t_0)\|_{\gamma} \tag{4.57}$$

Similarly, we compute the operator norm:

$$\sup_{p \subseteq u} \|\partial_p \Delta^n(\cdot, t_0)\|_{\gamma} \tag{4.58}$$

$$\leqslant \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sup_{p \subseteq u} \sum_{w \subseteq p} \|(\partial_{p \setminus w} a \ \partial_w \Delta^{n-1})(t, t_0)\| \tag{4.59}$$

$$+ \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sup_{p \subseteq u} \sum_{w \subseteq p} \|(\partial_{p \setminus w} b \ \partial_w \Delta^{n-2})(t, t_0)\| \tag{4.60}$$

$$\leqslant \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sup_{p \subseteq u} \sum_{w \subseteq p} \int_{t_0}^{t} dt' \|\partial_{p \setminus w} a(t, t') \ \partial_w \Delta^{n-1}(t', t_0)\| \tag{4.61}$$

$$+ \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sup_{p \subseteq u} \sum_{w \subseteq u} \int_{t_0}^{t} dt' \|\partial_{p \setminus w} b(t, t') \ \partial_w \Delta^{n-2}(t', t_0)\| \tag{4.62}$$

$$\leqslant \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sup_{p \subseteq u} \sum_{w \subseteq p} \int_{t_0}^{t} dt' \|\partial_{p \setminus w} a(t, t')\| \|\partial_w \Delta^{n-1}(t', t_0)\| \tag{4.63}$$

$$+ \sup_{t \in [t_0, t_1]} e^{-\gamma(t-t_0)} \sup_{p \subseteq u} \sum_{w \subseteq p} \int_{t_0}^{t} dt' \|\partial_{p \setminus w} b(t, t')\| \|\partial_w \Delta^{n-2}(t', t_0)\| \tag{4.64}$$

$$\leqslant \sup_{t \in [t_0, t_1]} e^{-\gamma t} \sup_{p \subseteq u} \sum_{w \subseteq p} \int_{t_0}^{t} dt' e^{\gamma t'} \|\partial_{p \setminus w} a\|_0 \|\partial_w \Delta^{n-1}(\cdot, t_0)\|_{\gamma} \tag{4.65}$$

$$+ \sup_{t\in[t_0,t_1]} e^{-\gamma t} \sup_{p\subseteq u} \sum_{w\subseteq p} \int_{t_0}^{t} dt' e^{\gamma t} \|\partial_{u\setminus p} b\|_0 \|\partial_w \Delta^{n-2}(\cdot, t_0)\|_\gamma \tag{4.66}$$

$$\leqslant \frac{1}{\gamma} \sup_{p\subseteq u} \sum_{w\subseteq p} \|\partial_{p\setminus w} a\|_0 \|\partial_w \Delta^{n-1}(\cdot, t_0)\|_\gamma \tag{4.67}$$

$$+ \frac{1}{\gamma} \sup_{p\subseteq u} \sum_{w\subseteq p} \|\partial_{p\setminus w} b\|_0 \|\partial_w \Delta^{n-2}(\cdot, t_0)\|_\gamma \tag{4.68}$$

$$\leqslant \frac{2^{|u|}}{\gamma} \sup_{u'\subseteq u} \|\partial_{u'} a\|_0 \sup_{w\subseteq u} \|\partial_w \Delta^{n-1}(\cdot, t_0)\|_\gamma \tag{4.69}$$

$$+ \frac{2^{|u|}}{\gamma} \sup_{u'\subseteq u} \|\partial_{u'} b\|_0 \sup_{w\subseteq u} \|\partial_w \Delta^{n-2}(\cdot, t_0)\|_\gamma \tag{4.70}$$

We can summarize the last calculations more briefly using the abbreviation

$$\alpha = \frac{2^{|u|+1}}{\gamma} \sup_{u'\subseteq u} \{\|\partial_{u'} a\|_0, \|\partial_{u'} b\|_{I_2,\infty}, \|\partial_{u'} b\|_0\}. \tag{4.71}$$

Here α is finite. This can be seen as follows: firstly, $\partial_u b = U^0 \partial_u \mathsf{F}^B_{\text{odd}} U^0 \mathsf{F}^B$ vanishes if $|u| \geqslant 7$ because the factors Q'^B, Q^B and Z^B are all linear in B and the longest product of such operators appearing in b has six factors, analogously all derivatives $\partial_u a = 0$ for $|u| \geqslant 4$. Secondly, each of the operators Q'^C, Q^C and Z^C are bounded for every $C \in \mathcal{V}$, hence the polynomials a and b of these operators are also bounded. This shows finiteness of the two operator norms appearing in the expression for α. For the Hilbert-Schmidt norm we see that $\partial_u b$ is always a sum of terms where each term has a factor $U^0 \partial_p \mathsf{F}^B_{\text{odd}} U^0$ with $p \subseteq u$. This factor has finite Hilbert-Schmidt norm due to the I_2 estimate lemma 91.

We can thus summarize the last two calculations

$$\begin{pmatrix} \sup_{p\subseteq u} \|\partial_p \Delta^n\|_{\text{op}+I_2,\gamma} \\ \sup_{p\subseteq u} \|\partial_p \Delta^{n-1}\|_{\text{op}+I_2,\gamma} \end{pmatrix} \tag{4.72}$$

$$\leqslant \begin{pmatrix} \alpha & \alpha \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sup_{p\subseteq u} \|\partial_p \Delta^{n-1}\|_{\text{op}+I_2,\gamma} \\ \sup_{p\subseteq u} \|\partial_p \Delta^{n-2}\|_{\text{op}+I_2,\gamma} \end{pmatrix} \tag{4.73}$$

$$\leqslant \begin{pmatrix} \alpha & \alpha \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} \sup_{p\subseteq u} \|\partial_p \Delta^{1}\|_{\text{op}+I_2,\gamma} \\ \sup_{p\subseteq u} \|\partial_p \Delta^{0}\|_{\text{op}+I_2,\gamma} \end{pmatrix}. \tag{4.74}$$

This matrix can be diagonalized, its eigenvalues are

$$\lambda_\pm = \frac{\alpha}{2}\left(1 \pm \sqrt{1 + \frac{4}{\alpha}}\right). \tag{4.75}$$

The larger eigenvalue λ_+ is less than 1 if and only if $0 < \alpha < 0.5$ holds true, as can be seen from a quick calculation:

$$\frac{\alpha}{2}\left(1 + \sqrt{1 + \frac{4}{\alpha}}\right) < 1 \tag{4.76}$$

$$\iff \sqrt{1 + \frac{4}{\alpha}} < \frac{2}{\alpha} - 1. \tag{4.77}$$

If $\alpha \geqslant \frac{1}{2}$ or $\alpha < 0$ this inequality is not satisfied, otherwise we may square both sides to find

$$1 + \frac{4}{\alpha} < (2/\alpha - 1)^2 = 4/\alpha^2 - 4/\alpha + 1 \tag{4.78}$$

$$\iff \alpha < \frac{1}{2}. \tag{4.79}$$

So we conclude that for γ large enough the right-hand side of (4.74) tends to zero as $c\lambda_+^n$ for $n \to \infty$, with

$$c = \sqrt{\sup_{p\subseteq u} \|\partial_p \Delta^1\|^2_{\text{op}+I_2,\gamma} + \sup_{p\subseteq u} \|\partial_p \Delta^0\|^2_{\text{op}+I_2,\gamma}}. \tag{4.80}$$

Concerning the norms of $\partial_p\Delta^1$ and $\partial_p\Delta^0$: the operator norms of these terms are finite, since they are polynomials of bounded operators linear in the external potential. The Hilbert-Schmidt norm of the odd part of these terms can be bounded using lemma 91.
That is, we have

$$\sup_{p\subseteq u} \|\partial_p \Delta^n\|_{\mathrm{op}+I_2,\gamma} \leqslant \lambda_+^n c \xrightarrow{n\to\infty} 0. \tag{4.81}$$

For $2 \leqslant m \leqslant n$ we obtain

$$\sup_{p\subseteq u} \|\partial_p R_n^B - \partial_p R_m^B\|_{\mathrm{op}+I_2,\gamma} \leqslant \sum_{k=m}^{n-1} \sup_{p\subseteq u} \|\partial_p \Delta^k\|_{\mathrm{op}+I_2,\gamma} \tag{4.82}$$

$$\leqslant \sum_{k=m}^{\infty} \lambda_+^k c = \frac{\lambda_+^m}{1-\lambda_+} c \xrightarrow{m\to\infty} 0 \tag{4.83}$$

since the norms $\|\cdot\|_{\mathrm{op}+I_2}$ and $\|\cdot\|_{\mathrm{op}+I_2,\gamma}$ are equivalent, we have just proven that $\partial_{[k]}R_m^B$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{\mathrm{op}+I_2}$ and hence convergence by lemma 89.

□

The following lemma is a necessary ingredient for theorem 90. Morally, it has already been proven in [13, Lemma 3.7]; however, as that paper was not concerned with multiple four-potentials the lemma was not formulated general enough for our needs here. So we restate it and show how to modify the original proof.

Lemma 91 (I_2 estimates)**.** *Let $k \in \mathbb{N}$ and $A, H_b \in \mathcal{V}$ for $b \leqslant k$. Using the abbreviations introduced in (4.38) and (4.39) we have for any $u \subset [k]$ the following bounds:*

$$\|\partial_u U^0 \mathsf{F}_{\mathrm{odd}}^{A+\sum_{b=1}^k H_b} U^0\|_{I_2,0} < \infty \tag{4.84}$$

$$\|\partial_u \mathsf{G}^{A+\sum_{b=1}^k H_b}\|_{I_2,0} < \infty. \tag{4.85}$$

Proof. For $B \in \mathcal{V}$ recall

$$\begin{aligned}\mathsf{F}^B_{\text{odd}} :=& ((-Q'^B + Z^B_{\text{ev}} - Q^B Z^B)(1+Q^B))_{\text{odd}} &(4.86)\\ =& -Q'^B + Z^B_{\text{ev}} Q^B - Q^B Z^B_{\text{ev}} - Q^B Z^B_{\text{odd}} Q^B, &(4.87)\\ \mathsf{G}^B :=& -U^0 Q^B Q^B &(4.88)\\ &+U^0(-Q'^B + Z_{\text{ev}} - Q^B Z^B) Q^B Q^B U^B (1+Q^B). &(4.89)\end{aligned}$$

Pick $k \in \mathbb{N}$ and $A, B, H_b \in \mathcal{V}$ for $b \leqslant k$.
According to [13, lemma 3.7] the operators $U^0 Z^B_{\text{ev}} Q^B U^0$, $U^0 Q^B Z^B_{\text{ev}} U^0$, $U^0 Q'^B U^0$, $Q^B Q^B, Q'^B Q^B$ and $Q^B Z^B Q^B$ are Hilbert-Schmidt operators. Additionally, their Hilbert-Schmidt norm is uniformly bounded in time and F and G fulfil the following norm bound:

$$\|U^0 \mathsf{F}^B_{\text{odd}} U^0\|_{I_2,0} < \infty \text{ and } \|\mathsf{G}^B\|_{I_2,0} < \infty. \tag{4.90}$$

In fact, the proof given in [13] also works for non-identical four-potentials $A, B, C \in \mathcal{V}$ proving

$$\begin{aligned}&\|U^0 Z^A_{\text{ev}} Q^B U^0\|_{I_2,0} < \infty, &(4.91)\\ &\|U^0 Q^A Z^B_{\text{ev}} U^0\|_{I_2,0} < \infty, &(4.92)\\ &\|U^0 Q'^B U^0\|_{I_2,0} < \infty, &(4.93)\\ &\|Q^B Q^A\|_{I_2,0} < \infty, &(4.94)\\ &\|Q'^B Q^A\|_{I_2,0} < \infty, &(4.95)\\ &\|Q^A Z^B Q^C\|_{I_2,0} < \infty &(4.96)\end{aligned}$$

and therefore also

$$\|\partial_u U^0 \mathsf{F}^{A+\sum_{b=0}^k H_b}_{\text{odd}} U^0\|_{I_2,0} < \infty \text{ and } \|\partial_u \mathsf{G}^{A+\sum_{b=0}^k H_b}\|_{I_2,0} < \infty \tag{4.97}$$

for any $u \subseteq [k]$.

For the benefit of the reader, we will reproduce the proof of the estimate

$$\|\partial_u \mathsf{G}^{A+\sum_{b=0}^{k} H_b}\|_{I_2,0} < \infty, \tag{4.98}$$

to make clear the structure of the entire proof.
The operator $\mathsf{G}^{A+\sum_{b=0}^{k} H_b}$ consists of two summands. Each summand is a product of operators with operator norm uniformly bounded in time and containing a factor of $Q^{A+\sum_{b=0}^{k} H_b} Q^{A+\sum_{b=0}^{k} H_b}$. All the other factors contributing to G stay bounded when differentiated and the map $Q : \mathcal{V} \to I_2$ is linear, so the bound

$$\|Q^B Q^D\|_{I_2,0} < \infty \tag{4.99}$$

for general $B, D \in \mathcal{V}$ will suffice to prove (4.98).
Pick $B, D \in \mathcal{V}$, we estimate

$$\sup_{t\in\mathbb{R}} \|Q^B(t) Q^D(t)\|_{I_2} \tag{4.100}$$

$$\leqslant \sum_{\mu,\nu=0}^{3} \Big(\sup_{p,k,q\in\mathbb{R}^3} |P_+(p)\gamma^0\gamma^\mu P_-(k)\gamma^0\gamma^\nu P_+(q)| \tag{4.101}$$

$$+ \sup_{p,k,q\in\mathbb{R}^3} |P_-(p)\gamma^0\gamma^\mu P_+(k)\gamma^0\gamma^\nu P_-(q)|\Big) \tag{4.102}$$

$$\sup_{t\in\mathbb{R}} \left\| \int_{\mathbb{R}^3} dk \frac{\hat{B}_\mu(t, p-k)\hat{D}_\nu(t, k-q)|}{(E(p)+E(k))(E(k)+E(q))} \right\|_{I_2,(p,q)}, \tag{4.103}$$

where the index in the norm indicates with respect to which variables the integral of the norm is to be performed. The prefactor is finite since $P_\pm(p) : \mathbb{C}^4 \to \mathbb{C}^4$ is a projector for any $p \in \mathbb{R}^3$. Abbreviating

$$\tilde{c} := \sum_{\mu,\nu=0}^{3} (\sup_{p,k,q\in\mathbb{R}^3} |P_+(p)\gamma^0\gamma^\mu P_-(k)\gamma^0\gamma^\nu P_+(q)| \tag{4.104}$$

$$+ \sup_{p,k,q\in\mathbb{R}^3} |P_-(p)\gamma^0\gamma^\mu P_+(k)\gamma^0\gamma^\nu P_-(q)|), \tag{4.105}$$

and using the integral estimate lemma [13, lemma 3.8 (iii)] we find

$$(4.100) \leqslant \tilde{c}\, C_{8,\text{of} \quad [13]} \sum_{\mu,\nu=0}^{3} \sup_{t\in\mathbb{R}} \|\hat{B}_\mu(t)\|_{I_1} \|\hat{D}_\nu(t)\|_{I_2}. \tag{4.106}$$

Because of $B, D \in C_c^\infty(\mathbb{R}^4)$, we have $\hat{B}(t), \hat{D}(t)$ are analytic functions decaying faster than any negative power at infinity for any t, so (4.106) is finite.

Also, in order to proof the first estimate in (4.84) the proof of [13, lemma 3.7] can be followed almost verbatim. First one dissects F_{odd} into the sum U^0 (4.87)U^0, next the summands have to be bounded individually. This is achieved by repeating the proof of the partial integration lemma [13, lemma 3.6], estimates of the form of (4.100)-(4.103) and making use of the integral estimate lemma [13, lemma 3.8].

□

Theorem 92 (Properties of Derivatives of S)**.** *Let* $A, H \in \mathcal{V}$*, pick* t_1 *after* $\operatorname{supp} A \cup \operatorname{supp} H$ *and* t_0 *before* $\operatorname{supp} A \cup \operatorname{supp} H$*, let* $T_1 \in I_2(\mathcal{F})$ *then the following equalities are satisfied:*

$$\begin{aligned} &\partial_H \operatorname{tr}(T_1 P^\pm U^A(t_0,t_1) U^{A+H}(t_1,t_0) P^\mp) \\ &= \operatorname{tr}(T_1 P^\pm U^A(t_0,t_1) \partial_H U^{A+H}(t_1,t_0) P^\mp) \end{aligned} \tag{4.107}$$

Proof. Let $A, H \in \mathcal{V}$ and $t_0, t_1 \in \mathbb{R}$ and T_1 be as in the theorem. The proof of the two equalities is analogous, so we only explicitly prove the first one. The trace is linear, so we have

$$\begin{aligned}
&\Bigg| \operatorname{tr}\left(T_1 P^+ U^A(t_0,t_1)\frac{1}{\varepsilon}(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0))P^-\right) \\
&\quad - \operatorname{tr}(T_1 P^+ U^A(t_0,t_1)\partial_H U^{A+H}(t_1,t_0)P^-)\Bigg| \quad (4.108)\\
&\leqslant \|T_1\|_{I_2} \Bigg\| P^+ U^A(t_0,t_1)\frac{1}{\varepsilon}\big(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\big)P^- \\
&\quad - P^+ U^A(t_0,t_1)\partial_H U^{A+H}(t_1,t_0)P^- \Bigg\|_{I_2} \quad (4.109)
\end{aligned}$$

For the first summand we insert the identity in the form $P^+ + P^-$ and obtain

$$\begin{aligned}
&P^+ U^A(t_0,t_1)\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^- \quad (4.110)\\
&= P^+ U^A(t_0,t_1)P^+\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^- \quad (4.111)\\
&+ P^+ U^A(t_0,t_1)P^-\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^-. \quad (4.112)
\end{aligned}$$

Analogously for the second summand. Now because of the Smoothness of S theorem 90 we know that

$$P^-\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^- \xrightarrow[\|\cdot\|]{\varepsilon\to 0} P^-\partial_H U^{A+H}(t_1,t_0)P^- \quad (4.113)$$

$$P^+\frac{1}{\varepsilon}\left(U^{A+\varepsilon H}(t_1,t_0) - U^A(t_1,t_0)\right)P^- \xrightarrow[\|\cdot\|_{I_2}]{\varepsilon\to 0} P^+\partial_H U^{A+H}(t_1,t_0)P^- \quad (4.114)$$

holds true. Hence, we find in total

$$\overset{(4.109)}{\overline{\|T_1\|_{I_2}}} \tag{4.115}$$

$$\leqslant \left\| P^+ U^A(t_0, t_1) \right\| \left\| P^+ \frac{1}{\varepsilon} \big(U^{A+\varepsilon H}(t_1, t_0) - U^A(t_1, t_0) \big) P^- - P^+ \partial_H U^{A+H}(t_1, t_0) P^- \right\|_{I_2} \tag{4.116}$$

$$+ \left\| P^+ U^A(t_0, t_1) P^- \right\|_{I_2} \left\| \frac{1}{\varepsilon} \big(U^{A+\varepsilon H}(t_1, t_0) - U^A(t_1, t_0) \big) P^- - \partial_H U^{A+H}(t_1, t_0) P^- \right\| \xrightarrow{\varepsilon \to 0} 0. \tag{4.117}$$

□

4.2 Lemma of Poincaré in infinite dimensions

In this section we give prove of the Poincaré lemma in infinite dimensions used in section 3.2. First recall the lemma itself.

Lemma 93 (Poincaré)**.** *Let $\omega \in \Omega^p(\mathcal{V})$ for $p \in \mathbb{N}$ be closed, i.e. $d\omega = 0$. Then ω is also exact, more precisely we have*

$$\omega = d \int_0^1 \iota_t^* i_X f^* \omega dt, \tag{4.118}$$

where X, ι_t for $t \in \mathbb{R}$ and f are given by

$$
\begin{aligned}
& X : \mathbb{R} \times \mathcal{V} \to \mathbb{R} \times \mathcal{V}, && (4.119)\\
& \qquad (t, B) \mapsto (1, 0) && (4.120)\\
& \forall t \in \mathbb{R} : \iota_t : \mathcal{V} \to \mathbb{R} \times \mathcal{V}, && (4.121)\\
& \qquad B \mapsto (t, B) && (4.122)\\
& f : \mathbb{R} \times \mathcal{V} \mapsto \mathcal{V}, && (4.123)\\
& \qquad (t, B) \mapsto tB && (4.124)\\
& i_X : \Omega^p(\mathcal{V}) \to \Omega^{p-1}(\mathcal{V}), && (4.125)\\
& \qquad \omega \mapsto ((A; Y_1, \ldots, Y_{p-1}) \mapsto \omega_A(X, Y_1, \ldots, Y_{p-1})) && (4.126)
\end{aligned}
$$

Proof. Pick some $\omega \in \Omega^p(\mathcal{V})$. We will first show the more general formula

$$f_b^*\omega - f_a^*\omega = d\int_a^b \iota_t^* i_X f^*\omega\ dt + \int_a^b \iota_t^* i_X f^* d\omega dt, \tag{4.127}$$

where f_t is defined as

$$\forall t \in \mathbb{R} : f_t := f(t, \cdot). \tag{4.128}$$

The lemma follows then by $b = 1, a = 0$, $f_1^*\omega = \omega$, $f_0^*\omega = 0$ and $d\omega = 0$ for a closed ω. We begin by rewriting the right-hand side of (4.127):

$$
\begin{aligned}
d\int_a^b \iota_t^* i_X f^*\omega\ dt + \int_a^b \iota_t^* i_X f^* d\omega dt& \\
= \int_a^b (d\iota_t^* i_X f^*\omega + \iota_t^* i_X f^* d\omega) dt.& \qquad (4.129)
\end{aligned}
$$

Next we look at both of these terms separately. Let therefore $p \in \mathbb{N}$, $t, s_k \in \mathbb{R}$ and $A, B_k \in \mathcal{V}$ for each $\mathbb{N} \ni k \leqslant p+1$. First, we calculate

$d\iota_t^* i_X f^*\omega$:

$$(f^*\omega)_{(t,A)}((s_1, B_1), \ldots, (s_p, B_p)) = \omega_{tA}(s_1 A + tB_1, \ldots, s_p A + tB_p) \tag{4.130}$$

$$\Rightarrow (i_X f^*\omega)_{(t,A)}((s_1, B_1), \ldots, (s_{p-1}, B_{p-1})) = \omega_{tA}(A, s_1 A + tB_1, \ldots, s_{p-1} A + tB_{p-1}) \tag{4.131}$$

$$\Rightarrow (\iota_t^* i_X f^*\omega)_A(B_1, \ldots, B_{p-1}) = t^{p-1}\omega_{tA}(A, B_1, \ldots, B_{p-1}) \tag{4.132}$$

$$\Rightarrow (d\iota_t^* i_X f^*\omega)_A(B_1, \ldots, B_p)$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^{k+1} t^{p-1} \omega_{tA+\varepsilon tB_k}(A, B_1, \ldots, \widehat{B_k}, \ldots, B_p) \tag{4.133}$$

$$+ \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} (-1)^{k+1} t^{p-1} \omega_{tA}(A + \varepsilon B_k, B_1, \ldots, \widehat{B_k}, \ldots, B_p) \tag{4.134}$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p} t^p (-1)^{k+1} \omega_{tA+\varepsilon B_k}(A, B_1, \ldots, \widehat{B_k}, \ldots, B_p) + pt^{p-1}\omega_{tA}(B_1, \ldots, B_p). \tag{4.135}$$

Now, we calculate $\iota_t^* i_X f^* d\omega$:

$$(d\omega)_A(B_1, \cdots, B_{p+1}) = \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p+1} (-1)^{k+1} \omega_{A+\varepsilon B_k}(B_1, \ldots, \widehat{B_k}, \ldots, B_{p+1}) \tag{4.136}$$

$$(f^* d\omega)(t, A)((s_1, B_1), \ldots, (s_{p+1}, B_{p+1})) \tag{4.137}$$

$$= (d\omega)_{tA}(s_1 A + tB_1, \ldots, s_{p+1} A + tB_{p+1})$$

$$= \partial_\varepsilon|_{\varepsilon=0} \sum_{k=1}^{p+1} (-1)^{k+1} \tag{4.138}$$

$$\times\, \omega_{tA+\varepsilon(s_kA+tB_k)}(s_1A+tB_1,\dots,s_k\widehat{A+t}B_k,\dots,s_pA+tB_p)$$

$$(i_Xf^*d\omega)_{(t,A)}((s_1,B_1),\dots,(s_p,B_p)) \tag{4.139}$$
$$= \partial_\varepsilon|_{\varepsilon=0}\omega_{(t+\varepsilon)A}(s_1A+tB_1,\dots,s_pA+tB_p)$$
$$+\,\partial_\varepsilon|_{\varepsilon=0}\sum_{k=1}^p(-1)^k$$
$$\times\, \omega_{tA+\varepsilon(s_kA+tB_k)}(A,s_1A+tB_1,\dots,s_k\widehat{A+t}B_k,\dots,s_pA+tB_p)$$
$$= t^p\partial_\varepsilon|_{\varepsilon=0}\omega_{(t+\varepsilon)A}(B_1,\dots,B_p)) \tag{4.140}$$
$$+\sum_{k=1}^p s_kt^{p-1}(-1)^{k+1}\partial_\varepsilon|_{\varepsilon=0}\omega_{(t+\varepsilon)A}(A,B_1,\dots,\widehat{B_k},\dots,B_p) \tag{4.141}$$
$$+\,\partial_\varepsilon|_{\varepsilon=0}\sum_{k=1}^p(-1)^kt^{p-1}(\omega_{(t+s_k\varepsilon)A}(A,B_1,\dots,\widehat{B_k},\dots,B_p) \tag{4.142}$$
$$+\,\omega_{tA+\varepsilon tB_k}(A,B_1,\dots,\widehat{B_k},\dots,B_p)) \tag{4.143}$$
$$= t^p\partial_\varepsilon|_{\varepsilon=0}\Big(\omega_{(t+\varepsilon)A}(B_1,\dots,B_p) \tag{4.144}$$
$$+\sum_{k=1}^p(-1)^k\omega_{tA+\varepsilon B_k}(A,B_1,\dots,\widehat{B_k},\dots,B_p)\Big)$$

$$(\iota_t^*i_Xf^*d\omega)_A(B_1,\dots,B_p) = t^p\partial_\varepsilon|_{\varepsilon=0}\Big(\omega_{(t+\varepsilon)A}(B_1,\dots,B_p) \tag{4.145}$$
$$+\sum_{k=1}^p(-1)^k\omega_{tA+\varepsilon B_k}(A,B_1,\dots,\widehat{B_k},\dots,B_p)\Big)$$

Adding (4.135) and (4.145) we find for (4.129):

$$\int_a^b(d\iota_t^*i_Xf^*\omega+\iota_t^*i_Xf^*d\omega)dt = \tag{4.146}$$
$$\int_a^b\Big(t^p\partial_\varepsilon|_{\varepsilon=0}\omega_{(t+\varepsilon)A}(B_1,\dots,B_p)+pt^{p-1}\omega_{tA}(B_1,\dots,B_p)\Big)dt \tag{4.147}$$

$$= \int_a^b \frac{d}{dt}(t^p \omega_{tA}(B_1, \ldots, B_p))dt = \int_a^b \frac{d}{dt}(f_t^* \omega)_A(B_1, \ldots, B_p)dt \quad (4.148)$$

$$= (f_b^* \omega)_A(B_1, \ldots, B_p) - (f_a^* \omega)_A(B_1, \ldots, B_p). \quad (4.149)$$

□

4.3 Heuristic Construction of S-Matrix expression

This section is dedicated to the heuristic construction of the expression for the scattering operator stated in theorem 84.
We start from the power series of the one-particle scattering operator S^A:

$$S^A = \sum_{k=0}^{\infty} \frac{1}{k!} Z_k(A), \quad (4.150)$$

where $Z_k(A)$ are bounded operators on $\mathcal{H}$, which are homogeneous of degree k in A. Our strategy in this section is to try an analogous formal power series ansatz for the second quantized scattering operator $\tilde{S}^A$

$$\tilde{S}^A = \sum_{k=0}^{\infty} \frac{1}{k!} T_k(A). \quad (4.151)$$

Here T_k are assumed to be homogeneous of degree k in A; however, they will only turn out to be bounded on fixed particle number subspaces $\mathcal{F}_{m,p}$ of Fock space. We will identify operators T_k such that (4.151) holds up to a global phase. In order to fully characterize $\tilde{S}^A$ it is enough to characterize all of the T_k operators. Using the (lift condition) one can derive commutation relations for the operators T_k by plugging in (4.150) and (4.151) into the (lift condition) and its adjoint and collecting all terms with the same degree of homogeneity. They are given by

$$\left[T_m(A), a^\#(\phi)\right] = \sum_{j=1}^{m} \binom{m}{j} a^\# \left(Z_j(A)\phi\right) T_{m-j}(A), \tag{4.152}$$

where $a^\#$ is either a or a^*. In the following we will derive a recursive equation for the coefficients of the expansion of the second quantized scattering operator. The starting point of this derivation is the commutator of T_m, equation (4.152).

4.3.1 Guessing Equations

Looking at equation (4.152) for a while, one comes to the conclusion that if one replaces T_m by

$$T_m - \frac{1}{2}\sum_{k=1}^{m-1} \binom{m}{k} T_k T_{m-k}, \tag{4.153}$$

no T_k with $k > m-2$ will occur on the right-hand side of the resulting equation. So if one subtracts the right polynomial in T_k for suitable k one might achieve a commutator which contains only the creation respectively annihilation operator concatenated with some one-particle operator.
So having this in mind we start with the ansatz

$$\Xi_m := \sum_{g=2}^{m} \sum_{\substack{b\in\mathbb{N}^g \\ |b|=m}} c_b \prod_{k=1}^{g} T_{b_k}. \tag{4.154}$$

Now in order to show that T_m and Ξ_m agree up to operators which have a commutation relation of the form (3.444), we calculate $\left[T_m - \Xi_m, a^\#(\varphi)\right]$ for arbitrary $\varphi \in \mathcal{H}$ and try to choose the coefficients c_b of (4.154) such that all contributions vanish which do not have the form $a^{\#}\left(\prod_k Z_{\alpha_k}\varphi\right)$ for any suitable $(\alpha_k)_k \subset \mathbb{N}$. If one does so, one is led to a system of equations of which the first few are written down to give an overview of its structure. The objects α_k, β_l in the system of equations can be any natural Number for any $k, l \in \mathbb{N}$.

$$
\begin{aligned}
0 &= c_{\alpha_1,\beta_1} + c_{\beta_1,\alpha_1} + \binom{\alpha_1+\beta_1}{\alpha_1} \\
0 &= c_{\alpha_1,\alpha_2,\beta_1} + c_{\beta_1,\alpha_1,\alpha_2} + c_{\alpha_2,\alpha_1,\beta_1} + \binom{\alpha_2+\beta_1}{\alpha_2} c_{\alpha_1,\alpha_2+\beta_1} \\
&\qquad + \binom{\alpha_1+\beta_1}{\alpha_1} c_{\alpha_1+\beta_1,\alpha_2} \\
0 &= c_{\alpha_1,\alpha_2,\alpha_3,\beta_1} + c_{\alpha_1,\alpha_2,\beta_1,\alpha_3} + c_{\alpha_1,\beta_1,\alpha_2,\alpha_3} + c_{\beta_1,\alpha_1,\alpha_2,\alpha_3} \\
&+ \binom{\alpha_1+\beta_1}{\beta_1} c_{\alpha_1+\beta_1,\alpha_2,\alpha_3} + \binom{\alpha_2+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2+\beta_1,\alpha_3} \\
&\qquad + \binom{\alpha_3+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2,\alpha_3+\beta_1} \\
0 &= c_{\alpha_1,\alpha_2,\beta_1,\beta_2} + c_{\alpha_1,\beta_1,\alpha_2,\beta_2} + c_{\beta_1,\alpha_1,\alpha_2,\beta_2} + c_{\alpha_1,\beta_1,\beta_2,\alpha_2} \\
&+ c_{\beta_1,\alpha_1,\beta_2,\alpha_2} + c_{\beta_1,\beta_2,\alpha_1,\alpha_2} + \binom{\alpha_1+\beta_1}{\alpha_1} \big(c_{\alpha_1+\beta_1,\alpha_2,\beta_2} \\
&+ c_{\alpha_1+\beta_1,\beta_2,\alpha_2} \big) + \binom{\alpha_1+\beta_2}{c}_{\beta_1,\alpha_1+\beta_2,\alpha_1} \\
&+ \binom{\alpha_2+\beta_1}{\alpha_2} c_{\alpha_1,\alpha_2+\beta_1,\beta_2} + \binom{\alpha_2+\beta_2}{\alpha_2} \big(c_{\alpha_1,\beta_1,\alpha_2+\beta_2} \\
&+ c_{\beta_1,\alpha_1,\alpha_2+\beta_2} \big) + \binom{\alpha_1+\beta_1}{\alpha_1} \binom{\alpha_2+\beta_2}{\alpha_2} c_{\alpha_1+\beta_1,\alpha_2+\beta_2} \\
0 &= c_{\alpha_1,\beta_1,\beta_2,\beta_3,\beta_4} + c_{\beta_1,\alpha_1,\beta_2,\beta_3,\beta_4} + c_{\beta_1,\beta_2,\alpha_1,\beta_3,\beta_4} \\
&\qquad + c_{\beta_1,\beta_2,\beta_3,\alpha_1,\beta_4} + c_{\beta_1,\beta_2,\beta_3,\beta_4,\alpha_1}
\end{aligned}
$$

$$
\begin{aligned}
&+ \binom{\alpha_1+\beta_1}{\alpha_1} c_{\alpha_1+\beta_1,\beta_2,\beta_3,\beta_4} + \binom{\alpha_1+\beta_2}{\alpha_1} c_{\beta_1,\alpha_1+\beta_2,\beta_3,\beta_4} \\
&+ \binom{\alpha_1+\beta_3}{\alpha_1} c_{\beta_1,\beta_2,\alpha_1+\beta_3,\beta_4} + \binom{\alpha_1+\beta_4}{\alpha_1} c_{\beta_1,\beta_2,\beta_3,\alpha_1+\beta_4} \\
&0 = c_{\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3} + c_{\alpha_1,\beta_1,\alpha_2,\beta_2,\beta_3} + c_{\beta_1,\alpha_1,\alpha_2,\beta_2,\beta_3} \\
&+ c_{\alpha_1,\beta_1,\beta_2,\alpha_2,\beta_3} + c_{\beta_1,\alpha_1,\beta_2,\alpha_2,\beta_3} + c_{\beta_1,\beta_2,\alpha_1,\alpha_2,\beta_3} \\
&+ c_{\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2} + c_{\beta_1,\alpha_1,\beta_2,\beta_3,\alpha_2} + c_{\beta_1,\beta_2,\alpha_1,\beta_3,\alpha_2} \\
&+ c_{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2} + \binom{\alpha_1+\beta_1}{\beta_1} \big(c_{\alpha_1+\beta_1,\alpha_2,\beta_2,\beta_3} \\
&+ c_{\alpha_1+\beta_1,\beta_2,\alpha_2,\beta_3} + c_{\alpha_1+\beta_1,\beta_2,\beta_3,\alpha_2} \big) \\
&+ \binom{\alpha_2+\beta_1}{\beta_1} c_{\alpha_1,\alpha_2+\beta_1,\beta_2,\beta_3} \\
&+ \binom{\alpha_2+\beta_2}{\beta_2} \big(c_{\beta_1,\alpha_1,\alpha_2+\beta_2,\beta_3} + c_{\alpha_1,\beta_1,\alpha_2+\beta_2,\beta_3} \big) \\
&+ \binom{\alpha_1+\beta_2}{\beta_2} \big(c_{\beta_1,\alpha_1+\beta_2,\alpha_2,\beta_3} + c_{\beta_1,\alpha_1+\beta_2,\beta_3,\alpha_2} \big) \\
&+ \binom{\alpha_2+\beta_3}{\beta_3} \big(c_{\alpha_1,\beta_1,\beta_2,\alpha_2+\beta_3} + c_{\beta_1,\alpha_1,\beta_2,\alpha_2+\beta_3} \\
&+ c_{\beta_1,\beta_2,\alpha_1,\alpha_2+\beta_3} \big) + \binom{\alpha_1+\beta_3}{\beta_3} c_{\beta_1,\beta_2,\alpha_1+\beta_3,\alpha_2} \\
&+ \binom{\alpha_1+\beta_1}{\alpha_1} \binom{\alpha_2+\beta_2}{\alpha_2} c_{\alpha_1+\beta_1,\alpha_2+\beta_2,\beta_3} \\
&+ \binom{\alpha_1+\beta_2}{\alpha_1} \binom{\alpha_2+\beta_3}{\alpha_2} c_{\beta_1,\alpha_1+\beta_2,\alpha_2+\beta_3} \\
&+ \binom{\alpha_1+\beta_1}{\alpha_1} \binom{\alpha_2+\beta_3}{\alpha_2} c_{\alpha_1+\beta_1,\beta_2,\alpha_2+\beta_3} \\
&\qquad\qquad \vdots
\end{aligned}
$$

Solving the first few equations and plugging the solution into the consecutive equations one can see that at least the first few equations are solved by

$$c_{\alpha_1,\ldots,\alpha_k} = \frac{(-1)^k}{k} \binom{\sum_{l=1}^{k} \alpha_l}{\alpha_1\ \alpha_2\ \cdots\ \alpha_k}, \tag{4.155}$$

where the last factor is the multinomial coefficient of the indices $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$.

4.3.2 Recursive equation for Coefficients of the second quantized scattering operator

We are going to use the following definition of binomial coefficients:

Definition 94. *For $a \in \mathbb{C}, b \in \mathbb{Z}$ we define*

$$\binom{a}{b} := \begin{cases} \prod_{l=0}^{b-1} \frac{a-l}{l+1} & \text{for } b \geqslant 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.156}$$

Defining the binomial coefficient for negative lower index to be zero has the merit, that one can extend the range of validity of many rules and sums involving binomial coefficients, also one does not have to worry about the range of summation in many cases.

The coefficients which we have already guessed result in the following

Conjecture 95. *For any $n \in \mathbb{N}$ the n-th expansion coefficient of the* *second quantized scattering operator T_n is given by*

$$T_n = \sum_{g=2}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l} + C_n \mathbb{1}_{\mathcal{F}}$$
$$+ d\Gamma\left(\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right), \tag{4.157}$$

for some $C_n \in \mathbb{C}$ which depends on the external field A. The last summand will henceforth be abbreviated by Γ_n.

Motivation: We compute the commutator of the difference between T_n and the first summand of (4.157) with the creation and annihilation operator of an element of the basis of $\mathcal{H}$. This will turn out to be exactly equal to the corresponding commutator of the second summand of (4.157), since two operators on Fock space only have the same commutator with general creation and annihilation operators if they agree up to multiples of the identity this will conclude the motivation of this conjecture.

In order to simplify the notation as much as possible, we will denote by $a^{\#}z$ either $a(z(\varphi_p))$ or $a^*(z(\varphi_p))$ for any one-particle operator z and any element φ_p of the orthonormal basis $(\varphi_p)_{p\in\mathbb{Z}\setminus\{0\}}$ of $\mathcal{H}$. (We need not decide between creation and annihilation operator, since the expressions all agree)

In order to organize the bookkeeping of all the summands which arise from iteratively making use of the commutation rule (4.152) we organize them by the looking at a spanning set of the possible terms that arise our choice is

$$\left\{ a^{\#} \prod_{k=1}^{m_1} Z_{\alpha_k} \prod_{k=1}^{m_2} T_{\beta_k} \middle| m_1 \in \mathbb{N}, m_2 \in \mathbb{N}_0, \alpha \in \mathbb{N}^{m_1}, \beta \in \mathbb{N}^{m_2}, |\alpha| + |\beta| = n \right\}. \tag{4.158}$$

As a first step of computing the commutator in question we look at the summand corresponding to a fixed value of the summation index g of

$$-\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l}. \tag{4.159}$$

We need to bring this object into the form of a sum of terms which are multiples of elements of the set (4.158). This we will commit ourselves to for the next few pages. First we apply the product rule for the commutator:

$$\begin{aligned}
&\left[\sum_{\substack{\vec{l}\in\mathbb{N}^g \\ |\vec{l}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{l}} \prod_{k=1}^{g} T_{l_k}, a^{\#} \right] \\
&= \sum_{\substack{\vec{l}\in\mathbb{N}^g \\ |\vec{l}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{l}} \sum_{\tilde{k}=1}^{g} \prod_{j=1}^{\tilde{k}-1} T_{l_j} \left[T_{l_{\tilde{k}}}, a^{\#} \right] \prod_{j=\tilde{k}+1}^{g} T_{l_j} \\
&= \sum_{\substack{\vec{l}\in\mathbb{N}^g \\ |\vec{l}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{l}} \sum_{\tilde{k}=1}^{g} \prod_{j=1}^{\tilde{k}-1} T_{l_j} \sum_{\sigma_{\tilde{k}}=1}^{l_{\tilde{k}}} \binom{l_{\tilde{k}}}{\sigma_{\tilde{k}}} a^{\#} Z_f T_{l_{\tilde{k}}-\sigma_{\tilde{k}}} \prod_{j=\tilde{k}+1}^{g} T_{l_j},
\end{aligned}$$

in the second step we used (4.152). Now we commute all the T_ls to the left of $a^{\#}$ to its right:

$$=\sum_{\substack{\vec{l}\in\mathbb{N}^g\\|l|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{l}}\sum_{\tilde{k}=1}^{g}\sum_{\substack{\forall 1\leqslant j<\tilde{k}\\0\leqslant\sigma_j\leqslant l_j}}\sum_{\sigma_{\tilde{k}}=1}^{l_{\tilde{k}}}\prod_{j=1}^{\tilde{k}}\binom{l_j}{\sigma_j}a^{\#}\prod_{j=1}^{\tilde{k}}Z_{\sigma_j}\prod_{j=1}^{\tilde{k}}T_{l_j-\sigma_j}\prod_{j=\tilde{k}+1}^{g}T_{l_j}. \tag{4.160}$$

At this point we notice that the multinomial coefficient can be combined with all the binomial coefficients to form a single multinomial coefficient of degree $g+\tilde{k}$. Incidentally this is also the amount of Z operators plus the amount of T operators in each product. Moreover, the indices of the multinomial index agree with the indices of the Z and T operators in the product. Because of this, we see that if we fix an element of the spanning set (4.158) $a^{\#}\prod_{k=1}^{m_1}Z_{\alpha_k}\prod_{k=1}^{m_2}T_{\beta_k}$, each summand of (4.160) which contributes to this element, has the prefactor

$$\frac{(-1)^g}{g}\binom{n}{\alpha_1\ \cdots\alpha_{m_1}\ \beta_1\cdots\beta_{m_2}} \tag{4.161}$$

no matter which summation index $l\in\mathbb{N}^g$ it corresponds to. In order to do the matching one may ignore the indices σ_j and $l_j-\sigma_j$ which vanish, because the corresponding operators Z_0 and T_0 are equal to the identity operator on $\mathcal{H}$ respectively Fock space.
Since we know that

$$\left[d\Gamma\left(\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}Z_{b_l}\right),a^{\#}\right]$$
$$=a^{\#}\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}Z_{b_l}$$

holds, all that is left to show is that

$$\left[-\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^g}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} T_{b_l}, a^{\#} \right] = a^{\#} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \tag{4.162}$$

also holds. For which we need to count the summands which are multiples of each element of (4.158) corresponding to each g in (4.159). So let us fix some element $K(m_1, m_2)$ of (4.158) corresponding to some $m_1 \in \mathbb{N}, m_2 \in \mathbb{N}_0, \alpha \in \mathbb{N}^{m_1}$ and $\beta \in \mathbb{N}^{m_2}$. Rephrasing this problem, we can ask which products

$$\prod_{l=1}^{g} T_{\gamma_l} \tag{4.163}$$

for suitable g and $(\gamma_l)_l$ produces, when commuted with a creation or annihilation operator, multiples of $K(m_1, m_2)$? We will call this number of total contributions weighted with the factor $-\frac{(-1)^g}{g}$ borrowed from (4.159) $\#K(m_1, m_2)$. Looking at the commutation relations (4.152) we split the set of indices $\{\gamma_1 \dots \gamma_g\}$ into three sets A, B and C, where the commutation relation has to be used in such a way, that

$$\begin{aligned} \forall k : \gamma_k \in A &\iff \exists j \leqslant m_1 : \gamma_k = \alpha_j, \\ \wedge \forall k : \gamma_k \in B &\iff \exists j \leqslant m_2 : \gamma_k = \beta_j \\ \wedge \forall k : \gamma_k \in C &\iff \exists j \leqslant m_1, l \leqslant m_2 : \gamma_k = \alpha_j + \beta_l \end{aligned}$$

holds. Unfortunately not every splitting corresponds to a contribution and not every order of multiplication of a legal splitting corresponds

to a contribution either. However, $\prod_j T_{\alpha_j} \prod_j T_{\beta_j}$ gives a contribution, and it is in fact the longest product that does. We may apply the commutation relations backwards to obtain any shorter valid combination and hence all combinations. Transforming the commutation rule for T_k read from right to left into a game results in the following rules.
Starting from the string

$$A_1 A_2 \dots A_{m_1} B_1 B_2 \dots B_{m_2}, \tag{4.164}$$

representing the longest product, where here and in the following A's represent operators T_k which will turn into Z_k by the commutation rule, B's represent operators T_k which will stay T_k after commutation and C's represent operators T_k which will produce both a Z_l in the creation/annihilation operator and a T_{k-l} behind that operator. The indices are merely there to keep track of which operator moved where. So the game consists in the answering how many strings can we produce by applying the following rules to the initial string?

1. You may replace any occurrence of $A_k B_j$ by $B_j A_k$ for any j and k.

2. You may replace any occurrence of $A_k B_j$ by $C_{k,j}$ for any j and k.

Where we have to count the number of times we applied the second rule, or equivalently the number $\#C$ of C's in the resulting string, because the summation index g in (4.159) corresponds to $m_1 + m_2 - \#C$.
Fix $\#C \in \{0, \dots, \min(m_1, m_2)\}$. A valid string has $m_1 + m_2 - \#C$ characters, because the number of its Cs is $\#C$, its number of As is $m_1 - \#C$ and its number of Bs is $m_2 - \#C$. Ignoring the labelling of the As, Bs and Cs there are $\binom{m_1+m_2-\#C}{\#C\ (m_1-\#C)\ (m_2-\#C)}$ such strings. Now if we consider one such string without labelling, e.g.

$$CAABACCBBACBBABBBB, \tag{4.165}$$

there is only one correct labelling to be restored, namely the one where each A and the first index of any C receive increasing labels from left to right and analogously for B and the second index of any C, resulting for our example in

$$C_{1,1}A_2A_3B_2A_4C_{5,3}C_{6,4}B_5B_6A_7C_{8,7}B_8B_9A_9B_{10}B_{11}B_{12}B_{13}. \tag{4.166}$$

So any unlabelled string corresponds to exactly one labelled string which in turn corresponds to exactly one choice of operator product $\prod T$. So returning to our Operators, we found the number $\#K(m_1, m_2)$ it is

$$\#K(m_1, m_2) = -\sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g} \binom{g}{(m_1+m_2-g)\ (g-m_1)\ (g-m_2)}, \tag{4.167}$$

where the total minus sign comes from the total minus sign in front of (4.162) with respect to (4.157).

Now since we introduced the slightly non-standard definition of binomial coefficients used in [41] we can make use of the rules for summing binomial coefficients derived there. As a first step to evaluate (4.167) we split the trinomial coefficient into binomial ones and make use of the absorption identity

$$\forall a \in \mathbb{C}\ \forall b \in \mathbb{Z} : b\binom{a}{b} = a\binom{a-1}{b-1} \tag{absorption}$$

for $m_2, m_1 \neq 0$ as follows

$$
\begin{aligned}
&\#K(m_1, m_2)\\
&= - \sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g} \binom{g}{(m_1+m_2-g)\ (g-m_1)\ (g-m_2)}\\
&= - \sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{g} \binom{g}{m_2} \binom{m_2}{g-m_1}\\
&\overset{\text{(absorption)}}{=} - \sum_{g=\max(m_1,m_2)}^{m_1+m_2} \frac{(-1)^g}{m_2} \binom{g-1}{m_2-1} \binom{m_2}{g-m_1}\\
&= \frac{-1}{m_2} \sum_{g=\max(m_1,m_2)}^{m_1+m_2} (-1)^g \binom{g-1}{m_2-1} \binom{m_2}{g-m_1}\\
&\overset{m_1>0}{=} \frac{-1}{m_2} \sum_{g\in\mathbb{Z}} (-1)^g \binom{m_2}{g-m_1} \binom{g-1}{m_2-1}\\
&\overset{*}{=} \frac{-1}{m_2} (-1)^{m_2-m_1} \binom{m_1-1}{-1} = 0,
\end{aligned}
$$

where for the second but last equality $m_1 > 0$ is needed for the $g = 0$ summand not to contribute and for the marked equality we used summation rule (5.24) of [41]. So all the coefficients vanish that fulfil $m_1, m_2 \neq 0$. The sum for the remaining cases is readily computed, since there is just one summand. Summarizing we find

$$
\#K(m_1, m_2) = \delta_{m_2,0} \frac{(-1)^{1+m_1}}{m_1} + \delta_{m_1,0} \frac{(-1)^{1+m_2}}{m_2},
$$

where the second summand can be ignored, since terms with $m_1 = 0$ are irrelevant for our considerations.
So the left-hand side of (4.162) can be evaluated

$$\left[-\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|\vec{b}|=n}}\frac{(-1)^g}{g}\binom{n}{\vec{b}}\prod_{l=1}^{g}T_{b_l},a^{\#}\right]$$
$$=\sum_{g=1}^{n}\sum_{\substack{\vec{b}\in\mathbb{N}^g\\|b|=n}}\frac{(-1)^{g+1}}{g}\binom{n}{\vec{b}}a^{\#}\prod_{l=1}^{g}Z_{b_l},$$

which is exactly equal to the right-hand side of (4.162). This ends the motivation of the conjecture.

4.3.3 Solution to Recursive Equation

So we found a recursive equation for the T_ns, now we need to solve it. In order to do so we need the following lemma about combinatorial distributions

Lemma 96. *For any $g \in \mathbb{N}, k \in \mathbb{N}$*

$$\sum_{\substack{\vec{g}\in\mathbb{N}^g\\|\vec{g}|=k}}\binom{k}{\vec{g}}=\sum_{l=0}^{g}(-1)^l(g-l)^k\binom{g}{l} \tag{4.168}$$

holds. The reader interested in terminology may be eager to know, that the right-hand side is equal to $g!$ times the Stirling number of the second kind $\left\{ k \atop g \right\}$.

Proof: We would like to apply the multinomial theorem, but there are all the summands missing where at least one of the entries of $\vec{g}$ is zero, so we add an appropriate expression of zero. We also give the expression in question a name, since we will later on arrive at a recursive expression.

$$F(g,k) := \sum_{\substack{\vec{g}\in\mathbb{N}^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} = \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} - \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k \\ \exists l:g_l=0}} \binom{k}{\vec{g}}$$
$$= g^k - \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k \\ \exists l:g_l=0}} \binom{k}{\vec{g}} = g^k - \sum_{n=1}^{g-1} \sum_{\substack{\vec{g}\in\mathbb{N}_0^g \\ |\vec{g}|=k}} \binom{k}{\vec{g}} 1_{\exists! i_1 \dots i_n:(\forall l\neq b: i_l\neq i_b)\wedge \forall l: g_{i_l}=0} \tag{4.169}$$

where in the last line the indicator function is to enforce there being exactly n different indices i_l for which $g_{i_l} = 0$ holds. Now since it does not matter which entries of the vector vanish because the multinomial coefficient is symmetric and its value is identical to the corresponding multinomial coefficient where the vanishing entries are omitted, we can further simplify the sum:

$$F(g,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} \sum_{\substack{\vec{g}\in\mathbb{N}^{g-n} \\ |\vec{g}|=k}} \binom{k}{\vec{g}}$$

The inner sum turns out to be $F(g-n,k)$, so we found the recursive relation for F:

$$F(g,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} F(g-n,k) = g^k - \sum_{n=1}^{g-1} \binom{g}{n} F(n,k), \tag{4.170}$$

where for the last equality we used the symmetry of binomial coefficients. By iteratively applying this equation, we find the following formula, which we will now prove by induction

$$\forall d \in \mathbb{N}_0 : F(g,k) = \sum_{l=0}^{d} (-1)^l (g-l)^k \binom{g}{l} + (-1)^{d+1} \sum_{n=1}^{g-d-1} \binom{n+d-1}{d} \binom{g}{n+d} F(g-d-n,k). \quad (4.171)$$

We already showed the start of the induction, so what's left is the induction step. Before we do so the following remark is in order: We are only interested in the case $d = g$ and the formula seems meaningless for $d > g$; however, the additional summands in the left sum vanish, whereas the right sum is empty for these values of d since the upper bound of the summation index is lower than its lower bound.
For the induction step, pick $d \in \mathbb{N}_0$, use (4.171) and pull the first summand out of the second sum, on this summand we apply the recursive relation (4.170) resulting in

$$\begin{aligned} F(g,k) &= \sum_{l=0}^{d} (-1)^l (g-l)^k \binom{g}{l} \\ &+ (-1)^{d+1} \sum_{n=2}^{g-d-1} \binom{n+d-1}{d} \binom{g}{n+d} F(g-d-n,k) \\ &+ (-1)^{d+1} \binom{d}{d} \binom{g}{d+1} F(g-d-1,k) \\ &\overset{(4.170)}{=} \sum_{l=0}^{d+1} (-1)^l (g-l)^k \binom{g}{l} \\ &+ (-1)^{d+1} \sum_{n=2}^{g-d-1} \binom{n+d-1}{d} \binom{g}{n+d} F(g-d-n,k) \end{aligned}$$

$$- (-1)^{d+1}\binom{g}{d+1}\sum_{n=1}^{g-d-2}\binom{g-d-1}{n}F(g-d-1-n,k)$$

$$= \sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l}$$

$$+ (-1)^{d+1}\sum_{n=1}^{g-d-2}\binom{n+d}{d}\binom{g}{n+d+1}F(g-d-1-n,k)$$

$$- (-1)^{d+1}\binom{g}{d+1}\sum_{n=1}^{g-d-2}\binom{g-d-1}{n}F(g-d-1-n,k). \quad (4.172)$$

After the index shift we can combine the last two sums.

$$F(g,k) = \sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l}$$

$$+ \sum_{n=1}^{g-d-2}\left[\binom{g}{d+1}\binom{g-d-1}{n} - \binom{n+d}{d}\binom{g}{n+d+1}\right]$$

$$(-1)^{d+2}F(g-d-1-n,k). \quad (4.173)$$

In order to combine the two binomials we reassemble $\binom{g}{d+1}\binom{g-d-1}{n}$ into $\binom{g}{n+d+1}\binom{n+d+1}{d+1}$, which can be seen to be possible by representing everything in terms of factorials. This results in

$$F(g,k) = \sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l}$$

$$+(-1)^{d+2}\sum_{n=1}^{g-d-2}\left[\binom{n+d+1}{d+1} - \binom{n+d}{d}\right]\binom{g}{n+d+1}F(g-d-1-n,k)$$

$$= \sum_{l=0}^{d+1}(-1)^l(g-l)^k\binom{g}{l}$$

$$+(-1)^{d+2}\sum_{n=1}^{g-d-2}\binom{n+d}{d+1}\binom{g}{n+d+1}F(g-d-1-n,k), \quad (4.174)$$

where we used the addition formula for binomials:

$$\forall n\in\mathbb{C}\forall k\in\mathbb{Z}:\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}. \quad (4.175)$$

This concludes the proof by induction. By setting $d=g$ in equation (4.171) we arrive at the desired result. □
Using the previous lemma, we are able to show the next

Lemma 97. *For any $k\in\mathbb{N}\backslash\{1\}$ the following equation holds*

$$\sum_{g=1}^{k}\frac{(-1)^g}{g}\sum_{\substack{\vec{g}\in\mathbb{N}^g\\|\vec{g}|=k}}\binom{k}{\vec{g}}=0. \quad (4.176)$$

Proof: Let $k\in\mathbb{N}\backslash\{1\}$, as a first step we apply lemma 96. We change the order of summation, use (absorption), extend the range of summation and shift summation index to arrive at

$$\begin{aligned}\sum_{g=1}^{k}\frac{(-1)^g}{g}\sum_{l=0}^{g}(-1)^l(g-l)^k\binom{g}{l}&=\sum_{g=1}^{k}\frac{1}{g}\sum_{l=0}^{g}(-1)^{g-l}(g-l)^k\binom{g}{g-l}\\&=\sum_{g=1}^{k}\sum_{p=0}^{g}(-1)^p p^k\frac{1}{g}\binom{g}{p}=\sum_{g=1}^{k}\sum_{p=0}^{g}(-1)^p p^k\frac{1}{p}\binom{g-1}{p-1}\\&=\sum_{g=1}^{k}\sum_{p\in\mathbb{Z}}(-1)^p p^{k-1}\binom{g-1}{p-1}=\sum_{p\in\mathbb{Z}}(-1)^p p^{k-1}\sum_{g=1}^{k}\binom{g-1}{p-1}\\&=\sum_{p\in\mathbb{Z}}(-1)^p p^{k-1}\sum_{g=0}^{k-1}\binom{g}{p-1}. \quad (4.177)\end{aligned}$$

Now we use equation (5.10) of [41]:

$$\forall m, n \in \mathbb{N}_0 : \sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}, \qquad \text{(upper summation)}$$

which can for example be proven by induction on n.
We furthermore rewrite the power of the summation index p in terms of the derivative of an exponential and change order summation and differentiation. This results in

$$\begin{aligned}
\sum_{g=1}^{k} \frac{(-1)^g}{g} \sum_{l=0}^{g} (-1)^l (g-l)^k \binom{g}{l} &= \sum_{p\in\mathbb{Z}} (-1)^p p^{k-1} \binom{k}{p} \\
&= \sum_{p=0}^{k} (-1)^p \left. \frac{\partial^{k-1}}{\partial \alpha^{k-1}} e^{\alpha p} \right|_{\alpha=0} \binom{k}{p} = \left. \frac{\partial^{k-1}}{\partial \alpha^{k-1}} \sum_{p=0}^{k} (-1)^p e^{\alpha p} \binom{k}{p} \right|_{\alpha=0} \\
&= \left. \frac{\partial^{k-1}}{\partial \alpha^{k-1}} (1 - e^{\alpha p})^k \right|_{\alpha=0} = (-1)^k \left. \frac{\partial^{k-1}}{\partial \alpha^{k-1}} \left(\sum_{l=1}^{\infty} \frac{(\alpha p)^l}{l!} \right)^k \right|_{\alpha=0} \\
&= (-1)^k \left. \frac{\partial^{k-1}}{\partial \alpha^{k-1}} \left((\alpha p)^k + \mathcal{O}((\alpha p)^{k+1}) \right) \right|_{\alpha=0} = 0.
\end{aligned}$$

□

We are now in a position to state the solution to the recursive equation (4.157) and motivate that it is in fact a solution.

Conjecture 98. *For $n \in \mathbb{N}$ the solution of the recursive equation (4.157) solely in terms of Γ_a and C_a is given by*

$$T_n = \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \sum_{\vec{d}\in\{0,1\}^g} \frac{1}{g!} \binom{n}{\vec{b}} \prod_{l=1}^{g} F_{b_l,d_l}, \tag{4.178}$$

where F is given by

$$F_{a,b} = \begin{cases} \Gamma_a & \text{for } b = 0 \\ C_a & \text{for } b = 1 \end{cases}. \tag{4.179}$$

For the reader's convenience we remind her, that Γ_a and the constants C_n are defined in conjecture 95.

Motivation: The structure of this proof will be induction over n. For $n = 1$ the whole expression on the right-hand side collapses to $C_1 + \Gamma_1$, which we already know to be equal to T_1. For arbitrary $n \in \mathbb{N}$ we apply for the induction step the recursive equation (4.157) once and use the induction hypothesis for all $k \leqslant n$ and thereby arrive at the rather convoluted expression

$$\begin{aligned} T_{n+1} &\overset{(4.157)}{=} \Gamma_{n+1} + C_{n+1} + \sum_{g=2}^{n+1} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n+1}} \frac{(-1)^g}{g} \binom{n+1}{\vec{b}} \prod_{l=1}^{g} T_{b_l} \\ &\overset{\text{induction hyp}}{=} \Gamma_{n+1} + C_{n+1} + \sum_{g=2}^{n+1} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n+1}} \frac{(-1)^g}{g} \binom{n+1}{\vec{b}} \prod_{l=1}^{g} \\ &\qquad \sum_{g_l=1}^{b_l} \sum_{\substack{\vec{c}_l\in\mathbb{N}^{g_l} \\ |\vec{c}_l|=b_l}} \sum_{\vec{e}_l\in\{0,1\}^{g_l}} \frac{1}{g_l!} \binom{b_l}{\vec{c}_l} \prod_{k=1}^{g_l} F_{c_{l,k},e_{l,k}}. \end{aligned} \tag{4.180}$$

If we were to count the contributions of this sum to a specific product $\prod F_{c_j,e_j}$ for some choice of $(c_j)_j, (e_j)_j$ we would first recognize that all the multinomial factors in (4.180) combine to a single one whose indices are given by the first indices of all the F factors involved. Other than this factor each contribution adds $\frac{(-1)^g}{g} \prod_{l=1}^{g} \frac{1}{g_l!}$ to the sum. So

we need to keep track of how many contributions there are and which distributions of g_l they belong to.
Fix some product $\prod F := \prod_{j=1}^{\tilde{g}} F_{\tilde{b}_j,\tilde{d}_j}$. In the sum (4.180) we pick some initial short product of length g and split each factor into g_l pieces to arrive at one of length $\tilde{g}$ if the product is to contribute to $\prod F$. So clearly $\sum_{l=1}^{g} g_l = \tilde{g}$ holds for any contribution to $\prod F$. The reverse is also true, for any g and $g_1,\ldots,g_g \in \mathbb{N}$ such that $\sum_{l=1}^{g} g_l = \tilde{g}$ holds the corresponding expression in (4.180) contributes to $\prod F$. Furthermore, $\prod F$ and g, $g_1,\ldots g_g$ is enough to uniquely determine the summand of (4.180) the contribution belongs to. For an illustration of this splitting see

$$\underbrace{\underbrace{F^1_{3,1}F^2_{2,0}F^3_{7,1}}_{g_1=3}\;\underbrace{F^4_{5,0}}_{g_2=1}\;\underbrace{F^5_{4,1}F^6_{2,1}}_{w_3=2}\;\underbrace{F^7_{1,1}F^8_{3,0}F^9_{4,1}}_{g_4=3}\;\underbrace{F^{10}_{4,1}F^{11}_{1,0}}_{g_5=2}}_{g=5}$$

$$g_1 + g_2 + g_3 + g_4 + g_5 = 11 = \tilde{g}.$$

We recognize that the sum we are about to perform is by no means unique for each order of n but only depends on the number of appearing factors and the number of splittings performed on them. By the preceding argument we need

$$\sum_{g=2}^{\tilde{g}} \frac{(-1)^g}{g} \sum_{\substack{\vec{g}\in\mathbb{N}^g \\ |\vec{g}|=\tilde{g}}} \prod_{l=1}^{g} \frac{1}{g_l!} = \frac{1}{\tilde{g}!} \tag{4.181}$$

to hold for $\tilde{g} > 1$, in order to find agreement with the proposed solution (4.179). Now proving (4.181) is done by realizing, that one can include the right-hand side into the sum as the $g = 1$ summand, dividing the equation by $\tilde{g}!$ and using lemma 97 with $k = \tilde{g}$. The remaining case, $\tilde{g} = 1$, can directly be read off of (4.180). This ends the motivation of this conjecture.

Conjecture 99. *For $n \in \mathbb{N}$, T_n can be written as*

$$\frac{1}{n!}T_n = \sum_{\substack{1 \leqslant c+g \leqslant n \\ c,g \in \mathbb{N}_0}} \sum_{\substack{\vec{g} \in \mathbb{N}^g \\ \vec{c} \in \mathbb{N}^c \\ |\vec{c}|+|\vec{g}|=n}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l}. \tag{4.182}$$

Please note that for ease of notation we defined $\mathbb{N}^0 := \{1\}$.

Motivation: By an argument completely analogous to the combinatorial argument in the motivation of conjecture (95) we see that we can disentangle the Fs in (4.178) into Γs and Cs if we multiply by a factor of $\binom{c+g}{c}$ where c is the number of Cs and g is the number of Γs giving

$$T_n = \sum_{\substack{1 \leqslant c+g \leqslant n \\ c,g \in \mathbb{N}_0}} \sum_{\substack{\vec{g} \in \mathbb{N}^g \\ \vec{c} \in \mathbb{N}^c \\ |\vec{c}|+|\vec{g}|=n}} \binom{c+g}{c} \frac{1}{(c+g)!} \binom{n}{\vec{g} \oplus \vec{c}} \prod_{l=1}^{c} C_{c_l} \prod_{l=1}^{g} \Gamma_{g_l}, \tag{4.183}$$

which directly reduces to the equation we wanted to prove, by plugging in the multinomials in terms of factorials.

Conjecture 100. *As a formal power series, the second quantized scattering operator can be written in the form*

$$S = e^{\sum_{l \in \mathbb{N}} \frac{C_l}{l!}} e^{\sum_{l \in \mathbb{N}} \frac{\Gamma_l}{l!}}. \tag{4.184}$$

Motivation: We plug conjecture 99 into the defining Series for the T_ns giving

$$
\begin{aligned}
S &= \sum_{n\in\mathbb{N}_0} \frac{1}{n!} T_n && (4.185)\\
&= \mathbb{1}_{\mathcal{F}} + \sum_{n\in\mathbb{N}} \sum_{\substack{1\leqslant c+g\leqslant n\\ c,g\in\mathbb{N}_0}} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c\\ |\vec{c}|+|\vec{g}|=n}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} && (4.186)\\
&= \mathbb{1}_{\mathcal{F}} + \sum_{\substack{1\leqslant c+g\\ c,g\in\mathbb{N}_0}} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} && (4.187)\\
&= \sum_{c,g\in\mathbb{N}_0} \sum_{\substack{\vec{g}\in\mathbb{N}^g\\ \vec{c}\in\mathbb{N}^c}} \frac{1}{c!g!} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} && (4.188)\\
&= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \sum_{\vec{c}\in\mathbb{N}^c} \prod_{l=1}^{c} \frac{1}{c_l!} C_{c_l} \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \sum_{\vec{g}\in\mathbb{N}^g} \prod_{l=1}^{g} \frac{1}{g_l!} \Gamma_{g_l} && (4.189)\\
&= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \prod_{l=1}^{c} \sum_{k\in\mathbb{N}} \frac{1}{k!} C_k \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \prod_{l=1}^{g} \sum_{b\in\mathbb{N}} \frac{1}{b!} \Gamma_b && (4.190)\\
&= \sum_{c\in\mathbb{N}_0} \frac{1}{c!} \left(\sum_{k\in\mathbb{N}} \frac{1}{k!} C_k \right)^c \sum_{g\in\mathbb{N}_0} \frac{1}{g!} \left(\sum_{b\in\mathbb{N}} \frac{1}{b!} \Gamma_b \right)^g && (4.191)\\
&= e^{\sum_{l\in\mathbb{N}} \frac{1}{l!} C_l} e^{\sum_{l\in\mathbb{N}} \frac{1}{l!} \Gamma_l}. && (4.192)
\end{aligned}
$$

Conjecture 101. *For A such that*

$$\|\mathbb{1} - U^A\| < 1. \tag{4.193}$$

The second quantized scattering operator fulfils

$$S = e^{\sum_{n\in\mathbb{N}} \frac{C_n}{n!}} e^{\mathrm{d}\Gamma(\ln(U))} \tag{4.194}$$

where C_n must be imaginary for any $n \in \mathbb{N}$ in order to satisfy unitarity.

Motivation: First the remark about $C_n \in i\mathbb{R}$ for any n is a direct consequence of the second factor of (4.194) begin unitary. This in turn follows directly from $\mathrm{d}\Gamma^*(K) = -\mathrm{d}\Gamma(K)$ for any K in the domain of $\mathrm{d}\Gamma$. That $\ln U$ is in the domain of $\mathrm{d}\Gamma$ follows from $(\ln U)^* = \ln U^* = \ln U^{-1} = -\ln U$ and $\|U - \mathbb{1}\| < 1$.
We are going to change the sum in the second exponential of (4.184), so let's take a closer look at that: by exchanging summation we can step by step simplify

$$
\begin{aligned}
\sum_{l\in\mathbb{N}} \frac{\Gamma_l}{l!} &= \sum_{n\in\mathbb{N}} \frac{1}{n!} \mathrm{d}\Gamma \left(\sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \binom{n}{\vec{b}} \prod_{l=1}^{g} Z_{b_l} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{n\in\mathbb{N}} \sum_{g=1}^{n} \sum_{\substack{\vec{b}\in\mathbb{N}^g \\ |\vec{b}|=n}} \frac{(-1)^{g+1}}{g} \prod_{l=1}^{g} \frac{Z_{b_l}}{b_l!} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{g\in\mathbb{N}} \sum_{\vec{b}\in\mathbb{N}^g} \frac{(-1)^{g+1}}{g} \prod_{l=1}^{g} \frac{Z_{b_l}}{b_l!} \right) \\
&= \mathrm{d}\Gamma \left(\sum_{g\in\mathbb{N}} \frac{(-1)^{g+1}}{g} \prod_{l=1}^{g} \left(\sum_{b_l\in\mathbb{N}} \frac{Z_{b_l}}{b_l!} \right) \right) \\
&= \mathrm{d}\Gamma \left(\sum_{g\in\mathbb{N}} \frac{(-1)^{g+1}}{g} \left(\sum_{b\in\mathbb{N}} \frac{Z_b}{b!} \right)^{g} \right)
\end{aligned}
$$

$$= \mathrm{d}\Gamma\left(\sum_{g\in\mathbb{N}}\frac{(-1)^{g+1}}{g}(U-\mathbb{1})^g\right) = \mathrm{d}\Gamma\left(-\sum_{g\in\mathbb{N}}\frac{1}{g}(\mathbb{1}-U)^g\right)$$

$$= \mathrm{d}\Gamma\left(\ln\left(\mathbb{1}-(\mathbb{1}-U)\right)\right) = \mathrm{d}\Gamma\left(\ln(U)\right). \quad (4.195)$$

The last conjecture is proven directly in subsection 3.3.2.

Bibliography

[1] Milton Abramowitz and Irene A Stegun.
Handbook of mathematical functions: with formulas, graphs, and mathematical tables, volume 55.
Courier Corporation, 1965.

[2] Luis Alvarez-Gaume and Miguel A Vazquez-Mozo.
Introductory lectures on quantum field theory.
arXiv preprint hep-th/0510040, 2005.

[3] Christian Bär, Nicolas Ginoux, and Frank Pfäffle.
Wave equations on Lorentzian manifolds and quantization, volume 3.
European Mathematical Society, 2007.

[4] F Bloch.
Die physikalische bedeutung mehrerer zeiten in der quantenelektrodynamik. phys. z. d. sowjetunion, 5: 301–315, 1934.

[5] Nikolai Nikolaevich Bogolyubov and Dmitry Shirkov.
Introduction to the theory of quantized fields.
Introduction to the theory of quantized fields, 1959.

[6] G Cosenza, L Sertorio, and M Toller.

Singular integral equations in the bound-state problem.
Il Nuovo Cimento (1955-1965), 35(3):913–932, 1965.

[7] Ovidiu Costin.
Asymptotics and Borel summability.
CRC press, 2008.

[8] DG Currie, TF Jordan, and ECG Sudarshan.
Relativistic invariance and hamiltonian theories of interacting particles.
Reviews of Modern Physics, 35(2):350, 1963.

[9] RE Cutkosky.
Solutions of a bethe-salpeter equation.
Physical Review, 96(4):1135, 1954.

[10] Eduardus Maria de Jager.
The Lorentz-invariant solutions of the Klein-Gordon equation.
Stichting Mathematisch Centrum, Afd. Toegepaste Wiskunde, 1962.

[11] EM de Jager.
The lorentz-invariant solutions of the klein–gordon equation. ii.
In *Indagationes Mathematicae (Proceedings)*, volume 66, pages 532–545. Elsevier, 1963.

[12] EM de Jager.
The lorentz-invariant solutions of the klein–gordon equation. iii.
In *Indagationes Mathematicae (Proceedings)*, volume 66, pages 546–558. North-Holland, 1963.

[13] D-A Deckert, D Dürr, F Merkl, and M Schottenloher.
Time-evolution of the external field problem in quantum electrodynamics.
Journal of Mathematical Physics, 51(12):122301, 2010.

[14] D-A Deckert and F Merkl.

External field qed on cauchy surfaces for varying electromagnetic fields.
Communications in Mathematical Physics, 345(3):973–1017, 2016.

[15] D-A Deckert and Franz Merkl.
Dirac equation with external potential and initial data on cauchy surfaces.
Journal of Mathematical Physics, 55(12):122305, 2014.

[16] Dirk-André Deckert and Franz Merkl.
A perspective on external field qed.
In *Quantum Mathematical Physics*, pages 381–399. Springer, 2016.

[17] Dirk-André Deckert and Lukas Nickel.
Consistency of multi-time dirac equations with general interaction potentials.
Journal of Mathematical Physics, 57(7):072301, 2016.

[18] Dirk-André Deckert and Lukas Nickel.
Multi-time dynamics of the dirac-fock-podolsky model of qed.
Journal of Mathematical Physics, 60(7):072301, 2019.

[19] Dirk-André Deckert and Martin Oelker.
Distinguished self-adjoint extension of the two-body dirac operator with coulomb interaction.
In *Annales Henri Poincaré*, volume 20, pages 2407–2445. Springer, 2019.

[20] Jan Dereziński and Christian Gérard.
Mathematics of quantization and quantum fields.
Cambridge University Press, 2013.

[21] J Dimock.
Dirac quantum fields on a manifold.

Transactions of the American mathematical Society, 269(1):133–147, 1982.

[22] PAM Dirac.
Théorie du positron.
Solvay report, 25:203–212, 1934.

[23] PAM Dirac, VA Fock, and B Podolsky.
Selected papers on quantum electrodynamics.
Ed.: Schwinger J., Dover Publi. Inc. NY, pages 29–40, 1958.

[24] Paul Adrien Maurice Dirac.
A theory of electrons and protons.
Proceedings of the Royal Society of London. Series A, Containing papers of a mathematical and physical character, 126(801):360–365, 1930.

[25] Paul Adrien Maurice Dirac.
Relativistic quantum mechanics.
Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, 136(829):453–464, 1932.

[26] Paul AM Dirac.
The quantum theory of the electron.
In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 117, pages 610–624. The Royal Society, 1928.

[27] Paul AM Dirac.
Discussion of the infinite distribution of electrons in the theory of the positron.
In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 30, pages 150–163. Cambridge University Press, 1934.

[28] NIST Digital Library of Mathematical Functions.

http://dlmf.nist.gov/, Release 1.1.1 of 2021-03-15.
F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[29] Ph Droz-Vincent.
Relativistic wave equations for a system of two particles with spin 1/2.
Lettere al Nuovo Cimento, 30(12):375–378, 1981.

[30] Yu A Dubinskii.
Sobolev spaces of infinite order.
Russian Mathematical Surveys, 46(6):107, 1991.

[31] Arthur Stanley Eddington.
The charge of an electron.
Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, 122(789):358–369, 1929.

[32] Henri Epstein and Vladimir Glaser.
The role of locality in perturbation theory.
In *Annales de l'IHP Physique théorique*, volume 19, pages 211–295, 1973.

[33] Lawrence C Evans.
Partial differential equations.
Graduate studies in mathematics, 19(2), 1998.

[34] Marián Fecko.
Differential geometry and Lie groups for physicists.
Cambridge University Press, 2006.

[35] Richard P Feynman.
The theory of positrons.
Physical Review, 76(6):749, 1949.

[36] Richard Phillips Feynman.
Space-time approach to quantum electrodynamics.
Physical Review, 76(6):769, 1949.

[37] Felix Finster, Simone Murro, and Christian Röken.
The fermionic projector in a time-dependent external potential: Mass oscillation property and hadamard states.
Journal of Mathematical Physics, 57(7):072303, 2016.

[38] Stephen A Fulling, Francis J Narcowich, and Robert M Wald.
Singularity structure of the two-point function in quantum field theory in curved spacetime, ii.
Annals of Physics, 136(2):243–272, 1981.

[39] Stephen A Fulling, Mark Sweeny, and Robert M Wald.
Singularity structure of the two-point function in quantum field theory in curved spacetime.
Communications in Mathematical Physics, 63(3):257–264, 1978.

[40] James Glimm and Arthur Jaffe.
Quantum physics: a functional integral point of view.
Springer Science & Business Media, 2012.

[41] Ronald L Graham, Donald E Knuth, and Oren Patashnik.
Concrete mathematics: a foundation for computer science.
Addison-Wesley, Reading, 1994.

[42] Philippe Gravejat, Christian Hainzl, Mathieu Lewin, and Eric Séré.
Construction of the pauli–villars-regulated dirac vacuum in electromagnetic fields.
Archive for Rational Mechanics and Analysis, 208(2):603–665, 2013.

[43] Philippe Gravejat, Mathieu Lewin, and Éric Séré.
Renormalization and asymptotic expansion of dirac's polarized vacuum.

Communications in mathematical physics, 306(1):1–33, 2011.

[44] Philippe Gravejat, Mathieu Lewin, and Éric Séré.
Derivation of the magnetic euler–heisenberg energy.
Journal de Mathématiques Pures et Appliquées, 117:59–93, 2018.

[45] HS Green.
Separability of a covariant wave equation.
Il Nuovo Cimento (1955-1965), 5(4):866–871, 1957.

[46] Walter Greiner et al.
Relativistic quantum mechanics, volume 2.
Springer, 2000.

[47] Marian Günther.
The relativistic configuration space formulation of the multi-electron problem.
Physical Review, 88(6):1411, 1952.

[48] Christian Hainzl, Mathieu Lewin, and Éric Séré.
Existence of a stable polarized vacuum in the bogoliubov-dirac-fock approximation.
Communications in mathematical physics, 257(3):515–562, 2005.

[49] Christian Hainzl, Mathieu Lewin, and Eric Séré.
Self-consistent solution for the polarized vacuum in a no-photon qed model.
Journal of Physics A: Mathematical and General, 38(20):4483, 2005.

[50] Christian Hainzl, Mathieu Lewin, and Éric Séré.
Existence of atoms and molecules in the mean-field approximation of no-photon quantum electrodynamics.
Archive for rational mechanics and analysis, 192(3):453–499, 2009.

[51] Christian Hainzl, Mathieu Lewin, Eric Sere, and Jan Philip Solovej.
Minimization method for relativistic electrons in a mean-field approximation of quantum electrodynamics.
Physical Review A, 76(5):052104, 2007.

[52] Christian Hainzl, Mathieu Lewin, and Jan Philip Solovej.
The mean-field approximation in quantum electrodynamics: The no-photon case.
Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 60(4):546–596, 2007.

[53] Christian Hainzl, Mathieu Lewinand, and Christof Sparber.
Existence of global-in-time solutions to a generalized dirac-fock type evolution equation.
Letters in Mathematical Physics, 72(2):99–113, 2005.

[54] Abdolhossein Hoorfar and Mehdi Hassani.
Inequalities on the lambert w function and hyperpower function.
J. Inequal. Pure and Appl. Math, 9(2):5–9, 2008.

[55] Lars Hörmander.
The analysis of linear partial differential operators. i. distribution theory and fourier analysis. reprint of the second (1990) edition, 2003.

[56] Michael Ibison.
On the conformal forms of the robertson-walker metric.
Journal of Mathematical Physics, 48(12):122501, 2007.

[57] Arthur Jaffe.
Constructive quantum field theory.
Mathematical physics, pages 111–127, 2000.

[58] RW John.

The hadamard construction of green's functions on a curved space-time with symmetries.
Annalen der Physik, 499(7):531–544, 1987.

[59] Edwin Langmann and Jouko Mickelsson.
Scattering matrix in external field problems.
Journal of Mathematical Physics, 37(8):3933–3953, 1996.

[60] H Leutwyler.
A no-interaction theorem in classical relativistic hamiltonian particle mechanics.
Il Nuovo Cimento (1955-1965), 37(2):556–567, 1965.

[61] M Lienert and L Nickel.
Multi-time formulation of creation and annihilation of particles via interior-boundary conditions.
Preprint: https://arxiv.org/abs/1808, 4192, 2018.

[62] Matthias Lienert.
On the question of current conservation for the two-body dirac equations of constraint theory.
Journal of Physics A: Mathematical and Theoretical, 48(32):325302, 2015.

[63] Matthias Lienert.
A relativistically interacting exactly solvable multi-time model for two massless dirac particles in 1+ 1 dimensions.
Journal of Mathematical Physics, 56(4):042301, 2015.

[64] Matthias Lienert.
Direct interaction along light cones at the quantum level.
Journal of Physics A: Mathematical and Theoretical, 51(43):435302, 2018.

[65] Matthias Lienert and Lukas Nickel.
A simple explicitly solvable interacting relativistic n-particle model.

Journal of Physics A: Mathematical and Theoretical, 48(32):325301, 2015.

[66] Matthias Lienert and Markus Nöth.
Singular light cone interactions of scalar particles in 1+3 dimensions.
2020.

[67] Matthias Lienert and Markus Nöth.
Existence of relativistic dynamics for two directly interacting dirac particles in 1+ 3 dimensions.
Reviews in Mathematical Physics, 2021.

[68] Matthias Lienert, Sören Petrat, and Roderich Tumulka.
Multi-time wave functions.
In *Journal of Physics: Conference Series*, volume 880, page 012006. IOP Publishing, 2017.

[69] Matthias Lienert, Sören Petrat, and Roderich Tumulka.
Multi-time wave functions versus multiple timelike dimensions.
Foundations of Physics, 47(12):1582–1590, 2017.

[70] Matthias Lienert and Roderich Tumulka.
Interacting relativistic quantum dynamics of two particles on spacetimes with a big bang singularity.
Journal of Mathematical Physics, 60(4), 2019.

[71] Matthias Lienert and Roderich Tumulka.
Born's rule for arbitrary cauchy surfaces.
Letters in Mathematical Physics, 110(4):753–804, 2020.

[72] Matthias Lienert, Roderich Tumulka, et al.
A new class of volterra-type integral equations from relativistic quantum physics.
Journal of Integral Equations and Applications, 31(4):535–569, 2019.

[73] Sascha Lill, Lukas Nickel, and Roderich Tumulka.
Consistency proof for multi-time schrodinger equations with particle creation and ultraviolet cut-off.
arXiv preprint arXiv:2001.05920, 2020.

[74] Egon Marx.
Many-times formalism and coulomb interaction.
International Journal of Theoretical Physics, 9(3):195–217, 1974.

[75] Jouko Mickelsson.
The phase of the scattering operator from the geometry of certain infinite-dimensional groups.
Letters in Mathematical Physics, 104(10):1189–1199, 2014.

[76] Noboru Nakanishi.
A general survey of the theory of the bethe-salpeter equation.
Progress of Theoretical Physics Supplement, 43:1–81, 1969.

[77] L. Nickel.
Phd thesis. on the dynamics of multi-time systems, 2019.

[78] DM O'Brien.
The wick rotation.
Australian Journal of Physics, 28(1):7–14, 1975.

[79] Blaise Pascal.
Lettres écrites à un provincial, seizième lettre.
Garnier, 1656.

[80] Roger Penrose and Wolfgang Rindler.
Spinors and space-time: Volume 1, Two-spinor calculus and relativistic fields, volume 1.
Cambridge University Press, 1984.

[81] Sören Petrat and Roderich Tumulka.
Multi-time formulation of pair creation.

Journal of Physics A: Mathematical and Theoretical, 47(11):112001, 2014.

[82] Sören Petrat and Roderich Tumulka.
Multi-time schrödinger equations cannot contain interaction potentials.
Journal of Mathematical Physics, 55(3):032302, 2014.

[83] Sören Petrat and Roderich Tumulka.
Multi-time wave functions for quantum field theory.
Annals of Physics, 345:17–54, 2014.

[84] Marek J Radzikowski.
Micro-local approach to the hadamard condition in quantum field theory on curved space-time.
Communications in mathematical physics, 179(3):529–553, 1996.

[85] Michael Reed and Barry Simon.
Methods of modern mathematical physics, vol. ii, 1975.

[86] SNM Ruijsenaars.
Charged particles in external fields. i. classical theory.
Journal of Mathematical Physics, 18(4):720–737, 1977.

[87] SNM Ruijsenaars.
Charged particles in external fields. ii. the quantized dirac and klein-gordon theories.
Communications in Mathematical Physics, 52(3):267–294, 1977.

[88] Edwin E Salpeter and Hans Albrecht Bethe.
A relativistic equation for bound-state problems.
Physical Review, 84(6):1232, 1951.

[89] Hagop Sazdjian.
Relativistic wave equations for the dynamics of two interacting particles.
Physical Review D, 33(11):3401, 1986.

[90] Gunter Scharf.
Finite quantum electrodynamics: the causal approach.
Courier Corporation, 2014.

[91] Jan Schlemmer and Jochen Zahn.
The current density in quantum electrodynamics in external potentials.
Annals of Physics, 359:31–45, 2015.

[92] Matthew D Schwartz.
Quantum field theory and the standard model.
Cambridge University Press, 2014.

[93] Silvan S Schweber.
An introduction to relativistic quantum field theory.
Courier Corporation, 2011.

[94] Julian Schwinger.
Quantum electrodynamics. i. a covariant formulation.
Physical Review, 74(10):1439, 1948.

[95] Julian Schwinger.
On gauge invariance and vacuum polarization.
Physical Review, 82(5):664, 1951.

[96] GB Segal.
Loop groups.
Springer, 1985.

[97] David Shale and W Forrest Stinespring.
Spinor representations of infinite orthogonal groups.
Journal of Mathematics and Mechanics, 14(2):315–322, 1965.

[98] Alexander J Silenko.
New symmetry properties of pointlike scalar and dirac particles.
Physical Review D, 91(6):065012, 2015.

[99] Bernd Thaller.

The dirac equation.
Springer Science & Business Media, 2013.

[100] Walter E Thirring.
A soluble relativistic field theory.
Annals of Physics, 3(1):91–112, 1958.

[101] George Tiktopoulos.
Note on positronium.
Journal of Mathematical Physics, 6(4):573–577, 1965.

[102] Sin-itiro Tomonaga.
On a relativistically invariant formulation of the quantum theory of wave fields.
Progress of Theoretical Physics, 1(2):27–42, 1946.

[103] Peter Van Alstine and Horace W Crater.
A tale of three equations: Breit, eddington—gaunt, and two-body dirac.
Foundations of Physics, 27(1):67–79, 1997.

[104] John Archibald Wheeler and Richard Phillips Feynman.
Interaction with the absorber as the mechanism of radiation.
Reviews of modern physics, 17(2-3):157, 1945.

[105] John Archibald Wheeler and Richard Phillips Feynman.
Classical electrodynamics in terms of direct interparticle action.
Reviews of modern physics, 21(3):425, 1949.

[106] Gian Carlo Wick.
Properties of bethe-salpeter wave functions.
Physical Review, 96(4):1124, 1954.

[107] Jochen Zahn.
The renormalized locally covariant dirac field.
Reviews in Mathematical Physics, 26(01):1330012, 2014.

Eidestattliche Versicherung

(siehe Promotionsordnung vom 12.07.11 § 8. Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eides statt, dass die Dissertation von mir selbstständig ohne unerlaubte Beihilfe angefertigt wurde.
Die Abschnitte 2.1 und 3.1 fassen Ergebnisse anderer Autoren zusammen um einen in sich geschlossenen Einstieg in das Thema zu ermöglichen.
Die Abschnitte 2.2 und 2.3 enthalten Resultate welche mit Matthias Lienert in Zusammenarbeit entstanden und zur Veröffentlichung eingereicht sind. Die Arbeit der in Abschnitt 2.3 präsentierten Ergebnisse ist erfolgreich veröffentlicht. Diese Resultate sind aus gemeinschaftlicher Arbeit entstanden und können den Autoren nicht einzeln zugeordnet werden. Die neuen Ergebnisse des Kapitels 3 sind in Zusammenarbeit mit meinen Betreuern entstanden, jedoch noch nicht veröffentlicht.

Nöth, Markus Hartmut
München, 06.08.2021