Resonances, Spectral Estimates and their Connection to Scattering Theory in the Spin-Boson Model

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Abstract

The main goal of this thesis is to understand the relation between resonances and scattering theory in a certain model of quantum field theory. We address the Spin-Boson model, which describes the interaction of a two-level quantum system with a second-quantized scalar field. We consider massive and massless scalar fields. In both cases an ultraviolet cut-off is imposed. For the massless model, we study a slightly infrared-regularized model but no infrared cut-off is imposed.

This thesis consists of three parts. In the first part, using the method of complex dilation, the resonance and the ground-state of the dilated Hamiltonian of the massless model are constructed and it is shown that they (and the corresponding eigenprojections) are analytic with respect to the dilation parameter and the coupling constant. Furthermore, it is proven that in neighborhoods of the resonance and the ground-state eigenvalue the spectrum of the dilated Hamiltonian is localized in two cones in the complex plane with vertices at the location of the resonance and the ground-state eigenvalue, respectively. In addition, certain relevant norm-estimates for the resolvent of the dilated Hamiltonian in regions close to the resonance and the ground-state eigenvalue are provided. These results are obtained by an extension of Pizzo's multiscale method for resonances.

In the second part, again for the massless model, a non-perturbative formula for the scattering coefficients of the one-boson scattering processes is derived. In particular, the integral kernel of a scattering matrix element is given as an explicit function of the resolvent, and we calculate the leading order term (with respect to the coupling constant). This establishes a precise relation between the scattering matrix elements and the resonance. The derivation of this formula strongly relies on a good control of the time-evolution operator in the scattering regime which is achieved by the technical results provided in the first part.

In the third part, the massive Spin-Boson model is considered. Similarly as in the massless case, a formula for leading order term (with respect to the coupling constant) of the one-boson scattering matrix elements is deduced. Here, we use a Mourre theory argument combined with a suitable application of the Feshbach-Schur map, instead of complex dilation, to study the spectral properties and the scattering matrix.

Zusammenfassung

Das Hauptziel dieser Arbeit ist es ein besseres Verständnis der Beziehung zwischen Resonanzen und Streutheorie in einem bestimmten Modell der Quantenfeldtheorie zu erlangen. Wir untersuchen das Spin-Boson Modell, welches die Wechselwirkung zwischen einem zwei-stufigen Atom und einem zweit-quantisierten Skalarfeld beschreibt. Es werden sowohl massive als auch masselose Skalarfelder betrachtet. In beiden Fällen wird ein Ultraviolett Cut-off angenommen. Im Falle des masselosen Modells wird ein Infrarot regularisiertes Modell analysiert, jedoch kein Infrarot Cut-off angenommen.

Diese Arbeit besteht aus drei Teilen. Im ersten Teil werden die Resonanz und der Grundzustand des Hamiltonians nach komplexer Dilatation für das masselose Modell konstruiert und gezeigt, dass diese (und die dazugehörigen Eigenprojektionen) analytisch bezüglich des Dilatationsparameters und der Kopplungskonstanten sind. Desweiteren wird bewiesen, dass das Spektrum des Hamiltonians nach komplexer Dilatation in Umgebungen der Resonanz und des Grundzustandes in zwei Kegeln in der komplexen Ebene, mit Scheitelpunkten an der Position der Resonanz beziehungsweise des Grundzustands, lokalisiert ist. Zusätzlich werden bestimmte Normabschätzungen der Resolvente nach komplexer Dilatation in Regionen nahe der Resonanz sowie des Grundzustands bereitgestellt. Diese Resultate wurden durch eine Erweiterung der Pizzo Multiskalen Methode für Resonanzen erzielt.

Im zweiten Teil wird eine nicht-perturbative Formel für die Koeffizienten der Streumatrix für Ein-Boson Streuprozesse hergeleitet. Die Integralkerne dieser Matrixelemente werden als Funktion der Resolvente dargestellt und wir berechnen den führenden Term (bezüglich der Kopplungskonstanten). Dadurch wird die genaue Beziehung zwischen der Streumatrix und der Resonanz erläutert. Die Herleitung dieser Formel beruht auf einer guten Kontrolle des Zeitentwicklungs-Operators im Streuregime, welche durch die technischen Resultate im ersten Teil der Arbeit erzielt wird.

Im dritten Teil wird das massive Spin-Boson Modell betrachtet und ähnlich wie im masselosen Fall eine Formel für den führenden Term (bezüglich der Kopplungskonstanten) der Ein-Boson Streumatrixelemente hergeleitet. Hier werden Argumente der Mourre Theorie zusammen mit einer passenden Anwendung der Feshbach-Schur Abbildung benutzt, anstatt die Methode der komplexen Dilatation zu verwenden.

Style of Writing

Although this doctoral thesis is written by only one author, the chosen form of writing employs the use of first person plural throughout the work for three reasons: First, research is never done by a single person alone. In this sense phrases like “we conclude” are used to recall all people who contributed to a “conclusion” in one way or another. Second, the particular results presented in this thesis are based on four publications (see [21, 23, 19, 22]) and one conference paper (see [20]) including more than one authors. Third, for an interested reader phrases like “we prove” are also meant in the sense that the author and the reader go through a “proof” together to check if it is correct.

Contents

1. Introduction and definition of the model

In the first part of this chapter, we formulate our main goals and present an outline of this thesis. In the second part, we introduce the Spin-Boson model and recall some basic results.

1.1. Main topics and outline of this thesis

All mathematical results and statements, presented in this thesis, are drawn from the articles [21, 23, 20, 22] and [19] which arose from collaborations with Miguel Ballesteros, Dirk-A. Deckert and the present author and Miguel Ballesteros, Dirk-A. Deckert, Jérémy Faupin and the present author, respectively. The purpose of this manuscript is twofold: first, it provides the opportunity to compare the results obtained in the works mentioned above. In particular, this allows us to state a *General Scattering Formula* presented in Theorem 3.0.1 below, which holds for both the massless and the massive scalar field and separate the physical import from the mathematical technicalities (see Chapter 3). Secondly, we present a self-comprehensive work in which all relevant results are collected so that they are accessible for newcomers and students.

1.1.1. Main goal

The main goal of this thesis is to understand the relation between resonances and scattering theory in the context of quantum field theory. The connection of these two fields is of great interest since it permits to interpret resonances as peaks at certain energy values in the measured scattering cross sections per solid angle. Thereby, we rigorously derive the typical intensity profiles obtained in scattering experiments between photons and atoms. More precisely, we show that poles of the dilated resolvent operator lead to poles in the scattering matrix. Note that, in this context, resonances are defined as poles of the complex dilated resolvent operator. In the case of quantum mechanics this problem has been studied extensively and lead to a vast number of works that culminated in the seminal paper [63]. In [63], an explicit formula for the transition matrix elements for n-body Schödinger operators is presented and it is shown that the integral kernel is meromorphic with poles at the positions of the resonances. However, in the realm of quantum field theory, even for the case of special models, an analogous result remained an open problem for several decades.

1.1.2. Outline of this thesis

Chapter 1 We briefly describe the main goals and the structure of this thesis. Moreover, we precisely define the Spin-Boson model, and collect some basic properties. Parts of this chapter are based on [21, 23, 22, 19].

Chapter 2 We recall some well-known results about scattering theory for the Spin-Boson model and derive an intermediate scattering formula. This chapter is based on [23].

Chapter 3 We present our main theorems: a perturbative formula for the one-boson scattering matrix elements (which holds true for the massive and the massless cases) – see Theorem 3.0.1, and a non-perturbative formula for the one-boson scattering matrix elements in the massless case – see Theorem 3.0.3. These results summarize Chapters 5 and 6, which are based on [22, 23, 20] and [19], respectively.

Chapter 4 We construct the ground-state and the resonance as eigenvalues of a dilated Hamiltonian and prove that they (and the corresponding eigenprojections) are analytic with respect to the coupling constant and the dilation parameter, for the massless case. Furthermore, we localize the spectrum of the dilated Hamiltonian in cones with vertices at the position of the ground-state energy and the resonance, respectively, and give resolvent estimates for the dilated Hamiltonian in a large subset of the complex plane. This chapter is based on [21].

Chapter 5 We derive a non-perturbative formula for the one-boson scattering matrix elements for the massless case. In particular, we present a formula for the scattering matrix elements as a function of the dilated resolvent operator and derive an explicit formula for the leading order term with respect to the coupling constant. This chapter is based on [22, 23] and a first announcement can be found in [20].

Chapter 6 We give a formula for the leading order term of the one-boson scattering matrix elements for the massive case. We provide spectral estimates without using the method of complex dilation, but using a Mourre type argument instead. This chapter is based on [19].

Chapter 7 We present an outlook of prospective open problems.

1.2. Definition of the Spin-Boson model

We present the Spin-Boson model which is a non-trivial model of quantum field theory. It can be seen as a model of a two-level atom interacting with its second-quantized scalar field, and hence, provides a widely employed model for quantum optics which gives insights into scattering processes between photons and atoms.

The non-interacting Spin-Boson Hamiltonian is defined as

$$H_0 := K + H_f, \qquad K := \begin{pmatrix} e_1 & 0 \\ 0 & e_0 \end{pmatrix}, \qquad H_f := \int \mathrm{d}^3k\, \omega(k) a(k)^* a(k). \tag{1.2.1}$$

We regard K as an idealized free Hamiltonian of a two-level atom. Its two energy levels are denoted by the real numbers $0 = e_0 < e_1$ and H_f denotes the free Hamiltonian of a scalar field having dispersion relation $\omega(k) = \sqrt{k^2 + m^2}$. The parameter m is referred to as the mass of the scalar field. Throughout this thesis, we consider two different scenarios: a massless scalar field ($m = 0$) in Chapters 4 and 5, and a massive field ($m > 0$) in Chapter 6. Furthermore, a, a^* are the annihilation and creation operators on the standard Fock space, they are defined in (1.2.10) and (1.2.11) below. We sometimes call K the atomic part, and H_f the free field part of the Hamiltonian. The sum of the free two-level atom Hamiltonian K and the free field Hamiltonian H_f is named "free Hamiltonian" H_0. The interaction term reads

$$V := \sigma_1 \otimes \Phi(f), \qquad \Phi(f) := \left(a(f) + a(f)^*\right), \qquad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.2.2}$$

where the boson form factor is given by

$$f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \qquad k \mapsto e^{-\frac{k^2}{\Lambda^2}} \omega(k)^{-\frac{1}{2}+\mu}. \tag{1.2.3}$$

Note that the relativistic form factor of a scalar field is $f(k) = (2\pi)^{-\frac{3}{2}}(2|k|)^{-\frac{1}{2}}$, which however renders the model ill-defined due to the fact that such an f would not be square integrable. This is referred to as ultraviolet divergence. In our case, the Gaussian factor in (1.2.3) acts as an ultraviolet cut-off for $\Lambda > 0$ being the ultraviolet cut-off parameter. In addition, for $m > 0$, we take $\mu = 0$, and for $m = 0$, we take

$$\mu \in (0, 1/2), \tag{1.2.4}$$

which is a regularization of the infrared singularity at $k = 0$. In [8], a method to construct the ground-state of the massless model also for the case $\mu = 0$ is provided. The missing factor of $2^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}}$ is absorbed in the coupling constant g in our notation. Note that the form factor f only depends on the radial part of k. To emphasize this, we often write $f(k) \equiv f(|k|)$. Our proofs in Chapter 6, where we analyze massive scalar fields ($m > 0$), allow for more general boson form factors f. The particular conditions on f in this case are specified at the beginning of Chapter 6 and in Chapter 3.

The full Spin-Boson Hamiltonian is defined as

$$H := H_0 + gV \tag{1.2.5}$$

for some coupling constant $g > 0$ on the Hilbert space

$$\mathcal{H} := \mathcal{K} \otimes \mathcal{F}[\mathfrak{h}], \qquad \mathcal{K} := \mathbb{C}^2, \tag{1.2.6}$$

where

$$\mathcal{F}[\mathfrak{h}] := \bigoplus_{n=0}^{\infty} \mathcal{F}_n[\mathfrak{h}], \qquad \mathcal{F}_n[\mathfrak{h}] := \mathfrak{h}^{\odot n}, \qquad \mathfrak{h} := L^2(\mathbb{R}^3, \mathbb{C}) \tag{1.2.7}$$

denotes the standard bosonic Fock space, and the superscript $\odot n$ denotes the n-th symmetric tensor product and, by convention, $\mathfrak{h}^{\odot 0} \equiv \mathbb{C}$. We identify $K \equiv K \otimes \mathbb{1}_{\mathcal{F}[\mathfrak{h}]}$ and $H_f \equiv \mathbb{1}_{\mathcal{K}} \otimes H_f$ in our notation (see Remark 1.2.1 below).

An element $\Psi \in \mathcal{F}[\mathfrak{h}]$ can be represented as a family $(\psi^{(n)})_{n\in\mathbb{N}_0}$ of wave functions $\psi^{(n)} \in \mathfrak{h}^{\odot n}$. The state Ψ with $\psi^{(0)} = 1$ and $\psi^{(n)} = 0$ for all $n \geq 1$ is called the vacuum and is denoted by

$$\Omega := (1, 0, 0, \dots) \in \mathcal{F}[\mathfrak{h}]. \tag{1.2.8}$$

We define

$$\mathcal{F}_0 := \left\{ \Psi = (\psi^{(n)})_{n\in\mathbb{N}_0} \in \mathcal{F}[\mathfrak{h}] \,\middle|\, \exists N \in \mathbb{N}_0 : \psi^{(n)} = 0 \, \forall n \geq N, \forall n \in \mathbb{N} : \psi^{(n)} \in S(\mathbb{R}^{3n}, \mathbb{C}) \right\}, \tag{1.2.9}$$

where $S(\mathbb{R}^{3n}, \mathbb{C})$ denotes the Schwartz space of infinitely differentiable functions with rapid decay.

Then, for any $h \in \mathfrak{h}$, we define the operator $a(h) : \mathcal{F}_0 \to \mathcal{F}_0$ by

$$\left(a(h)\Psi\right)^{(n)}(k_1, ..., k_n) = \sqrt{n+1} \int \mathrm{d}^3k \, \overline{h(k)} \psi^{(n+1)}(k, k_1, ..., k_n) \tag{1.2.10}$$

and $a(h)\Omega = 0$. The operator $a(h)$ is closable and, using a slight abuse of notation, we denote its closure by the same symbol $a(h)$ in the following. The operator $a(h)$ is called the annihilation operator. The creation operator is defined as the adjoint of $a(h)$ and we denote it by $a(h)^*$. For $\Psi = (\psi^{(n)})_{n\in\mathbb{N}_0} \in \mathcal{F}_0$, we find that

$$\left(a(h)^*\Psi\right)^{(n)}(k_1, ..., k_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h(k_i) \psi^{(n-1)}(k_1, ..., \tilde{k}_i, ..., k_n), \tag{1.2.11}$$

where the notation $\tilde{\cdot}$ means that the corresponding variable is omitted.

Occasionally, we shall also use the physics notation and define the point-wise creation and annihilation operators. The action of the latter in the n boson sector is to be understood as:

$$\left(a(k)\Psi\right)^{(n)}(k_1, ..., k_n) = \sqrt{n+1} \psi^{(n+1)}(k, k_1, ..., k_n), \tag{1.2.12}$$

for $\Psi = (\psi^{(n)})_{n\in\mathbb{N}_0} \in \mathcal{F}_0$. The operator $a(k)$ is not closable. The point-wise creation operator $a(k)^*$ is only defined as a quadratic form on $\mathcal{F}_0$ in the following sense:

$$\langle \Phi, a(k)^*\Psi \rangle = \langle a(k)\Phi, \Psi \rangle, \qquad \forall \Phi, \Psi \in \mathcal{F}_0. \tag{1.2.13}$$

Moreover, we define quadratic forms:

$$\mathcal{F}_0 \times \mathcal{F}_0 \to \mathbb{C}, \quad (\Phi, \Psi) \mapsto \int \mathrm{d}^3k \, \overline{h(k)} \, \langle \Phi, a(k)\Psi \rangle \tag{1.2.14}$$

and

$$\mathcal{F}_0 \times \mathcal{F}_0 \to \mathbb{C}, \quad (\Phi, \Psi) \mapsto \int \mathrm{d}^3k \, h(k) \, \langle \Phi, a(k)^*\Psi \rangle \,. \tag{1.2.15}$$

It is not difficult to see that these quantities are equal to $\langle \Phi, a(h)\Psi \rangle$ and $\langle \Phi, a(h)^*\Psi \rangle$, respectively. The point-wise creation operator $a(k)^*$ is not defined as an operator but, formally, we can express it in the following way:

$$\left(a(k)^*\Psi\right)^{(n)}(k_1, ..., k_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta^{(3)}(k - k_i)\psi^{(n-1)}(k_1, ..., \tilde{k}_i, ..., k_n). \tag{1.2.16}$$

This is the usual formula that physicists use. Here, δ denotes the Dirac's delta tempered distribution acting on the Schwartz space of test functions. Note that a and a^* fulfill the canonical commutation relations:

$$\forall h, l \in \mathfrak{h}, \qquad [a(h), a^*(l)] = \langle h, l \rangle_2 \,, \qquad [a(h), a(l)] = 0, \qquad [a^*(h), a^*(l)] = 0. \tag{1.2.17}$$

Remark 1.2.1. *In this work we omit spelling out identity operators whenever unambiguous. For every vector spaces V_1, V_2 and operators A_1 and A_2 defined on V_1 and V_2, respectively, we identify*

$$A_1 \equiv A_1 \otimes \mathbb{1}_{V_2}, \qquad A_2 \equiv \mathbb{1}_{V_1} \otimes A_2. \tag{1.2.18}$$

In order to simplify our notation further, and whenever unambiguous, we do not utilize specific notations for every inner product or norm that we employ.

1.3. Collection of well-known properties of the Spin-Boson model

We collect some well-known facts which are frequently used in the remainder of this work. The following properties hold for massless scalar fields ($m = 0$ and $\mu \in (0, 1/2)$) as well as the massive scalar fields ($m > 0$ and $\mu = 0$).

1.3.1. Standard estimates

In the following we shall use the well-known standard inequalities

$$\begin{aligned} \|a(h)\Psi\| &\leq \|h/\sqrt{\omega}\|_2 \, \|H_f^{1/2}\Psi\| \\ \|a(h)^*\Psi\| &\leq \|h/\sqrt{\omega}\|_2 \, \|H_f^{1/2}\Psi\| + \|h\|_2 \, \|\Psi\| \end{aligned} \tag{1.3.1}$$

which hold for all $h, h/\sqrt{\omega} \in \mathfrak{h}$ and $\Psi \in \mathcal{H}$ such that the left- and right-hand sides are well-defined; see [64, Eq. (13.70)].

Lemma 1.3.1. *Let $h, h/\sqrt{\omega} \in \mathfrak{h}$. Then, we have the following estimates:*

$$\left\| a(h)^*(H_f+1)^{-\frac{1}{2}} \right\| \leq \|h\|_2 + \|h/\sqrt{\omega}\|_2, \tag{1.3.2}$$

$$\left\| a(h)(H_f+1)^{-\frac{1}{2}} \right\| \leq \|h/\sqrt{\omega}\|_2, \tag{1.3.3}$$

$$\left\| V\,(H_f+1)^{-\frac{1}{2}} \right\| \leq \|f\|_2 + 2\|f/\sqrt{\omega}\|_2. \tag{1.3.4}$$

Proof. Let $\Psi \in \mathcal{F}[\mathfrak{h}]$ with $\|\Psi\|_{\mathcal{H}} = 1$. Applying (1.3.1) and the spectral theorem, we find

$$\begin{aligned} \|a(h)^*(H_f+1)^{-\frac{1}{2}}\Psi\| &\leq \|h\|_2\|(H_f+1)^{-\frac{1}{2}}\Psi\| + \|h/\sqrt{\omega}\|_2\|H_f^{\frac{1}{2}}(H_f+1)^{-\frac{1}{2}}\Psi\| \\ &\leq \|h\|_2 + \|h/\sqrt{\omega}\|_2, \end{aligned} \tag{1.3.5}$$

$$\|a(h)(H_f+1)^{-\frac{1}{2}}\Psi\| \leq \|h/\sqrt{\omega}\|_2\|H_f^{\frac{1}{2}}(H_f+1)^{-\frac{1}{2}}\Psi\| \leq \|h/\sqrt{\omega}\|_2. \tag{1.3.6}$$

The inequality (1.3.4) is implied by the boundedness of σ_1 and the triangle inequality:

$$\begin{aligned} \left\| V\,(H_f+1)^{-\frac{1}{2}} \right\| &\leq \left\| \sigma_1 \otimes a(f)\,(H_f+1)^{-\frac{1}{2}} \right\| + \left\| \sigma_1 \otimes a(f)^*\,(H_f+1)^{-\frac{1}{2}} \right\| \\ &\leq \left\| a(f)\,(H_f+1)^{-\frac{1}{2}} \right\| + \left\| a(f)^*\,(H_f+1)^{-\frac{1}{2}} \right\| \leq \|f\|_2 + 2\|f/\sqrt{\omega}\|_2. \end{aligned} \tag{1.3.7}$$

This completes the proof. □

As preparation of the proof of Lemma 2.2.1 below, we recall that the Hamiltonians H, c.f. (1.2.5), as well as H_f, c.f. (1.2.1), are self-adjoint on the common domain $D(H) = \mathcal{K} \otimes \mathcal{D}(H_f)$ and bounded below by the constant $b \in \mathbb{R}$; c.f. Proposition 1.3.3 and (1.3.17) below. By spectral calculus we can define the operators $H_f^{1/2}$, $(H-b+1)^{1/2}$ and $(H_f+1)^{-1/2}$, $(H-b+1)^{-1/2}$ which are closed and densely defined and the latter two are even bounded with norms bounded by 1.

Lemma 1.3.2. *The following operators are bounded:*

$$H_f^{\frac{1}{2}}(H-b+1)^{-\frac{1}{2}}, \tag{1.3.8}$$

$$(H-b+1)^{\frac{1}{2}}(H_f+1)^{-\frac{1}{2}}. \tag{1.3.9}$$

Proof. Let $\Psi \in \mathcal{H}$ with $\|\Psi\| = 1$. The boundedness of (1.3.8) follows from the equality

$$\begin{aligned} \|H_f^{\frac{1}{2}}(H-b+1)^{-\frac{1}{2}}\Psi\|^2 &= \langle (H-b+1)^{-\frac{1}{2}}\Psi, H_f(H-b+1)^{-\frac{1}{2}}\Psi\rangle \\ &= \langle (H-b+1)^{-\frac{1}{2}}\Psi, (H-K-gV)(H-b+1)^{-\frac{1}{2}}\Psi\rangle \end{aligned} \tag{1.3.10}$$

and the fact that K is bounded by $|e_1|$ and that for all $\epsilon > 0$

$$\begin{aligned} &|\langle (H-b+1)^{-\frac{1}{2}}\Psi, gV(H-b+1)^{-\frac{1}{2}}\Psi\rangle| \leq \|(H-b+1)^{-\frac{1}{2}}\Psi\|\,\|gV(H-b+1)^{-\frac{1}{2}}\Psi\| \\ &\leq \frac{g}{\epsilon}2\|f/\sqrt{\omega}\|_2\,\epsilon\|H_f^{\frac{1}{2}}(H-b+1)^{-\frac{1}{2}}\Psi\| + \|f\|_2\|\Psi\| \\ &\leq \left(\frac{g}{\epsilon}2\|f/\sqrt{\omega}\|_2\right)^2 + \epsilon^2\|H_f^{\frac{1}{2}}(H-b+1)^{-\frac{1}{2}}\Psi\|^2 + \|f\|_2 \end{aligned} \tag{1.3.11}$$

holds, which is a consequence of (1.3.1). Choosing $0 < \epsilon < 1$ an explicit bound is

$$\|H_f^{\frac{1}{2}}(H-b+1)^{-\frac{1}{2}}\Psi\|^2 \leq \frac{1+|e_1|+\left(\frac{g}{\epsilon}2\|f/\sqrt{\omega}\|_2\right)^2+\|f\|_2}{1-\epsilon^2} < \infty. \tag{1.3.12}$$

The boundedness of (1.3.9) is implied by

$$\|(H-b+1)^{\frac{1}{2}}(H_f+1)^{-\frac{1}{2}}\Psi\|^2 = \langle (H_f+1)^{-\frac{1}{2}}\Psi, (K+H_f+gV-b+1)(H_f+1)^{-\frac{1}{2}}\Psi\rangle \tag{1.3.13}$$

and, again as a consequence of (1.3.1),

$$\begin{aligned}|\langle (H_f+1)^{-\frac{1}{2}}\Psi, gV(H_f+1)^{-\frac{1}{2}}\Psi\rangle| &\leq g2\|f/\sqrt{\omega}\|_2\,\|H_f^{\frac{1}{2}}(H_f+1)^{-\frac{1}{2}}\Psi\|+\|f\|_2 \\ &\leq \|f\|_2+2\|f/\sqrt{\omega}\|_2.\end{aligned} \tag{1.3.14}$$

□

1.3.2. Self-adjointness and spectral properties

Clearly, K is self-adjoint on $\mathcal{K}$ and its spectrum consists of two eigenvalues e_0 and e_1. The corresponding eigenvectors are

$$\varphi_0 = (0,1)^T \qquad \text{and} \qquad \varphi_1 = (1,0)^T \qquad \text{with} \qquad K\varphi_i = e_i\varphi_i, \quad i=0,1. \tag{1.3.15}$$

Moreover, H_f is self-adjoint on its natural domain $\mathcal{D}(H_f) \subset \mathcal{F}[\mathfrak{h}]$ and its spectrum is given by $\sigma(H_f) = \{0\} \cup [m,\infty)$ and its absolutely continuous is given by $[m,\infty)$ (see [61]). Consequently, the spectrum of H_0 is given by $\sigma(H_0) = [e_0,\infty)$ (see [60]).

The self-adjointness of the full Hamiltonian H is well-known (see, e.g., [51]) and it can be shown using the standard estimate in Lemma 1.3.1.

Proposition 1.3.3. *The operator gV is relatively bounded by H_0 with infinitesimal bound, and consequently, H is self-adjoint and bounded below on the domain*

$$\mathcal{D}(H) = D(H_0) = \mathcal{K} \otimes \mathcal{D}(H_f), \tag{1.3.16}$$

i.e., there is a constant $b \in \mathbb{R}$ such that

$$b \leq H. \tag{1.3.17}$$

Proof. It follows from Lemma 1.3.1 that

$$\begin{aligned}\left\|V(H_0+1)^{-\frac{1}{2}}\right\| &\leq \left\|V(H_f+1)^{-\frac{1}{2}}\right\|\left\|(H_f+1)^{\frac{1}{2}}(H_0+1)^{-\frac{1}{2}}\right\| \\ &\leq \left(\|f\|_2+2\|f/\sqrt{\omega}\|_2\right)\left\|(H_f+1)^{\frac{1}{2}}(H_0+1)^{\frac{1}{2}}\right\|,\end{aligned} \tag{1.3.18}$$

and moreover, we obtain from the spectral calculus that

$$\left\|(H_f+1)^{\frac{1}{2}}(H_0+1)^{-\frac{1}{2}}\right\| \leq \left\|(H_f+1)(H_0+1)^{-1}\right\| \leq \sup_{r\in[0,\infty), i\in\{0,1\}} \frac{r+1}{e_i+r+1} \leq 1. \tag{1.3.19}$$

This together with (1.3.18) yields that

$$\left\| V(H_0+1)^{-\frac{1}{2}} \right\| \leq \|f\|_2 + 2\|f/\sqrt{\omega}\|_2. \tag{1.3.20}$$

and thereby, V is relatively bounded by H_0 with infinitesimal bound. We conclude the proof by Kato's theorem. □

1.3.3. Complex dilation

In this section we consider massless scalar fields, i.e., $m = 0$ since we employ the method of complex dilation only for this case (in Chapters 4 and 5).

It is known (see, e.g., [51]) that the only eigenvalue in the spectrum of H is

$$\lambda_0 := \inf \sigma(H) \tag{1.3.21}$$

while the rest of the spectrum is absolutely continuous. This implies that there is no stable excited state in the massless Spin-Boson model. Heuristically, the reason for this is that the atomic energy of the excited state e_1 turns into what can be seen as a complex "energy" λ_1 with strictly negative imaginary part once the interaction is switched on (see e.g. [9, 10]). This complex energy λ_1 is referred to as resonance energy and its imaginary part is responsible for the decay of the excited state (see e.g. [2, 49]).

Note that the ground state Ψ_{λ_0} of H corresponding to ground state energy λ_0, i.e.,

$$H\Psi_{\lambda_0} = \lambda_0 \Psi_{\lambda_0}, \tag{1.3.22}$$

has already been constructed, e.g., in [51, Theorem 1], [48, Theorem 1] and [8, Theorem 3.5]. The ground state of the massive model can be constructed by regular perturbation theory (see Proposition 6.2.1) and we denote it by the same symbol Ψ_{λ_0}. Since H on $\mathcal{H}$ is a self-adjoint operator, λ_1 should rather be thought of as a complex eigenvalue of H on a bigger space than $\mathcal{H}$. This prevents us from being able to calculate the resonance energy directly by regular perturbation theory on $\mathcal{H}$. The standard way to nevertheless get access to such a resonance without leaving the underlying Hilbert space is the method of complex dilation which will be introduced next. We start by defining a family of unitary operators on $\mathcal{H}$ indexed by $\theta \in \mathbb{R}$.

Definition 1.3.4. *For $\theta \in \mathbb{R}$, we define the unitary transformation*

$$u_\theta : \mathfrak{h} \to \mathfrak{h}, \qquad \psi(k) \mapsto e^{-\frac{3\theta}{2}} \psi(e^{-\theta} k). \tag{1.3.23}$$

Similarly, we define its canonical lift $U_\theta : \mathcal{F}[\mathfrak{h}] \to \mathcal{F}[\mathfrak{h}]$ by the lift condition $U_\theta a(h)^* U_\theta^{-1} = a(u_\theta h)^*$, $h \in \mathfrak{h}$, and $U_\theta \Omega = \Omega$. With slight abuse of notation, we also denote $\mathbb{1}_{\mathcal{K}} \otimes U_\theta$ on $\mathcal{H}$ by the same symbol U_θ.

We define the family of transformed Hamiltonians, for $\theta \in \mathbb{R}$,

$$H^\theta := U_\theta H U_\theta^{-1} = H_0^\theta + gV^\theta, \quad \text{where} \quad H_0^\theta := K + H_f^\theta \tag{1.3.24}$$

and

$$H_f^\theta := \int \mathrm{d}^3k\, \omega^\theta(k) a^*(k) a(k), \qquad V^\theta := \sigma_1 \otimes \left(a(\overline{f^\theta}) + a(f^\theta)^* \right) \tag{1.3.25}$$

with

$$\omega^\theta(k) := e^{-\theta}|k|, \qquad f^\theta : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \quad k \mapsto e^{-\theta(1+\mu)} e^{-e^{2\theta}\frac{k^2}{\Lambda^2}} |k|^{-\frac{1}{2}+\mu}. \tag{1.3.26}$$

Eqs. (1.3.26), (1.3.25) and the right hand side of (1.3.24) can be defined for complex θ. If $|\theta|$ is small enough, $K + H_f^\theta + gV^\theta$ is a closed (non self-adjoint) operator. However, the middle term in (1.3.24) is not necessarily correct because, although U_θ can be defined for complex θ, it turns out to be an unbounded operator, and $U_\theta H U_\theta^{-1}$ might not be densely defined.

We say that Ψ is an analytic vector if the map $\theta \mapsto \Psi^\theta := U_\theta \Psi$ has an analytic continuation from an open connected set in the real line to a (connected) domain in the complex plane. In general we will not specify their domains of analyticity (it will be clear from the context). It is well-known that there is a dense set of entire vectors (they are analytic in $\mathbb{C}$). This result has been proven in a variety of similar models, for example, in [9, 53]. The set the example

$$\mathcal{D} = \left\{ \chi_{[-R,R]}(A)\Psi : \Psi \in \mathcal{H}, R > 0 \right\}, \tag{1.3.27}$$

with A being the generator of U_θ and χ the corresponding spectral projection (c.f. [9, 53]), is a dense set of entire vectors.

Furthermore, we define the open disc

$$D(x, r) := \{ z \in \mathbb{C} : |z - x| < r \} \qquad x \in \mathbb{C}, r > 0, \tag{1.3.28}$$

and note that, for $\theta \in D(0, \pi/16)$, we have

$$\left\| V^\theta \left(H_0 + 1 \right)^{-\frac{1}{2}} \right\| \leq \left\| f^\theta \right\|_2 + 2 \left\| f^\theta / \sqrt{\omega} \right\|_2 \tag{1.3.29}$$

which is guaranteed by the standard estimate (1.3.4), since (1.3.26) together with the special choice $\theta \in D(0, \pi/16)$ implies that $f^\theta, f^\theta/\sqrt{\omega} \in \mathfrak{h}$. Hence, for $\theta \in D(0, \pi/16)$ the operators H^θ are densely defined and closed. Moreover, the following holds true:

Lemma 1.3.5. *The family* $\left\{ H^\theta \right\}_{\theta \in \mathbb{R}}$ *of unitary equivalent, self-adjoint operators with* $\mathcal{D}(H^\theta) = \mathcal{D}(H)$ *extends to an analytic family of type A for* $\theta \in D(0, \pi/16)$.

The above result was proven for the Pauli-Fierz model in [9, Theorem 4.4], and with small effort that proof can be adapted to our setting.

Lemma 1.3.6. *Let* $\theta \in \mathbb{C}$. *Then,* $\sigma(H_0^\theta) = \left\{ e_i + e^{-\theta} r : r \geq 0, i = 0, 1 \right\}$.

Proof. Let $\theta \in \mathbb{C}$. Definition in (1.2.1) implies that $H_0^\theta = K \otimes \mathbb{1}_{\mathcal{F}[\mathfrak{h}]} + \mathbb{1}_{\mathcal{K}} \otimes H_f^\theta$ is a sum of commuting self-adjoint operators and $\sigma(K) = \{e_0, e_1\}$. As shown in [61], we have $\sigma(H_f) = \mathbb{R}_0^+$ and it follows from the definition of $H_f^\theta = e^{-\theta} H_f$ in (1.3.25) that $\sigma(H_f^\theta) = \left\{ e^{-\theta} r : r \geq 0 \right\}$. The claim then follows from the spectral theorem for two commuting normal operators. □

For sufficiently small coupling constants and for $\theta \in \mathcal{S}$, where $\mathcal{S}$ is a certain subset of the complex plane defined in (4.1.1) below, it has been shown that H^θ has two non-degenerate eigenvalues λ_0^θ and λ_1^θ with corresponding rank one projectors denoted by P_0^θ and P_1^θ, respectively; see, e.g., Proposition 4.2.1 below. The corresponding dilated eigenstates can, therefore, be written as

$$\Psi_{\lambda_i}^\theta := P_i^\theta \varphi_i \otimes \Omega, \qquad i = 0, 1, \tag{1.3.30}$$

where the eigenstates φ_i of the free atomic system are given in (1.3.15), and Ω is the bosonic vacuum defined in (1.2.8). In our notation $\Psi_{\lambda_i}^\theta$ is not necessarily normalized. We know from Theorem 4.2.3 that the eigenvalues λ_i^θ are independent of θ as long as θ belongs to the set $\mathcal{S}$, and therefore, we suppress it in our notation writing $\lambda_i^\theta \equiv \lambda_i$. Note that this is not true for the eigenstates $\Psi_{\lambda_i}^\theta$. In [21] (as well as in Definition 5.2.1 below) we choose an open connected set $\mathcal{S}$ that does not include 0 (the imaginary parts of the points in this set are bounded from below by a fixed positive constant). We chose such a set in order to have a single set $\mathcal{S}$ for the cases $i = 0$ and $i = 1$, because we want to keep our notation as simple as possible (otherwise a two cases formulation would propagate all over this manuscript). However, the fact that 0 is not contained in $\mathcal{S}$ is only necessary for the case $i = 1$ (the resonance - due to the self-adjointness of H the state $\Psi_{\lambda_1}^\theta$ can not even exist for $\theta = 0$). For the case $i = 0$ (the ground-state) we can choose instead a connected open set containing 0. For θ in this set, it is still valid that λ_0^θ does not depend on θ, and therefore, it equals the ground state energy, and $\Psi_{\lambda_0}^{\theta=0} = \Psi_{\lambda_0}$ - as introduced above. For a further explanation we refer to Remark 4.2.4 below.

2. Scattering theory in the Spin-Boson model

In the first part, Section 2.1, we collect some well-known facts about scattering theory for the Spin-Boson model. In the proceeding part, Section 2.2, we use these results in order to derive an intermediate scattering formula which will be used to prove one of our main results, the scattering formulas for both the massive and the massless case; see Theorems 5.1.1, 5.1.3 and 6.3.2. Note that all results presented in this chapter hold true for massless scalar fields ($m = 0$ and $\mu \in (0, 1/2)$) as well as massive scalar fields ($m > 0$ and $\mu = 0$).

2.1. Collection of well-known properties about scattering theory in the Spin-Boson model

Let us recall some important results of scattering theory which will be necessary to state our main results in Chapters 3, 5 and 6.

The first obstacle in formulating scattering theory for a second-quantized system lies in the definition of the wave operators. Unlike in first-quantized quantum theory, where one defines the scattering operator to be $S := \Omega_+^* \Omega_-$ with the wave operators $\Omega_\pm$ given by the strong limits $\Omega_\pm := s\text{-}\lim_{t\to\pm\infty} e^{itH} e^{-itH_0}$, in quantum field theory, the corresponding wave operators usually do not exist in a straight-forward sense. Instead, one establishes the existence of the asymptotic annihilation and creation operators first, which can then be used to define the wave operators.

Definition 2.1.1 (Basic components of scattering theory)**.** *We denote by*

$$\mathfrak{h}_0 \tag{2.1.1}$$

the set of smooth complex-valued functions on $\mathbb{R}^3$ with compact support contained in $\mathbb{R}^3 \setminus \{0\}$.

Furthermore, we define the following objects:

(i) For $h \in \mathfrak{h}_0$ and $\Psi \in \mathcal{K} \otimes \mathcal{D}(H_f^{1/2})$, the asymptotic annihilation operators

$$a_\pm(h)\Psi := \lim_{t\to\pm\infty} a_t(h)\Psi, \quad a_t(h) := e^{itH} a(h_t) e^{-itH}, \quad h_t(k) := h(k) e^{-it\omega(k)}. \tag{2.1.2}$$

The existence of this limit is proven in Lemma 2.2.1 (i) below. Moreover, we define the asymptotic creation operators $a_\pm^*(h)$ as the respective adjoints.

(ii) The asymptotic Hilbert spaces

$$\mathcal{H}^\pm := \mathcal{K}^\pm \otimes \mathcal{F}[\mathfrak{h}] \quad \textit{where} \quad \mathcal{K}^\pm := \{\Psi \in \mathcal{H} : a_\pm(h)\Psi = 0 \;\; \forall h \in \mathfrak{h}_0\}. \tag{2.1.3}$$

(iii) The wave operators

$$\Omega_\pm : \mathcal{H}^\pm \to \mathcal{H} \tag{2.1.4}$$
$$\Omega_\pm \Psi \otimes a^*(h_1)...a^*(h_n)\Omega := a^*_\pm(h_1)...a^*_\pm(h_n)\Psi, \quad h_1, ..., h_n \in \mathfrak{h}_0, \quad \Psi \in \mathcal{K}^\pm.$$

(iv) The scattering operator $S := \Omega_+^* \Omega_-$.

The limit operators $a_\pm$ and $a^*_\pm$ are called asymptotic outgoing/ingoing annihilation and creation operators. The existence of the limits in (2.1.2) and their properties, especially that $\Psi_{\lambda_0} \in \mathcal{K}^\pm$ and $\Omega_\pm$ are well-defined, are well-known facts (see, e.g., [38, 37, 26, 41, 40] for various models of quantum field theory and [30, 31, 32, 34, 17] for the Spin-Boson model). For the convenience of the reader, Lemma 2.2.1 below collects all relevant facts. We can thus define the following two-body scattering matrix coefficients:

$$S(h, l) = \|\Psi_{\lambda_0}\|^{-2} \left\langle a^*_+(h)\Psi_{\lambda_0}, a^*_-(l)\Psi_{\lambda_0} \right\rangle, \qquad \forall h, l \in \mathfrak{h}_0, \tag{2.1.5}$$

where the factor $\|\Psi_{\lambda_0}\|^{-2}$ appears due to the fact that, as already mentioned above, in our notation, the ground state Ψ_{λ_0} is not necessarily normalized. In addition, it will be convenient to work with the corresponding two-body transition matrix coefficients given by

$$T(h, l) = S(h, l) - \langle h, l \rangle_2 \qquad \forall h, l \in \mathfrak{h}_0. \tag{2.1.6}$$

These matrix coefficients carry a ready physical interpretation as transition amplitudes of the scattering process in which an incoming boson with wave function l is scattered at the two-level atom into an outgoing boson with wave function h. Notice that the transition matrix coefficients of multi-photon processes can be defined likewise but in this work we focus on one-photon processes only.

It has been shown in [51] that the spectrum of H contains only one eigenvalue λ_0 (and it is non-degenerate), namely the ground state energy, and the rest of the spectrum of H is absolutely continuous. In case that asymptotic completeness holds, i.e.,

$$\mathcal{K}^\pm = \mathrm{Ran}\left(\chi_{\mathrm{pp}}(H)\right), \tag{2.1.7}$$

all one-boson processes are of the form (2.1.5). Here, $\mathrm{Ran}\left(\chi_{\mathrm{pp}}(H)\right)$ denotes the states associated with pure points in the spectrum of H.

Asymptotic completeness has actually been proven in [30, 31, 32] for the Hamiltonian H defined in (1.2.5), however, with coupling functions $f \in C^3_c(\mathbb{R}^3 \setminus \{0\}, \mathbb{C})$, i.e., the functions that are three times continuously differentiable and have compact support. The boson form factor f defined in (1.2.3) does not fulfill this property. When considering massless scalar fields in Chapters 4 and 5, we need an analytic continuation of our Hamiltonian in order to study resonances. This implies that the coupling function f cannot be compactly supported (see (1.2.3)), however, it belongs to the Schwartz space. We expect asymptotic completeness also to hold in our case, although our results do not depend on it.

2.2. Intermediate scattering formula

In Theorem 2.2.2 below we derive an intermediate formula for scattering processes with one incoming and outgoing asymptotic photon. A related formula was already employed in [50]. In order to derive it rigorously we need several properties of the asymptotic creation and annihilation operators. The necessary properties are collected in Lemma 2.2.1. They have already been proven for a range of models in several works [38, 37, 26, 41, 40, 30, 31, 32, 34, 17]. For convenience of the reader we provide a self-contained proof.

Lemma 2.2.1. *Let $\Psi \in \mathcal{K} \otimes D(H_f^{1/2})$ and $h, l \in \mathfrak{h}_0$. The asymptotic creation and annihilation operators $a_\pm^*, a_\pm$ defined in Definition 2.1.1 have the following properties:*

(i) The limits $a_\pm^\#(h)\Psi = \lim_{t\to\pm\infty} a_t^\#(h)\Psi$ exist, where $a^\#$ stands for a or a^*.

(ii) The next equalities holds true:

$$a_+(h)\Psi = a(h)\Psi - ig\int_0^\infty \mathrm{d}s\, e^{isH}\langle h_s, f\rangle_2\, \sigma_1 e^{-isH}\Psi \tag{2.2.1}$$

$$a_-(h)\Psi = a(h)\Psi + ig\int_{-\infty}^0 \mathrm{d}s\, e^{isH}\langle h_s, f\rangle_2\, \sigma_1 e^{-isH}\Psi. \tag{2.2.2}$$

We point out to the reader that the integrals above are convergent since it can be shown by integration by parts that there is constant C such that $|\langle h_s, f\rangle_2| \le C/(1+s^2)$ for $s \in \mathbb{R}$ (see (2.2.11) below).

(iii) The following pull-through formula holds true:

$$e^{-isH}a_-(h)^*\Psi = a_-(h_s)^* e^{-isH}\Psi. \tag{2.2.3}$$

(iv) The equality $a_\pm(h)\Psi_{\lambda_0} = 0$ holds true, i.e., $\Psi_{\lambda_0} \in \mathcal{K}^\pm$.

(v) The following commutation relation holds: $\langle a_\pm(h)^*\Psi_{\lambda_0}, a_\pm(l)^*\Psi_{\lambda_0}\rangle = \langle h, l\rangle_2 \|\Psi_{\lambda_0}\|^2$.

(vi) There is a finite constant $C(h) > 0$ such that for all $t \in \mathbb{R}$

$$\left\|a_t(h)^*(H_f+1)^{-\frac{1}{2}}\right\|, \left\|a_t(h)(H_f+1)^{-\frac{1}{2}}\right\| \le C(h). \tag{2.2.4}$$

Proof. We prove the statements only for the case of a massless scalar field, i.e., for the dispersion relation $\omega(k) = |k|$. The proof for massive fields with dispersion relation $\omega(k) = \sqrt{k^2+m^2}$ and $m > 0$ follows analogously.

Let $h, l \in \mathfrak{h}_0$ and $\Psi \in \mathcal{K} \otimes \mathcal{D}(H_f^{1/2})$. Thanks to Lemma 1.3.2 we have $\mathcal{K} \otimes \mathcal{D}(H_f^{\frac{1}{2}}) = \mathcal{D}((H - b + 1)^{\frac{1}{2}})$. We prove claims (i)-(vi) separately:

(ii) The subspace of $\mathcal{H}_0$, defined in (2.2.36), is dense in the domain of $(H-b+1)^{1/2}$ w.r.t. the graph norm $\|\cdot\|_{(H-b+1)^{1/2}}$ of $(H-b+1)^{\frac{1}{2}}$ so that there is a sequence $(\Psi_n)_{n\in\mathbb{N}}$ in $\mathcal{K}\otimes\mathcal{F}_{\text{fin}}[\mathfrak{h}_0]$ with $\Psi_n\to\Psi$ in this norm as $n\to\infty$. For all $n\in\mathbb{N}$, the definition in (2.1.2) together with the group properties $(e^{-itH})_{t\in\mathbb{R}}$, in particularly, the strong continuous differentiability on $D(H)$, justify

$$
\begin{aligned}
a_t(h)\Psi_n = e^{itH}a(h_t)e^{-itH} &= a(h)\Psi_n + \int_0^t ds\,\frac{d}{ds}e^{isH}a(h_s)e^{-isH}\Psi_n \\
&= a(h)\Psi_n - ig\int_0^t ds\,\langle h_s, f\rangle_2 e^{isH}\sigma_1 e^{-isH}\Psi_n,
\end{aligned}
\tag{2.2.5}
$$

where the last integrand was computed by observing the CCR (c.f. (1.2.17))

$$
[V, a(h_s)] = \sigma_1\otimes[a(f)+a(f)^*, a(h_s)] = -\sigma_1\langle h_s, f\rangle_2\,. \tag{2.2.6}
$$

We may now take the limit $n\to\infty$ of identity (2.2.5) and find

$$
a_t(h)\Psi = a(h)\Psi - ig\int_0^t ds\,\langle h_s, f\rangle_2\, e^{isH}\sigma_1 e^{-isH}\Psi \tag{2.2.7}
$$

because of the following two ingredients: First, by definition (2.1.2), the standard estimate (1.3.1) and Lemma 1.3.2, for all $m\in\mathfrak{h}_0$, there is a finite constant $C_{(2.2.8)}$ such that

$$
\begin{aligned}
\|a_t(m)(\Psi-\Psi_n)\| &= \|a(m_t)(H-b+1)^{-\frac{1}{2}}e^{-itH}(H-b+1)^{\frac{1}{2}}(\Psi-\Psi_n)\| \\
&\le \|m/\sqrt{\omega}\|_2\,\|H_f^{\frac{1}{2}}(H-b+1)^{-\frac{1}{2}}\|\,\|(H-b+1)^{\frac{1}{2}}(\Psi-\Psi_n)\| \\
&= C_{(2.2.8)}\|\Psi-\Psi_n\|_{(H-b+1)^{1/2}},
\end{aligned}
\tag{2.2.8}
$$

and likewise

$$
\begin{aligned}
\|a(m)(\Psi-\Psi_n)\| &= \|a(m)(H-b+1)^{-\frac{1}{2}}(H-b+1)^{\frac{1}{2}}(\Psi-\Psi_n)\| \\
&\le \|m/\sqrt{\omega}\|_2\,\|H_f^{\frac{1}{2}}(H-b+1)^{-\frac{1}{2}}\|\,\|(H-b+1)^{\frac{1}{2}}(\Psi-\Psi_n)\| \\
&= C_{(2.2.8)}\|\Psi-\Psi_n\|_{(H-b+1)^{1/2}}.
\end{aligned}
\tag{2.2.9}
$$

Second, the integrand in (2.2.5) is continuous in s and, for sufficiently large n, fulfills an n-independent bound

$$
\|e^{isH}\sigma_1 e^{-isH}(\Psi-\Psi_n)\| \le \|\sigma_1\|\,\|\Psi-\Psi_n\| \le 1 \tag{2.2.10}
$$

so dominated convergence can be applied to interchanging the integral and the $n\to\infty$ limit to prove (2.2.7).

Finally, a stationary phase argument in $\omega(k)=|k|$ as well as the facts that $h\in\mathfrak{h}_0$ and $f\in\mathcal{C}^\infty(\mathbb{R}\setminus\{0\})$, c.f. (1.2.3), provide the estimate

$$
\langle h_s, f\rangle = C\frac{1}{1+|s|^2} \tag{2.2.11}
$$

for all $s \in \mathbb{R}$, thanks to a two-fold partial integration. Hence, me way finally carry out the limit $t \to \pm\infty$ to find

$$a_\pm(h)\Psi = \lim_{t\to\pm\infty} a_t(h)\Psi = a(h)\Psi - ig\int_0^{\pm\infty} ds\, \langle h_s, f\rangle_2 e^{isH}\sigma_1 e^{-isH}\Psi \qquad (2.2.12)$$

as the indefinite integral exists thanks to (2.2.11) and the continuity of the integrand in s. We omit the proof for the asymptotic creation operator $a^*_\pm$ as the argument is almost the same.

(i) This follows from (ii).

(iii) Next, we calculate

$$\begin{aligned} e^{-isH}a_-(h)^*\psi &= \lim_{t\to-\infty} e^{-isH}e^{itH}a(h_t)^*e^{-itH}\psi \\ &= \lim_{t\to-\infty} e^{i(t-s)H}a(h_{(t-s)+s})^*e^{-i(t-s)H}e^{-isH}\psi \\ &= \lim_{t'\to-\infty} e^{it'H}a(h_{t'+s})^*e^{-it'H}e^{-isH}\psi = a_-(h_s)^*e^{-isH}\psi \qquad (2.2.13) \end{aligned}$$

which proves the pull-through formula in (iii).

(iv) First, for all $t \in \mathbb{R}$ we observe

$$\|a_t(h)\Psi_{\lambda_0}\| = \|e^{itH}a(h_t)e^{-itH}\Psi_{\lambda_0}\| = \|a(h_t)\Psi_{\lambda_0}\| \qquad (2.2.14)$$

due to the ground state property in (1.3.22). Second, for $\Psi = \Psi_{\lambda_0} \in \mathcal{D}(H) \subset \mathcal{K}\otimes\mathcal{D}(H_f^{1/2})$, we employ the same sequence $(\Psi_n)_{n\in\mathbb{N}}$ as in (ii) to compute

$$\|a(h_t)\Psi_n\|^2 = \sum_{l\in\mathbb{N}} \sqrt{l+1}\int d^3k_1\ldots d^3k_l \left|\int d^3k\, e^{it\omega(k)}\overline{h(k)}\psi_n^{(l+1)}(k,k_1,\ldots,k_l)\right|^2, \qquad (2.2.15)$$

where we used the Fock vector representation $\Psi_n = (\psi_n^{(l)})_{l\in\mathbb{N}_0}$. We observe that $\Psi_n \in \mathcal{H}_0$ implies $\psi_n^{(l)} \in \mathcal{K}\otimes C_0^\infty(\mathbb{R}^{3l}\setminus\{0\})$ and, by definition of $\mathcal{H}_0$, c.f. (2.2.36), there is a constant L such that $\psi_n^{(l)} = 0$ for $l \geq L$. A stationary phase argument in $\omega(k) = |k|$ and a partial integration in k gives

$$\begin{aligned} &\left|\int d^3k\, e^{it\omega(k)}\overline{h(k)}\psi_n^{(l+1)}(k,k_1,\ldots,k_l)\right| \\ &\leq \frac{1}{t}\int d^3k\, |k|^{-2}|\partial_{|k|}(|k|^2\overline{h(|k|,\Sigma)}\psi_n^{(l+1)}(|k|,\Sigma,|k_1|,\Sigma_1,\ldots,|k_l|,\Sigma_l))|, \qquad (2.2.16) \end{aligned}$$

where we use spherical coordinates $k = (|k|,\Sigma)$ and $k_i = (|k_i|,\Sigma_i)$. Here, Σ and Σ_i denote the solid angles. Then, we find

$$\begin{aligned} (2.2.15) \leq &\frac{1}{t}\sum_{0\leq l<L}\sqrt{l+1}\int d^3k_1\ldots d^3k_l \qquad (2.2.17) \\ &\times\left(\int d^3k\, |k|^{-2}|\partial_{|k|}(|k|^2\overline{h(|k|,\Sigma)}\Psi_n^{(l+1)}(|k|,\Sigma,|k_1|,\Sigma_1,\ldots,|k_l|,\Sigma_l)|\right)^2 \end{aligned}$$

which converges to zero for $t \to \pm\infty$. In conclusion, for all $n \in \mathbb{R}$ we have

$$\lim_{t\to\pm\infty} a(h_t)\Psi_n = 0. \tag{2.2.18}$$

Moreover, there is a t-independent, finite constant $C_{(2.2.19)}(h)$ such that

$$\begin{aligned}\|a_t(h)(\Psi_{\lambda_0} - \Psi_n)\| &= \|e^{itH} a(h_t) e^{-itH} (\Psi_{\lambda_0} - \Psi_n)\| \\ &= \|a(h_t)(H-b+1)^{-\frac{1}{2}} e^{-itH} (H-b+1)^{\frac{1}{2}} (\Psi - \Psi_n)\| \\ &\leq \||h|/\sqrt{\omega}\|_2 \, \|H_f^{\frac{1}{2}} (H-b+1)^{-\frac{1}{2}}\| \|\Psi - \Psi_n\|_{(H-b+1)^{1/2}} \\ &= C_{(2.2.19)}(h) \|\Psi - \Psi_n\|_{(H-b+1)^{1/2}} \end{aligned} \tag{2.2.19}$$

and

$$\begin{aligned}\|a_\pm(h)\Psi_{\lambda_0}\| &\leq \lim_{t\to\pm\infty} (\|a_t(h)(\Psi_{\lambda_0} - \Psi_n)\| + \|a_t(h)\Psi_n\|) \\ &\leq C_{(2.2.19)}(h)\|\Psi - \Psi_n\|_{(H-b+1)^{1/2}} \end{aligned} \tag{2.2.20}$$

holds true for all $n \in \mathbb{N}$, where we have use the standard inequalities (1.3.1), Lemma 1.3.2 and (2.2.18). Taking the limit $n \to \infty$ proves the claim (iv).

(v) We consider the same sequence $(\Psi_n)_{n\in\mathbb{N}}$ as in (iv) and, for all $n \in \mathbb{N}$, we observe that, by (i) and definition in (2.1.2), it holds

$$\langle a(h)^*_\pm \Psi_{\lambda_0}, a(l)^*_\pm \Psi_{\lambda_0}\rangle = \lim_{t\to\pm\infty} \langle a(h_t)^* \Psi_{\lambda_0}, a(l_t)^* \Psi_{\lambda_0}\rangle. \tag{2.2.21}$$

Furthermore, using the CCR in (1.2.17), we find for all $n \in \mathbb{N}$ that

$$\begin{aligned}&\langle a(h_t)^* \Psi_{\lambda_0}, a(l_t)^* \Psi_n\rangle = \langle \Psi_{\lambda_0}, a(h_t) a(l_t)^* \Psi_n\rangle \qquad (2.2.22)\\ &= \langle \Psi_{\lambda_0}, (a(l_t)^* a(h_t) + [a(h_t), a(l_t)^*])\, \Psi_n\rangle = \langle a(l_t)\Psi_{\lambda_0}, a(h_t)\Psi_n\rangle + \langle \Psi_{\lambda_0}, \Psi_n\rangle \langle h, l\rangle_2 \end{aligned}$$

holds. We may control the limit $n \to \infty$ of this identity by

$$\begin{aligned}&|\langle a(h_t)^* \Psi_{\lambda_0}, a(l_t)^* (\Psi_{\lambda_0} - \Psi_n)\rangle| \leq \|a(h_t)^* \Psi_{\lambda_0}\| \, \|a(l_t)^* (\Psi_{\lambda_0} - \Psi_n)\| \qquad (2.2.23)\\ &\leq (\|h\|_2 + \|h/\sqrt{\omega}\|_2) \|\Psi_{\lambda_0}\|_{(H-b+1)^{1/2}} (\|l\|_2 + \|l/\sqrt{\omega}\|_2) \|\Psi_{\lambda_0} - \Psi_n\|_{(H-b+1)^{1/2}}, \end{aligned}$$

and likewise,

$$\begin{aligned}&|\langle a(l_t)\Psi_{\lambda_0}, a(h_t)(\Psi_{\lambda_0} - \Psi_n)\rangle| \leq \|a(l_t)\Psi_{\lambda_0}\| \, \|a(h_t)(\Psi_{\lambda_0} - \Psi_n)\| \qquad (2.2.24)\\ &\leq (\|l\|_2 + \|l/\sqrt{\omega}\|_2) \|\Psi_{\lambda_0}\|_{(H-b+1)^{1/2}} (\|h\|_2 + \|h/\sqrt{\omega}\|_2) \|\Psi_{\lambda_0} - \Psi_n\|_{(H-b+1)^{1/2}}, \end{aligned}$$

which are ensured by the standard estimates (1.3.1) and Lemma 1.3.2. These bounds allow to take the limit $n \to \infty$ of identity (2.2.23) which yields

$$\langle a(h_t)^* \Psi_{\lambda_0}, a(l_t)^* \Psi_{\lambda_0}\rangle = \langle a(l_t)\Psi_{\lambda_0}, a(h_t)\Psi_{\lambda_0}\rangle + \langle \Psi_{\lambda_0}, \Psi_{\lambda_0}\rangle \langle h, l\rangle_2$$

Finally, recalling (2.2.21) and exploiting (iv) that states $a_\pm(h)\Psi_{\lambda_0} = 0$, we find

$$\langle a(h)^*_\pm \Psi_{\lambda_0}, a(l)^*_\pm \Psi_{\lambda_0}\rangle = \lim_{t\to\pm\infty} \langle a(h_t)^* \Psi_{\lambda_0} a(l_t)^* \Psi_{\lambda_0}\rangle = \langle \Psi_{\lambda_0}, \Psi_{\lambda_0}\rangle \langle h, l\rangle_2$$

which concludes the proof of (v).

(vi) Let $t \in \mathbb{R}$. Thanks to the standard estimate (1.3.1), we find

$$\begin{aligned}
&\|a_t(h)(H_f+1)^{-\frac{1}{2}}\| = \|e^{itH}a(h_t)(H-b+1)^{-\frac{1}{2}}e^{-itH}(H-b+1)^{\frac{1}{2}}(H_f+1)^{-\frac{1}{2}}\| \\
&\leq \|a(h_t)(H-b+1)^{-\frac{1}{2}}\|\,\|(H-b+1)^{\frac{1}{2}}(H_f+1)^{-\frac{1}{2}}\| \\
&\leq \|h/\sqrt{\omega}\|_2\,\|H_f^{\frac{1}{2}}(H-b+1)^{-\frac{1}{2}}\|\,\|(H-b+1)^{\frac{1}{2}}(H_f+1)^{-\frac{1}{2}}\|. \qquad (2.2.25)
\end{aligned}$$

Lemma 1.3.2 ensures that the right-hand side of (2.2.25) is bounded by a finite constant $C(h)$ which depends only on h. This proves the first inequality of (vi). The proof of the second is omitted here as it is almost identical.

□

Theorem 2.2.2 (Intermediate Scattering Formula)**.** *For $h, l \in \mathfrak{h}_0$, the two-body transition matrix coefficient $T(h, l)$ defined in* (2.1.6) *fulfills*

$$T(h,l) = -2\pi i g\|\Psi_{\lambda_0}\|^{-2}\Big\langle \sigma_1\Psi_{\lambda_0}, a_-(W)^*\Psi_{\lambda_0}\Big\rangle, \qquad (2.2.26)$$

where $W \in \mathfrak{h}_0$ is given by

$$\mathbb{R}^3 \ni k \mapsto W(k) := |k|^2 l(k) \int \mathrm{d}\Sigma\, \overline{h(|k|,\Sigma)} f(|k|,\Sigma). \qquad (2.2.27)$$

Here, we use spherical coordinates $k = (|k|, \Sigma)$ with Σ being the solid angle.

Proof. Again, we prove the statement only for the case of a massless scalar field, i.e., for the dispersion relation $\omega(k) = |k|$. The proof for massive fields with dispersion relation $\omega(k) = \sqrt{k^2 + m^2}$ and $m > 0$ follows analogously.

Let $h, l \in \mathfrak{h}_0$. Thanks to Lemma 2.2.1 (i) and the fact that the ground state Ψ_{λ_0} lies in $\mathcal{D}(H) = \mathcal{K} \otimes \mathcal{D}(H_f)$, c.f. [51, Theorem 1] and Proposition 1.3.3, the transmission matrix coefficient given in (2.1.6), i.e.,

$$T(h,l) = S(h,l) - \langle h,l\rangle_2 = \|\Psi_{\lambda_0}\|^{-2}\langle a_+(h)^*\Psi_{\lambda_0}, a_-(l)^*\Psi_{\lambda_0}\rangle - \langle h,l\rangle_2 \qquad (2.2.28)$$

is well-defined. Lemma 2.2.1 (iv) and (v) implies that

$$(2.2.28) = \|\Psi_{\lambda_0}\|^{-2}\langle [a_+(h)^* - a_-(h)^*]\Psi_{\lambda_0}, a_-(l)^*\Psi_{\lambda_0}\rangle. \qquad (2.2.29)$$

Using Lemma 2.2.1 (ii), we obtain

$$(2.2.28) = -ig\|\Psi_{\lambda_0}\|^{-2}\int_{-\infty}^{\infty} \mathrm{d}s\langle \Psi_{\lambda_0}, e^{isH}\sigma_1 e^{-isH}a_-(l)^*\Psi_{\lambda_0}\rangle\langle h_s, f\rangle_2. \qquad (2.2.30)$$

Finally, we use Lemma 2.2.1 (iii) to get

$$\begin{aligned}
(2.2.28) &= -ig\|\Psi_{\lambda_0}\|^{-2}\int_{-\infty}^{\infty} \mathrm{d}s\,\Big\langle e^{-isH}\Psi_{\lambda_0}, \sigma_1 a_-(l_s)^* e^{-isH}\Psi_{\lambda_0}\Big\rangle\,\langle h_s, f\rangle_2 \\
&= -ig\|\Psi_{\lambda_0}\|^{-2}\int_{-\infty}^{\infty} \mathrm{d}s\,\langle \sigma_1\Psi_{\lambda_0}, a_-(l_s)^*\Psi_{\lambda_0}\rangle\,\langle h_s, f\rangle_2. \qquad (2.2.31)
\end{aligned}$$

We insert the definition of the asymptotic creation operator in (2.1.2) to find

$$(2.2.28) = -ig\|\Psi_{\lambda_0}\|^{-2} \int_{-\infty}^{\infty} \mathrm{d}s \lim_{t\to-\infty} \langle \sigma_1 \Psi_{\lambda_0}, a_t(l_s)^* \Psi_{\lambda_0} \rangle \langle h_s, f \rangle_2. \tag{2.2.32}$$

Next, it is possible to interchange the ds integral and the limit $t \to -\infty$. This can be seen as follows. A two-fold partial integration implies that there is a constant C such that, for all $s \in \mathbb{R}$, we get

$$|\langle h_s, f \rangle_2| \leq C \frac{1}{1+|s|^2}. \tag{2.2.33}$$

By applying Lemma 2.2.1 (vi), we infer that there is a finite constant $C_{(2.2.34)}(l) > 0$ such that for all $s \in \mathbb{R}$

$$\begin{aligned} |\langle \sigma_1 \Psi_{\lambda_0}, a_t(l_s)^* \Psi_{\lambda_0} \rangle| &\leq \|\sigma_1 \Psi_{\lambda_0}\| \, \|a_t(l_s)^* (H_f+1)^{-\frac{1}{2}}\| \, \|(H_f+1)^{\frac{1}{2}} \Psi_{\lambda_0}\| \\ &\leq C_{(2.2.34)}(l) \|\Psi_{\lambda_0}\| \|\Psi_{\lambda_0}\|_{H_f} \end{aligned} \tag{2.2.34}$$

holds true. Both estimates, (2.2.33) and (2.2.34), give an integrable bound of the ds-integrand in (2.2.32) that is uniform in t. Hence, by dominated convergence, we have the equality

$$\begin{aligned} (2.2.28) &= -ig\|\Psi_{\lambda_0}\|^{-2} \lim_{t\to-\infty} \int_{-\infty}^{\infty} \mathrm{d}s \, \langle \sigma_1 \Psi_{\lambda_0}, a_t(l_s)^* \Psi_{\lambda_0} \rangle \langle h_s, f \rangle_2 \\ &= -ig\|\Psi_{\lambda_0}\|^{-2} \lim_{t\to-\infty} e^{-it\lambda_0} \int_{-\infty}^{\infty} \mathrm{d}s \left\langle e^{-itH} \sigma_1 \Psi_{\lambda_0}, a(l_{s+t})^* \Psi_{\lambda_0} \right\rangle \langle h_s, f \rangle_2, \end{aligned} \tag{2.2.35}$$

where in the last step we have inserted definition (2.1.2) and exploited the ground state property (1.3.22).

In order to rewrite this integral in form of (2.2.26)-(2.2.27), we shall use the following approximation argument. Let

$$\mathcal{H}_0 := \mathcal{K} \otimes \mathcal{F}_{\text{fin}}[\mathfrak{h}_0] \tag{2.2.36}$$

be the set of states with only finitely many bosons, i.e.,

$$\begin{aligned} \mathcal{F}_{\text{fin}}[\mathfrak{h}_0] := \Big\{ \Psi = (\psi^{(n)})_{n\in\mathbb{N}_0} \in \mathcal{F}[\mathfrak{h}] \,\big|\, \exists N \in \mathbb{N}_0 : \psi^{(n)} = 0 \, \forall n \geq N, \\ \forall n \in \mathbb{N} : \psi^{(n)} \in C_c^\infty(\mathbb{R}^{3n} \setminus \{0\}, \mathbb{C}) \Big\}. \end{aligned} \tag{2.2.37}$$

Note that $\mathcal{H}_0$ is a dense subset of $\mathcal{H}$ with respect to the norm in $\mathcal{H}$ and it is dense in the domain of H_f with respect to the graph norm of the operator H_f defined by $\|\cdot\|_{H_f} := \|H_f \cdot\| + \|\cdot\|$. Hence, for $t \in \mathbb{R}$, there are sequences $(\Psi_m)_{m\in\mathbb{N}}$, $(\Phi_m^t)_{m\in\mathbb{N}}$ in $\mathcal{H}_0$ with $\|\Psi_m - \Psi_{\lambda_0}\|_{H_f} \to 0$, as $m \to \infty$, and $\left\|\Phi_m^t - e^{-itH}\sigma_1\Psi_{\lambda_0}\right\| \to 0$, as $m \to \infty$. Then, Lemma 1.3.1, applied in the same fashion as in (2.2.34), implies that

$$\lim_{m\to\infty} \left\langle \Phi_m^t, a(l_{s+t})^* \Psi_m \right\rangle = \left\langle e^{-itH} \sigma_1 \Psi_{\lambda_0}, a(l_{s+t})^* \Psi_{\lambda_0} \right\rangle, \tag{2.2.38}$$

uniformly in s. Thanks to the bound (2.2.33), we may apply dominated convergence theorem to conclude that

$$\lim_{m\to\infty}\int_{-\infty}^{\infty}\mathrm{d}s\left\langle\Phi_m^t, a(l_{s+t})^*\Psi_m\right\rangle\langle h_s, f\rangle_2 = \int_{-\infty}^{\infty}\mathrm{d}s\left\langle e^{-itH}\sigma_1\Psi_{\lambda_0}, a(l_{s+t})^*\Psi_{\lambda_0}\right\rangle\langle h_s, f\rangle_2. \tag{2.2.39}$$

Now, we study the integrals in the left hand side of Eq. (2.2.39). The advantage of the sequences $(\Psi_m)_{m\in\mathbb{N}}$, $(\Phi_m^t)_{m\in\mathbb{N}}$ is that they allow to use point-wise annihilation operators in the following manner:

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}s\left\langle\Phi_m^t, a(l_{s+t})^*\Psi_m\right\rangle\langle h_s, f\rangle_2 \qquad (2.2.40)\\ &\quad=\int_{-\infty}^{\infty}\mathrm{d}s\int d^3k' e^{-is\omega(k')}e^{-it\omega(k')}l(k')\left\langle a(k')\Phi_m^t,\Psi_m\right\rangle\int \mathrm{d}^3k\,\overline{h(k)}f(k)e^{is\omega(k)}\\ &\quad=\int_{-\infty}^{\infty}ds\Big[\Big(\int_{-\infty}^{\infty}\mathrm{d}r\,e^{isr}\Theta(r)u(r)\Big)\Big(\int_{-\infty}^{\infty}\mathrm{d}r'\,e^{-isr'}\Theta(r')v_m^t(r')\Big)\Big],\end{aligned}$$

where Θ is the Heaviside function and we use spherical coordinates and the abbreviations

$$u(r) := r^2\int\mathrm{d}\Sigma\,\overline{h(r,\Sigma)}f(r,\Sigma)\quad\text{and}\quad v_m^t(r') := e^{-itr'}r'^2\int\mathrm{d}\Sigma'\,l(r',\Sigma')\left\langle a(r',\Sigma')\Phi_m^t,\Psi_m\right\rangle.$$

By definition, v_m^t and u belong to $C_c^\infty(\mathbb{R}\setminus\{0\})$ so that the integrals with respect to r and r' above can be regarded as Fourier transform, introduced in Definition 5.3.3 below, i.e.,

$$(2.2.40) = \int_{-\infty}^{\infty}\mathrm{d}s\,\overline{\mathfrak{F}\left[\overline{\Theta u}\right]}(s)\mathfrak{F}\left[\Theta v_m^t\right](s) \tag{2.2.41}$$

holds true. Plancherel's identity yields for all $t\in\mathbb{R}$

$$\begin{aligned}(2.2.40) &= 2\pi\int_{-\infty}^{\infty}dr'\,\Theta u\Theta v_m^t(r')\\ &= 2\pi\int_0^{\infty}dr'\,r'^2\int\mathrm{d}\Sigma\,\overline{h(r',\Sigma)}f(r',\Sigma)e^{-itr'}r'^2\int\mathrm{d}\Sigma'\,l(r',\Sigma')\left\langle a(r',\Sigma')\Phi_m^t,\Psi_m\right\rangle\\ &= 2\pi\left\langle a(W_t)\Phi_m^t,\Psi_m\right\rangle = 2\pi\left\langle\Phi_m^t, a(W_t)^*\Psi_m\right\rangle \qquad (2.2.42)\end{aligned}$$

where we have used the definition of W in (2.2.27) and the definition (2.1.2), in particular, the notation $W_t(k) = W(k)e^{-it\omega(k)}$. Using Lemma 1.3.1, applied in the same fashion as in (2.2.34), allows to carry out the limit $m\to\infty$ which results in

$$(2.2.39) = \lim_{m\to\infty}(2.2.42) = 2\pi\left\langle e^{-itH}\sigma_1\Psi_{\lambda_0},\ a(W_t)^*\Psi_{\lambda_0}\right\rangle. \tag{2.2.43}$$

This together with (2.2.35) and Lemma 2.2.1 guarantees

$$\begin{aligned}(2.2.28) &= -ig\|\Psi_{\lambda_0}\|^{-2}\lim_{t\to-\infty}e^{-it\lambda_0}2\pi\left\langle e^{-itH}\sigma_1\Psi_{\lambda_0}, a(W_t)^*\Psi_{\lambda_0}\right\rangle \qquad (2.2.44)\\ &= -2\pi ig\|\Psi_{\lambda_0}\|^{-2}\lim_{t\to-\infty}\langle\sigma_1\Psi_{\lambda_0}, a_t(W)^*\Psi_{\lambda_0}\rangle \qquad (2.2.45)\\ &= -2\pi ig\|\Psi_{\lambda_0}\|^{-2}\langle\sigma_1\Psi_{\lambda_0}, a_-(W)^*\Psi_{\lambda_0}\rangle, \qquad (2.2.46)\end{aligned}$$

which concludes the proof. □

3. Main results and comparison

This chapter summarizes our central results. We derive a formula for one-boson scattering processes which relates the corresponding matrix elements to the resonance of the model. In particular, we present a general (perturbative) formula which holds true for massless as well as massive scalar fields and a non-perturbative formula for massless scalar fields.

We consider a class of Spin-Boson models fulfilling one of the following sets of conditions:

(A) For massless scalar fields ($m = 0$), we take $\omega(k) = |k|$ and $\mu \in (0, 1/2)$ (see (1.2.4)). The boson form factor f is defined in (1.2.3).

(B) For massive scalar fields, we take $e_0 = 0 < m < e_1$ with $e_1 - e_0 \notin m\mathbb{N}$ (see Assumption 6.0.1 below) and $\omega(k) = \sqrt{k^2 + m^2}$. Moreover, we choose the boson form factor f to be spherical symmetric (in order to simplify our notation), satisfying $f, Df, D^2 f \in L^2(\mathbb{R}^3)$, where D is the generator of dilations introduced in Definition 6.4.1 (ii) below, and $f(\sqrt{e_1^2 - m^2}) > 0$ (see (6.0.1) below). Here, we use a slight abuse notation and identify $f(k) \equiv f(|k|)$. In particular, f does not have to be analytic and the infrared singularity is not an issue here. Note that (1.2.3) meets these conditions.

Theorem 3.0.1 (General Scattering Formula)**.** *We assume that either condition (A) or (B) holds true. Then, for sufficiently small g, θ in the set $\mathcal{S}$ (defined in* (5.2.2) *below), and for all $h, l \in \mathfrak{h}_0$, there is a complex-valued function $\lambda \equiv \lambda(g)$ such that the transition matrix coefficients for one-boson processes are given by*

$$T(h, l) = T_P(h, l) + R(h, l), \tag{3.0.1}$$

where

$$T_P(h, l) = \int \mathrm{d}^3k\mathrm{d}^3k'\, \delta(\omega(k) - \omega(k')) \frac{|k|}{\omega(k)} T_P(k, k') \tag{3.0.2}$$

and

$$T_P(k, k') = 4\pi i g^2 \|\Psi_{\lambda_0}\|^{-2} \overline{h(k)} l(k') f(k) f(k') \frac{\operatorname{Re}\lambda - \lambda_0}{(\omega(k') + \lambda_0 - \lambda)\left(\omega(k') - \lambda_0 + \overline{\lambda}\right)}. \tag{3.0.3}$$

Moreover, there is a constant $C(h, l)$ (that does not depend on g) such that

$$|R(h, l)| \leq \begin{cases} C(h, l) g^2 g |\log g| & \textit{if condition (A) is fulfilled.} \\ C(h, l) g^2 g^{1/3} |\log g| & \textit{if condition (B) is fulfilled.} \end{cases}, \tag{3.0.4}$$

$T_P(h,l)$ is the leading term in terms of powers of g for small g, and $R(h,l)$ is regarded as the error term. This is justified by Remark 5.1.5 below.

This statement is a direct consequence of Theorems 5.1.3 and 6.3.2 below (c.f. Remark 5.1.6).

Remark 3.0.2. *(i) If condition (A) holds true, $\lambda \equiv \lambda(g)$ (defined in Theorem 3.0.1) is the resonance of the model which is constructed in Theorem 4.2.1. Moreover, if condition (B) is fulfilled, $\lambda \equiv \lambda(g)$ is defined in* (6.3.5) *below. With similar techniques as described in Chapter 4, it can be shown that, in this case, $\lambda \equiv \lambda(g)$ is the resonance of the model up to an error term of order $g^{2+\iota}$ for some $\iota > 0$. For both cases, there is a constant $\mathbf{c}$ such that $\operatorname{Im} \lambda \leq -g^2\mathbf{c} < 0$. This can be seen from* (5.1.9) *and* (6.3.5)*, respectively.*

(ii) The difference in the error term given in (3.0.4) *for the two settings (A) and (B) is solely due to the different techniques exploited in the proofs. As it will be discussed in detail in the remainder of this work, in setting (A) a multiscale analysis and complex dilation is used while in setting (B) we rely on Mourre theory. In Section 6.3, we give an overview over the advantages and disadvantages of both methods.*

We emphasize that the *General Scattering Formula* stated in Theorem 3.0.1 covers both massless and massive scalar fields. For the massless model, e_0 and e_1 are not isolated points in the spectrum of the dilated free Hamiltonian H_0^θ (defined in (1.3.24)), and hence, one can not apply regular perturbation theory in order to construct the ground-state and the resonance of the full Hamiltonian H^θ (defined in (1.3.24)). We treat this so-called infrared problem with an extension of Pizzo's multiscale method in Chapter 4 below. We write the matrix elements of the time-evolution operator in terms of the complex dilated resolvent and use the properties, obtained by the multiscale analysis mentioned above, in order to control it in the scattering regime (see Chapter 5 below). The massive model lacks an infrared problem, and hence, the construction of the resonance and the ground-state can be obtained by regular perturbation theory. On the other hand, the non-zero boson mass introduces a new complication as the spectrum of the complex dilated free Hamiltonian exhibits lines of spectrum attached to every multiple of the boson rest mass energy starting from the ground and excited state energies. This leads to an absence of decay of the complex dilated resolvent close to the real line. Hence, the control of the time-evolution operator in the scattering regime has to be achieved by a different method when considering massive scalar fields (see Chapter 6 below).

In conclusion, the structure of the *General Scattering Formula* presented in Theorem 3.0.1 is not affected by the infrared problem which occurs for massless fields, however, the technicalities in the proofs are completely different in each scenario. For a more detailed discussion of the different methods we refer to Section 6.3 below.

In addition to the perturbative result presented in Theorem 3.0.1, we obtain a non-perturbative formula for the one-boson scattering matrix elements in the massless Spin-Boson model.

Theorem 3.0.3 (Exact Scattering Formula for the massless Spin-Boson model)**.** *We assume that condition (A) holds true. Then, for sufficiently small g, θ in the set $\mathcal{S}$ (defined in* (5.2.2) *below), and for all $h, l \in \mathfrak{h}_0$, the transition matrix coefficients for one-boson processes are given by*

$$T(h,l) = \int \mathrm{d}^3k \mathrm{d}^3k' \, \delta(\omega(k) - \omega(k')) T(k,k'), \tag{3.0.5}$$

where

$$\begin{aligned} T(k,k') = & -2\pi i g^2 \overline{h(k)} l(k') f(k) f(k') \|\Psi_{\lambda_0}\|^{-2} \\ & \times \left(\left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - \lambda_0 - |k'|\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle + \left\langle \sigma_1 \Psi_{\lambda_0}^{\theta}, \left(H^{\overline{\theta}} - \lambda_0 + |k'|\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}} \right\rangle \right). \end{aligned} \tag{3.0.6}$$

The statement above is proven Chapter 5 – see Theorem 5.1.1. We point out that, in comparison to Theorem 3.0.1, this is a major improvement since it connects the integral kernel of the scattering matrix elements for one-boson processes to the dilated resolvent in a precise formula without using perturbation theory. This result for the class of quantum field models at hand can be seen as an analogue of the one obtained in [63] for the case of n-body Schrödinger operators.

To the best of our knowledge Theorems 3.0.1 and 3.0.3 are among the first results towards a clarification of the relation between resonances and scattering theory in models of quantum field theory. It has to be emphasized, however, that the relation between the imaginary value of the resonance and the decay rate of the unstable excited state has been established rigorously in various models of quantum field theory in several articles [2, 49, 3, 18]. In [12], a rigorous mathematical justification of Bohr's frequency condition is derived, using an expansion of the scattering amplitudes with respect to powers the fine-structure constant for the Pauli-Fierz model. In [17], the photoelectric effect is studied for a model of an atom with a single bound state, coupled to the quantized electromagnetic field.

4. Ground state, resonance and spectral properties in the massless Spin-Boson model

We analyze the massless Spin-Boson model, as introduced in Chapter 1.2, where we set the mass of the scalar field $m = 0$ and fix an infrared regularization parameter $\mu \in (0, 1/2)$ (see (1.2.4)). Note that this yields the relativistic dispersion relation $\omega(k) = |k|$. We construct the ground state and the resonance of the model and prove analytic properties of them. In addition, we derive resolvent estimates, which allow for spectral localization in cuspidal domains.

In the massless case, neither e_0 nor e_1 are isolated points in the spectrum of the free Hamiltonian. This is why the interacting ground-state and the resonance (λ_0 and λ_1) cannot be constructed using standard results from regular perturbation theory. Several technologies were developed to overcome this difficulty. Two succesful methods that recently received a lot of attention are: they are referred to as Pizzo's multiscale method (see e.g. [58, 59, 10, 9]) and the renormalization group method (see, e.g., [13, 15, 14, 11, 16, 7, 43, 45, 62, 35, 24]). In both cases, a family of spectrally dilated Hamiltonians is analyzed since this allows for complex eigenvalues. In this chapter we employ the Pizzo's multiscale method. This technique invokes an infrared cut-off which is then removed using an inductive scheme. In each step of the induction, lower and lower boson momenta are added to the interaction and regular perturbation theory is used to construct the respective ground-state and resonance. In order to reach the limit of no infrared cut-off good control over the closing gap is essential. Note that such a procedure has been introduced for the construction of resonances in the Pauli-Fierz model [9, 10]. In this work we also construct resonances (and ground-state eigenvalues) but this is not our main purpose. Our main purpose is to prove that resonances are analytic with respect to the dilation parameter and coupling constant, and furthermore, to provide certain spectral and resolvent estimates that allow for the control of the dynamics including the scattering regime. In Chapter 5, these estimates are employed to address scattering theory for the model at hand. In [18], the time evolution of this model is studied using the spectral renormalization method. Some results derived therein are similar to some of ours, however, utilizing different methods, respectively.

What we call resonances and ground-states multiscale analysis is an inductive construction of a sequence of Hamiltonians that enjoy infrared cut-offs and satisfy certain properties. As the parameter of the sequence tends to infinity these cut-offs are removed. Our mutliscale analysis is the content of Theorem 4.4.5. Its basic scheme is presented in Section 4.4.2 and the proof of Theorem 4.4.5 is carried out in Section 4.4.3. Our proofs of analyticity and of resolvent and spectral estimates are not part of our multiscale analysis employed in the contruction of the resonance and ground-state energy, they only use it as mathematical input. The latter results are presented in Section 4.5 (spectral and resolvent estimates) and Section 4.6 (analyticity). Theorem 4.4.5 is only an intermediate but necessary step. As we mention above, the method of Pizzo multiscale analysis for resonances is introduced in [9, 10]. However, the results in [9, 10] cannot be used directly to prove analyticity because many of the estimations therein consider the dilation parameter, θ, to be purely imaginary whereas analyticity requires estimates that are uniform for θ in an open set. Thus, although it does not involve major obstacles, for the sake of analyticity, many of the given calculations and some of the proofs need to be redone. For the convenience of the reader and in order to keep this work self-contained we provide them in the proof of Theorem 4.4.5. Note that Theorem 4.4.5 is applied to the Spin-Boson model while [9, 10] address the Pauli-Fierz model. This gives us the opportunity to review the Pizzo multiscale technique for a non-trivial but more tractable model.

In Section 4.5 (in particular, in Subsection 4.5.1) we introduce a new inductive scheme that is used to study resolvent and spectral estimates. This scheme is independent and different from the scheme used in Section 4.4 to construct resonances. It allows to localize the spectrum in two cones with vertices at the location of the resonance and ground-state energy, respectively, and allows for arbitrary small apex angles provided the coupling constant is sufficiently small. We want to emphasize that such a result requires a more subtle analysis than localizing the spectrum in cuspidal domains. Additionally, we provide estimates for the resolvent operator in the vicinity of the cones.

The study of analytic properties of resonances and ground-state eigenvalues in the context of non-relativistic quantum field theory has been the source of several studies. These papers use the method of spectral renormalization. In [46], a large class of models of quantum field theory was analyzed and analyticity of the ground-state with respect to the coupling constant was proven under the assumption that this ground-state is non-degenerate. The existence of a unique ground-state and its analyticity with respect to the coupling constant was shown in [48] for the Spin-Boson model without an infrared regularization and, in [47], for the Pauli-Fierz model. Furthermore, in [24], a model describing the interaction of an atom with its quantized electromagnetic field was studied and it was proven that the excited states are analytic functions of the momentum of the atom and of the coupling constant. Likewise, in [1], it is shown that the ground-state energy of the translationally invariant Nelson model is an analytic function of the coupling constant and the total momentum.

In [21], to the best of our knowledge, we give the first extension of the Pizzo multiscale method that provides a ready access to analyticity properties that essentially amounts to proving it for isolated eigenvalues only and exploiting that uniform limits of analytic functions are analytic.

In [13, 15, 14], the renormalized group technique was invented and applied in order to construct the ground state and resonances for the confined Pauli-Fierz model. Moreover, resolvent and spectral estimates were obtained therein. Based on this new method, several simplifications and applications were developed in a variety of works [11, 16, 24, 45, 43, 7, 24, 62, 35, 7]. The Pizzo multiscale analysis was first invented in [58, 59] and then adapted in order to gain access to spectral and resolvent estimates and the construction of ground-states in [9, 10]. In [18], resolvent and spectral estimates are derived in order to control the time-evolution operator of the Spin-Boson model and, in [25], smoothness of the resolvent and local decay of the photon dynamics for quantum states in a spectral interval just above the ground state energy was proven.

4.1. Infrared cut-offs and multiscale scheme

As discussed at the beginning of Chapter 4 it is not possible to construct the ground-state and the resonance using regular perturbation theory since e_0 and e_1 are not isolated points in the spectrum of H_0^θ. One way to circumvent this problem is to employ a multiscale analysis. For this purpose, we introduce a family of Hamiltonians $H^{(n),\theta}$ which have two isolated (complex) eigenvalues $\lambda_i^{(n)}$ in small neighborhoods of e_i, $i \in \{0,1\}$. For every $n \in \mathbb{N}$, $H^{(n),\theta}$ enjoys an infared cut-off which is removed as n tends to infinity.

Definition 4.1.1. *We fix a real number* $\boldsymbol{\nu} \in (0, \pi/16)$ *and for every* $\theta \in \mathbb{C}$ *we set* $\nu := \operatorname{Im}\theta$*. We define*

$$\mathcal{S} := \Big\{\theta \in \mathbb{C} : -10^{-3} < \operatorname{Re}\theta < 10^{-3} \quad \textit{and} \quad \boldsymbol{\nu} < \operatorname{Im}\theta < \pi/16\Big\}. \tag{4.1.1}$$

For $\theta \in \mathcal{S}$ *and* $n \in \mathbb{N}$*, we define:*

(i) The sequence of infrared cut-offs $\{\rho_n\}_{n\in\mathbb{N}}$ *with* $\rho_n := \rho_0\rho^n$ *for real* $0 < \rho_0 < \min(1, e_1/4)$ *and* $0 < \rho < 1$*. In Definition 4.4.2 below we specify additional properties of it.*

(ii) The cut-off Hilbert space of one particle, $\mathfrak{h}^{(n)}$*:*

$$\mathfrak{h}^{(n)} := L^2(\mathbb{R}^3 \setminus \mathcal{B}_{\rho_n}), \quad \mathcal{B}_{\rho_n} := \Big\{x \in \mathbb{R}^3 : |x| < \rho_n\Big\}. \tag{4.1.2}$$

The Fock space with one particle sector $\mathfrak{h}^{(n)}$ *is defined as in* (1.2.7)*, and we denote it by* $\mathcal{F}[\mathfrak{h}^{(n)}]$ *and its vacuum state by* $\Omega^{(n)}$*. We set*

$$\mathcal{H}^{(n)} := \mathcal{K} \otimes \mathcal{F}[\mathfrak{h}^{(n)}]. \tag{4.1.3}$$

The free boson energy operator with an infrared cutoff is defined on $\mathcal{F}[\mathfrak{h}^{(n)}]$ *by* (1.2.1)*, we denote it by* $H_f^{(n),0} \equiv H_f^{(n)}$*. We set*

$$H_f^{(n),\theta} := e^{-\theta} H_f^{(n),0}. \tag{4.1.4}$$

For every function $h \in \mathfrak{h}^{(n)}$*, we define creation and annihilation operators,* $a_n(h)$*,* $a_n^*(h)$*, on* $\mathcal{F}[\mathfrak{h}^{(n)}]$ *according to* (1.2.10) *and* (1.2.11)*. We use the same formula for functions* $h \in \mathfrak{h}$*, then it is understood that we take the restriction of* h *to* $\mathbb{R}^3 \setminus \mathcal{B}_{\rho_n}$*.*

We define the following family of Hamiltonians (densely defined on $\mathcal{H}^{(n)}$ - see Remark 1.2.1)

$$H_0^{(n),\theta} := K + H_f^{(n),\theta}, \qquad V^{(n),\theta} := \sigma_1 \otimes \Big(a_n(f^{\overline{\theta}}) + a_n(f^{\theta})^* \Big) \tag{4.1.5}$$

and

$$H^{(n),\theta} := H_0^{(n),\theta} + gV^{(n),\theta}. \tag{4.1.6}$$

The Hamiltonians $H^{(n),\theta}$ turn out to have gaps between the eigenvalues $\lambda_i^{(n)}$ and the rest of the spectrum of $H^{(n),\theta}$. This allows us to define Riesz projections, $P_i^{(n)}$, corresponding to the eigenvalues $\lambda_i^{(n)}$ and use regular perturbation theory for each $n \in \mathbb{N}$. In an inductive scheme, one can obtain explicit estimates on the resolvents and the eigenvalues in each step. Below, we prove that the sequences $(\lambda_i^{(n)})_{n\in\mathbb{N}}$ converge and the interacting ground-state energy λ_0 and resonance energy λ_1 of H^θ are the limits

$$\lambda_i := \lim_{n\to\infty} \lambda_i^{(n)}, \qquad i = 0, 1. \tag{4.1.7}$$

We define

$$\mathfrak{h}^{(n,\infty)} := L^2(\mathcal{B}_{\rho_n}). \tag{4.1.8}$$

We denote the corresponding Fock space by $\mathcal{F}[\mathfrak{h}^{(n,\infty)}]$ (it is defined as in (1.2.7)), with vacuum state $\Omega^{(n,\infty)}$. It is straightforward to verify that $\mathcal{H}$ is isomorphic to $\mathcal{H}^{(n)} \otimes \mathcal{F}[\mathfrak{h}^{(n,\infty)}]$, and therefore, we identify

$$\mathcal{H} \equiv \mathcal{H}^{(n)} \otimes \mathcal{F}[\mathfrak{h}^{(n,\infty)}]. \tag{4.1.9}$$

We prove below that the sequence $(P_i^{(n)} \otimes P_{\Omega^{(n,\infty)}})_{n\in\mathbb{N}}$, where $P_{\Omega^{(n,\infty)}}$ is the orthogonal projection onto the vector space generated by $\Omega^{(n,\infty)}$, converges to an eigenprojection corresponding to the eigenvalue λ_i.

4.2. Main results of this chapter

Here, we state the main results of this chapter. All proofs are presented in the next sections. In Proposition 4.2.1 below we state the existence of the ground-state eigenvalue and the resonance of H^θ. A similar result, for a more complicated model (Pauli-Fierz), is proved in [9]. The strategy of proof of Proposition 4.2.1 is based on the methods introduced in [9] but it differs from the proof therein because, here, all our estimates must be independent of $\theta \in \mathcal{S}$. As emphasized earlier, the existence of the resonance and the ground-state is not our focus but is only provided in order for this work to be self-contained.

The next proposition is proved in Section 4.5.3 below.

Proposition 4.2.1 (Construction of the ground-state and the resonance)**.** *For every ρ, ρ_0 sufficiently small (see Definition 4.4.2) there is a constant $g_0 > 0$ (that depends on ρ, ρ_0 and $\boldsymbol{\nu}$) such that, for every $\theta \in \mathcal{S}$ (see (4.1.1)) and every $g \in D(0, g_0)$, the (complex) number*

$$\lambda_i := \lim_{n\to\infty} \lambda_i^{(n)}, \quad i = 0, 1 \tag{4.2.1}$$

is an eigenvalue of H^θ and the range of

$$P_i := \lim_{n\to\infty} P_i^{(n)} \otimes P_{\Omega^{(n,\infty)}}, \quad i = 0, 1 \tag{4.2.2}$$

consists of eigenvectors corresponding to λ_i. An explicit formula for g_0 is presented in Definition 4.4.3 below.

The non-degeneracy of the eigenvalues in Proposition 4.2.1 as well as estimates for the imaginary part of the resonance can be derived from the corresponding results for the Pauli-Fierz model in [9] and [10]. Since their proofs do not need the new features of our multiscale scheme and they are not relevant for our main results, we only state them without proofs and refer to [9].

Remark 4.2.2 (Fermi golden rule)**.** *The eigenvalues λ_0 and λ_1 are non-degenerate, this follows from Section 6.4.3 in [9] (we do not repeat the proof here). The leading order of the imaginary part of the resonance λ_1 can be explicitly calculated. This is presented in Theorem 5.6 in [9] for the Pauli-Fierz model and, using a different method, in [16]. We do not include a proof here because it follows, for the model at hand, without much change from the proof in [9].*

We assume that $|g| > 0$ is small enough and define

$$E_I := -4\pi^2(e_1 - e_0)^2 |f(e_1 - e_0)|^2. \tag{4.2.3}$$

Then, there is a constant $C_{(4.2.4)} > 0$ and a constant $\epsilon > 0$ such that for all $n \in \mathbb{N}$ large enough

$$\left| \operatorname{Im} \lambda_1^{(n)} - g^2 E_I \right| \leq g^{2+\epsilon} C_{(4.2.4)}. \tag{4.2.4}$$

The next theorems are our main results of this chapter. We prove analyticity of the resonance and the ground-state, and the corresponding eigen-projections, with respect to the dilation parameter and coupling constant.

The next theorem is proved in Section 4.6 (see Theorem 4.6.9).

Theorem 4.2.3 (Analyticity with respect to the dilation parameter)**.** *For ρ, ρ_0 sufficiently small and $g \in D(0, g_0)$ (see Proposition 4.2.1), the functions*

$$\mathcal{S} \ni \theta \mapsto P_i, \qquad \mathcal{S} \ni \theta \mapsto \lambda_i \tag{4.2.5}$$

are analytic. Moreover, this implies that $\lambda_i(\theta) \equiv \lambda_i$ is constant for $\theta \in \mathcal{S}$ (see (4.1.1)).

Remark 4.2.4. *Our bounds in the inductive scheme (see Theorem 4.4.5 below) which are used to prove Theorem 4.2.3 blow up as we take $\nu \to 0$. We study simultaneously the cases $i = 0$ and $i = 1$, and therefore, our estimations blow up also for $i = 0$. However, it is easy to see from our method that, for $i = 0$ alone, we can take θ in a neighborhood of* 0 *and prove analyticity in this neighborhood. This implies that λ_0 is real, because H^θ is self-adjoint for $\theta = 0$. It is the ground-state energy constructed in [8, 51].*

The next theorem is proved in Section 4.6 (Theorem 4.6.9).

Theorem 4.2.5 (Analyticity with respect to the coupling constant)**.** *For every ρ, ρ_0 sufficiently small and $g \in D(0, g_0)$, the functions*

$$g \mapsto P_i, \qquad g \mapsto \lambda_i \tag{4.2.6}$$

are analytic.

Our next two theorems provide an estimate for the spectrum of H^θ in neighborhoods of λ_0 and λ_1, and resolvent estimates in these neighborhoods. As discussed in the introduction, similar results on spectral estimates can be found in [13, 15, 14, 16] in which the spectrum is located in cuspidal domains using the spectral renormalization method based on the Feshbach-Schur map. Here, we localize the spectrum in cones. For every $z \in \mathbb{C}$, we define

$$\mathcal{C}_m(z) := \left\{ z + xe^{-i\alpha} : x \geq 0, |\alpha - \nu| \leq \nu/m \right\}, \tag{4.2.7}$$

where we assume that $m \geq 4$, allover this work.

The next theorem is proved in Theorems 4.5.9 and 4.5.10 below.

Theorem 4.2.6 (Resolvent estimates)**.** *There is a constant* **C** *(see Definition 4.4.1 and* (4.5.58)*) that depends on* $\boldsymbol{\nu}$ *but not on g nor in ρ and ρ_0 such that for every $m \geq 4$ and ρ, ρ_0 sufficiently small, there exists $g^{(m)} > 0$ with the following properties: for every $\theta \in \mathcal{S}$ and $g \in D(0, g_0)$ (see Proposition 4.2.1) with $|g| \leq g^{(m)}$,*

$$\left\| \frac{1}{H^\theta - z} \right\| \leq 16 \boldsymbol{C}^{n+1} \frac{1}{\mathrm{dist}(z, \mathcal{C}_m(\lambda_i))}, \tag{4.2.8}$$

for every $z \in B_i^{(1)} \setminus \mathcal{C}_m \left(\lambda_i - 2\rho_n^{1+\mu/4} e^{-i\nu} \right)$ and

$$\left\| \frac{1}{H^\theta - z} \right\| \leq 8 \boldsymbol{C}^{n+1} \frac{1}{\mathrm{dist}(z, \mathcal{C}_m(\lambda_i^{(n)}))}, \tag{4.2.9}$$

for every $z \in B_i^{(1)} \setminus \mathcal{C}_m \left(\lambda_i^{(n)} - \rho_n^{1+\mu/4} e^{-i\nu} \right)$. Here, the symbol dist *denotes the Euclidean distance in $\mathbb{C}$ and $B_i^{(1)}$ is defined in* (4.3.5)*.*

Explicit bounds for $\boldsymbol{C}$, ρ_0 and ρ, g_0 and $g^{(m)}$ are given in Definitions 4.4.1, 4.4.2, 4.4.3 and (4.5.58), respectively. We remark that we intentionally do not provide optimal estimates because these would render the proof unnecessary opaque.

The next theorem is proved in the proof of Theorem 4.5.10 below.

Theorem 4.2.7 (Spectral estimates)**.** *For every ρ, ρ_0 sufficiently small, $\theta \in \mathcal{S}$ and $g \in D(0, g_0)$ with $|g| \leq g^{(m)}$, there is a neighborhood $B_i^{(1)}$ of λ_i (that depends on ν but not on g) such that the spectrum of H^θ in $B_i^{(1)}$ is contained in $\mathcal{C}(\lambda_i)$ (recall that ν is the imaginary part of θ). An explicit formula for $B_i^{(1)}$ is given in* (4.3.5)*.*

4.3. Resolvent estimates far away from the spectrum and detailed analysis of $H^{(1),\theta}$

In this subsection we derive resolvent estimates for $H^{(n),\theta}$ and H^θ for complex numbers z that are far away from their respective spectra. For the first Hamiltonian, $H^{(1),\theta}$, having an infrared cut-off, we present resolvent estimates for points that are close to its spectrum. Here, we do not need any restrictions on the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ other than $0 < \rho_0 < \min(1, e_1/4)$, $0 < \rho < 1$. In the forthcoming sections we need to assume other properties for the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ (see Definition 4.4.2). We emphasize that the particular choice of numbers ρ_n does not imply any physical constraint, it only specifies the rate at which the infrared cut-off is removed.

In this section and (in the whole thesis) we denote by $c > 0$ any generic (indeterminate) constant (it can change from line to line) that is independent of the parameters n, θ, ρ_0, ρ, θ, $\boldsymbol{\nu}$ and g. It might depend on the set $\mathcal{S}$, as a whole, but not on its elements $\theta \in \mathcal{S}$ and neither on the parameter $\boldsymbol{\nu}$. Moreover, by stating that $|g|$ is small enough, we mean that there is a constant such that uniformly for $|g|$ smaller than this constant the referred statement holds true. We employ that such a constant does not depend on θ and n but it depends on the set $\mathcal{S}$ and on the remaining parameters (e_1, ρ_0, ρ, μ and Λ).

4.3.1. Resolvent estimates far away from the spectrum

We define regions in the complex plane in which we derive resolvent estimates.

Definition 4.3.1. *We set $\delta := e_1 - e_0 = e_1$ and define the region*

$$A := A_1 \cup A_2 \cup A_3, \tag{4.3.1}$$

where

$$A_1 := \left\{ z \in \mathbb{C} : \operatorname{Re} z < e_0 - \frac{1}{2}\delta \right\} \tag{4.3.2}$$

$$A_2 := \left\{ z \in \mathbb{C} : \operatorname{Im} z > \frac{1}{8}\delta \sin(\nu) \right\} \tag{4.3.3}$$

$$A_3 := \left\{ z \in \mathbb{C} : \operatorname{Re} z > e_1 + \frac{1}{2}\delta, \operatorname{Im} z \geq -\sin\left(\frac{\nu}{2}\right)\left(\operatorname{Re}(z) - (e_1 + \frac{1}{2}\delta)\right) \right\}, \tag{4.3.4}$$

and for $i \in \{0, 1\}$,

$$B_i^{(1)} := \left\{ z \in \mathbb{C} : |\operatorname{Re} z - e_i| \leq \frac{1}{2}\delta, -\frac{1}{2}\rho_1 \sin(\nu) \leq \operatorname{Im} z \leq \frac{1}{8}\delta \sin(\nu) \right\}. \tag{4.3.5}$$

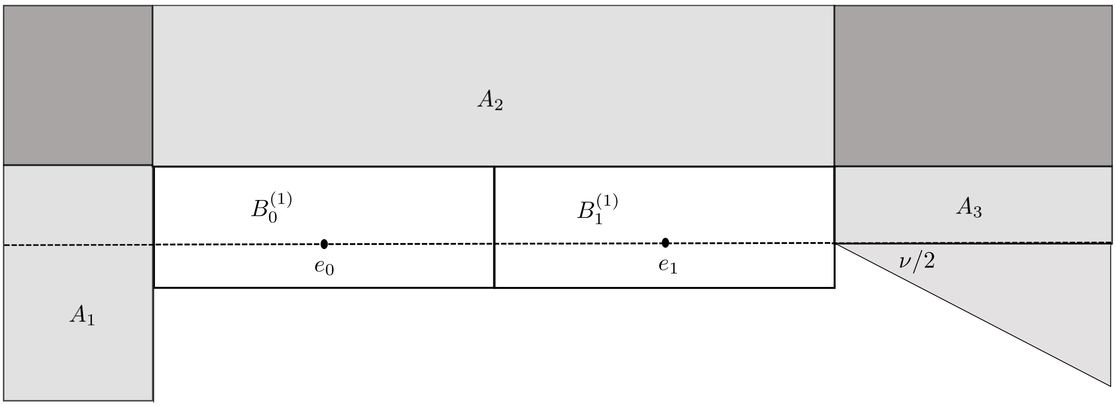

Figure 4.1.: Subsets of the complex plane (see Definition 4.3.1)

In this subsection, we estimate the resolvent of $H^{(n),\theta}$ and H^θ far away from their spectra, namely in the region A defined in (4.3.1). These estimates are applied for the induction basis in our inductive scheme described in Section 4.4.2.

Lemma 4.3.2. *Let $\theta \in \mathcal{S}$ (see (4.1.1)) and $n \in \mathbb{N}$. There is a constant $C_{(4.3.6)}$ (independent of θ, n, g, ρ_0, ρ and $\boldsymbol{\nu}$) such that for small enough $|g|$ (depending on $\boldsymbol{\nu}$), for every $i \in \{0,1\}$:*

$$\left\| \frac{1}{H^{(n),\theta} - z} \right\| \leq \frac{C_{(4.3.6)}}{\sin(\boldsymbol{\nu}/2)} \frac{1}{|e_i - z|}, \qquad \left\| \frac{1}{H^{\theta} - z} \right\| \leq \frac{C_{(4.3.6)}}{\sin(\boldsymbol{\nu}/2)} \frac{1}{|e_i - z|}, \qquad \forall z \in A. \tag{4.3.6}$$

Proof. Let $z \in A$ and $n \in \mathbb{N}$. Then, arguing as in Lemma 1.3.1 and using functional calculus, we obtain that

$$\begin{aligned} \left\| V^{(n),\theta} \frac{1}{H_0^{(n),\theta} - z} \right\| &\leq \left\| V^{(n),\theta} \frac{1}{(H_0^{(n)} + 1)^{\frac{1}{2}}} \right\| \left\| \frac{H_0^{(n)} + 1}{H_0^{(n),\theta} - z} \right\| \\ &\leq \left(\left\| f^\theta \right\|_2 + 2 \left\| f^\theta / \sqrt{\omega} \right\|_2 \right) \sup_{y \in [0,\infty), i=0,1} \left| \frac{e_i + y + 1}{e_i + e^{-\theta} y - z} \right|. \end{aligned} \tag{4.3.7}$$

Geometrical considerations imply that there is a constant $c > 0$ such that

$$\operatorname{dist}\Big(\{ e_i + e^{-\theta} y : \, i = 0,1 \}, A \Big) \geq \frac{c}{\sin(\boldsymbol{\nu}/2)} (1 + y) \qquad \forall y \geq 0, \tag{4.3.8}$$

and hence, there there is a constant $c > 0$ such that

$$\left\| V^{(n),\theta} \frac{1}{H_0^{(n),\theta} - z} \right\| \leq \frac{c}{\sin(\boldsymbol{\nu}/2)}, \qquad \forall z \in A. \tag{4.3.9}$$

Then, we choose $|g|$ small enough such that

$$\left\| gV^{(n),\theta} \frac{1}{H_0^{(n),\theta} - z} \right\| \le \frac{1}{2} \tag{4.3.10}$$

and hence,

$$H^{(n),\theta} - z = \left(1 + gV^{(n),\theta} \frac{1}{H_0^{(n),\theta} - z}\right) \left(H_0^{(n),\theta} - z\right) \tag{4.3.11}$$

is invertible for all $z \in A$, since $A \cap \sigma(H_0^{(n),\theta}) = \emptyset$. Thanks to the particular geometry, there is a constant $c > 0$ such that $|e_j + ye^{-\theta} - z| \ge c \sin(\boldsymbol{\nu}/2)|e_i - z|$, for every $z \in A$, every $j \in \{0, 1\}$, and every positive number y. This and (4.3.11) imply

$$\left\| \frac{1}{H^{(n),\theta} - z} \right\| \le 2 \left\| \frac{1}{H_0^{(n),\theta} - z} \right\| = \sup_{i=0,1} \sup_{y \ge \rho_n} \frac{2}{|e_i + ye^{-\theta} - z|} \le \frac{c}{\sin(\boldsymbol{\nu}/2)|e_i - z|} \tag{4.3.12}$$

for all $z \in A$, $i = 0, 1$ and some constant $c > 0$. This completes the proof for the first equation in (4.3.6). Since the second equation can be shown in a very similar fashion we omit the proof here. □

4.3.2. Analysis of $H^{(1),\theta}$

Lemma 4.3.3. *Let $\theta \in \mathcal{S}$ (see (4.1.1)) and $|g|$ small enough (depending on $\boldsymbol{\nu}$ and ρ_1). Then,*

$$\left\| \frac{1}{H^{(1),\theta} - z} \right\| \le 2 \left\| \frac{1}{H_0^{(1),\theta} - z} \right\| \qquad \forall z \in E_i^{(1)}, i = 0, 1, \tag{4.3.13}$$

where

$$E_i^{(1)} := B_i^{(1)} \setminus D\left(e_i, \frac{\rho_1 \sin(\nu)}{8}\right). \tag{4.3.14}$$

Proof. Let $z \in E_i^{(1)}$ and $i = 0, 1$. Then, we have, arguing as in Lemma 1.3.1,

$$\begin{aligned} \left\| V^{(1),\theta} \frac{1}{H_0^{(1),\theta} - z} \right\| &\le \left\| V^{(1),\theta} \frac{1}{(H_0^{(1)} + 1)^{\frac{1}{2}}} \right\| \left\| \frac{H_0^{(1)} + 1}{H_0^{(1),\theta} - z} \right\| \\ &\le \left(\left\| f^\theta \right\|_2 + 2 \left\| f^\theta / \sqrt{\omega} \right\|_2 \right) \sup_{y \in \{0\} \cup [\rho_1, \infty), i=0,1} \left| \frac{e_i + y + 1}{e_i + e^{-\theta} y - z} \right|. \end{aligned} \tag{4.3.15}$$

Take $y \in \{0\} \cup [\rho_1, \infty)$ and $i \in \{0, 1\}$. It follows that

$$\begin{aligned} \left| \frac{e_i + y + 1}{e_i + e^{-\theta} y - z} \right| &= \left| \frac{e_i + e^\theta (e_i + e^{-\theta} y - z) + 1 - e^\theta (e_i - z)}{e_i + e^{-\theta} y - z} \right| \\ &\le |e^\theta| + c \frac{1}{|e_i + e^{-\theta} y - z|} \le \frac{c}{\rho_1 \sin(\nu)}, \end{aligned} \tag{4.3.16}$$

where the last inequality is due to the considered geometry.

From (4.3.15) and (4.3.16) we obtain that there is a finite constant $c > 0$ such that

$$\left\| V^{(1),\theta} \frac{1}{H_0^{(1),\theta} - z} \right\| \leq \frac{c}{\rho_1 \sin(\nu)} \leq \frac{c}{\rho_1 \sin(\boldsymbol{\nu})}. \tag{4.3.17}$$

For $|g|$ small enough (depending on ρ_1 and $\boldsymbol{\nu}$), we arrive at

$$\left\| g V^{(1),\theta} \frac{1}{H_0^{(1),\theta} - z} \right\| \leq \frac{1}{2}, \tag{4.3.18}$$

and hence,

$$H^{(1),\theta} - z = \left(1 + g V^{(1),\theta} \frac{1}{H_0^{(1),\theta} - z} \right) \left(H_0^{(1),\theta} - z \right) \tag{4.3.19}$$

is invertible for all $z \in E_i^{(1)}$, since $E_i^{(1)} \cap \sigma(H_0^{(1),\theta}) = \emptyset$. Then, we obtain

$$\left\| \frac{1}{H^{(1),\theta} - z} \right\| \leq 2 \left\| \frac{1}{H_0^{(1),\theta} - z} \right\|, \tag{4.3.20}$$

which completes the proof. □

Definition 4.3.4. *We define the projections*

$$P_i^{(1)} := -\frac{1}{2\pi i} \int_{\hat{\gamma}_i^{(1)}} \mathrm{d}z \, \frac{1}{H^{(1),\theta} - z} \tag{4.3.21}$$

and

$$P_{at,i}^{(1)} := -\frac{1}{2\pi i} \int_{\hat{\gamma}_i^{(1)}} \mathrm{d}z \, \frac{1}{H_0^{(1),\theta} - z} = P_{e_i} \otimes P_{\Omega^{(1)}} \tag{4.3.22}$$

where

$$\hat{\gamma}_i^{(1)} : [0, 2\pi] \to \mathbb{C}, \quad t \mapsto e_i + \frac{1}{4} \rho_1 \sin(\nu) e^{it}, \tag{4.3.23}$$

P_{e_i} is the projection onto the eigenspace space corresponding to e_i of the Hamiltonian K and $P_{\Omega^{(1)}}$ is the projection onto the vector space generated by the vacuum, $\Omega^{(1)}$, of $\mathcal{F}[\mathfrak{h}^{(1)}]$.

Remark 4.3.5. *The right-hand side of* (4.3.22) *follows from the following computation:*

$$\begin{aligned} -\frac{1}{2\pi i} \int_{\hat{\gamma}_i^{(1)}} \mathrm{d}z \, \frac{1}{H_0^{(1),\theta} - z} &= -\frac{1}{2\pi i} \int_{\hat{\gamma}_i^{(1)}} \mathrm{d}z \, \frac{1}{H_0^{(1),\theta} - z} (P_{e_i} \otimes P_{\Omega^{(1)}} + \overline{P_{e_i} \otimes P_{\Omega^{(1)}}}) \\ &= P_{e_i} \otimes P_{\Omega^{(1)}} - \frac{1}{2\pi i} \int_{\hat{\gamma}_i^{(1)}} \mathrm{d}z \, \frac{1}{H_0^{(1),\theta} - z} \overline{P_{e_i} \otimes P_{\Omega^{(1)}}}, \end{aligned} \tag{4.3.24}$$

where

$$\overline{P_{e_i} \otimes P_{\Omega^{(1)}}} = \overline{P_{e_i}} \otimes 1 + P_{e_i} \otimes \overline{P_{\Omega^{(1)}}} \tag{4.3.25}$$

implies that $-\frac{1}{2\pi i} \int_{\hat{\gamma}_i^{(1)}} \mathrm{d}z \, \frac{1}{H_0^{(1),\theta} - z} \overline{P_{e_i} \otimes P_{\Omega^{(1)}}} = 0$.

Lemma 4.3.6. *Let $\theta \in \mathcal{S}$ (see (4.1.1)) and let $|g|$ be small enough (depending on $\boldsymbol{\nu}$ and ρ_1). Take $i \in \{0,1\}$. Then, there is a constant $C_{(4.3.26)} > 0$ (independent of θ, n, g, ρ_0, ρ and $\boldsymbol{\nu}$) such that*

$$\left\| P_i^{(1)} - P_{at,i}^{(1)} \right\| \leq |g| \frac{C_{(4.3.26)}}{\rho_1 \sin(\boldsymbol{\nu})} < 1 \qquad \text{and} \qquad \left\| P_i^{(1)} \right\| \leq 1 + |g| \frac{C_{(4.3.26)}}{\rho_1 \sin(\boldsymbol{\nu})} < 2, \quad (4.3.26)$$

where $\hat{\gamma}_i^{(1)}$, $P_i^{(1)}$ and $P_{at,i}^{(1)}$ are introduced in Definition 4.3.4.

Proof. First, we observe that

$$\left\| P_i^{(1)} - P_{\mathrm{at},i}^{(1)} \right\| \leq \frac{1}{2\pi} \left\| \int_{\hat{\gamma}_i^{(1)}} \mathrm{d}z \left(\frac{1}{H^{(1),\theta} - z} - \frac{1}{H_0^{(1),\theta} - z} \right) \right\|. \quad (4.3.27)$$

Note that $\hat{\gamma}_i^{(1)} \subset E_i^{(1)}$ (see (4.3.14)). Eq. (4.3.17) implies that there is a finite constant $c > 0$ such that for every z in the (image of the) curve $\hat{\gamma}_i^{(1)}$

$$\left\| gV^{(1),\theta} \frac{1}{H_0^{(1),\theta} - z} \right\| < |g| \frac{c}{\rho_1 \sin(\boldsymbol{\nu})} \leq \frac{1}{2}, \quad (4.3.28)$$

for $|g|$ small enough (depending on $\boldsymbol{\nu}$ and ρ_1). Next, we obtain

$$\begin{aligned} \left\| P_i^{(1)} - P_{\mathrm{at},i}^{(1)} \right\| &\leq \frac{1}{2\pi} \left\| \int_{\hat{\gamma}_i^{(1)}} \mathrm{d}z \, \frac{1}{H_0^{(1),\theta} - z} \sum_{l=1}^{\infty} \left(-gV^{(1),\theta} \frac{1}{H_0^{(1),\theta} - z} \right)^l \right\| \\ &\leq \frac{\rho_1 \sin(\nu)}{4} \sup_{|z - e_i| = \frac{\rho_1 \sin(\nu)}{4}} \left(\left\| \frac{1}{H_0^{(1),\theta} - z} \right\| \right) |g| \frac{c}{\rho_1 \sin(\boldsymbol{\nu})} \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)^l < |g| \frac{c}{\rho_1 \sin(\boldsymbol{\nu})}. \quad (4.3.29) \end{aligned}$$

This proves the first part of the lemma. Furthermore, it follows from (4.3.22) that $\left\| P_{\mathrm{at},i}^{(1)} \right\| = 1$, and hence,

$$\left\| P_i^{(1)} \right\| \leq \left\| P_{\mathrm{at},i}^{(1)} \right\| + \left\| \hat{P}_i^{(1)} - P_{\mathrm{at},i}^{(1)} \right\| \leq 1 + |g| \frac{c}{\rho_1 \sin(\boldsymbol{\nu})} < 2, \quad (4.3.30)$$

for sufficiently small $|g|$. This proves proves the second part of the lemma. ☐

Remark 4.3.7. *Let $\theta \in \mathcal{S}$ (see (4.1.1)) and $i \in \{0,1\}$. Suppose that $|g|$ is small enough (depending on $\boldsymbol{\nu}$ and ρ_1). Then, it follows from Lemma 4.3.6 together with the fact that $P_{at,i}^{(1)}$ is a rank-one projection that also $P_i^{(1)}$ is a rank-one projection. Lemma 4.3.3 implies that $H^{(1),\theta}$ has no spectral points in $E_i^{(1)} = B_i^{(1)} \setminus D\left(e_i, \frac{\rho_1 \sin(\nu)}{8}\right)$. Since the contour of integration for the projection $P_i^{(1)}$ is contained in $B_i^{(1)}$ and its interior contains $D\left(e_i, \frac{\rho_1 \sin(\nu)}{8}\right)$, we conclude that there is a unique spectral point $\lambda_i^{(1)}$ of $H^{(1),\theta}$ in $B_i^{(1)}$, it is a simple eigenvalue and it is contained in $D\left(e_i, \frac{\rho_1 \sin(\nu)}{8}\right)$.*

Lemma 4.3.3 together with Lemma 4.3.6 yield a resolvent estimate in the whole region $B_i^{(1)} \setminus \{\lambda_i^{(1)}\}$ by making use of the maximum modulus principle of complex analysis.

Lemma 4.3.8. *Let $\theta \in \mathcal{S}$ (see (4.1.1)) let $|g|$ be small enough (depending on $\boldsymbol{\nu}$ and ρ_1). Take $i \in \{0,1\}$. Then, there is a constant $C_{(4.3.31)} > 0$ (independent of θ, n, g, ρ_0, ρ and $\boldsymbol{\nu}$) such that*

$$\left\| \frac{1}{H^{(1),\theta} - z} \overline{P_i^{(1)}} \right\| \leq \frac{C_{(4.3.31)}}{\rho_1 \sin(\nu)} \leq \frac{C_{(4.3.31)}}{\rho_1 \sin(\boldsymbol{\nu})}, \qquad \forall z \in B_i^{(1)}. \tag{4.3.31}$$

Proof. Note that the function

$$\overline{D\left(e_i, \frac{1}{8}\rho_1 \sin(\nu)\right)} \ni z \mapsto G_{\phi,\psi}(z) := \left\langle \phi, \frac{1}{H^{(1),\theta} - z} \overline{P_i^{(1)}} \psi \right\rangle \tag{4.3.32}$$

is continuous, and furthermore, analytic on $D\left(e_i, \frac{1}{8}\rho_1 \sin(\nu)\right)$, for all $\phi, \psi \in \mathcal{H}$. Then, it follows from the maximum modulus principle that this function attains its maximum on the boundary of its domain. This together with the Cauchy-Schwarz inequality and Lemma 4.3.3 and 4.3.6 implies that there is a finite constant $c > 0$ such that

$$|G_{\phi,\psi}(z)| \leq \frac{c}{\rho_1 \sin(\nu)} \|\phi\| \|\psi\|, \qquad \forall z \in \overline{D\left(e_i, \frac{1}{8}\rho_1 \sin(\nu)\right)}. \tag{4.3.33}$$

Consequently, there is a finite constant $c > 0$ such that

$$\left\| \frac{1}{H^{(1),\theta} - z} \overline{P_i^{(1)}} \right\| \leq \frac{c}{\rho_1 \sin(\nu)}, \qquad \forall z \in \overline{D\left(e_i, \frac{1}{8}\rho_1 \sin(\nu)\right)}, \tag{4.3.34}$$

and moreover, Lemma 4.3.3 (again) guarantees that there is a finite constant $c > 0$ such that

$$\left\| \frac{1}{H^{(1),\theta} - z} \right\| \leq 2 \left\| \frac{1}{H_0^{(1),\theta} - z} \right\| \leq \frac{c}{\rho_1 \sin(\nu)}, \tag{4.3.35}$$

for all $z \in B_i^{(1)} \setminus D\left(e_i, \frac{1}{8}\rho_1 \sin(\nu)\right)$. This together with (4.3.34) completes the proof. $\square$

Applying Lemma 4.3.8 to our particular geometry allows to formulate the following corollary.

Corollary 4.3.9. *We define $q_i^{(1)} := \lambda_i^{(1)} + \frac{1}{4}\rho_1 e^{-i\nu}$ and recall (4.2.7). Let $\theta \in \mathcal{S}$ (see (4.1.1)) let $|g|$ be small enough (depending on $\boldsymbol{\nu}$ and ρ_1). Take $i \in \{0,1\}$. Then, there is a constant $C_{(4.3.36)} > 0$ (independent of θ, n, g, ρ_0, ρ and $\boldsymbol{\nu}$) such that*

$$\left\| \frac{1}{H^{(1),\theta} - z} \overline{P_i^{(1)}} \right\| \leq \frac{1}{\sin(\nu/m)} \frac{C_{(4.3.31)}}{\operatorname{dist}(z, \mathcal{C}_m(q_1^{(1)}))}, \qquad \forall z \in B_i^{(1)} \setminus \mathcal{C}_m(q_i^{(1)}). \tag{4.3.36}$$

Lemma 4.3.10. *Let $\theta \in \mathcal{S}$ (see (4.1.1)) and let $|g|$ be small enough (depending on $\boldsymbol{\nu}$ and ρ_1). Then, there is a constant $C_{(4.3.37)} > 0$ (independent of θ, n, g, ρ_0, ρ and $\boldsymbol{\nu}$) such that for every $i \in \{0,1\}$*

$$\left|\lambda_i^{(1)} - e_i\right| \leq |g| C_{(4.3.37)}. \tag{4.3.37}$$

Proof. It follows from Lemma 4.3.6 that $\left| \left\langle \varphi_i \otimes \Omega, P_i^{(1)} \varphi_i \otimes \Omega \right\rangle \right| > \frac{1}{2}$ for $|g|$ small enough (depending on $\boldsymbol{\nu}$ and ρ_1). We calculate

$$\lambda_i^{(1)} = \frac{\left\langle \varphi_i \otimes \Omega^{(1)}, H^{(1),\theta} P_i^{(1)} \varphi_i \otimes \Omega^{(1)} \right\rangle}{\left\langle \varphi_i \otimes \Omega^{(1)}, P_i^{(1)} \varphi_i \otimes \Omega^{(1)} \right\rangle} = e_i + g \frac{\left\langle V^{(1),\overline{\theta}} \varphi_i \otimes \Omega^{(1)}, P_i^{(1)} \varphi_i \otimes \Omega^{(1)} \right\rangle}{\left\langle \varphi_i \otimes \Omega^{(1)}, P_i^{(1)} \varphi_i \otimes \Omega^{(1)} \right\rangle}. \tag{4.3.38}$$

Let now $z \in \mathbb{C}$ such that $|e_i - z| = \frac{1}{4}\rho_1 \sin(\nu)$. Eq. (4.3.26) (which requires $|g|$ to be small enough -depending on $\boldsymbol{\nu}$ and ρ_1) then allows to obtain

$$\left|\lambda_i^{(1)} - e_i\right| \leq 4 \left\| g V^{(1),\overline{\theta}} \varphi_i \otimes \Omega^{(1)} \right\| \leq 4\, |e_i - z| \left\| g V^{(1),\overline{\theta}} \frac{1}{H_0^{(1),\overline{\theta}} - z} \right\| \leq |g| c, \tag{4.3.39}$$

for some constant c (independent of θ, n, g, ρ_0, ρ and $\boldsymbol{\nu}$) . Here, we have used (4.3.17) from Lemma 4.3.3 in the last step. Notice that in this work we assume that the imaginary part of θ, ν, is positive. Then, strictly speaking, we do not have the right to use our results for $\overline{\theta}$. However, the restriction we impose by assuming that ν is not negative is irrelevant. This is assumed only for convenience in order to simplify our notation. Of course, the same results hold true if we take $-\pi/16 < \nu < -\boldsymbol{\nu}$. In this case the spectrum of $H_0^{(1),\theta}$ is just mirrored with respect to the real line. Hence, one has to mirror also the definition of $B_i^{(1)}$ with respect to the real line in order to obtain the same estimates. Afterwards, the proofs are just the same. □

Remark 4.3.11. *Let $\theta \in \mathcal{S}$ (see (4.1.1)) and let $|g|$ be small enough (depending on $\boldsymbol{\nu}$ and ρ_1). Then*

$$P_i^{(1)} = -\frac{1}{2\pi i} \int_{\gamma_i^{(1)}} \mathrm{d}z\, \frac{1}{H^{(1),\theta} - z}, \tag{4.3.40}$$

where $\gamma_i^{(1)} : [0, 2\pi] \to \mathbb{C}, \quad t \mapsto \lambda_i^{(1)} + \frac{1}{4}\rho_1 \sin(\nu) e^{it}$. This follows from Remark 4.3.7, because, for small enough $|g|$, $\gamma_i^{(1)}, \hat{\gamma}_i^{(1)} \subset B_i^{(1)} \setminus \{\lambda_i^{(1)}\}$, see Lemma 4.3.10.

Lemma 4.3.12. *Let $\theta \in \mathcal{S}$ (see (4.1.1)) and let $|g|$ be small enough (depending on $\boldsymbol{\nu}$ and ρ_1). Take $i \in \{0,1\}$. Then, there is a constant $C_{(4.3.41)} > 0$ (independent of θ, n, g, ρ_0, ρ and $\boldsymbol{\nu}$) such that*

$$\left\| \frac{1}{H^{(1),\theta} - z} \overline{P_i^{(1)}} \right\| \leq \frac{C_{(4.3.41)}}{\sin(\boldsymbol{\nu})} \frac{1}{\rho_1 + \left|\lambda_i^{(1)} - z\right|}, \qquad \forall z \in B_i^{(1)}. \tag{4.3.41}$$

Proof. We use Lemma 4.3.8 and calculate, for $|z - e_i| \leq \rho_1$,

$$\frac{1}{\rho_1 \sin(\nu)} = \frac{\rho_1 + |\lambda_i^{(1)} - z|}{\rho_1 \sin(\nu)} \frac{1}{\rho_1 + |\lambda_i^{(1)} - z|} \leq \frac{c}{\sin(\boldsymbol{\nu})} \frac{1}{\rho_1 + |\lambda_i^{(1)} - z|}, \tag{4.3.42}$$

where we use Lemma 4.3.10 and choose $|g|$ small enough. For $|z-e_i| > \rho_1$ we use Lemma 4.3.3. The spectral theorem and the explicit form of the spectrum of non-interacting Hamiltonian $H_0^{(1),\theta}$ allow us to estimate the norm of its resolvent. Then, similar estimates as above imply the desired result. □

Lemma 4.3.13. *Let $\theta \in \mathcal{S}$ (see (4.1.1)) and $n \in \mathbb{N}$. There is a constant $C_{(4.3.43)}$ (independent of θ, n, g, ρ_0, ρ and $\boldsymbol{\nu}$) such that for small enough $|g|$ (depending on $\boldsymbol{\nu}$), for every $i \in \{0, 1\}$:*

$$\left\| \frac{1}{H^{(n),\theta} - z} \right\| \leq \frac{C_{(4.3.43)}}{\sin(\boldsymbol{\nu}/2)} \frac{1}{\rho_l + \left|\widetilde{\lambda}_i - z\right|}, \quad \left\| \frac{1}{H^{\theta} - z} \right\| \leq \frac{C_{(4.3.43)}}{\sin(\boldsymbol{\nu}/2)} \frac{1}{\rho_l + \left|\widetilde{\lambda}_i - z\right|}, \tag{4.3.43}$$

for every $z \in A \cup \left(B_1^{(1)} - [0, \infty)e^{-i\nu}\right) \setminus B_1^{(1)}$, every $l \in \mathbb{N}$ and every $\widetilde{\lambda}_i \in D(\lambda_i^{(1)}, 3g)$.

Moreover,

$$\left\| \frac{1}{H^{(n),\theta} - z} \overline{P_i^{(1)}} \right\| \leq \frac{1}{\sin(\nu/m)} \frac{C_{(4.3.44)}}{\operatorname{dist}(z, \mathcal{C}_m(q_1^{(n)}))}, \tag{4.3.44}$$

for every $z \in \left(B_1^{(1)} - [0, \infty)e^{-i\nu}\right) \setminus B_1^{(1)}$.

Proof. We take $z \in A$. Lemma 4.3.10 implies that $|\widetilde{\lambda}_i - e_i| \leq |g|(C_{(4.3.37)} + 3)$. Notice that

$$\begin{aligned} \frac{1}{|e_i - z|} \leq \frac{\rho_m + |\widetilde{\lambda}_i - z|}{|e_i - z|} \frac{1}{\rho_m + |\widetilde{\lambda}_i - z|} &\leq \frac{\rho_m + |\widetilde{\lambda}_i - e_i| + |e_i - z|}{|e_i - z|} \frac{1}{\rho_m + \left|\widetilde{\lambda}_i - z\right|} \\ &\leq c \frac{1}{\rho_m + \left|\widetilde{\lambda}_i - z\right|}, \end{aligned} \tag{4.3.45}$$

since $|e_i - z|$ is bounded from below uniformly for $z \in A$. Then, the result follows from Lemma 4.3.2. The result for $z \in \left(B_1^{(1)} - [0, \infty)e^{-i\nu}\right)$ can be found similarly which is why we omit the proof. The proof of (4.3.44) follows from a similar argument as in Corollary 4.3.9,and therefore, it is also omitted. □

Definition 4.3.14. *In this subsection (Section 4.3) we assumed a finite number of times that $|g|$ is small enough (depending on $\boldsymbol{\nu}$ and ρ_1). We set $\boldsymbol{g} > 0$ such that for every $|g| \leq \boldsymbol{g}$ all results of this section hold true. Similarly, in the statements of our results we use a finite number of times constants (that are independent of θ, n, g, ρ_0, ρ and $\boldsymbol{\nu}$) in order to estimate from above norms of various entities. We denote by $\boldsymbol{c} \geq 1$ a, fixed, upper bound of the set of all these constants. We additionally take $\boldsymbol{g}$ small enough such that*

$$\left\| P_i^{(1)} - P_{at,i}^{(1)} \right\| < 10^{-3},$$

see Lemma 4.3.6.

4.4. Resonance and ground-state multiscale analysis

4.4.1. Notation: the sequence $(\rho_n)_{n\in\mathbb{N}\cup\{0\}}$ and the coupling constant g

Next, we introduce a constant, $\boldsymbol{D}$, that includes all constants involved in estimations for our multiscale construction. This constant does not depend on $\theta \in \mathcal{S}$, g, $\boldsymbol{\nu}$, n, ρ and ρ_0. Our bounds do depend on $\boldsymbol{\nu}$. They blow up as $\boldsymbol{\nu}$ tends to zero with a rate that is not worse than $\sin(\boldsymbol{\nu}/2)^{-3}$. The constant $\boldsymbol{D}$ does not depend on $\boldsymbol{\nu}$. The dependence on $\boldsymbol{\nu}$ in our bounds is reflected in a function $\boldsymbol{C} \equiv \boldsymbol{C}(\boldsymbol{\nu})$ that is bounded from below by the constant $\boldsymbol{D}$ multiplied by the factor $\sin(\boldsymbol{\nu}/2)^{-3}$. As already explained, the constant $\boldsymbol{D}$ and the blow-up rate $\sin(\boldsymbol{\nu}/2)^{-3}$ are intentionally not optimal but often estimated from above to increase the readability of the proofs.

Definition 4.4.1 (The function $\boldsymbol{C} \equiv \boldsymbol{C}(\boldsymbol{\nu})$)**.** *We fix a constant, $\boldsymbol{D}$, that does not depend on $g \in D(0, \boldsymbol{g})$, $\theta \in \mathcal{S}$, and $\boldsymbol{\nu}$. The only property that it must satisfy is the following (see Definition 4.3.14)*

$$\boldsymbol{D} \geq 10^6 + 10\boldsymbol{c}. \tag{4.4.1}$$

Next, we fix a function $\boldsymbol{C} \equiv \boldsymbol{C}(\boldsymbol{\nu})$ satisfying

$$\boldsymbol{C} \equiv \boldsymbol{C}(\boldsymbol{\nu}) \geq \boldsymbol{D} \sin(\boldsymbol{\nu}/2)^{-3}. \tag{4.4.2}$$

The sequence $(\rho_n)_{n\in\mathbb{N}_0}$ which we introduced above is defined in the following manner.

Definition 4.4.2 (Parameters ρ_0 and ρ)**.** *The parameters ρ_0 and ρ in the definition of the sequence $\rho_n = \rho_0\rho^n, n \in \mathbb{N}_0$, see Definition 4.1.1(i), have to fulfill the following constraints:*

$$\boldsymbol{C}^8 \rho_0^\mu \leq 1, \qquad \boldsymbol{C}^4 \rho^\mu \leq 1/4. \tag{4.4.3}$$

We recall that in this work we require $|g|$ to be small enough. The next definition summarizes all requirements that it must satisfy.

Definition 4.4.3 (The coupling constant g)**.** *We set a constant $g_0 \leq \boldsymbol{g}$ satisfying the following (see Definition 4.3.14):*

$$g_0 \leq \frac{\rho_1 \sin(\boldsymbol{\nu}/2)^2}{10^4 \boldsymbol{c}}. \tag{4.4.4}$$

Henceforth, we always require $|g| \leq g_0$.

Remark 4.4.4. *The selection of $\boldsymbol{C}$, the sequence $(\rho_n)_{n\in\mathbb{N}_0}$ and the constant g_0 will later allow to set up the infrared induction scheme, and is therefore, rather involved. This remark is intended to help the reader to understand this procedure. Below (in this section), we use boldface fonts whenever we use the properties of $\boldsymbol{C}$ and g (or g_0) that we specified above.*

The requirements for $(\rho_n)_{n\in\mathbb{N}_0}$ are only present in order to satisfy the last inequalities in (4.4.7) *and* (4.4.11) *below. Then, it will turn out that it is only necessary to assume that $\boldsymbol{C}^4\rho_0^\mu \leq 1$ and $\boldsymbol{C}^2\rho^\mu \leq 1/2$ in order to close our induction. We assume stronger conditions again, for notational convenience, and because it implies a faster convergence rate in* (4.4.11)*, which will be used in Chapter 5 (see Remark 4.5.11).*

4.4.2. Induction scheme and the strategy of our multiscale construction

We denote by

$$\mathfrak{h}^{(n,n+1)} := L^2(\mathcal{B}_{\rho_n} \setminus \mathcal{B}_{\rho_{n+1}}) \tag{4.4.5}$$

the Hilbert space of one-particle (boson) states with energies in the interval $[\rho_{n+1}, \rho_n)$. We denote the corresponding Fock space by $\mathcal{F}[\mathfrak{h}^{(n,n+1)}]$ (it is defined as in (1.2.7)). Note that $\mathcal{H}^{(n+1)}$ is isomorphic to $\mathcal{H}^{(n)} \otimes \mathcal{F}[\mathfrak{h}^{(n,n+1)}]$, and therefore, we identify

$$\mathcal{H}^{(n+1)} \equiv \mathcal{H}^{(n)} \otimes \mathcal{F}[\mathfrak{h}^{(n,n+1)}]. \tag{4.4.6}$$

For $i = 0, 1$, we inductively (and simultaneously) construct sequences $\{\lambda_i^{(n)}\}_{n\in\mathbb{N}_0}$ of complex numbers, sequences $\{B_i^{(n)}\}_{n\in\mathbb{N}}$ of subsets of the complex plane and sequences $\{P_i^{(n)}\}_{n\in\mathbb{N}_0}$ of operators that satisfy the properties listed below.

($\mathcal{P}1$) We set $\lambda_i^{(0)} \equiv \lambda_i^{(1)}$. For $n \in \mathbb{N}$, $\lambda_i^{(n)}$ is a simple eigenvalue of $H^{(n),\theta}$ and

$$\left|\lambda_i^{(n)} - \lambda_i^{(n-1)}\right| < |g|\mathbf{C}^{n+1}(\rho_{n-1})^{1+\mu} \leq |g| \left(\frac{1}{2}\right)^{n-1} \rho_{n-1}. \tag{4.4.7}$$

The second inequality follows from Definition 4.4.2.

($\mathcal{P}2$) For $n \in \mathbb{N}$, we define (recall that $\nu = \operatorname{Im}\theta$)

$$B_i^{(n)} := B_i^{(1)} \setminus \left\{ z \in \mathbb{C} : \operatorname{Im} z < \operatorname{Im} \lambda_i^{(n)} - \frac{1}{4}\rho_n \sin(\nu) \right\}. \tag{4.4.8}$$

$\lambda_i^{(n)}$ is the only point in the spectrum of $H^{(n),\theta}$ intersected with $B_i^{(n)}$.

($\mathcal{P}3$) We set $P_i^{(0)} \equiv P_i^{(1)}$. For $n \in \mathbb{N}$, we define

$$P_i^{(n)} = -\frac{1}{2\pi i} \int_{\gamma_i^{(n)}} \mathrm{d}z \, \frac{1}{H^{(n),\theta} - z}, \tag{4.4.9}$$

where

$$\gamma_i^{(n)} : [0, 2\pi] \to \mathbb{C}, \quad t \mapsto \lambda_i^{(n)} + \frac{1}{4}\rho_n \sin(\nu) e^{it}. \tag{4.4.10}$$

The projections $P_i^{(n)}$ satisfy

$$\left\| P_i^{(n)} - P_i^{(n-1)} \otimes P_{\Omega^{(n-1,n)}} \right\| \leq \frac{|g|}{\rho} \mathbf{C}^{2n+2} \rho_{n-1}^{\mu} \leq \frac{|g|}{\rho} \left(\frac{1}{2}\right)^{n-1}, \tag{4.4.11}$$

where $P_{\Omega^{(n-1,n)}}$ is the projection onto the vacuum vector $\Omega^{(n-1,n)} \in \mathcal{F}[\mathfrak{h}^{(n-1,n)}]$ (see (4.4.5)-(4.4.6)). In (4.4.11) we omit the tensor product for $n = 1$. The second inequality follows from Definition 4.4.2.

($\mathcal{P}4$) Set $n \in \mathbb{N}$. For any $z \in B_i^{(n)}$, we have that

$$\left\| \frac{1}{H^{(n),\theta} - z} \overline{P_i^{(n)}} \right\| \leq \frac{\mathbf{C}^{n+1}}{\rho_n + \left| z - \lambda_i^{(n)} \right|}, \tag{4.4.12}$$

where $\overline{P_i^{(n)}} = \mathbb{1}_{\mathcal{H}^{(n)}} - P_i^{(n)}$.

Theorem 4.4.5 (Multiscale analysis for resonances and ground state eigenvalues)**.** *For every $i \in \{0,1\}$ and $\theta \in \mathcal{S}$ (see (4.1.1)), there exist sequences of complex numbers $\{\lambda_i^{(n)}\}_{n\in\mathbb{N}_0}$, subsets of the complex plane $\{B_i^{(n)}\}_{n\in\mathbb{N}}$ and projection operators $\{P_i^{(n)}\}_{n\in\mathbb{N}_0}$ satisfying Properties ($\mathcal{P}1$)-($\mathcal{P}4$). Recall that we assume that $|g| \leq g_0$.*

The proof for this theorem is given in Section 4.4.3.

Similar results, for the Pauli-Fierz model, are derived in [10]. In the present chapter we need uniform estimates with respect to $\theta \in \mathcal{S}$ and $g \in D(0, g_0)$, in order to obtain uniform convergence with respect to these parameters (which is an important ingredient for the proof of analyticity). This is not the case in [10] where analyticity is not an issue at stake.

Remark 4.4.6. *Note that ($\mathcal{P}1$) and ($\mathcal{P}3$) hold true for $n = 1$ by definition. Remark 4.3.7 implies that ($\mathcal{P}2$) holds true for $n = 1$. Moreover, ($\mathcal{P}4$), for $n = 1$, follows from Lemma 4.3.12 (where we recall Definitions 4.3.14, 4.4.1 and 4.4.3).*

4.4.3. Proof of Theorem 4.4.5

We recall that in the remainder of this work we always assume that $|g| \leq g_0$ (see Definition 4.4.3) and $\theta \in \mathcal{S}$ (see (4.1.1)). In Section 4.4.3, we prove some key ingredients which are then used in Section 4.4.3 in order to conclude the induction step.

Key estimates for the induction step

Here, we assume that ($\mathcal{P}1$)-($\mathcal{P}4$) hold true for all $j \leq n \in \mathbb{N}$ and we derive some key estimates which we apply in the next section in order to show the induction step in the proof of Theorem 4.4.5.

By (1.2.1), we define free boson energy operator restricted to $\mathcal{F}[\mathfrak{h}^{(n,n+1)}]$ and denote it by $H_f^{(n,n+1),0} \equiv H_f^{(n,n+1)}$ (see (4.4.5)-(4.4.6)). We set

$$H_f^{(n,n+1),\theta} := e^{-\theta} H_f^{(n,n+1),0}. \tag{4.4.13}$$

For every function $h \in \mathfrak{h}^{(n,n+1)}$, we denote the creation and annihilation operators, $a_{n,n+1}(h)$, $a_{n,n+1}^*(h)$, on $\mathcal{F}[\mathfrak{h}^{(n,n+1)}]$ according to (1.2.10) and (1.2.11). We use the same notation for functions $h \in \mathfrak{h}$ but then understand the argument as h restricted to $\mathfrak{h}^{(n,n+1)}$. Furthermore, we fix the following operator (defined on $\mathcal{K} \otimes \mathcal{F}[\mathfrak{h}^{(n,n+1)}]$, and hence, on $\mathcal{H}^{(n+1)}$ - see Remark 1.2.1)

$$V^{(n,n+1),\theta} := \sigma_1 \otimes \left(a_{n,n+1}(f^{\overline{\theta}}) + a_{n,n+1}(f^{\theta})^* \right). \tag{4.4.14}$$

In this notation we obtain (see Remark 1.2.1):

$$H^{(n+1),\theta} = H^{(n),\theta} + H_f^{(n,n+1),\theta} + gV^{(n,n+1),\theta}. \tag{4.4.15}$$

Lemma 4.4.7. *Suppose that ($\mathcal{P}1$)-($\mathcal{P}4$) hold true for all $m \in \mathbb{N}$ such that $m \leq n$. Then,*

$$\left\| P_i^{(n)} \right\| \leq 2 + \frac{2|g|}{\rho} \leq 3, \qquad i = 0, 1 \tag{4.4.16}$$

***(notice that** $|g| \leq \rho/2$**, see Definition 4.4.3- recall Remark 4.4.4)** and*

$$|\lambda_i^{(n)} - \lambda_i^{(1)}| \leq 2|g|. \tag{4.4.17}$$

Proof. Eq. (4.4.17) is a consequence of Property ($\mathcal{P}1$). We estimate

$$\begin{aligned} \left\| P_i^{(n)} \right\| &\leq \left\| P_i^{(1)} \right\| + \sum_{j=2}^{n} \left\| P_i^{(j)} - P_i^{(j-1)} \otimes P_{\Omega^{(j-1,j)}} \right\| \\ &\leq 2 + \frac{|g|}{\rho} \sum_{j=0}^{n-1} \left(\frac{1}{2} \right)^j \leq 2 + \frac{2|g|}{\rho}, \end{aligned} \tag{4.4.18}$$

where we apply the induction hypothesis ($\mathcal{P}3$) for $j \leq n$ and use Definition 4.3.14 **and Definition 4.4.3**. □

Definition 4.4.8. *Let $n \in \mathbb{N}$ and $i \in \{0, 1\}$. We define the region*

$$M_i^{(n)} := B_i^{(n)} \setminus \left\{ z \in \mathbb{C} : \operatorname{Im}(z) \in \left(-\infty, \operatorname{Im}(\lambda_i^{(n)}) - \frac{2}{5}\rho_{n+1} \sin(\nu) \right) \right\}. \tag{4.4.19}$$

Lemma 4.4.9. *Suppose that ($\mathcal{P}1$)-($\mathcal{P}4$) hold true for all $m \in \mathbb{N}$ such that $m \leq n$. Then, for $i \in \{0, 1\}$:*

$$\left\| \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \overline{P_i^{(n,n+1)}} \right\| \leq \frac{24 + 4\mathbf{C}^{n+1}}{\sin(\boldsymbol{\nu})} \frac{1}{\rho_{n+1} + \left| z - \lambda_i^{(n)} \right|}, \tag{4.4.20}$$

for all $z \in M_i^{(n)}$, where we have used the notation $P_i^{(n,n+1)} := P_i^{(n)} \otimes P_{\Omega^{(n,n+1)}}$.

Proof. Let $z \in M_i^{(n)}$. Note that (see Remark 1.2.1)

$$\begin{aligned} \overline{P_i^{(n)}} + P_i^{(n)} \otimes \overline{P_{\Omega^{(n,n+1)}}} &= 1 - P_i^{(n)} + P_i^{(n)} \otimes (1 - P_{\Omega^{(n,n+1)}}) = 1 - P_i^{(n)} \otimes P_{\Omega^{(n,n+1)}} \\ &= \overline{P_i^{(n,n+1)}}, \end{aligned} \tag{4.4.21}$$

and consequently, we obtain from functional calculus (notice that $\left[H^{(n),\theta}, H_f^{(n,n+1),\theta}\right] = 0$) that

$$\begin{aligned}
&\left\| \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \overline{P_i^{(n,n+1)}} \right\| \\
&\leq \left\| \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \overline{P_i^{(n)}} \right\| + \left\| \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} P_i^{(n)} \otimes \overline{P_{\Omega^{(n,n+1)}}} \right\| \\
&= \sup_{s \in \{0\} \cup [\rho_{n+1}, \infty)} \left\| \frac{1}{H^{(n),\theta} - (z - e^{-\theta} s)} \overline{P_i^{(n)}} \right\| + \sup_{s \in [\rho_{n+1}, \infty)} \frac{\left\| P_i^{(n)} \right\|}{|\lambda_i^{(n)} - (z - e^{-\theta} s)|}. \qquad (4.4.22)
\end{aligned}$$

Lemma 4.3.13, Definition 4.3.14 and induction hypothesis ($\mathcal{P}4$), together with Lemma 4.4.7 **and the Definition of $\boldsymbol{C}$, see Remark 4.4.4, in Definition 4.4.1 (notice that $\boldsymbol{C} \geq \frac{4\boldsymbol{c}}{\sin(\boldsymbol{\nu}/2)} \geq \frac{4C_{(4.3.43)}}{\sin(\boldsymbol{\nu}/2)}$ and** $\|\overline{P_i^{(n)}}\| \leq 4$**)**, imply that

$$\left\| \frac{1}{H^{(n),\theta} - (z - e^{-\theta} s)} \overline{P_i^{(n)}} \right\| \leq \frac{\mathbf{C}^{n+1}}{\rho_n + \left| \lambda_i^{(n)} - (z - e^{-\theta} s) \right|}, \qquad (4.4.23)$$

for every $s \in \{0\} \cup [\rho_{n+1}, \infty)$.

From the definitions of the sets $M_i^{(n)}$ and $\mathcal{S}$, it follows that

$$|\lambda_i^{(n)} - (z - e^{-\theta} s)| \geq \frac{1}{4} \rho_{n+1} \sin(\nu) \qquad (4.4.24)$$

for all $z \in M_i^{(n)}$ and $s \in [\rho_{n+1}, \infty)$. Moreover, we define the sets

$$G_i^{(n)} := \left\{ z \in M_i^{(n)} : \mathrm{Re}(z) \geq \mathrm{Re}(\lambda_i^{(n)}) \right\}, \quad i = 0, 1, \qquad (4.4.25)$$

and for $d \geq 0$

$$L_i^{(n),d} := \left\{ \lambda_i^{(n)} + e^{-\theta}(x + id) : x \in \mathbb{R} \right\}, \quad i = 0, 1. \qquad (4.4.26)$$

Furthermore, we define

$$L_i^{(n)} := \bigcup_{d \geq 0} L_i^{(n),d} \cap G_i^{(n)}, \quad i = 0, 1. \qquad (4.4.27)$$

Note that, by construction, we have

$$\mathrm{dist}\left(L_i^{(n),d}, \lambda_i^{(n)} \right) = e^{-\mathrm{Re}\,\theta} d, \quad i = 0, 1, \qquad (4.4.28)$$

and, by definition of the sets $M_i^{(n)}$ and $\mathcal{S}$, it follows that

$$\left| z - \lambda_i^{(n)} \right| \leq \left| z - \lambda_i^{(n)} - e^{-\theta} s \right|, \qquad \forall z \in M_i^{(n)} \setminus L_i^{(n)}, \quad \forall s \in [\rho_{n+1}, \infty), \qquad (4.4.29)$$

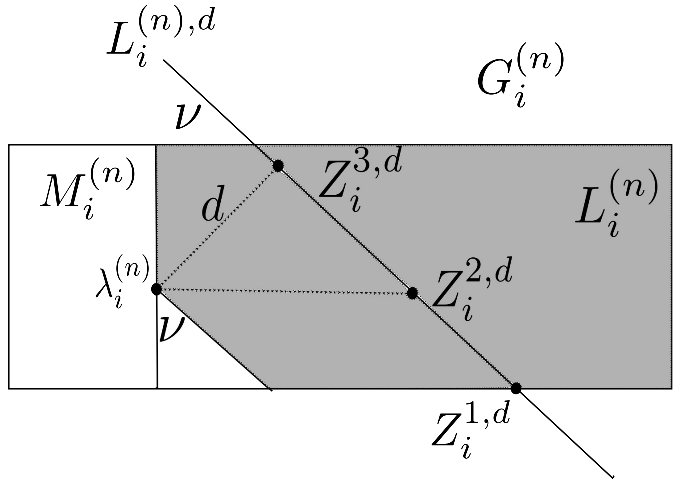

Figure 4.2.: Subsets of $M_i^{(n)}$

where we have used the factor $\frac{2}{5}$ in the definition of $M_i^{(n)}$. Let $Z_i^{1,d}$ and $Z_i^{2,d}$ be the intersections of $L_i^{(n),d}$ with the lines

$$\lambda_i^{(n)} - i\frac{2}{5}\rho_{n+1}\sin(\nu) + \mathbb{R} \quad \text{and} \quad \lambda_i^{(n)} + \mathbb{R}, \tag{4.4.30}$$

respectively. Furthermore, we define $Z_i^{3,d} := \lambda_i^{(n)} + de^{i\frac{\pi}{2}-\theta}$ and recall that $\nu < \pi/16$. Then, we obtain

$$\begin{aligned}\sup_{z\in L_i^{(n),d}\cap G_i^{(n)}} \left|z-\lambda_i^{(n)}\right|^2 &= \left|Z_i^{1,d}-\lambda_i^{(n)}\right|^2 = e^{-2\operatorname{Re}\theta}d^2 + \left|Z_i^{3,d}-Z_i^{1,d}\right|^2 \\ &= e^{-2\operatorname{Re}\theta}d^2 + \left(\left|Z_i^{3,d}-Z_i^{2,d}\right| + \left|Z_i^{2,d}-Z_i^{1,d}\right|\right)^2 = e^{-2\operatorname{Re}\theta}d^2 + \left(\frac{e^{-\operatorname{Re}\theta}d}{\tan(\nu)} + \frac{2}{5}\rho_{n+1}\right)^2.\end{aligned} \tag{4.4.31}$$

This yields the bound

$$\begin{aligned}&\frac{\left|z-\lambda_i^{(n)}\right|}{\left|z-\lambda_i^{(n)}-e^{-\theta}s\right|} \\ &\le \left[\frac{e^{-2\operatorname{Re}\theta}d^2}{\left|z-\lambda_i^{(n)}-e^{-\theta}s\right|^2} + \left(\frac{e^{-\operatorname{Re}\theta}d}{\tan(\nu)\left|z-\lambda_i^{(n)}-e^{-\theta}s\right|} + \frac{2\rho_{n+1}}{5\left|z-\lambda_i^{(n)}-e^{-\theta}s\right|}\right)^2\right]^{\frac{1}{2}}\end{aligned} \tag{4.4.32}$$

for $s \in [\rho_{n+1}, \infty)$ and $z \in L_i^{(n),d} \cap G_i^{(n)}$. Note that $\left|z-\lambda_i^{(n)}-e^{-\theta}s\right| \ge e^{-\operatorname{Re}\theta}d$ for all

$z \in L_i^{(n),d}$ and together with (4.4.24) we obtain

$$\frac{\left|z-\lambda_i^{(n)}\right|}{\left|z-\lambda_i^{(n)}-e^{-\theta}s\right|} \leq \left[1+\left(\frac{\cos(\nu)}{\sin(\nu)}+\frac{8}{5\sin(\nu)}\right)^2\right]^{\frac{1}{2}} \leq \frac{4}{\sin(\boldsymbol{\nu})}, \tag{4.4.33}$$

for all $z \in L_i^{(n)} \cap G_i^{(n)}$. This and (4.4.29) guarantees

$$\frac{1}{\left|z-\lambda_i^{(n)}-e^{-\theta}s\right|} \leq \frac{4}{\sin(\boldsymbol{\nu})}\frac{1}{\left|z-\lambda_i^{(n)}\right|} \qquad \forall z \in M_i^{(n)}, \quad \forall s \in [\rho_{n+1},\infty), \quad i=0,1. \tag{4.4.34}$$

Now, if $\left|z-\lambda_i^{(n)}\right| \geq \rho_{n+1}$, we use (4.4.34) and compute

$$\begin{aligned}\frac{1}{\left|z-\lambda_i^{(n)}-e^{-\theta}s\right|} &\leq \frac{4}{\sin(\boldsymbol{\nu})}\frac{\rho_{n+1}+\left|z-\lambda_i^{(n)}\right|}{\left|z-\lambda_i^{(n)}\right|}\frac{1}{\rho_{n+1}+\left|z-\lambda_i^{(n)}\right|}\\ &\leq \frac{8}{\sin(\boldsymbol{\nu})}\frac{1}{\rho_{n+1}+\left|z-\lambda_i^{(n)}\right|},\end{aligned} \tag{4.4.35}$$

and if $\left|z-\lambda_i^{(n)}\right| < \rho_{n+1}$, we use (4.4.24) and find

$$\begin{aligned}\frac{1}{\left|z-\lambda_i^{(n)}-e^{-\theta}s\right|} &\leq \frac{4}{\sin(\boldsymbol{\nu})}\frac{\rho_{n+1}+\left|z-\lambda_i^{(n)}\right|}{\rho_{n+1}}\frac{1}{\rho_{n+1}+\left|z-\lambda_i^{(n)}\right|}\\ &\leq \frac{8}{\sin(\boldsymbol{\nu})}\frac{1}{\rho_{n+1}+\left|z-\lambda_i^{(n)}\right|}.\end{aligned} \tag{4.4.36}$$

We conclude from (4.4.35) and (4.4.36) that for $i=0,1$

$$\frac{1}{\left|z-\lambda_i^{(n)}-e^{-\theta}s\right|} \leq \frac{8}{\sin(\boldsymbol{\nu})}\frac{1}{\rho_{n+1}+\left|z-\lambda_i^{(n)}\right|} \qquad \forall z \in M_i^{(n)}, \quad \forall s \in [\rho_{n+1},\infty) \tag{4.4.37}$$

holds true. Eqs. (4.4.22), (4.4.23), (4.4.34), together with Lemma 4.4.7 and (4.4.37) yield

$$\left\|\frac{1}{H^{(n),\theta}+H_f^{(n,n+1),\theta}-z}\overline{P_i^{(n,n+1)}}\right\| \leq \frac{24+4\mathbf{C}^{n+1}}{\sin(\boldsymbol{\nu})}\frac{1}{\rho_{n+1}+\left|z-\lambda_i^{(n)}\right|}. \tag{4.4.38}$$

This completes the proof. □

Lemma 4.4.10. *For all $z \in M_i^{(n)} \setminus \{\lambda_i^{(n)}\}$, all $0 \leq r \leq \left|z-\lambda_i^{(n)}\right|$ and every $i \in \{0,1\}$:*

$$\left\|\frac{H_f^{(n,n+1)}+r}{H_f^{(n,n+1),\theta}-\left(z-\lambda_i^{(n)}\right)}\right\| \leq \frac{10}{\sin(\boldsymbol{\nu})}. \tag{4.4.39}$$

Proof. We calculate:

$$\left\| \frac{H_f^{(n,n+1)} + r}{H_f^{(n,n+1),\theta} - (z - \lambda_i^{(n)})} \right\| = \sup_{y \in \{0\} \cup [\rho_{n+1}, \infty)} \left| \frac{y + r}{e^{-\theta} y + \lambda_i^{(n)} - z} \right|$$
$$\leq |e^{\theta}| + |e^{\theta}| \sup_{y \in \{0\} \cup [\rho_{n+1}, \infty)} \left| \frac{e^{-\theta} r - \lambda_i^{(n)} + z}{e^{-\theta} y + \lambda_i^{(n)} - z} \right| \leq |e^{\theta}| + (1 + |e^{\theta}|) \sup_{y \in \{0\} \cup [\rho_{n+1}, \infty)} \frac{\left| z - \lambda_i^{(n)} \right|}{\left| e^{-\theta} y + \lambda_i^{(n)} - z \right|}$$
$$\leq |e^{\theta}| + \frac{4(1 + |e^{\theta}|)}{\sin(\boldsymbol{\nu})} \leq \frac{10}{\sin(\boldsymbol{\nu})}, \tag{4.4.40}$$

where we have used (4.4.34) in the second last step. □

Lemma 4.4.11. *Suppose that ($\mathcal{P}1$)-($\mathcal{P}4$) hold true for all $m \in \mathbb{N}$ such that $m \leq n$. Then,*

$$\left\| V^{(n,n+1),\theta} \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \right\| \leq \frac{2500}{\rho \sin(\boldsymbol{\nu})^2} \mathbf{C}^{n+1} \rho_n^{\mu} \tag{4.4.41}$$

for all $z \in M_i^{(n)} \setminus \{\lambda_i^{(n)}\}$ such that $\left| z - \lambda_i^{(n)} \right| \geq \frac{1}{10} \rho_{n+1} \sin(\nu)$ and for all $i \in \{0, 1\}$.

Proof. Set $r = \left| z - \lambda_i^{(n)} \right| \geq \frac{1}{10} \rho_{n+1} \sin(\nu)$. We observe that

$$\left\| V^{(n,n+1),\theta} \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \right\| \leq \left\| V^{(n,n+1),\theta} \frac{1}{H_f^{(n,n+1)} + r} \right\| \left\| \frac{H_f^{(n,n+1)} + r}{H_f^{(n,n+1),\theta} - (z - \lambda_i^{(n)})} \right\|$$
$$\cdot \left\| \left(H_f^{(n,n+1),\theta} - (z - \lambda_i^{(n)}) \right) \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \right\|. \tag{4.4.42}$$

Lemma 4.4.10 yields

$$\left\| \frac{H_f^{(n,n+1)} + r}{H_f^{(n,n+1),\theta} - (z - \lambda_i^{(n)})} \right\| \leq \frac{10}{\sin(\boldsymbol{\nu})}, \tag{4.4.43}$$

and furthermore, we obtain from functional calculus that

$$\left\| \left(H_f^{(n,n+1),\theta} - (z - \lambda_i^{(n)}) \right) \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \right\| \leq \sup_{y \in \{0\} \cup [\rho_{n+1}, \infty)} \left\| \frac{e^{-\theta} y + \lambda_i^{(n)} - z}{H^{(n),\theta} + e^{-\theta} y - z} \right\|$$
$$\leq \left\| P_i^{(n)} \right\| + \sup_{y \in \{0\} \cup [\rho_{n+1}, \infty)} \left\| \frac{\lambda_i^{(n)} - (z - e^{-\theta} y)}{H^{(n),\theta} - (z - e^{-\theta} y)} \overline{P_i^{(n)}} \right\| \leq 3 + \mathbf{C}^{n+1} \leq 4\mathbf{C}^{n+1}. \tag{4.4.44}$$

In the last step, we use Lemma 4.4.7 for the first term. For the second term, we utilize Lemma 4.3.13, Definition 4.3.14 and induction hypothesis ($\mathcal{P}4$), together with Lemma

4.4.7 **and the Definition of $\boldsymbol{C}$ in Definition 4.4.1 - see Remark 4.4.4** (notice that $\boldsymbol{C} \geq \frac{4\boldsymbol{c}}{\sin(\boldsymbol{\nu}/2)} \geq \frac{4C_{(4.3.43)}}{\sin(\boldsymbol{\nu}/2)}$ and $\|\overline{P_i^{(n)}}\| \leq 4$).

Using the proof of Lemma 1.3.1, we obtain

$$\begin{aligned}
&\left\| V^{(n,n+1),\theta} \frac{1}{H_f^{(n,n+1)} + r} \right\| \\
&\leq \frac{1}{\sqrt{r}} \left(\left\| a_{n,n+1}(f^{\overline{\theta}}) \left(H_f^{(n,n+1)} + r \right)^{-\frac{1}{2}} \right\| + \left\| a_{n,n+1}(f^{\theta})^* \left(H_f^{(n,n+1)} + r \right)^{-\frac{1}{2}} \right\| \right) \\
&\leq \frac{1}{r} \left\| f^\theta \right\|_{\mathfrak{h}^{(n,n+1)}} + \frac{2}{\sqrt{r}} \left\| f^\theta / \sqrt{\omega} \right\|_{\mathfrak{h}^{(n,n+1)}}. \qquad (4.4.45)
\end{aligned}$$

We estimate

$$\begin{aligned}
\left\| f^\theta \right\|_{\mathfrak{h}^{(n,n+1)}} &= \sqrt{\int_{\mathcal{B}_{\rho_n} \setminus \mathcal{B}_{\rho_{n+1}}} \mathrm{d}^3k \, |f^\theta(k)|^2} = |e^{-\theta(1+\mu)}| \sqrt{4\pi \int_{\rho_{n+1}}^{\rho_n} \mathrm{d}u \, u^{1+2\mu} |e^{-2e^{2\theta} \frac{u^2}{\Lambda^2}}|} \\
&\leq |e^{-\theta(1+\mu)}| \sqrt{4\pi} \rho_n^\mu \rho_n, \qquad (4.4.46)
\end{aligned}$$

and similarly,

$$\left\| f^\theta / \sqrt{\omega} \right\|_{\mathfrak{h}^{(n,n+1)}} \leq |e^{-\theta(1+\mu)}| \sqrt{4\pi} \rho_n^\mu \rho_n^{\frac{1}{2}}. \qquad (4.4.47)$$

From our choice of r, it follows that $\sqrt{\frac{\rho_n}{r}} \leq \frac{\sqrt{10}}{\sqrt{\rho \sin(\nu)}}$ and, consequently, we obtain

$$\left\| V^{(n,n+1),\theta} \frac{1}{H_f^{(n,n+1)} + r} \right\| \leq |e^{-\theta(1+\mu)}| \sqrt{4\pi} \left(\frac{\rho_n}{r} + 2\sqrt{\frac{\rho_n}{r}} \right) \rho_n^\mu \leq |e^{-\theta(1+\mu)}| \frac{60}{\rho \sin(\boldsymbol{\nu})} \rho_n^\mu. \qquad (4.4.48)$$

Plugging (4.4.43), (4.4.44) and (4.4.48) into (4.4.42) yields (we recall that $\mu \in (0, 1/2)$)

$$\left\| V^{(n,n+1),\theta} \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \right\| \leq \frac{2500}{\rho \sin(\boldsymbol{\nu})^2} \mathbf{C}^{n+1} \rho_n^\mu. \qquad (4.4.49)$$

This completes the proof. □

Induction step

Here, we apply the results from Section 4.4.3 in order to show the induction step, i.e., we assume that ($\mathcal{P}1$)-($\mathcal{P}4$) hold true for all $j \leq n \in \mathbb{N}$, and prove that ($\mathcal{P}1$)-($\mathcal{P}4$) hold true for $n+1$. This together with Remark 4.4.6 then completes the proof of Theorem 4.4.5.

We first employ the estimates of Section 4.4.3 in order to prove Property ($\mathcal{P}2$) and ($\mathcal{P}3$). After this, we prove ($\mathcal{P}1$). Finally, ($\mathcal{P}4$) follows again from the results of Section 4.4.3 together with the maximum modulus principle.

Proof of ($\mathcal{P}2$) and ($\mathcal{P}3$) :

Proposition 4.4.12. *Suppose that ($\mathcal{P}$1)-($\mathcal{P}$4) hold true for all $m \in \mathbb{N}$ with $m \leq n$. Then,*

$$\left\| \frac{1}{H^{(n+1),\theta} - z} \right\| \leq \frac{168 + 8\mathbf{C}^{n+1}}{\sin(\boldsymbol{\nu})} \frac{1}{\rho_{n+1} + \left| z - \lambda_i^{(n)} \right|}, \tag{4.4.50}$$

for all $z \in M_i^{(n)} \setminus \{\lambda_i^{(n)}\}$ (see Definition 4.4.8) such that $\left| z - \lambda_i^{(n)} \right| \geq \frac{1}{10}\rho_{n+1}\sin(\nu)$ and for all $i \in \{0,1\}$.

Proof. Let $z \in M_i^{(n)} \setminus \{\lambda_i^{(n)}\}$ such that $\left| z - \lambda_i^{(n)} \right| \geq \frac{1}{10}\rho_{n+1}\sin(\nu)$ and $i \in \{0,1\}$. Then, it follows from Lemma 4.4.11 that

$$\left\| V^{(n,n+1),\theta} \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \right\| \leq \frac{2500}{\rho \sin(\boldsymbol{\nu})^2} \mathbf{C}^{n+1} \rho_n^{\mu}. \tag{4.4.51}$$

Our assumption on g in Definition 4.4.3 together with (4.4.51) **yield that (notice that Definition 4.4.2 implies that $C^{n+1}\rho_n^{\mu} \leq 1$ and Definition 4.4.3 implies that $\frac{g2500}{\rho\sin(\boldsymbol{\nu})^2} \leq \frac{1}{2}$, see also Remark 4.4.4)**

$$\left\| gV^{(n,n+1),\theta} \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \right\| \leq \frac{1}{2}. \tag{4.4.52}$$

This and Lemma 4.4.9 imply that

$$H^{(n+1),\theta} - z = \left(1 + gV^{(n,n+1),\theta} \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \right) \left(H^{(n),\theta} + H_f^{(n,n+1),\theta} - z \right) \tag{4.4.53}$$

is invertible, and we estimate

$$\begin{aligned} \left\| \frac{1}{H^{(n+1),\theta} - z} \right\| &\leq 2 \left\| \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \right\| \\ &\leq 2 \frac{\left\| P_i^{(n,n+1)} \right\|}{\left| z - \lambda_i^{(n)} \right|} + 2 \left\| \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \overline{P_i^{(n,n+1)}} \right\| \\ &\leq 2 \frac{\left\| P_i^{(n)} \right\|}{\left| z - \lambda_i^{(n)} \right|} + \frac{48 + 8\mathbf{C}^{n+1}}{\sin(\boldsymbol{\nu})} \frac{1}{\rho_{n+1} + \left| z - \lambda_i^{(n)} \right|}, \end{aligned} \tag{4.4.54}$$

where we apply Lemma 4.4.9. Moreover, Lemma 4.4.7 implies that $\left\| P_i^{(n)} \right\| \leq 3$ and it follows from $\left| z - \lambda_i^{(n)} \right| \geq \frac{1}{10}\rho_{n+1}\sin(\nu)$ that

$$\frac{1}{\left| z - \lambda_i^{(n)} \right|} \leq \frac{20}{\sin(\boldsymbol{\nu})} \frac{1}{\rho_{n+1} + \left| z - \lambda_i^{(n)} \right|}. \tag{4.4.55}$$

Altogether, we obtain

$$\left\| \frac{1}{H^{(n+1),\theta} - z} \right\| \leq \frac{168 + 8\mathbf{C}^{n+1}}{\sin(\boldsymbol{\nu})} \frac{1}{\rho_{n+1} + \left| z - \lambda_i^{(n)} \right|}, \tag{4.4.56}$$

and thereby, complete the proof. □

Lemma 4.4.13. *Suppose that ($\mathcal{P}$1)-($\mathcal{P}$4) hold true for all $m \in \mathbb{N}$ with $m \leq n$. We define*

$$\hat{P}_i^{(n+1)} := -\frac{1}{2\pi i} \int_{\hat{\gamma}_i^{(n)}} \mathrm{d}z \, \frac{1}{H^{(n+1),\theta} - z}, \tag{4.4.57}$$

where

$$\hat{\gamma}_i^{(n)} : [0, 2\pi] \to \mathbb{C}, \quad t \mapsto \lambda_i^{(n)} + \frac{1}{8}\rho_{n+1} \sin(\nu) e^{it}. \tag{4.4.58}$$

Then,

$$\left\| \hat{P}_i^{(n+1)} - P_i^{(n)} \otimes P_{\Omega^{(n,n+1)}} \right\| \leq \frac{|g|}{\rho} \mathbf{C}^{2(n+1)+2} \rho_n^{\mu} \leq \frac{|g|}{\rho} \left(\frac{1}{2} \right)^n. \tag{4.4.59}$$

(The last inequality follows from Definition 4.4.2.)

Proof. Recall that the definition of $P_i^{(n)}$ is introduced in (4.4.9). We notice that the function

$$B_i^{(n)} \setminus \{\lambda_i^{(n)}\} \ni z \mapsto \frac{1}{H^{(n),\theta} - z} \tag{4.4.60}$$

is analytic as an operator valued function and the region between $\hat{\gamma}_i^{(n)}$ and $\gamma_i^{(n)}$ is contained in the domain of (4.4.60). We obtain from the Cauchy integral theorem that

$$P_i^{(n)} = -\frac{1}{2\pi i} \int_{\gamma_i^{(n)}} \mathrm{d}z \, \frac{1}{H^{(n),\theta} - z} = -\frac{1}{2\pi i} \int_{\hat{\gamma}_i^{(n)}} \mathrm{d}z \, \frac{1}{H^{(n),\theta} - z}. \tag{4.4.61}$$

As in Remark 4.3.5, it turns out that (see Remark 1.2.1)

$$P_i^{(n)} \otimes P_{\Omega^{(n,n+1)}} = -\frac{1}{2\pi i} \int_{\hat{\gamma}_i^{(n)}} \mathrm{d}z \, \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z}. \tag{4.4.62}$$

We calculate

$$\begin{aligned} \|\hat{P}_i^{(n+1)} - P_i^{(n)} \otimes P_{\Omega^{(n,n+1)}}\| &= \frac{1}{2\pi} \Big\| \int_{\hat{\gamma}_i^{(n)}} \mathrm{d}z \, \frac{1}{H^{(n+1),\theta} - z} - \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \Big\| \\ = \frac{1}{2\pi} \Big\| \int_{\hat{\gamma}_i^{(n)}} \mathrm{d}z \, \frac{1}{H^{(n+1),\theta} - z} g V^{(n,n+1),\theta} &\frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \Big\|. \end{aligned} \tag{4.4.63}$$

Furthermore, Lemma 4.4.11 implies that for z in the curve $\hat{\gamma}_i^{(n)}$

$$\left\| gV^{(n,n+1),\theta} \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \right\| \leq |g| \frac{2500}{\rho \sin(\boldsymbol{\nu})^2} \mathbf{C}^{n+1} \rho_n^{\mu}, \tag{4.4.64}$$

and Proposition 4.4.12 ensures that

$$\left\| \frac{1}{H^{(n+1),\theta} - z} \right\| \leq \frac{168 + 8\mathbf{C}^{n+1}}{\sin(\boldsymbol{\nu})} \frac{1}{\rho_{n+1}}. \tag{4.4.65}$$

Eqs (4.4.63)-(4.4.65) imply

$$\left\| \hat{P}_i^{(n+1)} - P_i^{(n)} \otimes P_{\Omega^{(n,n+1)}} \right\| \leq |g| \frac{2500}{\rho \sin(\boldsymbol{\nu})^2} \mathbf{C}^{n+1} \rho_n^{\mu} \frac{168 + 8\mathbf{C}^{n+1}}{\sin(\boldsymbol{\nu})} \leq \frac{|g| 2500}{\rho \sin(\boldsymbol{\nu})^2} \mathbf{C}^{2n+2} \rho_n^{\mu} \frac{200}{\sin(\boldsymbol{\nu})}, \tag{4.4.66}$$

which together with the definition of C in Definitions 4.4.1 imply the desired result (Definition 4.4.1 imply that $\frac{500\,000}{\sin(\boldsymbol{\nu})^3} \leq \mathbf{C}^2$, see also Remark 4.4.4). □

Proposition 4.4.14 (Proof of Properties ($\mathcal{P}2$) and ($\mathcal{P}3$))**.** *Suppose that ($\mathcal{P}1$)-($\mathcal{P}4$) hold true for all $m \in \mathbb{N}$ with $m \leq n$, then ($\mathcal{P}2$) and ($\mathcal{P}3$) hold true for $n+1$.*

Proof. Lemma 4.4.13 implies that $\left\| \hat{P}_i^{(n+1)} - P_i^{(n)} \otimes P_{\Omega^{(n,n+1)}} \right\| < 1$ **(see Definition 4.4.3 and recall Remark 4.4.4)**. From the induction hypothesis it follows that $P_i^{(n)} \otimes P_{\Omega^{(n,n+1)}}$ is a rank-one projection. Therefore, $\hat{P}_i^{(n+1)}$ is also a rank-one projection. Proposition 4.4.12 implies that $H^{(n+1),\theta}$ has no spectral points in $M_i^{(n)} \backslash D\Big(\lambda_i^{(n)}, \frac{1}{10}\rho_{n+1}\sin(\nu)\Big)$. Since the contour of integration for $\hat{P}_i^{(n+1)}$ is contained in $M_i^{(n)}$ and its interior contains $D\Big(\lambda_i^{(n)}, \frac{1}{10}\rho_{n+1}\sin(\nu)\Big)$, we obtain that there is only one point in $M_i^{(n)}$ contained in the spectrum of $H_i^{(n+1)}$. This point is the eigenvalue $\lambda_i^{(n+1)}$ that we introduced above. Lemma 4.4.12 implies that $|\lambda_i^{(n+1)} - \lambda_i^{(n)}| \leq \frac{1}{10}\rho_{n+1}\sin(\nu)$. This in turn implies that $B_i^{(n+1)} \subset M_i^{(n)}$. Then, $\lambda_i^{(n+1)}$ is the only spectral point of $H_i^{(n+1),\theta}$ in $B_i^{(n+1)}$, which is Property ($\mathcal{P}2$). A deformation in the integration contour in the definitions of $\hat{P}_i^{(n+1)}$ and $P_i^{(n+1)}$ implies that these projections coincide and, therefore, Property ($\mathcal{P}3$) is a consequence of Lemma 4.4.13. □

Proof of Property ($\mathcal{P}1$):

Proposition 4.4.15 (Proof of Property ($\mathcal{P}1$))**.** *Suppose that ($\mathcal{P}1$)-($\mathcal{P}4$) hold true for all $m \in \mathbb{N}$ with $m \leq n$. Then, we obtain for $i = 0, 1$ that*

$$\left| \lambda_i^{(n+1)} - \lambda_i^{(n)} \right| \leq |g| \mathbf{C}^{(n+1)+1} \rho_n^{1+\mu} \leq |g| \left(\frac{1}{2}\right)^n \rho_n \tag{4.4.67}$$

holds true. Notice that the last inequality follows from Definition 4.4.2.

Proof. In this proof we explicitly emphasize the dependence of $P_i^{(n)}$ on θ and write $P_i^{(n)} \equiv P_i^{(n),\theta}$. We define $\Psi_i^{(n),\theta} := P_i^{(n),\theta}\varphi_i \otimes \Omega^{(n+1)}$, see Remark 1.2.1. Proposition 4.4.14, Property $\mathcal{P}3$, Definition 4.3.14 (see Remark 4.3.11) **and the restrictions for** g **in Definition 4.4.3 imply that** $\|\Psi_i^{(n),\theta} - \varphi_i \otimes \Omega^{(n+1)}\| \leq \frac{1}{10^2}$**. This guarantees that**

$$\left\|\Psi_i^{(n),\theta}\right\| \leq 2 \qquad \text{and} \qquad \left|\left\langle \Psi_i^{(n),\overline{\theta}}, P_i^{(n+1),\theta}\Psi_i^{(n),\theta}\right\rangle\right| \geq \frac{1}{2}. \tag{4.4.68}$$

Notice that in this work we assume that the imaginary part of θ, ν, is positive. Then, strictly speaking, we do not have the right to use the symbol $\Psi_i^{(n),\overline{\theta}} := P_i^{(n),\overline{\theta}}\varphi_i \otimes \Omega^{(n+1)}$. However, the restriction we impose by assuming that ν is not negative is irrelevant. This is assumed only for convenience in order to simplify our notation. Of course, the same results hold true if we take $-\pi/16 < \nu < -\boldsymbol{\nu}$ (we use this fact in the present proof as well as $P_i^{(n),\overline{\theta}} = (P_i^{(n),\theta})^*$). Then, we obtain

$$\begin{aligned}\lambda_i^{(n+1)} &= \frac{\left\langle \Psi_i^{(n),\overline{\theta}}, H^{(n+1),\theta}P_i^{(n+1),\theta}\Psi_i^{(n),\theta}\right\rangle}{\left\langle \Psi_i^{(n),\overline{\theta}}, P_i^{(n+1),\theta}\Psi_i^{(n),\theta}\right\rangle} = \frac{\left\langle H^{(n+1),\overline{\theta}}\Psi_i^{(n),\overline{\theta}}, P_i^{(n+1),\theta}\Psi_i^{(n),\theta}\right\rangle}{\left\langle \Psi_i^{(n),\overline{\theta}}, P_i^{(n+1),\theta}\Psi_i^{(n),\theta}\right\rangle} \\ &= \lambda_i^{(n)} + g\frac{\left\langle V^{(n,n+1),\overline{\theta}}\Psi_i^{(n),\overline{\theta}}, P_i^{(n+1),\theta}\Psi_i^{(n),\theta}\right\rangle}{\left\langle \Psi_i^{(n),\overline{\theta}}, P_i^{(n+1),\theta}\Psi_i^{(n),\theta}\right\rangle}.\end{aligned} \tag{4.4.69}$$

Now we choose $z \in \mathbb{C}$ such that $|z - \lambda_i^{(n)}| = \frac{\rho_{n+1}\sin(\nu)}{10}$. We get that

$$\begin{aligned}&\left|\lambda_i^{(n+1)} - \lambda_i^{(n)}\right| \leq 2\left\|gV^{(n,n+1),\overline{\theta}}P_i^{(n),\overline{\theta}}\varphi_i \otimes \Omega^{(n+1)}\right\|\left\|P_i^{(n+1),\theta}\Psi_i^{(n),\theta}\right\| \\ &\leq 54\left|z - \lambda_i^{(n)}\right|\left\|gV^{(n,n+1),\overline{\theta}}\frac{1}{H^{(n),\overline{\theta}} + H_f^{(n,n+1),\overline{\theta}} - z}\right\| \\ &\leq g54\frac{\rho_{n+1}\sin(\nu)}{10}\frac{2500}{\rho\sin(\boldsymbol{\nu})^2}\mathbf{C}^{n+1}\rho_n^{\mu} \leq |g|\mathbf{C}^{(n+1)+1}\rho_n^{1+\mu} \leq |g|\left(\frac{1}{2}\right)^n\rho_n,\end{aligned} \tag{4.4.70}$$

where we use Lemmas 4.4.11 and 4.4.7 **and the definition of** $\boldsymbol{C}$ **in Definition 4.4.1 (it implies that** $54\frac{2500}{\sin(\boldsymbol{\nu})^2} \leq \boldsymbol{C}$**, see also Remark 4.4.4).** □

Proof of Property ($\mathcal{P}4$):

Lemma 4.4.16. *Suppose that ($\mathcal{P}1$)-($\mathcal{P}4$) hold true for all $m \in \mathbb{N}$ with $m \leq n$. Then, for $i \in \{0,1\}$:*

$$\left\|\frac{1}{H^{(n+1),\theta} - z}\overline{P_i^{(n+1)}}\right\| \leq 3\frac{168 + 8\mathbf{C}^{n+1}}{\sin(\boldsymbol{\nu})}\frac{1}{\rho_{n+1} + \left|z - \lambda_i^{(n)}\right|}, \qquad \forall z \in M_i^{(m)}, \tag{4.4.71}$$

and hence, for all $z \in B_i^{(n+1)}$ (recall $B_i^{(n+1)} \subset M_i^{(n)}$ from the proof of Proposition 4.4.14).

Proof. Let $z \in M_i^{(n)}$ such that $\left|z - \lambda_i^{(n)}\right| \geq \frac{\rho_{n+1}}{10}\sin(\nu)$ and $i \in \{0,1\}$. Then, (4.4.71) follows from Proposition 4.4.12 and the fact that $\|\overline{P_i^{(n+1)}}\| \leq 3$ (see the proof of Lemma 4.4.7 and Proposition 4.4.14). Furthermore, we observe that $M_i^{(n)} \ni z \mapsto \frac{1}{H^{(n+1),\theta} - z}\overline{P_i^{(n+1)}}$ is analytic (see the proof of Proposition 4.4.14), and hence, (4.4.71) follows for $\left|z - \lambda_i^{(n)}\right| \leq \frac{\rho_{n+1}}{10}\sin(\nu)$ from the maximum modulus principle of complex analysis. □

Proposition 4.4.17 (Proof of Property ($\mathcal{P}4$))**.** *Suppose that ($\mathcal{P}1$)-($\mathcal{P}4$) hold true for all $m \in \mathbb{N}$ with $m \leq n$ and take $i \in \{0,1\}$. Then,*

$$\left\| \frac{1}{H^{(n+1),\theta} - z}\overline{P_i^{(n+1)}} \right\| \leq 4\frac{168 + 8\mathbf{C}^{n+1}}{|\sin(\boldsymbol{\nu})|}\frac{1}{\rho_{n+1} + \left|z - \lambda_i^{(n+1)}\right|} \leq \frac{\mathbf{C}^{(n+1)+1}}{\rho_{n+1} + \left|z - \lambda_i^{(n+1)}\right|} \tag{4.4.72}$$

for all $z \in B_i^{(n+1)}$.

Proof. Let $i \in \{0,1\}$ and $z \in B_i^{(n+1)}$. It follows from Proposition 4.4.16 that

$$\left\| \frac{1}{H^{(n+1),\theta} - z}\overline{P_i^{(n+1)}} \right\| \leq 3\frac{168 + 8C(\nu)^{n+1}}{\sin(\boldsymbol{\nu})}\frac{1}{\rho_{n+1} + \left|z - \lambda_i^{(n)}\right|} \tag{4.4.73}$$

holds true. Lemma 4.4.12 implies that $|\lambda_i^{(n+1)} - \lambda_i^{(n)}| \leq \frac{1}{10}\rho_{n+1}\sin(\nu) \leq \frac{1}{10}\rho_{n+1}$. Therefore,

$$\frac{1}{\rho_{n+1} + \left|z - \lambda_i^{(n)}\right|} \leq \frac{10}{9}\frac{1}{\rho_{n+1} + \left|z - \lambda_i^{(n+1)}\right|}. \tag{4.4.74}$$

This together with (4.4.73) yields

$$\left\| \frac{1}{H^{(n+1),\theta} - z}\overline{P_i^{(n+1)}} \right\| \leq 4\frac{168 + 8\mathbf{C}^{n+1}}{\sin(\boldsymbol{\nu})}\frac{1}{\rho_{n+1} + \left|z - \lambda_i^{(n+1)}\right|} \leq \frac{\mathbf{C}^{(n+1)+1}}{\rho_{n+1} + \left|z - \lambda_i^{(n+1)}\right|}, \tag{4.4.75}$$

where in the last line **we use the definition of C in Definition 4.4.1 (it implies that $4\frac{168+8}{\sin(\nu)} \leq \mathbf{C}$, see also Remark 4.4.4).** □

4.5. Resolvent and spectral estimates

In this section we prove Theorems 4.2.6, 4.2.7 and Proposition 4.2.1. The resolvent and spectral estimates that we provide are essentially different from the ones presented in Section 4.4 and [10]. The reason is the following: in [10] the construction of the resonance is based on a sequence of infrared cut-off Hamiltonians. As the parameter, n, of the sequence tends to infinity the cut-off is removed. Each cut-off Hamiltonian has a resonance that is isolated from the rest of the spectrum and they tend to the resonance of the Hamiltonian without cut-off. The delicate point is to estimate spectra of the cut-off Hamiltonians in such a way that these estimates do not require conditions in the coupling constant that depend on n. This implies a selection of spectral regions to be analyzed at each step n. In [10], these regions are chosen in neighborhoods of the resonances and far away from the rest of the spectrum, because the interest lies in constructing the resonance of the full Hamiltonian. Here, we need more subtle estimates in regions that are not only close to the resonances but to other parts of the spectrum of the cut-off Hamiltonians. Then, we get resolvent estimates in terms of the distance to the spectrum rather than the distance to the resonance, as it is done in [10]. The regions that we choose are complements of cones with vertexes in neighborhoods of the resonances. Some parts of the cones are closer to the resonances than to the rest of the spectrum and other parts of the cones are closer to other spectral points. This makes our analysis harder than in [10]. Our analysis requires a geometric construction that controls spectra and resolvents outside cones at step n using the same information for the exterior of cones at step $n-1$. In Section 4.5.1 we analyze the infrared cut-off Hamiltonians and prove spectral and resolvent estimates about them (Theorem 4.5.5). Geometric aspects of the cones are presented in Lemmas 4.5.1 and 4.5.3 below. In Lemma 4.5.2 we give resolvent (and hence spectral) estimates of a Hamiltonian that is obtained by adding to the Hamiltonian at step n the free energy of step $n+1$, using the information we have at step n. From this last Hamiltonian we obtain the Hamiltonian at step $n+1$ by adding the interacting energy at step $n+1$, the analysis of this is presented in Lemma 4.5.4. Theorem 4.5.5 is a consequence of Lemma 4.5.4. The study of the full Hamiltonian is carried out in Theorems 4.5.9 and 4.5.10 in Section 4.5.2, using Theorem 4.5.5, in a similar manner as in Section 4.5.1. First, we add the full free energy to the Hamiltonian at step n in Lemma 4.5.7, and then, we add the full interacting energy, using Lemma 4.5.8, in Theorems 4.5.9 and 4.5.10. These theorems imply Theorems 4.2.6 and 4.2.7. The proof of Proposition 4.2.1 is not difficult and it is presented in Section 4.5.3.

We start with introducing some notation. In this section we assume that Definitions 4.4.1, 4.4.2 and 4.4.3 hold true. We additionally assume that $\boldsymbol{C} \geq \boldsymbol{D} \sin(\nu/m)^{-1}$, in order to freely apply Corollary 4.3.9 and (4.3.44). We fix the Hamiltonians (see Remark 1.2.1)

$$\tilde{H}^{(n),\theta} := H_0^\theta + gV^{(n),\theta}, \tag{4.5.1}$$

which are densely defined on the Hilbert space $\mathcal{H}$. We recall that we already defined

$$\mathfrak{h}^{(n,\infty)} = L^2(\mathcal{B}_{\rho_n}) \tag{4.5.2}$$

and the corresponding Fock space $\mathcal{F}[\mathfrak{h}^{(n,\infty)}]$ (it is defined in (1.2.7)), with vacuum state $\Omega^{(n,\infty)}$. We identify, as above,

$$\mathcal{H} \equiv \mathcal{H}^{(n)} \otimes \mathcal{F}[\mathfrak{h}^{(n,\infty)}]. \tag{4.5.3}$$

We define the free boson energy operator on $\mathcal{F}[\mathfrak{h}^{(n,\infty)}]$ by restricting the definition in (1.2.1) accordingly and denote it by the symbol $H_f^{(n,\infty),0} \equiv H_f^{(n,\infty)}$. We set

$$H_f^{(n,\infty),\theta} := e^{-\theta} H_f^{(n,\infty),0}. \tag{4.5.4}$$

For every function $h \in \mathfrak{h}^{(n,\infty)}$, we define the creation and annihilation operators, $a_{n,\infty}(h)$ and $a^*_{n,\infty}(h)$, on $\mathcal{F}[\mathfrak{h}^{(n,\infty)}]$ according to (1.2.10) and (1.2.11). Again, we use the same notation also for $h \in \mathfrak{h}$ but then understand h as its restriction to $\mathfrak{h}^{(n,n+1)}$.
We fix the following operator (defined on $\mathcal{K} \otimes \mathcal{F}[\mathfrak{h}^{(n,\infty)}]$, and hence, on $\mathcal{H}$ - see Remark 1.2.1)

$$V^{(n,\infty),\theta} := \sigma_1 \otimes \Big(a_{n,\infty}(f^{\overline{\theta}}) + a_{n,\infty}(f^{\theta})^* \Big), \tag{4.5.5}$$

and further, we obtain (see Remark 1.2.1):

$$H^\theta = H^{(n),\theta} + H_f^{(n,\infty),\theta} + gV^{(n,\infty),\theta} = \tilde{H}^{(n),\theta} + gV^{(n,\infty),\theta}. \tag{4.5.6}$$

4.5.1. Resolvent and spectral estimates multiscale analysis

In this section we analyze the infrared cut-off Hamiltonians and prove spectral and resolvent estimates about them (Theorem 4.5.5). Geometric aspects of the cones are presented in Lemmas 4.5.1 and 4.5.3 below. In Lemma 4.5.2 we give resolvent (and hence spectral) estimates of a Hamiltonian that is obtained by adding to the Hamiltonian at step n the free energy of step $n+1$, using the information we have at step n. From this last Hamiltonian we obtain the Hamiltonian at step $n+1$ by adding the interacting energy at step $n+1$, the analysis of this is presented in Lemma 4.5.4. Theorem 4.5.5 is a consequence of Lemma 4.5.4.

Lemma 4.5.1. *Suppose that $|g| \le \rho \frac{1}{10} \sin(\nu/2m)$. We define for $i = 0, 1$*

$$q_i^{(n)} := \lambda_i^{(n)} + \frac{1}{4}\rho_n e^{-i\nu}, \qquad q_i^{(n,n+1)} := \lambda_i^{(n)} + \Big(\frac{2}{5} - \frac{1}{100}\Big)\rho_{n+1} e^{-i\nu}. \tag{4.5.7}$$

It follows that

$$|\lambda_i - \lambda_i^{(n)}| \le 2|g|\rho_n^{1+\mu/2} \tag{4.5.8}$$

and

$$\mathcal{C}_m(q_i^{(n)}) \subset \mathcal{C}_m(q_i^{(n,n+1)}) \subset \mathcal{C}_m(q_i^{(n+1)}), \tag{4.5.9}$$

where the set $\mathcal{C}_m(\cdot)$ is defined in (4.2.7) (see Figure 4.3). Moreover,

$$\operatorname{dist}\Big(\mathcal{C}_m(q_i^{(n,n+1)}), \mathbb{C} \setminus \mathcal{C}_m(q_i^{(n+1)})\Big) \ge \sin(\nu/2m)\frac{1}{10}\rho_{n+1}. \tag{4.5.10}$$

and

$$\operatorname{dist}\Big(\mathcal{C}_m(q_i^{(n)}), \mathbb{C} \setminus \mathcal{C}_m(q_i^{(n,n+1)})\Big) \ge \sin(\nu/m)\frac{1}{10}\rho_{n+1}. \tag{4.5.11}$$

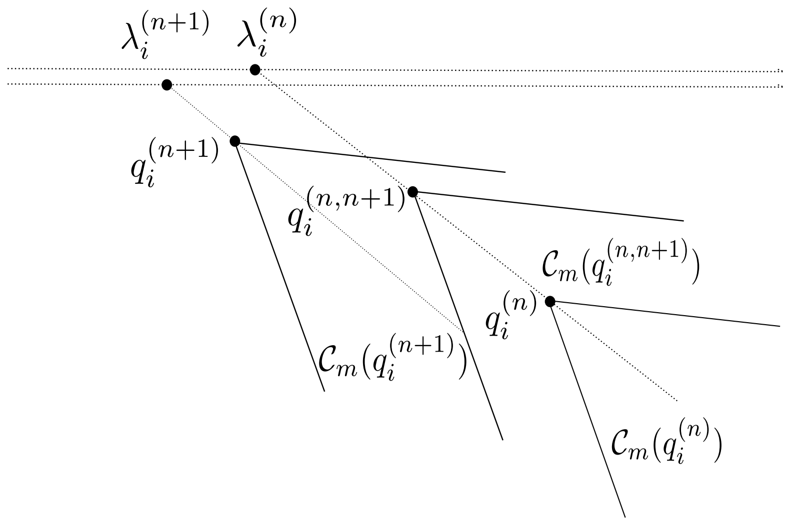

Figure 4.3.: Cones

Proof. That $\mathcal{C}_m(q_i^{(n)}) \subset \mathcal{C}_m(q_i^{(n,n+1)})$ is immediate. From Theorem 4.4.5 (Property $\mathcal{P}1$) and Definition 4.4.2 it follows that

$$|\lambda_i^{(n+1)} - \lambda_i^{(n)}| \leq |g|\big(\mathbf{C}^4\rho_0^{\mu}\big)^{1/2}\Big((C^2\rho^{\mu})^n\Big)^{1/2}\rho_n^{1+\mu/2} \leq |g|\frac{1}{2^n}\rho_n^{1+\mu/2}. \tag{4.5.12}$$

This and a geometric series argument prove (4.5.8). We write

$$q_i^{(n,n+1)} = q_i^{(n+1)} + \xi_1 e^{-i\nu} + \xi_2 i e^{-i\nu}. \tag{4.5.13}$$

Eq. (4.5.12) implies that

$$|\xi_2| \leq |g|\rho_n, \qquad \xi_1 \geq \Big(\frac{2}{5} - \frac{1}{100} - \frac{1}{4}\Big)\rho_{n+1} - |g|\rho_n > \frac{1}{10}\rho_{n+1}. \tag{4.5.14}$$

The last step follows for $g > 0$ sufficiently small (see Definition 4.4.3). To prove that $\mathcal{C}_m(q_i^{(n,n+1)}) \subset \mathcal{C}_m(q_i^{(n+1)})$, it is enough to show that $q_i^{(n,n+1)} \in \mathcal{C}_m(q_i^{(n+1)})$. We shall prove that

$$|\xi_2|/\xi_1 < \tan(\nu/2m), \tag{4.5.15}$$

which holds true if $|g| \leq \rho\frac{1}{10}\sin(\nu/2m) \leq \rho\frac{1}{10}\tan(\nu/2m)$.

Eq. (4.5.11) is is implied by the particular geometry considered here because both cones have the same axis, see also Definition 4.4.2.

Eq. (4.5.15) implies that the angle between the axis of the cone $\mathcal{C}_m(q_i^{(n+1)})$ and the complex number $q_i^{(n,n+1)} - q_i^{(n+1)}$ is smaller than $\nu/2m$ and, therefore, the angle between this complex number and the closest edge of the cone must be larger than $\nu/2m$. Then, the distance between the referred complex number and the edge is larger than

$$|q_i^{(n,n+1)} - q_i^{(n+1)}|\sin(\nu/2m) \geq \xi_1\sin(\nu/2m) \geq \sin(\nu/2m)\frac{1}{10}\rho_{n+1},$$

this implies (4.5.10). □

Lemma 4.5.2. *Assume that for all* $z \in B_i^{(1)} \setminus \mathcal{C}_m(q_i^{(n)})$

$$\left\| \frac{1}{H^{(n),\theta} - z} \overline{P_i^{(n)}} \right\| \leq \boldsymbol{C}^{n+1} \frac{1}{\mathrm{dist}\Big(\mathrm{z}, \mathcal{C}_\mathrm{m}(\mathrm{q_i^{(n)}})\Big)}, \tag{4.5.16}$$

then

$$\left\| \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \overline{P_i^{(n,n+1)}} \right\| \leq 4\boldsymbol{C}^{n+1} \frac{1}{\mathrm{dist}\Big(\mathrm{z}, \mathcal{C}_\mathrm{m}(\mathrm{q_i^{(n,n+1)}})\Big)}, \tag{4.5.17}$$

and

$$\left\| \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} H_f^{(n,n+1)} \overline{P_i^{(n,n+1)}} \right\| \leq \frac{100}{\sin(\nu/m)} \boldsymbol{C}^{n+1}, \tag{4.5.18}$$

for all $z \in B_i^{(1)} \setminus \mathcal{C}_m(q_i^{(n,n+1)})$.

Proof. Take $z \in B_i^{(1)} \setminus \mathcal{C}_m(q_i^{(n,n+1)})$. We use the spectral theorem and that (see (4.4.21))

$$\overline{P_i^{(n)}} + P_i^{(n)} \otimes \overline{P_{\Omega^{(n,n+1)}}} = \overline{P_i^{(n,n+1)}}, \tag{4.5.19}$$

to calculate

$$\begin{aligned} &\left\| \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \overline{P_i^{(n,n+1)}} \right\| \\ &\leq \left\| \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} \overline{P_i^{(n)}} \right\| + \left\| \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} P_i^{(n)} \otimes \overline{P_{\Omega^{(n,n+1)}}} \right\| \\ &= \sup_{s \in \{0\} \cup [\rho_{n+1}, \infty)} \left\| \frac{1}{H^{(n),\theta} - (z - e^{-\theta} s)} \overline{P_i^{(n)}} \right\| + \sup_{s \in [\rho_{n+1}, \infty)} \frac{\left\| P_i^{(n)} \right\|}{|\lambda_i^{(n)} - (z - e^{-\theta} s)|}. \end{aligned} \tag{4.5.20}$$

Thanks to the geometry, for all $s \geq 0$, we have

$$\mathrm{dist}\Big(z - e^{-\theta} s, \mathcal{C}_m(q_i^{(n)})\Big) \geq \mathrm{dist}\Big(z, \mathcal{C}_m(q_i^{(n)})\Big) \geq \mathrm{dist}\Big(z, \mathcal{C}_m(q_i^{(n,n+1)})\Big). \tag{4.5.21}$$

Eq. (4.5.21), our hypothesis, Lemma 4.3.13 and Definitions 4.3.14 and 4.4.1 imply that for $s \geq 0$

$$\left\| \frac{1}{H^{(n),\theta} - (z - e^{-\theta} s)} \overline{P_i^{(n)}} \right\| \leq \boldsymbol{C}^{n+1} \frac{1}{\mathrm{dist}\Big(\mathrm{z}, \mathcal{C}_\mathrm{m}(\mathrm{q_i^{(n,n+1)}})\Big)}. \tag{4.5.22}$$

Notice that for $s \geq \rho_{n+1}$,

$$z - e^{-\theta} s \notin \mathcal{C}_m(\lambda_i^{(n)}) \tag{4.5.23}$$

and, therefore,

$$|\lambda_i^{(n)} - (z - e^{-\theta}s)| \geq \mathrm{dist}\Big(z - e^{-\theta}s, \mathcal{C}_m(\lambda_i^{(n)})\Big) = \mathrm{dist}\Big(z, \mathcal{C}_m(\lambda_i^{(n)} + e^{-\theta}s)\Big) \geq \mathrm{dist}\Big(z, \mathcal{C}_m(q_i^{(n,n+1)})\Big). \tag{4.5.24}$$

Eqs. (4.5.21), (4.5.22) and (4.5.24), and Lemma 4.4.7 together with Definition 4.4.1 imply (4.5.17).

Now, we prove (4.5.18). As in (4.5.20) and (4.5.22), we have

$$\left\| \frac{1}{H^{(n),\theta} + H_f^{(n,n+1),\theta} - z} H_f^{(n,n+1)} \overline{P_i^{(n,n+1)}} \right\| \leq \sup_{s \in \{0\} \cup [\rho_{n+1}, \infty)} \boldsymbol{C}^{n+1} \frac{s}{\mathrm{dist}\Big(z - e^{-\theta}s, \mathcal{C}_m(q_i^{(n,n+1)})\Big)} + \sup_{s \in [\rho_{n+1}, \infty)} \frac{\left\|P_i^{(n)}\right\| s}{|\lambda_i^{(n)} - (z - e^{-\theta}s)|}. \tag{4.5.25}$$

Notice that for $z \notin \mathcal{C}_m(q_i^{(n,n+1)})$,

$$\mathrm{dist}\Big(z - e^{-\theta}s, \mathcal{C}_m(q_i^{(n,n+1)})\Big) \geq \frac{1}{2} s \sin(\nu/m). \tag{4.5.26}$$

Now we argue as in (4.5.24) and obtain, for $s \geq \rho_{n+1}$,

$$|\lambda_i^{(n)} - (z - e^{-\theta}s)| \geq \mathrm{dist}\Big(z - e^{-\theta}s, \mathcal{C}_m(\lambda_i^{(n)})\Big) = \mathrm{dist}\Big(z - \frac{1}{10}e^{-\theta}s, \mathcal{C}_m(\lambda_i^{(n)} + \frac{9}{10}e^{-\theta}s)\Big) \tag{4.5.27}$$

$$\geq \mathrm{dist}\Big(z - \frac{1}{10}e^{-\theta}s, \mathcal{C}_m(q_i^{(n,n+1)})\Big) \geq \frac{1}{20} s \sin(\nu/m).$$

Eqs. (4.5.25), (4.5.26) and (4.5.27) together with Lemma 4.4.7 imply (4.5.18). □

Lemma 4.5.3. *Let $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \mathcal{C}^{(3)}$ be cones in $\mathbb{C}$, such that $C^{(1)} \subsetneq \mathcal{C}^{(2)} \subsetneq \mathcal{C}^{(3)}$, of the form* (4.2.7) *- with the same m. Assume that*

$$\max_{y \in \partial\mathcal{C}^{(2)}} \mathrm{dist}\Big(y, \mathcal{C}^{(1)}\Big) \leq \frac{1}{2} \mathrm{dist}\Big(\mathbb{C} \setminus \mathcal{C}^{(3)}, \mathcal{C}^{(1)}\Big). \tag{4.5.28}$$

Then, for every $z \notin \mathcal{C}^{(3)}$:

$$\mathrm{dist}\Big(z, \mathcal{C}^{(2)}\Big) \geq \frac{1}{2} \mathrm{dist}\Big(z, \mathcal{C}^{(1)}\Big). \tag{4.5.29}$$

Proof. We take $z \notin \mathcal{C}^{(3)}$, $y \in \partial\mathcal{C}^{(2)}$, and $x \in \mathcal{C}^{(1)}$ such that $|y-x| = \mathrm{dist}\Big(y, \mathcal{C}^{(1)}\Big)$. We calculate

$$|z-y| \geq |z-x| - |x-y| = |z-x| - \mathrm{dist}\Big(y, \mathcal{C}^{(1)}\Big) \tag{4.5.30}$$

$$\geq |z-x| - \frac{1}{2}\mathrm{dist}\Big(\mathbb{C}\setminus\mathcal{C}^{(3)}, \mathcal{C}^{(1)}\Big). \tag{4.5.31}$$

Next, we use that

$$\mathrm{dist}\Big(\mathbb{C}\setminus\mathcal{C}^{(3)}, \mathcal{C}^{(1)}\Big) \leq |z-x| \tag{4.5.32}$$

to obtain:

$$|z-y| \geq \frac{1}{2}|z-x| \geq \frac{1}{2}\mathrm{dist}\Big(z, \mathcal{C}^{(1)}\Big), \tag{4.5.33}$$

and therefore,

$$\mathrm{dist}\Big(z, \mathcal{C}^{(2)}\Big) \geq \frac{1}{2}\mathrm{dist}\Big(z, \mathcal{C}^{(1)}\Big). \tag{4.5.34}$$

□

Lemma 4.5.4. *Assume that $|g| \leq \frac{\sin(\nu/2m)^3\rho}{10^8}$ and $\rho \leq 10^{-3}\sin(\nu/m)e_1$ and that for all $z \in B_i^{(1)} \setminus \mathcal{C}_m(q_i^{(n)})$*

$$\Big\| \frac{1}{H^{(n),\theta} - z}\overline{P_i^{(n)}} \Big\| \leq \boldsymbol{C}^{n+1}\frac{1}{\mathrm{dist}\Big(\mathrm{z}, \mathcal{C}_{\mathrm{m}}(\mathrm{q}_{\mathrm{i}}^{(\mathrm{n})})\Big)}, \tag{4.5.35}$$

then $\Big(B_i^{(1)} \setminus \mathcal{C}_m(q_i^{(n+1)})\Big) \setminus \{\lambda_i^{(n+1)}\}$ is contained in the resolvent set of $H^{(n+1),\theta}$ and

$$\Big\| \frac{1}{H^{(n+1),\theta} - z}\overline{P_i^{(n+1)}} \Big\| \leq \frac{10^5}{\sin(\nu/m)^2}\boldsymbol{C}^{n+1}\frac{1}{\mathrm{dist}\Big(\mathrm{z}, \mathcal{C}_{\mathrm{m}}(\mathrm{q}_{\mathrm{i}}^{(\mathrm{n,n+1})})\Big).}, \tag{4.5.36}$$

for every $z \in B_i^{(1)} \setminus \mathcal{C}_m(q_i^{(n+1)})$. Moreover, assuming that $\boldsymbol{C} \geq \frac{10^5}{\sin(\nu/m)^2}$, for all $z \in B_i^{(1)} \setminus \mathcal{C}_m(q_i^{(n+1)})$

$$\Big\| \frac{1}{H^{(n+1),\theta} - z}\overline{P_i^{(n)}} \Big\| \leq \boldsymbol{C}^{n+2}\frac{1}{\mathrm{dist}\Big(\mathrm{z}, \mathcal{C}_{\mathrm{m}}(\mathrm{q}_{\mathrm{i}}^{(\mathrm{n+1})})\Big)}. \tag{4.5.37}$$

Proof. Eq. (4.5.37) is a consequence of (4.5.36) and (4.5.9) together with $\boldsymbol{C} \geq \frac{10^5}{\sin(\nu/m)^2}$. We fix the cones:

$$\mathcal{C}^{(1)} = \mathcal{C}_m(q_i^{(n,n+1)}), \quad \mathcal{C}^{(2)} = \mathcal{C}_m(q_i^{(n,n+1)} - \rho_{n+1}e^{-i\nu}), \tag{4.5.38}$$

$$\mathcal{C}^{(3)} = \mathcal{C}_m(q_i^{(n,n+1)} - 2\frac{1}{\sin(\nu/m)}\rho_{n+1}e^{-i\nu}).$$

Note that the cones we just defined fulfill the hypothesis of Lemma 4.5.3. They satisfy the following properties (see Lemma 4.5.3). For all $z \notin C^{(3)}$ and for all $s \geq 0$:

$$\lambda_i^{(n)} \in \mathcal{C}^{(2)}, \quad |(z - se^{-\theta}) - \lambda_i^{(n)}| \geq \text{dist}((z - se^{-\theta}), \mathcal{C}^{(2)}) \geq \text{dist}(z, \mathcal{C}^{(2)}) \geq \frac{1}{2}\text{dist}(z, \mathcal{C}^{(1)}), \tag{4.5.39}$$

where we use that $z - se^{-\theta} \notin C^{(3)}$. We define $z_1 = x_1^{(1)} + ix_2^{(1)}$ $(x_1^{(1)}, x_2^{(1)} \in \mathbb{R})$ to be the point in the intersection of $q_i^{(n,n+1)} + \frac{1}{100}\rho_{n+1}e^{-i\nu} + \mathbb{R}$ with $\partial\mathcal{C}^{(1)}$ with smaller $x_1^{(1)}$, and similarly, $z_2 = x_1^{(2)} + ix_2^{(2)}$ the point in the intersection of $q_i^{(n,n+1)} + \frac{1}{100}\rho_{n+1}e^{-i\nu} + i\mathbb{R}$ with $\mathcal{C}^{(1)}$ with bigger $x_2^{(2)}$. We recall that

$$q_i^{(n,n+1)} + \frac{1}{100}\rho_{n+1}e^{-i\nu} = \lambda_i^{(n)} + \frac{2}{5}\rho_{n+1}e^{-i\nu} \in M_i^{(n)}, \tag{4.5.40}$$

see Definition 4.4.8, and therefore,

$$z_1, z_2 \in M_i^{(n)} \tag{4.5.41}$$

(the factor $\frac{1}{100}$ is chosen for this reason). Now, we set

$$\mathcal{U} := \left(\overline{\mathcal{C}^{(3)} \setminus \mathcal{C}^{(1)}}\right) \cap \bigcup_{t\in[0,1]} \{tz_1 + (1-t)z_2 + e^{-i\nu}\mathbb{R}\}. \tag{4.5.42}$$

Our restrictions on ρ together with (4.5.40) and (4.5.41) imply that

$$\mathcal{U} \subset M_i^{(n)}. \tag{4.5.43}$$

It follow from the particular considered geometry at hand that the distance between the boundary of $\mathcal{U}$ and $\lambda_i^{(n)}$ is bigger or equal than the distance between the point z_2 and the line $q_i^{(n,n+1)} + \mathbb{R}e^{-i\nu}$, which equals $\tan(\nu/m)\iota$, where ι is the distance between $q_i^{(n,n+1)}$ and the intersection of the line $z_2 + ie^{-i\nu}\mathbb{R}$ with $q_i^{(n,n+1)} + \mathbb{R}e^{-i\nu}$. Then ι is bigger or equal to the distance between $q_i^{(n,n+1)}$ and the intersection of the line $\tilde{z}_2 + ie^{-i\nu}\mathbb{R}$ with $q_i^{(n,n+1)} + \mathbb{R}e^{-i\nu}$, where $\tilde{z}_2$ is the intersection of $z_2 + i\mathbb{R}$ with $q_i^{(n,n+1)} + \mathbb{R}$. Then, $\iota \geq \frac{1}{100}\rho_{n+1}\cos(\nu)\cos(\nu)$. We obtain that

$$\text{dist}(\partial\mathcal{U}, \lambda_i^{(n)}) \geq \frac{1}{100}\rho_{n+1}\cos(\nu)\cos(\nu)\tan(\nu/m) \geq \frac{1}{200}\sin(\nu/m)\rho_{n+1}, \tag{4.5.44}$$

where we use that $\theta \in \mathcal{S}$.

For every $z \in \overline{\mathcal{C}^{(3)} \setminus \mathcal{C}^{(1)} \setminus \mathcal{U}}$ and $s \geq 0$, we have that

$$|\lambda_i^{(n)} - (z - se^{-\theta})| \geq \frac{1}{200}\sin(\nu/m)\rho_{n+1} \tag{4.5.45}$$

and

$$\text{dist}(z, \mathcal{C}^{(1)}) \leq 2\frac{1}{\sin(\nu/m)}\rho_{n+1}. \tag{4.5.46}$$

It follows form (4.5.45), (4.5.46) together with (4.5.39) that

$$\frac{\mathrm{dist}(z,\mathcal{C}^{(1)})}{|\lambda_i^{(n)}-(z-se^{-\theta})|}\leq 400\frac{1}{\sin(\nu/m)^2}, \tag{4.5.47}$$

for every $z\in\left(B_i^{(1)}\setminus\mathcal{C}^{(1)}\right)\setminus\mathcal{U}$. This implies, we also use Lemma 4.5.2 and the spectral theorem that (actually we only need $s=0$ above),

$$\Big\|\frac{1}{H^{(n),\theta}+H_f^{(n,n+1),\theta}-z}\Big\|\leq\frac{10^4}{\sin(\nu/m)^2}\boldsymbol{C}^{n+1}\frac{1}{\mathrm{dist}\Big(\mathrm{z},\mathcal{C}_{\mathrm{m}}(\mathrm{q_i^{(n,n+1)}})\Big)}, \tag{4.5.48}$$

and for every positive number r

$$\begin{aligned}\Big\|\frac{1}{H^{(n),\theta}+H_f^{(n,n+1),\theta}-z}(H_f^{(n,n+1)}+r)\Big\|\leq&\frac{100}{\sin(\nu/m)}\boldsymbol{C}^{n+1}\\&+\frac{10^4}{\sin(\nu/m)^2}\boldsymbol{C}^{n+1}\frac{r}{\mathrm{dist}\Big(\mathrm{z},\mathcal{C}_{\mathrm{m}}(\mathrm{q_i^{(n,n+1)}})\Big)},\end{aligned} \tag{4.5.49}$$

where we use that $H_f^{(n,n+1)}P_i^{(n,n+1)}=0$, for every $z\in\left(B_i^{(1)}\setminus\mathcal{C}^{(1)}\right)\setminus\mathcal{U}$. Choosing $r=\rho_{n+1}$, and additionally, $z\notin\mathcal{C}_m(q_i^{(n+1)})$, we get from (4.5.10) and (4.5.49) that

$$\Big\|\frac{1}{H^{(n),\theta}+H_f^{(n,n+1),\theta}-z}(H_f^{(n,n+1)}+r)\Big\|\leq\boldsymbol{C}^{n+1}\frac{10^6}{\sin(\nu/2m)^3}. \tag{4.5.50}$$

We observe that

$$\begin{aligned}\left\|V^{(n,n+1),\theta}\frac{1}{H^{(n),\theta}+H_f^{(n,n+1),\theta}-z}\right\|\leq&\left\|V^{(n,n+1),\theta}\frac{1}{H_f^{(n,n+1)}+r}\right\|\\&\times\left\|\left(H_f^{(n,n+1)}+r\right)\frac{1}{H^{(n),\theta}+H_f^{(n,n+1),\theta}-z}\right\|.\end{aligned} \tag{4.5.51}$$

Then, we have (see also (4.4.48))

$$\begin{aligned}\|gV^{(n,n+1),\theta}\frac{1}{H^{(n),\theta}+H_f^{(n,n+1),\theta}-z}\Big\|&\leq|g|\frac{10^6}{\sin(\nu/2m)^3}\boldsymbol{C}^{n+1}\left\|V^{(n,n+1),\theta}\frac{1}{H_f^{(n,n+1)}+r}\right\|\\\leq|g|\frac{10^6}{\sin(\nu/2m)^3}\boldsymbol{C}^{n+1}|e^{-\theta(1+\mu)}|\sqrt{4\pi}\left(\frac{\rho_n}{r}+2\sqrt{\frac{\rho_n}{r}}\right)\rho_n^\mu&\leq\frac{10^8|g|}{2\sin(\nu/2m)^3\rho}\boldsymbol{C}^{n+1}\rho_n^\mu\leq\frac{1}{2},\end{aligned} \tag{4.5.52}$$

because Definition 4.4.2 implies that $\boldsymbol{C}^{n+1}\rho_n^\mu\leq 1$ (we use as well our restrictions in $|g|$).

Eq. (4.5.52) and a Neumann series argument implies that $\left(B_i^{(1)} \setminus \mathcal{C}_m(q_i^{(n+1)})\right) \setminus \mathcal{U}$ is contained in the resolvent set of $H^{(n+1),\theta}$ and for all z in this set (see also (4.5.48))

$$\left\| \frac{1}{H^{(n+1),\theta} - z} \right\| \leq 2 \frac{10^4}{\sin(\nu/m)^2} \boldsymbol{C}^{n+1} \frac{1}{\mathrm{dist}\left(\mathrm{z}, \mathcal{C}_{\mathrm{m}}(\mathrm{q}_{\mathrm{i}}^{(\mathrm{n,n+1})})\right)}. \tag{4.5.53}$$

Lemma 4.4.16 ensures that $\lambda_i^{(n+1)}$ is the only spectral point of $H^{(n+1),\theta}$ in $M_i^{(n)}$. Hence, the function

$$\mathcal{U} \ni z \mapsto \frac{1}{H^{(n+1),\theta} - z} \overline{P_i^{(n+1)}} \tag{4.5.54}$$

is analytic. The maximum modulus principle implies that it attains its maximum on the boundary of $\mathcal{U}$, then we have (see Definition 4.4.1 and Lemma 4.4.16)

$$\left\| \frac{1}{H^{(n+1),\theta} - z} \overline{P_i^{(n+1)}} \right\| \leq \boldsymbol{C}^{n+1} \frac{1}{\rho_{n+1}} \tag{4.5.55}$$

for every $z \in \mathcal{U}$. Next, notice that, for $z \in \mathcal{U}$, $\mathrm{dist}\left(z, \mathcal{C}_m(q_i^{(n,n+1)})\right) \leq \frac{2}{\sin(\nu/m)} \rho_{n+1}$. Then, we obtain

$$\begin{aligned} \left\| \frac{1}{H^{(n+1),\theta} - z} \overline{P_i^{(n+1)}} \right\| &\leq \frac{1}{\sin(\nu/m)} \boldsymbol{C}^{n+1} \frac{2}{\mathrm{dist}\left(z, \mathcal{C}_m(q_i^{(n,n+1)})\right)} \\ &\leq \frac{1}{\sin(\nu/m)} \boldsymbol{C}^{n+1} \frac{2}{\mathrm{dist}\left(z, \mathcal{C}_m(q_i^{(n+1)})\right)}, \end{aligned} \tag{4.5.56}$$

Eqs. (4.5.53) and (4.5.56) together with Lemma 4.4.7 imply the desired result. □

The next theorem is proved inductively using Corollary 4.3.9 and Lemma 4.5.4. This is the main theorem of the present subsection.

Theorem 4.5.5. *Assume that* $|g| \leq \frac{\sin(\nu/2m)^3 \rho}{10^8}$, $\rho \leq 10^{-3} \sin(\nu/m) e_1$ *and* $\boldsymbol{C} \geq \frac{10^5}{\sin(\nu/m)^2}$. *Then, for all* $n \in \mathbb{N}$ *and for all* $z \in B_i^{(1)} \setminus \mathcal{C}_m(q_i^{(n)})$*:*

$$\left\| \frac{1}{H^{(n),\theta} - z} \overline{P_i^{(n)}} \right\| \leq \boldsymbol{C}^{n+1} \frac{1}{\mathrm{dist}\left(\mathrm{z}, \mathcal{C}_{\mathrm{m}}(\mathrm{q}_{\mathrm{i}}^{(\mathrm{n})})\right)}. \tag{4.5.57}$$

4.5.2. Resolvent estimates

In this section we study the spectrum and resolvent of the full Hamiltonian, it is carried out in Theorems 4.5.9 and 4.5.10, using Theorem 4.5.5, in a similar manner as in Section 4.5.1. First we add the full free energy to the Hamiltonian at step n in Lemma 4.5.7, and then, we add the full interacting energy, using Lemma 4.5.8, in Theorems 4.5.9 and 4.5.10. These theorems imply Theorems 4.2.6 and 4.2.7.

In this section we assume, in addition to Definitions 4.4.1 4.4.2 and 4.4.3 (and $\boldsymbol{C} \geq \boldsymbol{D}\sin(\nu/m)^{-1}$), that

$$|g| \leq \frac{\sin(\nu/2m)^3\rho}{10^8}, \qquad \rho \leq 10^{-3}\sin(\nu/m), \qquad \boldsymbol{C} \geq \frac{10^5}{\sin(\nu/m)^2}. \tag{4.5.58}$$

Lemma 4.5.6. *Let $z \notin \mathcal{C}_m(\lambda_i^{(n)})$ and $0 \leq r \leq |z-\lambda_i^{(n)}|$, $s \geq 0$. It follows that*

$$\Big|\frac{s+r}{\mathrm{dist}\Big(z-e^{-\theta}s, \mathcal{C}_m(\lambda_i^{(n)})\Big)}\Big| \leq 2\frac{1}{\sin(\nu/m)} + \frac{r}{\mathrm{dist}\Big(z, \mathcal{C}_m(\lambda_i^{(n)})\Big)}, \tag{4.5.59}$$
$$\Big|\frac{s+r}{e^{-\theta}s-\Big(z-\lambda_i^{(n)}\Big)}\Big| \leq \frac{6}{\sin(\nu/m)}.$$

Proof. We use coordinates in $\mathbb{C} \equiv \mathbb{R}^2$ with origin at $\lambda_i^{(n)}$, the first coordinate axis with direction $e^{-i\nu}$ and the second coordinate axis with direction $ie^{-i\nu}$. Notice that for every point $z = \lambda_i^{(n)} + \xi_1 e^{-i\nu} + \xi_2 i e^{-i\nu} \notin \mathcal{C}_m(\lambda_i^{(n)})$ and every $s \geq 0$, the following facts are implied by the considered geometry:

$$\xi_1 \leq 0 \Longrightarrow \Big|\lambda_i^{(n)} - (z - se^{-\theta})\Big| \geq |\lambda_i^{(n)} - z\Big|, \tag{4.5.60}$$

$$\xi_1 > 0 \Longrightarrow |\xi_2| \geq |\xi_1|\tan(\nu/m). \tag{4.5.61}$$

Eq. (4.5.61) implies that for $\xi_1 > 0$

$$|z-\lambda_i^{(n)}| \leq |\xi_2|\sqrt{1+\tan(\nu/m)^{-2}}, \tag{4.5.62}$$

and because $|\lambda_i^{(n)} - (z-se^{-\theta})\Big| \geq |\xi_2|$, we obtain that (we also use (4.5.60))

$$\frac{2}{\sin(\nu/m)}|\lambda_i^{(n)} - (z-se^{-\theta})\Big| \geq |\lambda_i^{(n)} - z\Big|, \tag{4.5.63}$$

for every $z \notin \mathcal{C}_m(\lambda_i^{(n)})$ and every $s \geq 0$. Take $z \notin \mathcal{C}_m(\lambda_i^{(n)})$ to obtain

$$\begin{aligned}\left|\frac{s+r}{e^{-\theta}s+\lambda_i^{(n)}-z}\right| &\leq |e^{\theta}| + |e^{\theta}|\left|\frac{e^{-\theta}r-\lambda_i^{(n)}+z}{e^{-\theta}s+\lambda_i^{(n)}-z}\right| \leq |e^{\theta}| + (1+|e^{\theta}|)\frac{\Big|z-\lambda_i^{(n)}\Big|}{\Big|e^{-\theta}s+\lambda_i^{(n)}-z\Big|} \\ &\leq |e^{\theta}| + \frac{2(1+|e^{\theta}|)}{\sin(\nu/m)} \leq \frac{6}{\sin(\nu/m)},\end{aligned} \tag{4.5.64}$$

which proves the second inequality in (4.5.59). The first inequality of the claim is again ensured thanks to our considered geometry that implies:

$$\mathrm{dist}\Big(z-e^{-\theta}s, \mathcal{C}_m(\lambda_i^{(n)})\Big) \geq \max\Big[\frac{\sin(\nu/m)}{2}s, \mathrm{dist}\Big(z, \mathcal{C}_m(\lambda_i^{(n)})\Big)\Big]; \tag{4.5.65}$$

recall that $\theta \in \mathcal{S}$. □

Lemma 4.5.7. *For $i = 0, 1$, the set $B_i^{(1)} \setminus \mathcal{C}_m(\lambda_i^{(n)})$ is contained in the resolvent set of $\tilde{H}^{(n),\theta}$ and for all z this set:*

$$\left\| \frac{1}{\tilde{H}^{(n),\theta} - z} \right\| \leq 4\boldsymbol{C}^{n+1} \frac{1}{\mathrm{dist}(z, \mathcal{C}_m(\lambda_i^{(n)}))}. \tag{4.5.66}$$

Proof. The spectral theorem, Lemma 4.3.13, Definitions 4.3.14 and 4.4.1, and Theorem 4.5.5 imply that (we also use Lemma 4.4.7, which is valid for every n because Theorem 4.4.5 is proved above)

$$\begin{aligned} \left\| \frac{1}{\tilde{H}^{(n),\theta} - z} \right\| =& \sup_{r \geq 0} \left\| \frac{1}{H^{(n),\theta} + e^{-\theta} r - z} (P_i^{(n)} + \overline{P_i^{(n)}}) \right\| \\ \leq& \boldsymbol{C}^{n+1} \frac{1}{\mathrm{dist}(z - e^{-\theta} r, \mathcal{C}_m(q_i^{(n)}))} + 3 \frac{1}{|\lambda_i^{(n)} + e^{-\theta} y - z|} \\ \leq& 4\boldsymbol{C}^{n+1} \frac{1}{\mathrm{dist}(z - e^{-\theta} r, \mathcal{C}_m(\lambda_i^{(n)}))} \leq 4\boldsymbol{C}^{n+1} \frac{1}{\mathrm{dist}(z, \mathcal{C}_m(\lambda_i^{(n)}))}, \end{aligned} \tag{4.5.67}$$

were we use that $\lambda_i^{(n)} \in \mathcal{C}_m(\lambda_i^{(n)})$ and the geometrical fact that $\mathrm{dist}(z - e^{-\theta} r, \mathcal{C}_m(\lambda_i^{(n)})) \geq \mathrm{dist}(z, \mathcal{C}_m(\lambda_i^{(n)}))$. □

Lemma 4.5.8. *For every $z \in B_i^{(1)} \setminus \mathcal{C}_m(\lambda_i^{(n)} - e^{-i\nu} \rho_n^{1+\mu/4})$ the following inequality holds true*

$$\left\| V^{(n,\infty),\theta} \frac{1}{\tilde{H}^{(n),\theta} - z} \right\| \leq \frac{10^3}{\sin(\nu/m)^2} \mathbf{C}^{n+1} \rho_n^{\frac{3\mu}{4}}. \tag{4.5.68}$$

Proof. We compute (see Remark 1.2.1):

$$\left\| V^{(n,\infty),\theta} \frac{1}{\tilde{H}^{(n),\theta} - z} \right\| \leq \left\| V^{(n,\infty),\theta} \frac{1}{H_f^{(n,\infty)} + r} \right\| \left\| (H_f^{(n,\infty)} + r) \frac{1}{\tilde{H}^{(n),\theta} - z} \right\|, \tag{4.5.69}$$

where we take

$$r = \mathrm{dist}\Big(\mathbb{C} \setminus \mathcal{C}_m(\lambda_i^{(n)} - e^{-i\nu} \rho_n^{1+\mu/4}), \mathcal{C}_m(\lambda_i^{(n)}) \Big) = \sin(\nu/m) \rho_n^{1+\mu/4} \ \leq \left| z - \lambda_i^{(n)} \right|. \tag{4.5.70}$$

As in (4.4.48) we obtain

$$\left\| V^{(n,\infty),\theta} \frac{1}{H_f^{(n,\infty)} + r} \right\| \leq |e^{-\theta(1+\mu)}| \sqrt{4\pi} \left(\frac{\rho_n}{r} + 2\sqrt{\frac{\rho_n}{r}} \right) \rho_n^{\mu} \leq \frac{50}{\sin(\nu/m)} \rho_n^{\frac{3\mu}{4}}. \tag{4.5.71}$$

The spectral theorem and Theorem 4.5.5 and Lemma 4.5.6 imply that

$$\begin{aligned}\Big\|\Big(H_f^{(n,\infty),\theta}+r\Big)\frac{1}{\tilde{H}^{(n),\theta}-z}\Big\| \leq & 4\boldsymbol{C}^{n+1}\sup_{s\geq 0}\Big|\frac{s+r}{\mathrm{dist}\Big(z-e^{-\theta}s,\mathcal{C}_m(\lambda_i^{(n)})\Big)}\Big| \\ \leq & 4\boldsymbol{C}^{n+1}\left[\frac{2}{\sin(\nu/m)}+\frac{r}{\mathrm{dist}\Big(z,\mathcal{C}_m(\lambda_i^{(n)})\Big)}\right] \\ \leq & 4\boldsymbol{C}^{n+1}\left[\frac{2}{\sin(\nu/m)}+\frac{r}{\sin(\nu/m)\rho_n^{1+\mu/4}}\right]\leq\frac{12\boldsymbol{C}^{n+1}}{\sin(\nu/m)}.\end{aligned} \tag{4.5.72}$$

We conclude the desired result by (4.5.69) together with (4.5.71) and (4.5.72). □

Theorem 4.5.9. *The set* $\in B_i^{(1)}\setminus\mathcal{C}_m(\lambda_i^{(n)}-e^{-i\nu}\rho_n^{1+\mu/4})$ *is contained in the resolvent set of* H^θ *and for all* $z\in B_i^{(1)}\setminus\mathcal{C}_m(\lambda_i^{(n)}-e^{-i\nu}\rho_n^{1+\mu/4})$*:*

$$\Big\|\frac{1}{H^\theta-z}\Big\|\leq 8\boldsymbol{C}^{n+1}\frac{1}{\mathrm{dist}(z,\mathcal{C}_m(\lambda_i^{(n)}))}, \tag{4.5.73}$$

and

$$\Big\|\frac{1}{H^\theta-z}-\frac{1}{\tilde{H}^{(n),\theta}-z}\Big\|\leq|g|\frac{10^5}{\sin(\nu/m)^2}\mathbf{C}^{2n+2}\rho_n^{\frac{3\mu}{4}}\frac{1}{\mathrm{dist}(z,\mathcal{C}_m(\lambda_i^{(n)}))}. \tag{4.5.74}$$

Proof. The result is a consequence of Neumann series and Lemmas 4.5.7 and 4.5.8 and (4.5.58). Notice that our assumptions on $\boldsymbol{C}$ in Definition 4.4.1 imply that $\mathbf{C}^{n+1}\rho_n^{\frac{3\mu}{4}}\leq 1$. □

Theorem 4.5.10. *For every* $n\in\mathbb{N}$ *and* $i\in\{0,1\}$*,*

$$\mathcal{C}_m(\lambda_i^{(n+1)}-\rho_{n+1}^{1+\mu/4}e^{-i\nu})\subset\mathcal{C}_m(\lambda_i^{(n)}-\rho_n^{1+\mu/4}e^{-i\nu})\subset\mathcal{C}_m(\lambda_i-2\rho_n^{1+\mu/4}e^{-i\nu}), \tag{4.5.75}$$

and thus, $B_i^{(1)}\setminus\mathcal{C}_m(\lambda_i-2\rho_n^{1+\mu/4}e^{-i\nu})$ *is contained in the resolvent set of* H^θ *(see Theorem 4.5.9). Moreover,* $B_i^{(1)}\setminus\mathcal{C}_m(\lambda_i)$ *is contained in the resolvent set of* H^θ*. Additionally, the following estimate holds true:*

$$\Big\|\frac{1}{H^\theta-z}\Big\|\leq 16\boldsymbol{C}^{n+1}\frac{1}{\mathrm{dist}(z,\mathcal{C}_m(\lambda_i))},\qquad\forall z\in\mathcal{C}_m(\lambda_i-2\rho_n^{1+\mu/4}e^{-i\nu}). \tag{4.5.76}$$

Proof. It follows from (4.5.12) that

$$|\lambda_i^{(n+1)}-\lambda_i^{(n)}|\leq|g|\rho_n^{1+\mu/2} \tag{4.5.77}$$

holds true. We write, for $\xi_1,\xi_2\in\mathbb{R}$,

$$\lambda_i^{(n+1)}-\rho_{n+1}^{1+\mu/4}e^{-i\nu}=\lambda_i^{(n)}-\rho_n^{1+\mu/4}e^{-i\nu}+\xi_1e^{-i\nu}+\xi_2ie^{-i\nu}. \tag{4.5.78}$$

Eq. (4.5.77) implies that

$$|\xi_2| \leq |g|\rho_n^{1+\mu/2}, \qquad \xi_1 \geq \rho_n^{1+\mu/4} - \rho_{n+1}^{1+\mu/4} - |g|\rho_n^{1+\mu/2} > \frac{1}{2}\rho_n^{1+\mu/4}, \tag{4.5.79}$$

see Definition 4.4.2 and Definition 4.4.3 (or (4.5.58) - notice that $|g|\rho_n^{1+\mu/2} \leq \frac{|g|}{\rho}\rho_{n+1}^{1+\mu/4}\rho_0^{\mu/4}$). To prove the first assertion in (4.5.75) it is enough to prove that $\lambda_i^{(n+1)} - \rho_{n+1}^{1+\mu/4}e^{-i\nu} \in \mathcal{C}_m(\lambda_i^{(n)} - \rho_n^{1+\mu/4}e^{-i\nu})$. Note that since $|g| \leq \frac{1}{2}\sin(\nu/m) \leq \frac{1}{2}\tan(\nu/m)$ (which is verified by (4.5.58)), we have

$$|\xi_2|/\xi_1 < \tan(\nu/m). \tag{4.5.80}$$

This proves the first assertion in (4.5.75).

The first part of (4.5.75) implies that, for all n,

$$\mathbb{C} \setminus \mathcal{C}_m(\lambda_i^{(n)} - \rho_n^{1+\mu/4}e^{-i\nu}) \subset \mathbb{C} \setminus \mathcal{C}_m(\lambda_i^{(n+1)} - \rho_{n+1}^{1+\mu/4}e^{-i\nu}) \tag{4.5.81}$$

and

$$\bigcup_n \mathbb{C} \setminus \mathcal{C}_m(\lambda_i^{(n)} - \rho_n^{1+\mu/4}e^{-i\nu}) = \mathbb{C} \setminus \mathcal{C}_m(\lambda_i) \tag{4.5.82}$$

belongs to the resolvent set of H^θ, see Theorem 4.5.9.

In a similar fashion as above we prove that

$$\mathcal{C}_m(\lambda_i^{(n)} - \rho_n^{1+\mu/4}e^{-i\nu}) \subset \mathcal{C}_m(\lambda_i - 2\rho_n^{1+\mu/4}e^{-i\nu}), \tag{4.5.83}$$

using (4.5.8). For every $z \notin \mathcal{C}_m(\lambda_i - 2\rho_n^{1+\mu/4}e^{-i\nu})$ and $a \in \mathcal{C}_m(\lambda_i^{(n)})$ we know that (see (4.5.8))

$$\mathrm{dist}(z, \mathcal{C}_m(\lambda_i)) \leq \mathrm{dist}(z, a) + \mathrm{dist}(a, \mathcal{C}_m(\lambda_i)) \leq \mathrm{dist}(z, a) + 2|g|\rho_n^{1+\mu/2}, \tag{4.5.84}$$

and hence, we obtain (see (4.5.58))

$$\mathrm{dist}(z, \mathcal{C}_m(\lambda_i)) \leq \mathrm{dist}(z, \mathcal{C}_m(\lambda_i^{(n)})) + \sin(\nu/m)\rho_n^{1+\mu/4}. \tag{4.5.85}$$

Moreover (see (4.5.83)),

$$\mathrm{dist}(z, \mathcal{C}_m(\lambda_i^{(n)})) \geq \mathrm{dist}\Big(\mathbb{C} \setminus \mathcal{C}_m(\lambda_i^{(n)} - \rho_n^{1+\mu/4}e^{-i\nu}), \mathcal{C}_m(\lambda_i^{(n)})\Big) \geq \sin(\nu/m)\rho_n^{1+\mu/4}. \tag{4.5.86}$$

Then, it follows that

$$\frac{\mathrm{dist}(z, \mathcal{C}_m(\lambda_i))}{\mathrm{dist}(z, \mathcal{C}_m(\lambda_i^{(n)}))} \leq \frac{\mathrm{dist}(z, \mathcal{C}_m(\lambda_i^{(n)}))}{\mathrm{dist}(z, \mathcal{C}_m(\lambda_i^{(n)}))} + \frac{\sin(\nu/m)\rho_n^{1+\mu/4}}{\sin(\nu/m)\rho_n^{1+\mu/4}} \leq 2. \tag{4.5.87}$$

This and Theorem 4.5.9 implies (4.5.76). □

We end this section with a remark about the convergence rate of the projections $P_i^n \otimes P_{\Omega(n,\infty)}$ that will be used in Chapter 5.

Remark 4.5.11. *It follows from Theorem 4.4.5, Property ($\mathcal{P}3$), that*

$$\left\| P_i - P_i^{(n)} \otimes P_{\Omega(n,\infty)} \right\| \le 2 \frac{|g|}{\rho} \frac{1}{2^n} \rho_n^{\mu/2}. \tag{4.5.88}$$

This is a consequence of a geometric series argument and Definition 4.4.2, since it implies that

$$\boldsymbol{C}^{2(n+1)+2} \rho_n^{\mu} = (\boldsymbol{C}^8 \rho_0^{\mu})^{1/2} (\boldsymbol{C}^4 \rho^{\mu})^{n/2} \rho_n^{\mu/2} \le \frac{1}{2^n} \rho_n^{\mu/2}. \tag{4.5.89}$$

4.5.3. Proof of Proposition 4.2.1

We define the sequence of vectors (see Remark 1.2.1)

$$\Psi_{\lambda_i}^{(n)} := P_i^{(n)} \varphi_i \otimes \Omega, \quad n \in \mathbb{N}. \tag{4.5.90}$$

Due to the Property ($\mathcal{P}3$) in Theorem 4.4.5, we know that the sequence above converges to the non-zero limit $\Psi_{\lambda_i} := P_i \varphi_i \otimes \Omega \neq 0$ (see the discussion above (4.4.68)). Note that (see Remark 1.2.1)

$$H^\theta = H^{(n),\theta} + H_f^{(n,\infty),\theta} + g V^{(n,\infty),\theta} = \tilde{H}^{(n),\theta} + g V^{(n,\infty),\theta} \tag{4.5.91}$$

and set $z = \lambda_i^{(n)} - 10 \rho_n e^{-i\nu}$. Then,

$$H^\theta \Psi_{\lambda_i}^{(n)} = \lambda_i^{(n)} \Psi_{\lambda_i}^{(n)} + g(\lambda_i^{(n)} - z) V^{(n,\infty),\theta} \frac{1}{\tilde{H}^{(n),\theta} - z} \Psi_{\lambda_i}^{(n)}. \tag{4.5.92}$$

Lemma 4.5.8 implies that $V^{(n,\infty),\theta} \frac{1}{\tilde{H}^{(n),\theta} - z}$ tends to zero as n tends to infinity. We conclude that

$$\lim_{n \to \infty} H^\theta \Psi_{\lambda_i}^{(n)} = \lambda_i \Psi_{\lambda_i}. \tag{4.5.93}$$

As H^θ is a closed operator, Ψ_{λ_i} belongs to its domain and is an eigenvector of H^θ corresponding to the eigenvalue λ_i. Furthermore, as P_i is rank-one, Ψ_{λ_i} spans its range.

4.6. Analyticity

In this section we prove Theorems 4.2.3 and 4.2.5, in particular, we show analyticity of the projections P_i and the eigenvalues λ_i with respect to the coupling constant g and the dilation parameter θ. We only prove in detail analyticity with respect to θ, the result for the coupling constant g follows the same line of arguments and it is actually simpler since g only appears in the interaction term and the dependence is linear. This result is achieved in two steps: At first, we prove these properties for $P_i^{(n)}$ and $\lambda_i^{(n)}$ which is straight-forward since there are spectral gaps between $\lambda_i^{(n)}$ and the rest of the spectrum of $H^{(n),\theta}$. For this purpose we collect several estimates leading to Lemma 4.6.5 where, among other things, we show that the resolvent is differentiable with respect to the dilation parameter θ. This together with the properties of the Riesz projection allows us to conclude analyticity of $P_i^{(n)}$ and $\lambda_i^{(n)}$ with respect to θ. Secondly, we take the uniform limit $n \to \infty$ in order to conclude that the statement also holds for P_i and λ_i (see Theorem 4.6.9 below). This is possible because our estimates are uniform in the parameter θ.

In this section we assume that Definitions 4.4.1, 4.4.2, 4.4.3 hold true. We recall that we use the symbol c to represent any generic (indeterminate) constant that does not depend on n, g, ρ, ρ_0 and dilation parameters (here do not only use θ, but also η and λ).

Lemma 4.6.1. *For every $r > 0$ and $\lambda, \eta \in D(0, \pi/16)$ (and every $n \in \mathbb{N}$):*

$$\Big\| \frac{H_0^{(n),\lambda} + r}{H_0^{(n),\eta} + r} \Big\| \leq 10. \tag{4.6.1}$$

Moreover, for large enough r (independent of n, g, ρ, ρ_0 and η and λ) and every z in the resolvent set of $H^{(n),\eta}$:

$$\Big\| H_0^{(n),\lambda} \frac{1}{H^{(n),\eta} - z} \Big\| \leq 20 + 20(|z| + r/2) \Big\| \frac{1}{H^{(n),\eta} - z} \Big\|. \tag{4.6.2}$$

Proof. Notice that for every $\eta, \lambda \in D(0, \pi/16)$

$$\Big\| \frac{H_0^{(n),\lambda} + r}{H_0^{(n),\eta} + r} \Big\| \leq \sup_{s \geq 0, i \in 0,1} \Big| \frac{e_i + e^{-\lambda} s + r}{e_i + e^{-\eta} s + r} \Big|. \tag{4.6.3}$$

For every $s \geq 0$ and $i \in \{0, 1\}$:

$$\Big| \frac{e_i + e^{-\lambda} s + r}{e_i + e^{-\eta} s + r} \Big| \leq |e^{\eta-\lambda}| + \Big| e^{\eta-\lambda} \frac{r + e_i}{e_i + e^{-\eta} s + r} \Big| + \Big| \frac{e_i + r}{e_i + e^{-\eta} s + r} \Big| \leq 10, \tag{4.6.4}$$

where we use that $\eta, \lambda \in D(0, \pi/16)$ and $e_i \geq 0$. This implies (4.6.1).

It follows from Lemma 1.3.1 that there is a constant c that does not depend on n, g, ρ, ρ_0 and η such that for every $r \geq 1$:

$$\Big\| V^{(n),\eta} \frac{1}{(H_0^{(n),0} + r)^{1/2}} \Big\| \leq c. \tag{4.6.5}$$

In conclusion,

$$\Big\| V^{(n),\eta} \frac{1}{H_0^{(n),0} + r} \Big\| \leq \frac{c}{r^{1/2}} \tag{4.6.6}$$

holds true. It follows that there is a constant $C_{(4.6.7)}$ that does not depend on n, g, ρ, ρ_0 and η such that for every $r \geq 1$:

$$\Big\| V^{(n),\eta} \frac{1}{H_0^{(n),\eta} + r} \Big\| \leq \frac{C_{(4.6.7)}}{r^{1/2}}. \tag{4.6.7}$$

Take ϕ in the domain of $H_0^{(n),\eta}$, $z \in \mathbb{C}$ and $r \geq 4C^2_{(4.6.7)}$. Then we have (recall that $|g| \leq 1$):

$$\begin{aligned} \|H_0^{(n),\eta}\phi\| \leq & \|(H^{(n),\eta} - z)\phi\| + \|V^{(n),\eta}\phi\| + |z|\|\phi\| \\ \leq & \|(H^{(n),\eta} - z)\phi\| + (1/2)\|H_0^{(n),\eta}\phi\| + (|z| + r/2)\|\phi\|. \end{aligned} \tag{4.6.8}$$

Then, we obtain, for z in the resolvent set of $H^{(n),\eta}$ and $s > 0$ (we take the term $(1/2)\|H_0^{(n),\eta}\phi\|$ in the previous equation to the other side and ϕ of the form $\frac{1}{H^{(n),\eta}-z}\psi$):

$$\Big\| (H_0^{(n),\eta} + s) \frac{1}{H^{(n),\eta} - z} \Big\| \leq 2 + 2(|z| + (r + 2s)/2) \Big\| \frac{1}{H^{(n),\eta} - z} \Big\|. \tag{4.6.9}$$

Using (4.6.1), we find

$$\begin{aligned} & \Big\| (H_0^{(n),\lambda} + s) \frac{1}{H^{(n),\eta} - z} \| \leq \Big\| \frac{H_0^{(n),\lambda} + s}{H_0^{(n),\eta} + s} (H_0^{(n),\eta} + s) \frac{1}{H^{(n),\eta} - z} \Big\| \\ & \leq 10 \Big\| (H_0^{(n),\eta} + s) \frac{1}{H^{(n),\eta} - z} \Big\| \leq 20 + 20(|z| + (r + 2s)/2) \Big\| \frac{1}{H^{(n),\eta} - z} \Big\|. \end{aligned} \tag{4.6.10}$$

Taking the limit s to zero, we arrive at (4.6.2). □

Lemma 4.6.2. *For every $\lambda, \eta, \theta \in D(0, \pi/16)$ (and every $n \in \mathbb{N}$), there is a constant c (independent of n, g, ρ, ρ_0, η, θ and λ) such that for every z in the resolvent set of $H^{(n),\theta}$:*

$$\Big\| (H^{(n),\eta} - H^{(n),\lambda}) \frac{1}{H^{(n),\theta} - z} \Big\| \leq c(1 + |z|)|\eta - \lambda| \Big(\Big\| \frac{1}{H^{(n),\theta} - z} \Big\| + 1 \Big). \tag{4.6.11}$$

Proof. We take a large enough $r > 0$ such that the results of Lemma 4.6.1 hold true. We calculate

$$\Big\| (H^{(n),\eta} - H^{(n),\lambda}) \frac{1}{H^{(n),\theta} - z} \Big\| \leq \Big\| (H^{(n),\eta} - H^{(n),\lambda}) \frac{1}{H_0^{(n),0} + r} \Big\| \Big\| (H_0^{(n),0} + r) \frac{1}{H^{(n),\theta} - z} \Big\|. \tag{4.6.12}$$

Next, we notice that

$$\Big\| (H_0^{(n),\eta} - H_0^{(n),\lambda}) \frac{1}{H_0^{(n),0} + r} \Big\| = \sup_{s \geq 0, i \in \{1,2\}} \Big\| (e^{-\eta} - e^{-\lambda}) s \frac{1}{e_i + s + r} \Big\| \leq |e^{-\eta} - e^{-\lambda}|. \tag{4.6.13}$$

Using Lemma 1.3.1, we find a constant c (independent of n, g, ρ, ρ_0 and η and λ) such that

$$\Big\|(V^{(n),\eta}-V^{(n),\lambda})\frac{1}{H_0^{(n),0}+r}\Big\|\leq c|\eta-\lambda| \tag{4.6.14}$$

Eqs. (4.6.12)-(4.6.14), together with Lemma 4.6.1, imply the desired result. □

Definition 4.6.3. *For every $\theta\in D(0,\pi/16)$, we set $h^\theta=\frac{\partial}{\partial\theta}f^\theta$ and*

$$\frac{\partial}{\partial\theta}V^{(n),\theta}:=\sigma_1\otimes\Big(a_n(h^{\overline{\theta}})+a_n(h^\theta)^*\Big) \tag{4.6.15}$$

and (see Remark 1.2.1)

$$\frac{\partial}{\partial\theta}H^{(n),\theta}:=-H_f^{(n),\theta}+g\frac{\partial}{\partial\theta}V^{(n),\theta}. \tag{4.6.16}$$

Lemma 4.6.4. *For every $\lambda,\eta,\theta\in D(0,\pi/16)$ (and every $n\in\mathbb{N}$), there is a constant c (independent of n, g, ρ, ρ_0, η, θ and λ) such that for every z in the resolvent set of $H^{(n),\theta}$:*

$$\Big\|\Big(\frac{1}{\eta-\lambda}(H^{(n),\eta}-H^{(n),\lambda})-\frac{\partial}{\partial\lambda}H^{(n),\lambda}\Big)\frac{1}{H^{(n),\theta}-z}\Big\|\leq c(1+|z|)|\eta-\lambda|\Big(\Big\|\frac{1}{H^{(n),\theta}-z}\Big\|+1\Big). \tag{4.6.17}$$

Proof. The proof is very similar to the proof of Lemma 4.6.2, and therefore, we omit it. □

Lemma 4.6.5. *For every $\lambda,\eta,\theta\in D(0,\pi/16)$ (and every $n\in\mathbb{N}$), there is a constant c (independent of n, g, ρ, ρ_0, η, θ and λ) such that for every z in the resolvent set of both $H^{(n),\eta}$ and $H^{(n),\lambda}$:*

$$\begin{aligned}\Big\|\frac{1}{\eta-\lambda}\Big(\frac{1}{H^{(n),\lambda}-z}-\frac{1}{H^{(n),\eta}-z}\Big)-\frac{1}{H^{(n),\lambda}-z}\frac{\partial}{\partial\lambda}H^{(n),\lambda}\frac{1}{H^{(n),\lambda}-z}\Big\|\\ \leq c(1+|z|)^2|\eta-\lambda|\Big(\Big\|\frac{1}{H^{(n),\lambda}-z}\Big\|+1\Big)^2\Big(\Big\|\frac{1}{H^{(n),\eta}-z}\Big\|+1\Big),\end{aligned} \tag{4.6.18}$$

and

$$\begin{aligned}\Big\|H_0^{(n),\theta}\Big(\frac{1}{\eta-\lambda}\Big(\frac{1}{H^{(n),\lambda}-z}-\frac{1}{H^{(n),\eta}-z}\Big)-\frac{1}{H^{(n),\lambda}-z}\frac{\partial}{\partial\lambda}H^{(n),\lambda}\frac{1}{H^{(n),\lambda}-z}\Big)\Big\|\\ \leq c(1+|z|)^2|\eta-\lambda|\Big(\Big\|\frac{1}{H^{(n),\lambda}-z}\Big\|+1\Big)^2\Big(\Big\|\frac{1}{H^{(n),\eta}-z}\Big\|+1\Big),\end{aligned} \tag{4.6.19}$$

Proof. First, we notice that Lemma 4.6.2 and the resolvent identity imply

$$\begin{aligned}\Big\|\frac{1}{H^{(n),\lambda}-z}(H^{(n),\eta}-H^{(n),\lambda})\Big(\frac{1}{H^{(n),\eta}-z}-\frac{1}{H^{(n),\lambda}-z}\Big)\Big\|\\ \leq c(1+|z|)^2|\eta-\lambda|^2\Big(\Big\|\frac{1}{H^{(n),\lambda}-z}\Big\|+1\Big)^2\Big(\Big\|\frac{1}{H^{(n),\eta}-z}\Big\|+1\Big).\end{aligned} \tag{4.6.20}$$

We use the resolvent identity again and also (4.6.20) to obtain

$$\Big\|\Big(\frac{1}{H^{(n),\lambda}-z}-\frac{1}{H^{(n),\eta}-z}\Big)-\Big(\frac{1}{H^{(n),\lambda}-z}(H^{(n),\eta}-H^{(n),\lambda})\frac{1}{H^{(n),\lambda}-z}\Big)\Big\| \qquad (4.6.21)$$
$$\leq c(1+|z|)^2|\eta-\lambda|^2\Big(\Big\|\frac{1}{H^{(n),\lambda}-z}\Big\|+1\Big)^2\Big(\Big\|\frac{1}{H^{(n),\eta}-z}\Big\|+1\Big).$$

Then, (4.6.18) follows from (4.6.21) and Lemma 4.6.4. The proof of (4.6.19) follows in a similar fashion as the one of (4.6.18), using Lemma 4.6.1. Therefore, we omit it. □

Proposition 4.6.6. *For every $\eta \in D(0,\pi/16)$, the operator valued functions*

$$\theta \in \mathcal{S} \mapsto P_i^{(n)}, \qquad \theta \in \mathcal{S} \mapsto H_0^{(n),\eta}P_i^{(n)} \qquad (4.6.22)$$

are analytic.

Proof. The proof is an obvious consequence of Lemma 4.6.5 and the formula for the Riesz projections as line integrals in the complex plane. □

Proposition 4.6.7. *The complex valued function*

$$\theta \in \mathcal{S} \mapsto \lambda_i^{(n)} \qquad (4.6.23)$$

is analytic.

Proof. We use the formalism of the proof of Proposition 4.4.15 and make explicit the dependence of $P_i^{(n)}$ on θ, i.e., $P_i^{(n)} \equiv P_i^{(n),\theta}$. We define $\Psi_i^{(n),\theta} = P_i^{(n),\theta}\varphi_i \otimes \Omega^{(n)}$ (here we use a slightly different notation from proof of Proposition 4.4.15). Notice that

$$\lambda_i^{(n)} = \frac{\langle \Psi_i^{(n),\overline{\theta}}, H^{(n),\theta}\Psi_i^{(n),\theta}\rangle}{\langle \Psi_i^{(n),\overline{\theta}}, \Psi_i^{(n),\theta}\rangle}, \qquad (4.6.24)$$

and that the denominator does not vanish (this follows as in (4.4.68)). Then, the result is a consequence of Proposition 4.6.6, because it implies that the functions

$$\theta \mapsto \Psi_i^{(n),\theta}, \qquad \theta \mapsto H^{(n),\theta}\Psi_i^{(n),\theta} = H^{(n),\theta}\frac{1}{H_0^{(n),0}+1}\Big((H_0^{(n),0}+1)P_i^{(n),\theta}\varphi_i \otimes \Omega^{(n)}\Big) \qquad (4.6.25)$$

are analytic. Notice that the function $\theta \mapsto H^{(n),\theta}\frac{1}{H_0^{(n),0}+1}$ is an operator valued analytic function (the proof of this fact is similar to the proof of Lemma 4.6.4, but much simpler). □

Proposition 4.6.8. *The maps*

$$g \in D(0,g_0) \mapsto P_i^{(n)}, \qquad g \in D(0,g_0) \mapsto \lambda_i^{(n)} \qquad (4.6.26)$$

are analytic.

Proof. The proof follows directly from the proofs of Propositions 4.6.6 and 4.6.7. In this case the proof is much simpler because the coupling constant is only present in the interaction term (and the interaction term depends linearly on the coupling constant). □

Theorem 4.6.9. *The functions*

$$\begin{aligned} \mathcal{S} \ni \theta &\mapsto P_i, \qquad & \mathcal{S} \ni \theta \mapsto \lambda_i \\ D(0,g_0) \ni g &\mapsto P_i, \qquad & D(0,g_0) \ni g \mapsto \lambda_i \end{aligned} \tag{4.6.27}$$

are analytic. Moreover, this implies that $\lambda_i(\theta) \equiv \lambda_i$ *is constant for* $\theta \in \mathcal{S}$ *(see* (4.1.1)*).*

Proof. Theorem 4.4.5, Properties ($\mathcal{P}1$) and ($\mathcal{P}3$) imply that the convergence rates of $\lambda_i^{(n)}$ to λ_i and $P_i^{(n)} \otimes P_{\Omega^{(n,\infty)}}$ to P_i do not depend on θ and g. Then λ_i and P_i are uniform limits of analytic functions (see Propositions 4.6.6, 4.6.7, 4.6.8). Therefore, they are analytic. That λ_i is constant with respect to θ follows from the fact that it does not depend of the real part of θ because a change in the real part of H^θ produces unitarily equivalent Hamiltonians: if θ and $\tilde{\theta}$ have the same imaginary part, then H^θ and $H^{\tilde{\theta}}$ are unitarily equivalent (thus, isospectral). Both $\lambda_i(\theta)$ and $\lambda_i(\tilde{\theta})$ are distinguished points in the spectrum because they are the vertex of the same cone (see Theorem 4.2.7), we conclude that $\lambda_i(\theta) = \lambda_i(\tilde{\theta})$. □

5. Scattering formula for the massless Spin-Boson model

In this chapter, we again analyze the massless Spin-Boson model introduced in Chapter 1.2 and we fix an infrared regularization parameter $\mu \in (0, 1/2)$ (see (1.2.4)). Note that this yields the relativistic dispersion relation $\omega(k) = |k|$. We derive an explicit formula for the two-body scattering matrix elements which reveals its dependence on the resonance (and ground-state) energy of the model. Theorem 5.1.1 is a improvement in comparison to [23, Theorem 2.2] since it provides an exact relation whereas [23, Theorem 2.2] only gives the leading term with respect to the coupling constant explicitly.

Scattering and resonance theories are well-established in the context of quantum field theory (see Chapters 2 and 4, respectively). The purpose of the present chapter is to bring these two well-developed fields together. For n-body Schrödinger operators, it has been shown that the singularities of the meromorphic continuation of the integral kernel of the scattering matrix are located precisely at the resonance energies (see [63]). To the best of our knowledge, this question has not yet been addressed in models of quantum field theory, which is most probably due to the fact that quantum field models involve new subtleties as compared to the quantum mechanical ones. These can however be addressed with the recently developed methods of multiscale analysis and spectral renormalization (while we rely on the former in this work; c.f. Chapter 4). We provide a representation of the scattering matrix in terms of an expectation value of the resolvent of a spectrally dilated Hamiltonian; see Theorem 5.1.1 below. The relation of the scattering matrix and the resonance can then be read of this formula; see Theorem 5.1.3 below. Loosely put, our results imply that, for the photon momenta $|k'|$ in a neighborhood of $\operatorname{Re}\lambda_1 - \lambda_0$, the leading order (in g for small g) of the integral kernel of the transition matrix fulfills

$$|T(k,k')|^2 \sim \frac{E_1^2 g^4}{(|k'| + \lambda_0 - \operatorname{Re}\lambda_1)^2 + g^4 E_1^2}, \tag{5.0.1}$$

where we define

$$E_1 := g^{-2} \operatorname{Im}\lambda_1 \tag{5.0.2}$$

and it turns out that there are constant numbers $E_I < 0$, $a > 0$ and a uniformly bounded function $\Delta \equiv \Delta(g)$ such that $E_1 = E_I + g^a \Delta$. Heuristically, for an experiment, in which a two-level atom is irradiated with monochromatic incoming light quanta of momentum $k' \in \mathbb{R}^3$, the relation (5.0.1) states that the intensity of the outgoing light quanta with momentum $k \in \mathbb{R}^3$ is proportional to $|T(k,k')|^2$, which is given as a Lorentzian function with maximum at $|k'| = \operatorname{Re}\lambda_1 - \lambda_0$ and width $2\operatorname{Im}\lambda_1$. This relation is already found as folklore knowledge in physics text-books. In this chapter we give a rigorous derivation in the model at hand. On the other hand, the relation between the imaginary value of the resonance and the decay rate of the unstable excited state was established rigorously in several articles [2, 49, 3, 18].

5.1. Main results

In this section we present the main results of Chapter 5. The corresponding proofs will be provided in Section 5.3.2. We point out that these proofs strongly rely on the results found in Chapter 4.

Our first main result of this chapter provides the precise relation between the scattering matrix element and the complex dilated resolvent of the Hamiltonian.

Theorem 5.1.1 (Exact Scattering Formula). *For sufficiently small g, θ in the set $\mathcal{S}$ (see* (5.2.2) *below), and for all $h, l \in \mathfrak{h}_0$, the transition matrix coefficients for one-boson processes are given by*

$$T(h,l) = \int \mathrm{d}^3k\mathrm{d}^3k'\, \overline{h(k)}l(k')\delta(\omega(k)-\omega(k'))T(k,k') \tag{5.1.1}$$

where

$$T(k,k') = -2\pi i g^2 f(k)f(k')\|\Psi_{\lambda_0}\|^{-2}\Bigg(\left\langle \sigma_1\Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - \lambda_0 - |k'|\right)^{-1}\sigma_1\Psi_{\lambda_0}^{\theta}\right\rangle + \left\langle \sigma_1\Psi_{\lambda_0}^{\theta}, \left(H^{\overline{\theta}} - \lambda_0 + |k'|\right)^{-1}\sigma_1\Psi_{\lambda_0}^{\overline{\theta}}\right\rangle\Bigg). \tag{5.1.2}$$

The integral with respect to the Dirac's delta distribution distribution δ in (5.1.1) is to be understood as

$$T(h,l) = \int_0^\infty \mathrm{d}|k| \int \mathrm{d}\Sigma\mathrm{d}\Sigma'\, \overline{h(|k|,\Sigma)}l(|k|,\Sigma')T(|k|,\Sigma,|k|,\Sigma'), \tag{5.1.3}$$

where we have introduced spherical coordinates $k = (|k|, \Sigma)$ with Σ being the solid angle and $T(k,k') \equiv T(|k|,\Sigma,|k|,\Sigma')$ is given by (5.1.2). Notice that (5.1.2) is not defined for $k = 0$ or $k' = 0$. However, since we take $h, l \in \mathfrak{h}_0$, the expression (5.1.1) is well-defined. Representing such matrix elements in terms of a distribution kernel is convenient (in our case, e.g., it makes the energy conservation apparent) and also frequently used in the literature. In particular, similar distribution kernels in a closely related model have been studied in [17, 12, 23].

Our second main result establishes the relation between resonance and scattering theory in our model. First, we state a definition that we use for our main result.

Definition 5.1.2. *Using solid angles $\mathrm{d}\Sigma, \mathrm{d}\Sigma'$, we define, for all $h, l \in \mathfrak{h}_0$,*

$$G : \mathbb{R} \to \mathbb{C}, \quad r \mapsto G(r) := \begin{cases} \int \mathrm{d}\Sigma\mathrm{d}\Sigma'\, r^4\overline{h(r,\Sigma)}l(r,\Sigma')f(r)^2 & \textit{for} \quad r \geq 0 \\ 0 & \textit{for} \quad r < 0. \end{cases} \tag{5.1.4}$$

Theorem 5.1.3. *For sufficiently small g, θ in the set $\mathcal{S}$ (see* (5.2.2) *below), and for all $h, l \in \mathfrak{h}_0$, the transition matrix coefficients for one-boson processes are given by*

$$T(h,l) = T_P(h,l) + R(h,l), \tag{5.1.5}$$

where

$$T_P(h,l) := Mg^2 \int \mathrm{d}r \frac{G(r)}{(r + \lambda_0 - \lambda_1)(r - \lambda_0 + \overline{\lambda_1})}, \tag{5.1.6}$$

and there is a constant $C(h,l)$ (that does not depend on g) such that

$$|R(h,l)| \leq C(h,l) g^3 |\log g|. \tag{5.1.7}$$

Here, we use the notation

$$M := 4\pi i (\operatorname{Re}\lambda_1 - \lambda_0) \|\Psi_{\lambda_0}\|^{-2}. \tag{5.1.8}$$

$T_P(h,l)$ is the leading term in terms of powers of g for small g, and $R(h,l)$ is regarded as the error term. This is justified by Remark 5.1.5 below.

Remark 5.1.4. *In the proof of Theorem 5.1.3 we show that the factor $\|\Psi_{\lambda_0}\|^{-2}$ can be dropped in* (5.1.8)*. We keep this factor in the statement in order to keep our notation consistent in Chapter 3.*

Remark 5.1.5. *The scattering processes described by the transition matrix in* (5.1.6) *clearly depend on the incoming and outgoing photon states, l and h. This is well understood from a physics as well as a mathematics perspective. For example, it can be read from* (5.1.2) *that, if l is supported in a ball of radius t and h is supported in its complement, then the principal term $T_P(h,l)$ vanishes and only higher order terms (with respect to powers of g) contribute to the scattering process. The quantity $T_P(h,l)$ is the only one that might produce scattering processes of order g^2 since the remainder is of order $g^3|\log(g)|$. If an experiment is appropriately prepared, then such a scattering process will be observed and the term describing this is $T_P(h,l)$. This justifies why we call it the leading order (or principal) term. In Appendix B we give an example of a large class of functions h and l that make $T_P(h,l)$ larger or equal than a strictly positive constant times g^2. In particular, we prove that this happens when the corresponding function G is positive and strictly positive at* $\operatorname{Re}\lambda_1 - \lambda_0$.

We recall the definition $E_1 = g^{-2} \operatorname{Im}\lambda_1$ given in (5.0.2). It follows from (5.2.11) and (4.2.4) below that $E_1 = E_I + g^a \Delta$ where $a > 0$, $\Delta \equiv \Delta(g)$ is uniformly bounded and $E_I < 0$ is the constant defined in (5.2.11). This implies that

$$E_1 \leq -\boldsymbol{c} < 0, \tag{5.1.9}$$

for some constant $\boldsymbol{c}$ that does not depend on g (for small enough g).

Remark 5.1.6. *By* (5.1.6) *and* (5.1.4)*, we can express the principal term* $T_P(h,l)$ *in terms of an integral kernel:*

$$T_P(h,l) = \int \mathrm{d}^3k \mathrm{d}^3k' \, \overline{h(k)} l(k') \delta(|k| - |k'|) T_P(k,k'), \tag{5.1.10}$$

where

$$T_P(k,k') = M f(k) f(k') \left(\frac{E_1 g^2}{(|k'| + \lambda_0 - \operatorname{Re}\lambda_1 - ig^2 E_1)(|k'| - \lambda_0 + \overline{\lambda_1})} \right) \tag{5.1.11}$$

and we recall (5.0.2)*. Eq.* (5.1.10) *is important, because it allows us to calculate the leading order of the scattering cross section. It is proportional to the modulus squared of* $T_P(k,k')$*:*

$$|T_P(k,k')|^2 = \left(\frac{|M|^2 |f(k)|^2 |f(k')|^2}{||k'| - \lambda_0 + \overline{\lambda_1}|^2} \right) \frac{E_1^2 g^4}{(|k'| + \lambda_0 - \operatorname{Re}\lambda_1)^2 + g^4 E_1^2}. \tag{5.1.12}$$

For momenta $|k'|$ *in a neighborhood of* $\operatorname{Re}\lambda_1 - \lambda_0$*, the behavior in the expression above is dominated by the Lorentzian function. As expected, there is a maximum when the energy of the incoming photons is close to the difference of the resonance and the ground-state energies of the system and the width of this peak is controlled by the imaginary part of the resonance* $\operatorname{Im}\lambda_1$*.*

Note that the Dirac's delta distribution in (5.1.10) is to be understood similarly in (6.3.3). Note that (5.1.11) is not defined for $k = 0$ or $k' = 0$. However, since we take $h, l \in \mathfrak{h}_0$, the expression (5.1.10) is well-defined.

Remark 5.1.7. *In this chapter we denote by* C *any generic (indeterminate) constant, that might change from line to line. These constants do not depend on the coupling constant and the auxiliary parameters* ρ_0*,* ρ *introduced in Section 5.2.1 and* n *introduced in Section 5.2.2.*

5.2. Known results on spectral properties and resolvent estimates

In this section we recall results about the spectrum of the dilated Spin-Boson Hamiltonian and resolvent estimates which have already been presented in Chapter 4. We repeat the relevant definitions, properties and estimates here in order to make this chapter more readable and self-contained.

Throughout this chapter we address the case of small coupling, i.e., we assume the coupling constant g to be sufficiently small. The restrictions on the coupling constant only stem from the requirements needed to prove the results reviewed in this section, i.e., the ones considered in [21] and Chapter 4 of the present work. We do not explicitly specify how small the coupling constant must be but give precise references from which such bounds can be inferred. This issue is addressed by the next definition:

Definition 5.2.1 (Coupling Constant)**.** *Throughout the remainder of this chapter we assume that* $g \in (0, \boldsymbol{g})$*, where* $0 < \boldsymbol{g}$ *satisfies Definition 4.4.3 and* (4.5.58)*, the Fermi-golden rule (see* (4.2.4) *and* (5.2.13) *below) and* (5.2.28) *below.*

We denote the imaginary part of the dilation parameter θ by

$$\nu := \operatorname{Im} \theta \tag{5.2.1}$$

and assume that θ belongs to the set

$$\mathcal{S} = \left\{ \theta \in \mathbb{C} : -10^{-3} < \operatorname{Re} \theta < 10^{-3} \text{ and } \boldsymbol{\nu} < \operatorname{Im} \theta < \pi/16 \right\}, \tag{5.2.2}$$

where $\boldsymbol{\nu} \in (0, \pi/16)$ is a fixed number, Here, we recall Definition 4.1.1.

5.2.1. Spectral estimates

We know from Proposition 4.2.1 that the Hamiltonian H^θ has two eigenvalues λ_0 and λ_1 in small neighborhoods of e_0 and e_1, respectively. Loosely put, e_0 turns into the ground state λ_0 and e_1 tuns into the resonance λ_1 once the interaction is tuned on. Both λ_0 and λ_1 do not depend on θ provided that $\theta \in \mathcal{S}$ and in the case of λ_0 we can take θ in a neighborhood of 0, and therefore, infer that λ_0 is real and gives the ground state energy. This is proven in Theorem 4.2.3 and Remark 4.2.4.

In Theorem 4.2.7, we give a very sharp estimate on the location of the spectrum of H^θ. We prove, among other things that, locally, in neighborhoods of λ_0 and λ_1, its spectrum is contained in cones with vertices at λ_0 and λ_1. To make this statement more precise we need to introduce some more concepts and notation. There are two auxiliary parameters that play an important role in our constructions:

$$\rho \in (0,1), \qquad \rho_0 \in (0, \min(1, e_1/4)), \tag{5.2.3}$$

which also satisfy the conditions in (5.2.31) below. In order to specify the spectral properties of H^θ we recall Definition 4.3.1, where we have defined some regions in the complex plane:

Definition 5.2.2. *For fixed* $\theta \in \mathcal{S}$*, we set* $\delta = e_1 - e_0 = e_1$ *and define the regions*

$$A := A_1 \cup A_2 \cup A_3, \tag{5.2.4}$$

where

$$A_1 := \{ z \in \mathbb{C} : \operatorname{Re} z < e_0 - \delta/2 \} \tag{5.2.5}$$

$$A_2 := \left\{ z \in \mathbb{C} : \operatorname{Im} z > \frac{1}{8} \delta \sin(\nu) \right\} \tag{5.2.6}$$

$$A_3 := \{ z \in \mathbb{C} : \operatorname{Re} z > e_1 + \delta/2, \operatorname{Im} z \geq -\sin(\nu/2) \left(\operatorname{Re}(z) - (e_1 + \delta/2) \right) \}, \tag{5.2.7}$$

and for $i = 0, 1$*, we define*

$$B_i^{(1)} := \left\{ z \in \mathbb{C} : |\operatorname{Re} z - e_i| \leq \frac{1}{2}\delta, -\frac{1}{2}\rho_1 \sin(\nu) \leq \operatorname{Im} z \leq \frac{1}{8} \delta \sin(\nu) \right\}. \tag{5.2.8}$$

These regions are depicted in Figure 5.1.

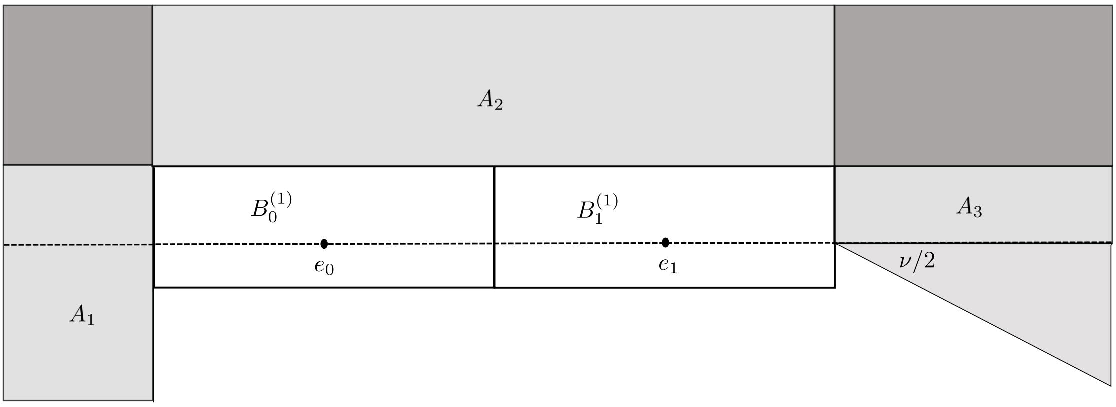

Figure 5.1.: An illustration of the subsets of the complex plane introduced in Definition 5.2.2.

For a fixed $m \in \mathbb{N}$ with $m \geq 4$, we define the cone (recalling (4.2.7))

$$\mathcal{C}_m(z) := \left\{ z + xe^{-i\alpha} : x \geq 0, |\alpha - \nu| \leq \nu/m \right\}. \tag{5.2.9}$$

It follows from the induction scheme in Section 4.4 that $\lambda_i \in B_i^{(1)}$, and moreover, Theorem 4.2.7 together with Lemma 4.3.13 yields

$$\sigma(H^\theta) \subset \mathbb{C} \setminus \left[A \cup (B_0^{(1)} \setminus \mathcal{C}_m(\lambda_0)) \cup (B_1^{(1)} \setminus \mathcal{C}_m(\lambda_1)) \right]. \tag{5.2.10}$$

As we mention above, we have $\lambda_0 \in \mathbb{R}$. The imaginary part of λ_1 can be also estimated (see Remark 4.2.2 – Fermi golden rule): Recalling (1.2.3), we define

$$E_I := -4\pi^2 (e_1 - e_0)^2 |f(e_1 - e_0)|^2. \tag{5.2.11}$$

Then, for g small enough, there are constants $C, a > 0$ such that

$$\left| \operatorname{Im} \lambda_1 - g^2 E_I \right| \leq g^{2+a} C. \tag{5.2.12}$$

This implies that, for g small enough, there is constant $\boldsymbol{c} > 0$ such that

$$\operatorname{Im} \lambda_1 < -g^2 \boldsymbol{c} < 0. \tag{5.2.13}$$

5.2.2. Auxiliary (infrared cut-off) Hamiltonians

Some of the bounds in Section 5.3 employ a certain approximation of the Hamiltonian H^θ by Hamiltonians with infrared cut-offs. The strategy will be the following: A mathematical expression that depends on H^θ is replaced by a corresponding one that depends on a particular infrared cut-off Hamiltonian. We then analyze the infrared cut-off expression and estimate the difference between both expressions. The construction of a sequence of infrared cut-off Hamiltonians $(H^{(n),\theta})$ such that, as n tends to infinity, the cut-off is removed is called multiscale analysis. In Chapter 4 (and [21]), we present the full details of this method and derive several results. Here, we only use some of those results and only present the notation necessary to review this part of Chapter 4. The infrared cut-off Hamiltonians $H^{(n),\theta}$ are parametrized by a sequence of numbers (see also (5.2.3) and (5.2.31))

$$\rho_n := \rho_0 \rho^n, \tag{5.2.14}$$

where the Hamiltonians $H^{(n),\theta}$ are defined by

$$H^{(n),\theta} : = K + H_f^{(n),\theta} + gV^{(n),\theta} =: H_0^{(n),\theta} + gV^{(n),\theta} \tag{5.2.15}$$

$$H_f^{(n),\theta} : = \int_{\mathbb{R}^3 \setminus \mathcal{B}_{\rho_n}} \mathrm{d}^3 k \, \omega^\theta(k) a^*(k) a(k), \quad \omega^\theta(k) = e^{-\theta}|k| \tag{5.2.16}$$

$$V^{(n),\theta} : = \sigma_1 \otimes \int_{\mathbb{R}^3 \setminus \mathcal{B}_{\rho_n}} \mathrm{d}^3 k \, \left(f^\theta(k) a(k) + f^\theta(k) a^*(k) \right), \tag{5.2.17}$$

$$f^\theta : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \quad k \mapsto e^{-\theta(1+\mu)} e^{-e^{2\theta} \frac{k^2}{\Lambda^2}} |k|^{-\frac{1}{2}+\mu}, \tag{5.2.18}$$

on the Hilbert space

$$\mathcal{H}^{(n)} := \mathcal{K} \otimes \mathcal{F}[\mathfrak{h}^{(n)}], \quad \mathfrak{h}^{(n)} := L^2(\mathbb{R}^3 \setminus \mathcal{B}_{\rho_n}, \mathbb{C}), \quad \mathcal{B}_{\rho_n} := \left\{ x \in \mathbb{R}^3 : |x| < \rho_n \right\}. \tag{5.2.19}$$

Additionally, we define

$$\tilde{H}^{(n),\theta} := H_0^\theta + gV^{(n),\theta} \tag{5.2.20}$$

and fix the Hilbert spaces

$$\mathfrak{h}^{(n,\infty)} := L^2(\mathcal{B}_{\rho_n}) \quad \text{and} \quad \mathcal{F}[\mathfrak{h}^{(n,\infty)}], \tag{5.2.21}$$

defined as in (1.2.7) with $\mathfrak{h}^{(n,\infty)}$ instead of $\mathfrak{h}$, with vacuum states $\Omega^{(n,\infty)}$ and corresponding orthogonal projections $P_{\Omega^{(n,\infty)}}$. Note that $\mathcal{H} \equiv \mathcal{H}^{(n)} \otimes \mathcal{F}[\mathfrak{h}^{(n,\infty)}]$.

In Proposition 4.2.1 and Theorem 4.4.5, we prove that, for each $n \in \mathbb{N}$, $H^{(n),\theta}$ has isolated eigenvalues $\lambda_i^{(n)}$ in certain neighborhoods of e_i, for $i \in \{0, 1\}$, respectively. The fact that these eigenvalues are isolated permits us to define their corresponding Riesz projections which are denoted by

$$P_i^{(n)} \equiv P_i^{(n),\theta}. \tag{5.2.22}$$

In Proposition 4.2.1, we prove that this sequence of projections converges to the projection associated to the eigenvalue λ_i, i.e.,

$$P_i^\theta \equiv P_i = \lim_{n \to \infty} P_i^{(n),\theta} \otimes P_{\Omega^{(n,\infty)}}, \tag{5.2.23}$$

and that the latter is analytic with respect to θ (see Theorem 4.2.3). Furthermore, it follows from Remark 4.5.11 that

$$\left\| P_i^\theta - P_i^{(n),\theta} \otimes P_{\Omega^{(n,\infty)}} \right\| \le 2\frac{g}{\rho}\rho_n^{\mu/2} \le \rho_n^{\mu/2}. \tag{5.2.24}$$

This together with Lemma 4.3.6 implies that there is a constant C such that

$$\left\| P_i^\theta - P_{\varphi_i} \otimes P_\Omega \right\| \le Cg, \tag{5.2.25}$$

and in addition, we know from Lemma 4.4.7 that

$$\left\| P_i^{(n),\theta} \right\| \le 3, \tag{5.2.26}$$

for every $n \in \mathbb{N}$. Finally, Lemma 4.5.1 yields that for all $n \in \mathbb{N}$

$$|\lambda_i - \lambda_i^{(n)}| \le 2g\rho_n^{1+\mu/2}. \tag{5.2.27}$$

This together with Lemma 4.3.10, which states that there is a constant C such that $|e_i - \lambda_i^{(1)}| < Cg$, proves that there is a constant C such that, for every $n \in \mathbb{N}$ and for g sufficiently small, we have

$$|\lambda_i^{(n)} - e_i| \le Cg \le 10^{-3}e_1, \qquad |\lambda_i - e_i| \le Cg \le 10^{-3}e_1. \tag{5.2.28}$$

5.2.3. Resolvent estimates

In Chapter 4 (and [21]), we derive bounds for the resolvent of H^θ in $\Big[A \cup (B_0^{(1)} \setminus \mathcal{C}_m(\lambda_0)) \cup (B_1^{(1)} \setminus \mathcal{C}_m(\lambda_1))\Big]$, see (5.2.10). The region A is far away from the spectrum, and therefore, resolvent estimates in this region are easy. In Lemma 4.3.2, we prove that there is a constant C such that

$$\left\| \frac{1}{H^\theta - z} \right\| \le C \frac{1}{|z - e_1|}, \qquad \forall z \in A. \tag{5.2.29}$$

Resolvent estimates in the regions $B_0^{(1)} \setminus \mathcal{C}_m(\lambda_0)$ and $B_1^{(1)} \setminus \mathcal{C}_m(\lambda_1)$ are much more complicated because these regions share boundaries with the spectrum.

In Theorem 4.5.5, we prove that, for $i \in \{0, 1\}$, $B_i^{(1)} \setminus \mathcal{C}_m\left(\lambda_i^{(n)} + (1/4)\rho_n e^{-i\nu}\right) \setminus \{\lambda_i^{(n)}\}$ is contained in the resolvent set of $H^{(n),\theta}$ and that there is a constant $\boldsymbol{C}$ such that

$$\left\| \frac{1}{H^{(n),\theta} - z} \overline{P_i^{(n),\theta}} \right\| \le \boldsymbol{C}^{n+1} \frac{1}{\mathrm{dist}(z, \mathcal{C}_m(\lambda_i^{(n)} + (1/4)\rho_n e^{-i\nu}))}, \tag{5.2.30}$$

for every $z \in B_i^{(1)} \setminus \mathcal{C}_m\left(\lambda_i^{(n)} + (1/4)\rho_n e^{-i\nu}\right)$, where $\overline{P_i^{(n),\theta}} = 1 - P_i^{(n),\theta}$. Here, the symbol dist denotes the Euclidean distance in $\mathbb{C}$. In [21], we select the auxiliary numbers ρ and

ρ_0 satisfying $\boldsymbol{C}^8\rho_0^\mu \leq 1$, and $\boldsymbol{C}^4\rho^\mu \leq 1/4$. In this chapter we assume the stronger conditions

$$\boldsymbol{C}^8\rho_0^\mu \leq 1, \qquad \boldsymbol{C}^8\rho^\mu \leq 1/4, \tag{5.2.31}$$

and observe that this implies

$$\boldsymbol{C}\rho^{\frac{1}{2}(1+\mu/4)} \leq 1. \tag{5.2.32}$$

The constant $\boldsymbol{C}$ is larger that 10^6, it is specified in Definition 4.4.1 and (4.5.58), however, its precise form is not relevant in this chapter (in Chapter 4 and [21], we do not intend to calculate optimal constants, because this would make the work harder to read). From the inequalities above and (5.2.3) we obtain that, for very $n \in \mathbb{N}$:

$$\rho_n \leq 10^{-6} e_1. \tag{5.2.33}$$

Finally, we prove in Theorem 4.5.9 that the set $\in B_i^{(1)} \setminus \mathcal{C}_m(\lambda_i^{(n)} - e^{-i\nu}\rho_n^{1+\mu/4})$ is contained in the resolvent set of both H^θ and $\tilde{H}^{(n),\theta}$ and for all z in this set there is a constant C such that:

$$\left\| \frac{1}{H^\theta - z} - \frac{1}{\tilde{H}^{(n),\theta} - z} \right\| \leq gC\mathbf{C}^{2n+2}\frac{1}{\rho_n}\rho_n^{\frac{\mu}{2}} \leq gC\frac{1}{\rho_n}\rho_n^{\frac{\mu}{4}}, \tag{5.2.34}$$

where we use (5.2.31). Notice that (5.2.30) implies that there is a constant C such that

$$\left\| \frac{1}{H^{(n),\theta} - z}\overline{P_i^{(n),\theta}} \right\| \leq C\boldsymbol{C}^{n+1}\frac{1}{\rho_n}, \tag{5.2.35}$$

for every $z \in B_i^{(1)} \setminus \mathcal{C}_m(\lambda_i^{(n)})$. Moreover, Theorem 4.2.6 implies that there is a constant C such that

$$\left\| \frac{1}{H^\theta - z} \right\| \leq C\boldsymbol{C}^{n+1}\frac{1}{\rho_n^{1+\mu/4}}, \tag{5.2.36}$$

for every $z \in B_i^{(1)} \setminus \mathcal{C}_m(\lambda_i^{(n)} - \rho_n^{1+\mu/4}e^{-i\nu})$, and

$$\left\| \frac{1}{H^\theta - z} \right\| \leq C\boldsymbol{C}^{n+1}\frac{1}{\mathrm{dist}(z, \mathcal{C}_m(\lambda_i))}, \tag{5.2.37}$$

for every $z \in B_i^{(1)} \setminus \mathcal{C}_m(\lambda_i - 2\rho_n^{1+\mu/4}e^{-i\nu})$.

5.3. Proof of our main results in this chapter

In the remainder of this chapter we provide the proofs of the main results Theorem 5.1.1 and 5.1.3. Note that we have already derived an intermediate formula for the scattering matrix coefficients (see Theorem 2.2.2). This formula together with several technical ingredients provided in Section 5.3.1 and the results summarized in Section 5.2 will pave the way for the proofs of our main results given in Section 5.3.2.

5.3.1. Technical ingredients

Here, we derive some technical results which will be applied in Section 5.3.2. Most of statements in this section will mostly be formulated without motivation, however, their importance will become clear later in Section 5.3.2.

General results

Lemma 5.3.1. *For $n \in \mathbb{N}$ and $\theta \in \mathcal{S}$, we have*

$$P_0^{(n),\theta} \sigma_1 P_0^{(n),\theta} = 0. \tag{5.3.1}$$

The statement has already been proven in [8, Lemma 2.1].

Next, we prove a representation formula of the time-evolution operator similar to the Laplace transform representation (see, e.g., [9]).

Lemma 5.3.2. *For $\epsilon > 0$ and sufficiently large $R > 0$, we consider the concatenated contour $\Gamma(\epsilon, R) := \Gamma_-(\epsilon, R) \cup \Gamma_c(\epsilon) \cup \Gamma_d(R)$ (see Figure 5.2), where*

$$\begin{aligned} \Gamma_-(\epsilon, R) &:= [-R, \lambda_0 - \epsilon] \cup [\lambda_0 + \epsilon, R], \\ \Gamma_d(R) &:= \left\{ -R - ue^{i\frac{\nu}{4}} : u \geq 0 \right\} \cup \left\{ R + ue^{-i\frac{\nu}{4}} : u \geq 0 \right\}, \\ \Gamma_c(\epsilon) &:= \left\{ \lambda_0 - \epsilon e^{-it} : t \in [0, \pi] \right\}. \end{aligned} \tag{5.3.2}$$

The orientations of the contours in (5.3.2) are given by the arrows depicted in Figure 5.2. Then, for all analytic vectors $\phi, \psi \in \mathcal{H}$ (analytic in a – connected – domain containing 0) and $t > 0$ the following identity holds true:

$$\left\langle \phi, e^{-itH} \psi \right\rangle = \frac{1}{2\pi i} \int_{\Gamma(\epsilon,R)} \mathrm{d}z \, e^{-itz} \left\langle \psi^{\overline{\theta}}, \left(H^\theta - z \right)^{-1} \phi^\theta \right\rangle. \tag{5.3.3}$$

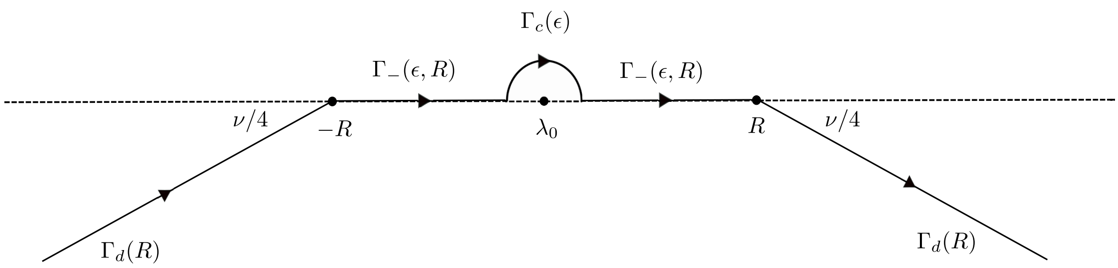

Figure 5.2.: An illustration of the contour $\Gamma(\epsilon, R) := \Gamma_-(\epsilon, R) \cup \Gamma_c(\epsilon) \cup \Gamma_d(R)$.

Proof. Let $t > 0$ and $\epsilon > 0$. We define a contour $\hat{\Gamma}(\epsilon) := \mathbb{R} + i\epsilon$ with a mathematical negative orientation if the contour were closed in the lower complex plane. As an application of the residue theorem closing the contour in the lower complex plane, we observe for all $E \in \mathbb{R}$

$$\frac{1}{2\pi i}\int_{\hat{\Gamma}(\epsilon)} \mathrm{d}z\, \frac{e^{-itz}}{it(E-z)^2} = e^{-itE} \tag{5.3.4}$$

holds true. Thanks to the spectral theorem we may write for all $\psi \in \mathcal{H}$

$$\left\langle \psi, e^{-itH}\psi \right\rangle = \int_{\sigma(H)} \langle \psi, \mathrm{d}P_E \psi\rangle\, e^{-itE} = \frac{1}{2\pi i}\int_{\sigma(H)}\int_{\hat{\Gamma}(\epsilon)} \mathrm{d}z\, \langle \psi, dP_E\psi\rangle \frac{e^{-itz}}{it(E-z)^2}. \tag{5.3.5}$$

Next, we may interchange the order of the integrals by the Fubini-Tonelli Theorem since the following integral is finite:

$$\int_{\sigma(H)} \langle \psi, dP_E\psi\rangle \int_{\hat{\Gamma}(\epsilon)} \mathrm{d}z \left| \frac{e^{-itz}}{it(E-z)^2}\right| \le \frac{e^{t\epsilon}}{t}\int_{\sigma(H)} \langle \psi, dP_E\psi\rangle \int_{-\infty}^{\infty} dx\, |x - i\epsilon|^{-2} < \infty. \tag{5.3.6}$$

Hence, after the interchange we may apply the spectral theorem again to find

$$(5.3.5) = \frac{1}{2\pi i}\int_{\hat{\Gamma}(\epsilon)} \mathrm{d}z \int_{\sigma(H)} \langle \psi, dP_E\psi\rangle \frac{e^{-itz}}{it(E-z)^2} = \frac{1}{2\pi i}\int_{\hat{\Gamma}(\epsilon)} \mathrm{d}z\, \frac{e^{-itz}}{it}\left\langle \psi, \frac{1}{(H-z)^2}\psi\right\rangle. \tag{5.3.7}$$

Exploiting the polarization identities we recover for all $\psi, \phi \in \mathcal{H}$ the identity

$$\left\langle \psi, e^{-itH}\phi \right\rangle = \frac{1}{2\pi i}\int_{\hat{\Gamma}(\epsilon)} \mathrm{d}z\, \frac{e^{-itz}}{it}\left\langle \psi, \frac{1}{(H-z)^2}\phi\right\rangle. \tag{5.3.8}$$

The fact that the family H^θ is an analytic family of type A implies that the operator valued function

$$\theta \mapsto \frac{1}{H^\theta - z} \tag{5.3.9}$$

is analytic for all z in the resolvent set of H^θ. A detailed and self-contained exposition of this topic is presented in Section 4.6. It is straight forward to prove that for real θ

$$\frac{1}{H^\theta - z} = U^\theta \frac{1}{H-z}(U^\theta)^{-1}. \tag{5.3.10}$$

For complex θ, however, this expression is not necessarily correct (due to a problem of domains of unbounded operators). Nevertheless, (5.3.9) and (5.3.10) imply that the function

$$\theta \mapsto \left\langle \psi^{\overline{\theta}}, \frac{1}{(H^\theta - z)^2}\phi^\theta \right\rangle \tag{5.3.11}$$

where $\phi^\theta = U^\theta\phi$, $\psi^{\overline{\theta}} = U^{\overline{\theta}}\psi$, is analytic and it coincides with $\left\langle \psi, \frac{1}{(H-z)^2}\phi \right\rangle$ for real θ, because in this case U^θ is unitary. Hence, we conclude that

$$\left\langle \psi^{\overline{\theta}}, \frac{1}{(H^\theta - z)^2}\phi^\theta \right\rangle = \left\langle \psi, \frac{1}{(H-z)^2}\phi \right\rangle \tag{5.3.12}$$

for every θ in a connected (open) domain containing 0 such that (5.3.11) is analytic in this domain. We obtain:

$$(5.3.8) = \frac{1}{2\pi i}\int_{\hat{\Gamma}(\epsilon)} \mathrm{d}z\, \frac{e^{-itz}}{it} \left\langle \psi^{\overline{\theta}}, \frac{1}{(H^\theta - z)^2}\phi^\theta \right\rangle \tag{5.3.13}$$

Eqs. (5.2.10) and (5.2.13) imply that the only spectral point of H^θ on the real line is λ_0^θ and all other spectral points have strictly negative imaginary part. Therefore, the operator valued function

$$A \cup \mathbb{C}^+ \ni z \mapsto \frac{1}{H^\theta - z}, \tag{5.3.14}$$

where $\mathbb{C}^+ = \{x + iy | x \in \mathbb{R}, y > 0\}$, is analytic. Moreover, for $R \geq e_1 + \delta = 2e_1$, $\Gamma_d(R)$ is contained in the region A, and hence, it follows from (5.2.29) that there is a constant C such that

$$\left\| \frac{1}{H^\theta - z} \right\| \leq \frac{C}{|z - e_1|} \qquad \forall z \in \Gamma_d. \tag{5.3.15}$$

Due to the analyticity, we may deform the integration contour from $\hat{\Gamma}(\epsilon)$ to $\Gamma(\epsilon, R)$ which gives:

$$(5.3.13) = \frac{1}{2\pi i}\int_{\Gamma(\epsilon,R)} \mathrm{d}z\, \frac{e^{-itz}}{it} \left\langle \psi^{\overline{\theta}}, \frac{1}{(H^\theta - z)^2}\phi^\theta \right\rangle. \tag{5.3.16}$$

Now we observe that the integrand on the right-hand side features an exponential decay for large $|\operatorname{Re} z|$ thanks to the factor e^{-itz} in the integrand and the definition of $\Gamma_d(\epsilon, R)$. In particular, the decay in $|z|$ provided by the resolvent, i.e., bound (5.3.15), is not necessary anymore to make the integral converge. We may therefore perform an integration by parts. Note that, for z in $A \cup \mathbb{C}^+$, we have

$$\frac{d}{dz}\left\langle \psi^{\overline{\theta}}, \frac{1}{(H^\theta - z)}\phi^\theta \right\rangle = \left\langle \psi^{\overline{\theta}}, \frac{1}{(H^\theta - z)^2}\phi^\theta \right\rangle \tag{5.3.17}$$

which is implied by the resolvent identity

$$\left\langle \psi^{\overline{\theta}}, \frac{1}{(H^\theta - (z+u))}\phi^\theta \right\rangle - \left\langle \psi^{\overline{\theta}}, \frac{1}{(H^\theta - z)}\phi^\theta \right\rangle = \left\langle \psi^{\overline{\theta}}, \frac{1}{(H^\theta - (z+u))} u \frac{1}{(H^\theta - z)}\phi^\theta \right\rangle. \tag{5.3.18}$$

Moreover, the boundary terms of the partial integration resulting from the piece-wise concatenation of contours, i.e., $\Gamma(\epsilon, R) = \Gamma_-(\epsilon, R) \cup \Gamma_c(\epsilon) \cup \Gamma_d(R)$, cancel and the ones at $|\operatorname{Re} z| \to \infty$ vanish because of the exponential decay. In conclusion, the identity

$$(5.3.13) = \frac{1}{2\pi i} \int_{\Gamma(\epsilon,R)} \mathrm{d}z\, e^{-itz} \left\langle \psi^{\bar{\theta}}, \frac{1}{H^\theta - z} \phi^\theta \right\rangle \tag{5.3.19}$$

holds true which proves the claim. □

Definition 5.3.3. *Let $S(\mathbb{R}, \mathbb{C})$ denote the Schwartz space of functions with rapid decay and let $S'(\mathbb{R}, \mathbb{C})$ denote its dual, i.e. the space of continuous linear functionals $L : S(\mathbb{R}, \mathbb{C}) \to \mathbb{C}$. For all $u \in S(\mathbb{R}, \mathbb{C})$, we define the Fourier transform of a function and its inverse*

$$\mathfrak{F}[u](x) := \int_{\mathbb{R}} \mathrm{d}s\, u(s) e^{-isx}, \qquad \mathfrak{F}^{-1}[u](x) := (2\pi)^{-1} \int_{\mathbb{R}} \mathrm{d}s\, u(s) e^{isx}. \tag{5.3.20}$$

and likewise for all $L \in S'(\mathbb{R}, \mathbb{C})$ and $u \in S(\mathbb{R}, \mathbb{C})$

$$\mathfrak{F}[L](u) := L(\mathfrak{F}[u]), \qquad \mathfrak{F}^{-1}[L](u) := L(\mathfrak{F}^{-1}[u]). \tag{5.3.21}$$

Note the factor $(2\pi)^{-1}$ which is not the normalization factor of the standard definition of the inverse Fourier transform, however, it is convenient in our setting (see e.g. [60]).

Definition 5.3.4. *We define*

(i) the Heaviside distribution $\Theta \in S'(\mathbb{R}, \mathbb{C})$ as

$$\Theta : S(\mathbb{R}, \mathbb{C}) \to \mathbb{C}, \qquad u \mapsto \Theta(u) := \int_0^\infty \mathrm{d}x\, u(x). \tag{5.3.22}$$

(ii) the Dirac-delta distribution $\delta \in S'(\mathbb{R}, \mathbb{C})$ as

$$\delta : S(\mathbb{R}, \mathbb{C}) \to \mathbb{C}, \qquad u \mapsto \delta(u) := u(0). \tag{5.3.23}$$

Remark 5.3.5. *We shall shall use the following conventions:*

(i) Every $f \in L^1_{loc}(\mathbb{R}, \mathbb{C})$ gives rise to a distribution

$$F : (\mathbb{R}, \mathbb{C}) \to \mathbb{C}, \qquad u \mapsto \int_{\mathbb{R}} \mathrm{d}x f(x) u(x) \tag{5.3.24}$$

in $C_c^\infty(\mathbb{R}, \mathbb{C})'$. With slight abuse of notation, we denote $f \in L^1_{loc}(\mathbb{R}, \mathbb{C})$ and $F \in C_c^\infty(\mathbb{R}, \mathbb{C})'$ by the same symbol f.

(ii) In a similar vain, we also introduce Θ as a function

$$\Theta : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \Theta(x) := \begin{cases} 1 & \textit{for} \quad x \geq 0 \\ 0 & \textit{for} \quad x < 0 \end{cases}. \tag{5.3.25}$$

(iii) In some locations we employ the physicists' notation and write $\delta(u) \equiv \int \mathrm{d}x\, \delta(x)u(x)$ *for all* $u \in \mathcal{S}(\mathbb{R}, \mathbb{C})$. *The right-hand side is just a symbolic expression and its meaning is defined in Definition 5.3.4.*

Lemma 5.3.6. *We denote by* $(PV(1/\bullet)) \in S'(\mathbb{R}, \mathbb{C})$ *the principal value:*

$$(PV(1/\bullet))(\varphi) \equiv PV\int_{\mathbb{R}} \mathrm{d}s\, \frac{1}{s}\varphi(s) := \lim_{\eta\to 0^+} \int_{\mathbb{R}\setminus[-\eta,\eta]} \mathrm{d}s\, \frac{1}{s}\varphi(s) \qquad \forall \varphi \in S(\mathbb{R}, \mathbb{C}). \quad (5.3.26)$$

It follows that

$$\mathfrak{F}[\Theta] = \pi\delta - iPV(1/\bullet)\,. \quad (5.3.27)$$

This statement is well-known, however, for the sake of completeness, we present a proof in Appendix A.

Key estimates

The next definition is motivated by a simple geometric argument which we give in the following for the convenience of the reader: take a cone of the form $\mathcal{C}_m(\lambda_0^{(n)} - xe^{-i\nu})$, $x > 0$, where m is a fixed (arbitrary) number greater or equal than 4. Although m is arbitrary, our estimates and constants depend on it. The distance between the vertex of the cone and the intersection of the line $\lambda_0^{(n)} - ix\sin(\nu) + \mathbb{R}$ with the cone is

$$\sqrt{\Big(\frac{2x\sin(\nu)}{\tan\left((1-1/m)\nu\right)}\Big)^2 + (2x\sin(\nu))^2} \leq 4x\frac{\sin(\nu)}{\sin\left((1-1/m)\nu\right)} \leq 8x.$$

To obtain the last inequality we use the sum of angles formula for $\sin(\nu)$, writing $\nu = (\nu - \nu/m) + \nu/m$. Then, we have that the distance between $\lambda_0^{(n)}$ and the line segment described above is smaller than $8x$.

Definition 5.3.7. *For every* $n \in \mathbb{N}$ *and recalling* (5.2.3), (5.2.14) *and* (5.2.31), *we define the sequence*

$$\epsilon_n := 20\rho_n^{1+\mu/4}. \quad (5.3.28)$$

It follows from (5.2.33) *and* (5.2.28) *that for every* $n \in \mathbb{N}$

$$D(\lambda_0, 2\epsilon_n) \subset B_0^{(1)}. \quad (5.3.29)$$

The geometric argument given above together with $|\lambda_0^{(n)} - \lambda_0| \leq 10^{-2}\rho_n^{1+\mu/2}$ *(see Definition 5.2.1 and* (5.2.27)*) yields that, for all* $n \in \mathbb{N}$ *and a fixed (arbitrary)* $m \geq 4$,

$$\mathcal{C}_m(\lambda_0^{(n)} - 2\rho_n^{1+\mu/4}e^{-i\nu}) \cap \Big(\overline{\mathbb{C}^+} + \lambda_0^{(n)} - i2\sin(\nu)\rho_n^{1+\mu/4}\Big) \subset D(\lambda_0, \epsilon_n) \subset D(\lambda_0, 2\epsilon_n) \subset B_0^{(1)} \quad (5.3.30)$$

and

$$\mathcal{C}_m(\lambda_0 - 2\rho_n^{1+\mu/4}e^{-i\nu}) \cap \left(\overline{\mathbb{C}^+} + \lambda_0 - i2\sin(\nu)\rho_n^{1+\mu/4}\right) \subset D(\lambda_0, \epsilon_n) \subset D(\lambda_0, 2\epsilon_n) \subset B_0^{(1)}. \tag{5.3.31}$$

Note that (5.2.27) *and the fact that* $\lambda_0 \in \mathbb{R}$ *imply that*

$$\operatorname{Im} \lambda_0^{(n)} - 2\sin(\nu)\rho_n^{1+\mu/4} \leq 2g\rho_n^{1+\mu/2} - 2\sin(\nu)\rho_n^{1+\mu/4} < 0, \qquad \forall n \in \mathbb{N}, \tag{5.3.32}$$

for small enough g *(see Definition 4.4.3). Eq.* (5.3.30) *implies that for every* $n \in \mathbb{N}$

$$\Gamma_c(\epsilon_n) \subset B_0^{(1)} \setminus \mathcal{C}_m(\lambda_0^{(n)} - 2\rho_n^{1+\mu/4}e^{-i\nu}). \tag{5.3.33}$$

Next, we formulate three essential technical ingredients for the proof of the main theorem.

Lemma 5.3.8. *For all* $n \in \mathbb{N}$*, a fixed (arbitrary)* $m \geq 4$ *and* $\theta \in \mathcal{S}$*, there is a constant* C *(that depends on* m*) such that*

$$\Big\| \frac{1}{H^\theta - z}\sigma_1 \Psi_{\lambda_0}^\theta \Big\| \leq C\boldsymbol{C}^{n+1}\frac{1}{\rho_n}, \tag{5.3.34}$$

for every $z \in B_0^{(1)} \setminus \mathcal{C}_m(\lambda_0^{(n)} - \rho_n^{1+\mu/4}e^{-i\nu})$*, and hence, for every* $z \in B_0^{(1)} \setminus \mathcal{C}_m(\lambda_0 - 2\rho_n^{1+\mu/4}e^{-i\nu})$*, see Theorem 4.5.10.*

Proof. We take $z \in B_0^{(1)} \setminus \mathcal{C}_m(\lambda_0^{(n)} - \rho_n^{1+\mu/4}e^{-i\nu})$ and recall the definition $\Psi_{\lambda_0}^\theta = P_0^\theta \varphi_0 \otimes \Omega$. Then, (5.2.24) yields

$$\|\Psi_{\lambda_0}^\theta - P_0^{(n),\theta} \otimes P_{\Omega(n,\infty)}\varphi_0 \otimes \Omega\| \leq \rho_n^{\mu/2}. \tag{5.3.35}$$

This together with (5.2.34), (5.2.36), (4.4.3) and (5.2.26) implies that there is a constant C such that (we use a telescopic sum argument)

$$\begin{aligned}
&\Big\| \frac{1}{H^\theta - z}\sigma_1 \Psi_{\lambda_0}^\theta - \frac{1}{\tilde{H}^{(n),\theta} - z}\sigma_1 P_0^{(n),\theta} \otimes P_{\Omega(n,\infty)}\varphi_0 \otimes \Omega \Big\| \\
&\leq \Big\| \frac{1}{H^\theta - z} - \frac{1}{\tilde{H}^{(n),\theta} - z} \Big\| \Big\| P_0^{(n),\theta} \Big\| + \Big\| \frac{1}{H^\theta - z} \Big\| \Big\| \Psi_{\lambda_0}^\theta - P_0^{(n),\theta} \otimes P_{\Omega(n,\infty)}\varphi_0 \otimes \Omega \Big\| \\
&\leq C\boldsymbol{C}^{n+1}\frac{1}{\rho_n}.
\end{aligned} \tag{5.3.36}$$

The fact (see Remark 1.2.1) that

$$\Big(\frac{1}{\tilde{H}^{(n),\theta} - z}\sigma_1\Big)\Big(P_0^{(n),\theta} \otimes P_{\Omega(n,\infty)}\Big)\varphi_0 \otimes \Omega = \Big(\big(\frac{1}{H^{(n),\theta} - z}\sigma_1\big) \otimes P_{\Omega(n,\infty)}\Big)P_0^{(n),\theta}\varphi_0 \otimes \Omega \tag{5.3.37}$$

guarantees that there is a constant C such that

$$\Big\| \frac{1}{\tilde{H}^{(n),\theta} - z}\sigma_1 P_0^{(n),\theta} \otimes P_{\Omega(n,\infty)}\varphi_0 \otimes \Omega \Big\| \leq \Big\| \Big(\overline{P_0^{(n),\theta} \frac{1}{H^{(n),\theta} - z}}\Big) \otimes P_{\Omega(n,\infty)} \Big\| \leq C\frac{\boldsymbol{C}^{n+1}}{\rho_n}. \tag{5.3.38}$$

Here, we use (5.2.35), (5.2.26) and Lemma 5.3.1. □

Lemma 5.3.9. *Set $G \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$ with support contained in $\mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$ large enough and $\eta > 0$ small enough such that $G(x) = 0$, for $|x| \le 2(\epsilon_n + \eta)$. We define*

$$T_{n,R}(\eta) := \int_{\Gamma_-(\epsilon_n,R)} \mathrm{d}z\, u(z) \int_{\mathbb{R}} \mathrm{d}r\, \frac{G(r)}{z - \lambda_0 - r} \left(1 - \mathbb{1}_{I_\eta(z)}(r)\right), \tag{5.3.39}$$

where $\mathbb{1}_{I_\eta(z)}$ is the characteristic function of the set $I_n(z) := [z - \lambda_0 - \eta, z - \lambda_0 + \eta]$, $\Gamma_-(\epsilon_n, R)$ is defined in (5.3.2) and

$$u : \overline{\mathbb{C}^+} \setminus \{\lambda_0\} \mapsto \mathbb{C}, \qquad z \to u(z) := \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle. \tag{5.3.40}$$

Then, for sufficiently large R (independent of n and $\theta \in \mathcal{S}$), there is a constant C (that does not depend on n, but it does depend on G and the other parameters) such that

$$\left| T_{n,R}(\eta) - \pi i \int_{\mathbb{R}} \mathrm{d}r\, G(r) u(r + \lambda_0) \right| \le C \left(\rho_n^{\mu/8} + \frac{1}{R} + \eta \right). \tag{5.3.41}$$

Proof. The integrand in (5.3.39) is absolutely integrable with respect to the variables z and r because the singularity is cut off by the characteristic function. We apply Fubini's theorem to get

$$T_{n,R}(\eta) = \int_{\mathbb{R}} \mathrm{d}r\, G(r) \int_{\Gamma_-(\epsilon_n,R)} \mathrm{d}z\, u(z) \frac{1}{z - \lambda_0 - r} \left(1 - \mathbb{1}_{I_\eta(z)}(r)\right). \tag{5.3.42}$$

Next, we analyze the inner integral above for r in the support of G. Set $\Gamma_{(r)}$ the half-circle in the upper half complex plane with center $r + \lambda_0$ and radius η. Moreover, set $\Gamma^{(R)}$ the half-circle in the upper half complex plane with center 0 and radius R. Then, $\Gamma_c(\epsilon_n)$ and $\Gamma_{(r)}$ do not intersect each other and both are contained in $\Gamma^{(R)}$, for large enough R (independent of n and $\theta \in \mathcal{S}$, but dependent on the support of G). Note that there is a constant C (that depends on the support of G, but not on n and $\theta \in S$) such that (see (5.2.29))

$$\left| u(z) \frac{1}{z - \lambda_0 - r} \right| \le \frac{C}{R^2}, \qquad \forall z \in \Gamma^{(R)}. \tag{5.3.43}$$

Moreover, there is a constant C (that depends on the support of G, but not on n and $\theta \in \mathcal{S}$) such that (see Lemma 5.3.8)

$$\left| u(z) \frac{1}{z - \lambda_0 - r} \right| \le C \boldsymbol{C}^{n+1} \frac{1}{\rho_n}, \qquad \forall z \in \Gamma_c(\epsilon_n), \tag{5.3.44}$$

where $\rho_n = \rho_0 \rho^n$ and $\rho_0 > 0$, $0 < \rho < 1$ and $\boldsymbol{C} > 0$ are specific numbers defined in Definitions 4.4.1 and 4.4.2. We know from (5.2.10) and (5.2.13) that the only spectral point of H^θ in $\overline{\mathbb{C}^+}$ is λ_0. Hence, there is a constant C (that depends on the support of G, but not on n) such that

$$|u(z) - u(\lambda_0 + r)| \le C\eta, \qquad \forall z \in \Gamma_{(r)}. \tag{5.3.45}$$

A direct calculation shows that

$$\int_{\Gamma_{(r)}} dz\, u(\lambda_0 + r)\, \frac{1}{z - \lambda_0 - r} = -u(\lambda_0 + r) i\pi. \tag{5.3.46}$$

It follows from Cauchy's integral formula that

$$\int_{\Gamma_-(\epsilon,R)} \mathrm{d}z\, \frac{u(z)}{z - \lambda_0 - r} \Big(1 - \mathbb{1}_{I_\eta(z)}(r)\Big) = -\int_{\Gamma^{(R)} \cup \Gamma(r) \cup \Gamma_c(\epsilon_n)} dz\, \frac{u(z)}{z - \lambda_0 - r}, \tag{5.3.47}$$

which together with (5.3.42)-(5.3.46) imply the desired result, we additionally use (5.2.31) to estimate the integral over $\Gamma_c(\epsilon_n)$. □

Lemma 5.3.10. *Let $n \geq 2$ and $R > 0$ be large enough. For $0 < q < 1 < Q < \infty$ and $\zeta \in S(\mathbb{R}, \mathbb{C})$, we define*

$$A(Q, n, R) := \int_q^Q \mathrm{d}s\, \zeta(s) \int_{\Gamma_-(\epsilon_n,R)} \mathrm{d}z\, e^{-is(z-\lambda_0)} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle. \tag{5.3.48}$$

Then, the limits $A(Q, \infty, \infty) := \lim_{n,R\to\infty} A(Q, n, R)$ and $A(\infty, n, R) := \lim_{Q\to\infty} A(Q, n, R)$ exist and they are uniform with respect to Q and (n, R), respectively. Moreover, there is a constant C (independent of n, q, Q and R) such that

$$|A(Q, n, R) - A(\infty, n, R)| \leq C/Q. \tag{5.3.49}$$

Additionally, the limits

$$\lim_{Q\to\infty} \lim_{n,R\to\infty} A(Q, n, R), \qquad \lim_{n,R\to\infty} A(\infty, n, R) \tag{5.3.50}$$

exist and they are equal.

Proof. For $0 < q < Q < \infty$, $n \in \mathbb{N}$ and $R \in \mathbb{R}^+$ sufficiently large, we write

$$A(Q, n, R) = A^{(1)}(Q, n, R) + A^{(2)}(Q, n, R), \tag{5.3.51}$$

where

$$A^{(1)}(Q, n) := \int_q^Q \mathrm{d}s\, \zeta(s) \int_{I_n} \mathrm{d}z\, e^{-is(z-\lambda_0)} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle, \tag{5.3.52}$$

$$A^{(2)}(Q, R) := \int_q^Q \mathrm{d}s\, \zeta(s) \int_{I_1} \mathrm{d}z\, e^{-is(z-\lambda_0)} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle. \tag{5.3.53}$$

Here, we split the the domain of integration $\Gamma_-(\epsilon_n, R) = I_1 \cup I_n$, where $I_1 := [-R, R] \setminus (\lambda_0 - \epsilon_1, \lambda_0 + \epsilon_1)$ and $I_n := [\lambda_0 - \epsilon_1, \lambda_0 + \epsilon_1] \setminus (\lambda_0 - \epsilon_n, \lambda_0 + \epsilon_n)$. We analyze first

(5.3.53). We obtain from the integration by parts formula (in the variable s) together with $e^{-is(z-\lambda_0)} = i(z-\lambda_0)^{-1}\partial_s e^{-is(z-\lambda_0)}$ that there is a constant C such that, for $\tilde{Q} > Q$,

$$\begin{aligned}
&A^{(2)}(\tilde{Q}, R) - A^{(2)}(Q, R) \\
&= i \int_Q^{\tilde{Q}} \mathrm{d}s\, \zeta(s) \int_{I_1} \mathrm{d}z\, (z-\lambda_0)^{-1} \partial_s e^{-is(z-\lambda_0)} \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle . \\
&= i \int_{I_1} \mathrm{d}z\, \left(\zeta(\tilde{Q}) e^{-i\tilde{Q}(z-\lambda_0)} - \zeta(Q) e^{-iQ(z-\lambda_0)}\right) (z-\lambda_0)^{-1} \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle \\
&- i \int_Q^{\tilde{Q}} \mathrm{d}s\, \left(\partial_s \zeta(s)\right) \int_{I_1} \mathrm{d}z\, (z-\lambda_0)^{-1} e^{-is(z-\lambda_0)} \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle . \qquad (5.3.54)
\end{aligned}$$

Since $\zeta \in S(\mathbb{R}, \mathbb{C})$, there is a constant C such that, for all $s \in \mathbb{R}$, $|\zeta(s)|, |\partial_s \zeta(s)| \leq C/(1+s^2)$, and hence, there is a constant C such that

$$\left| A^{(2)}(\tilde{Q}, R) - A^{(2)}(Q, R) \right| \leq C Q^{-1} \int_{I_1} \mathrm{d}z\, |z-\lambda_0|^{-1} \left| \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle \right| . \qquad (5.3.55)$$

It follows from (5.2.29) and Lemma 5.3.8 that there is a constant C (independent of n, R, q and Q) such that

$$\left| A^{(2)}(\tilde{Q}, R) - A^{(2)}(Q, R) \right| \leq C/Q. \qquad (5.3.56)$$

Similarly, again employing $\zeta \in S(\mathbb{R}, \mathbb{C})$, we find a constant C (independent of n, R, q and Q) such that

$$\begin{aligned}
\left| A^{(1)}(\tilde{Q}, n) - A^{(1)}(Q, n) \right| &\leq C Q^{-1} \int_{I_n} \mathrm{d}z\, \left| \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle \right| \\
&\leq C Q^{-1} \sum_{j=1}^{n-1} \int_{I_{j,j+1}} \mathrm{d}z\, \left| \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle \right| , \qquad (5.3.57)
\end{aligned}$$

where $I_{j,j+1} := [\lambda_0 - \epsilon_j, \lambda_0 + \epsilon_j] \setminus (\lambda_0 - \epsilon_{j+1}, \lambda_0 + \epsilon_{j+1})$. We observe from Lemma 5.3.8 together with Definition 5.3.7 that there is a constant C (independent of n, R, q and Q) such that

$$\left| A^{(1)}(\tilde{Q}, n) - A^{(1)}(Q, n) \right| \leq C Q^{-1} \sum_{j=1}^{\infty} \int_{I_{j,j+1}} \mathrm{d}z\, \frac{\boldsymbol{C}^{j+2}}{\rho_{j+1}} \leq C Q^{-1} \sum_{j=1}^{\infty} \frac{\boldsymbol{C}^{j+2} \epsilon_j}{\rho_{j+1}} . \qquad (5.3.58)$$

From Definition 5.3.7 together with (5.2.31), we obtain that

$$\left| A^{(1)}(\tilde{Q}, n) - A^{(1)}(Q, n) \right| \leq C/Q. \qquad (5.3.59)$$

This together with (5.3.56) implies that there is a constant C such that

$$\left| A(\tilde{Q}, n, R) - A(Q, n, R) \right| \leq C/Q. \qquad (5.3.60)$$

Consequently, the limit $\lim_{\tilde{Q}\to\infty} A(\tilde{Q}, n, R)$ exists and it converges uniformly with respect to n and R. We denote the limit by $A(\infty, n, R) = \lim_{Q\to\infty} A(Q, n, R)$. It follows that (5.3.49) holds true.

For fixed Q and $\tilde{n} > n$ and $\tilde{R} > R$, we have

$$\left|A(Q, \tilde{n}, \tilde{R}) - A(Q, n, R)\right| \leq \left|A(Q, \tilde{n}, \tilde{R}) - A(Q, \tilde{n}, R)\right| + |A(Q, \tilde{n}, R) - A(Q, n, R)| . \tag{5.3.61}$$

For $\tilde{n}$ and $\tilde{R}$ large enough, employing a similar calculation as in (5.3.55), we get from (5.3.51), (5.3.52), (5.3.53) that there is a constant C (that does not depend on Q) such that

$$\begin{aligned} &\left|A(Q, \tilde{n}, \tilde{R}) - A(Q, \tilde{n}, R)\right| = \left|A^{(2)}(Q, \tilde{R}) - A^{(2)}(Q, R)\right| \\ &\leq C' \int_{[-\tilde{R},-R]\cup[R,\tilde{R}]} \mathrm{d}z \ |z - \lambda_0|^{-1} \left|\left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi^{\theta}_{\lambda_0} \right\rangle\right| \leq C/R, \end{aligned} \tag{5.3.62}$$

and furthermore, similarly as in (5.3.58), we obtain that there is a constant C such that

$$|A(Q, \tilde{n}, R) - A(Q, n, R)| = \left|A^{(1)}(Q, \tilde{n}) - A^{(1)}(Q, n)\right| \leq C \sum_{j=n}^{\tilde{n}-1} \frac{\boldsymbol{C}^{j+2}\epsilon_j}{\rho_{j+1}}, \tag{5.3.63}$$

and consequently, it follows from Definition 5.3.7 together with (5.2.31) that there is a constant C (that does not depend on Q) such that

$$|A(Q, \tilde{n}, R) - A(Q, n, R)| \leq C/n. \tag{5.3.64}$$

This together with (5.3.61) and (5.3.62) yields that there there is a constant C (that does not depend on Q) such that

$$\left|A(Q, \tilde{n}, \tilde{R}) - A(Q, n, R)\right| \leq C(R^{-1} + n^{-1}). \tag{5.3.65}$$

We conclude that the limit $A(Q, \infty, \infty) := \lim_{n,R\to\infty} A(Q, n, R)$ exists (uniformly with respect to Q). This completes the first part of the lemma.

Now we prove the second part of the lemma. At first, we show the existence of the limit $\lim_{n,R\to\infty} A(\infty, n, R)$. For $\tilde{n} > n$ and $\tilde{R} > R$, we estimate

$$\begin{aligned} &\left|A(\infty, \tilde{n}, \tilde{R}) - A(\infty, n, R)\right| \\ &\leq \left|A(\infty, \tilde{n}, \tilde{R}) - A(Q, \tilde{n}, \tilde{R})\right| + \left|A(Q, \tilde{n}, \tilde{R}) - A(Q, n, R)\right| + |A(Q, n, R) - A(\infty, n, R)| . \end{aligned} \tag{5.3.66}$$

For $\epsilon > 0$, we take $Q_0 > 0$ such that for all $Q \geq Q_0$

$$\left|A(\infty, \tilde{n}, \tilde{R}) - A(Q, \tilde{n}, \tilde{R})\right| \leq \epsilon/3 \quad \text{and} \quad |A(\infty, n, R) - A(Q, n, R)| \leq \epsilon/3. \tag{5.3.67}$$

We obtain from (5.3.65) that, for $\epsilon > 0$, there are constants $n_0, R_0 > 0$ such that, for all $n, \tilde{n} \geq n_0$ and $R, \tilde{R} \geq R_0$,

$$\left| A(Q, \tilde{n}, \tilde{R}) - A(Q, n, R) \right| \leq \epsilon/3. \tag{5.3.68}$$

This together with (5.3.67) and (5.3.66) yields that, for $\epsilon > 0$, there are $n_0 > 0$ and $R_0 > 0$ such that, for $n \geq n_0$ and $R \geq R_0$, we have

$$\left| A(\infty, \tilde{n}, \tilde{R}) - A(\infty, n, R) \right| \leq \epsilon. \tag{5.3.69}$$

This implies the existence of the limit $\lim_{n,R\to\infty} A(\infty, n, R) =: A(\infty, \infty, \infty)$. We fix $\epsilon > 0$. According to (5.3.69) we obtain that for large enough n, R, $|A(\infty,\infty,\infty) - A(\infty,\ n, R)| < \epsilon/3$. Since $\lim_{Q\to\infty} A(Q, n, R) = A(\infty, n, R)$ uniformly with respect to n, R, then for large enough Q (independently of n, R) $|A(\infty, n, R) - A(Q, n, R)| < \epsilon/3$. Moreover, because $A(Q, \infty, \infty) := \lim_{n,R\to\infty} A(Q, n, R)$ (uniformly with respect to Q), for large enough n, R (independently of Q) we have that $|A(Q, n, R) - A(Q, \infty, \infty)| < \epsilon/3$. We conclude that there are $\boldsymbol{n} \in \mathbb{N}$, $\boldsymbol{R} > 0$ and $\boldsymbol{Q} > 0$ such that, for $n \geq \boldsymbol{n}$, $Q \geq \boldsymbol{Q}$ and $R \geq \boldsymbol{R}$, we have

$$\begin{aligned} |A(\infty,\infty,\infty) - A(Q,\infty,\infty)| \leq & |A(\infty,\infty,\infty) - A(\infty,n,R)| + |A(\infty,n,R) - A(Q,n,R)| \\ & + |A(Q,n,R) - A(Q,\infty,\infty)| < \epsilon. \end{aligned} \tag{5.3.70}$$

This proves that $\lim_{Q\to\infty} A(Q, \infty, \infty) = A(\infty, \infty, \infty)$ and completes the proof of the second part of the lemma. □

5.3.2. Proof of Theorems 5.1.1 and 5.1.3

In the section, we give the proof of the main theorems based on the previous results.

Proof of Theorem 5.1.1. Let $h, l \in \mathfrak{h}_0$; c.f. (2.1.1). Recall the definition of W given in (2.2.27) and the form factor f in (1.2.3). Thanks to the fact that $f \in C^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C})$, we find that

$$hf, lf, W \in \mathfrak{h}_0. \tag{5.3.71}$$

Theorem 2.2.2, i.e., Equation (2.2.26) together with Lemma 2.2.1 (iv) yields

$$T(h,l) = -2\pi i g \|\Psi_{\lambda_0}\|^{-2} \langle a_-(W)\sigma_1\Psi_{\lambda_0}, \Psi_{\lambda_0}\rangle = -2\pi i g \|\Psi_{\lambda_0}\|^{-2} \langle [a_-(W), \sigma_1]\Psi_{\lambda_0}, \Psi_{\lambda_0}\rangle, \tag{5.3.72}$$

and furthermore, recalling $\omega(k) = |k|$, and Lemma 2.2.1 (ii), we obtain that

$$\begin{aligned} T(h,l) &= -2\pi (ig)^2 \|\Psi_{\lambda_0}\|^{-2} \int_{-\infty}^{0} \mathrm{d}s\, \overline{\langle W_s, f\rangle_2} \left\langle \left[e^{isH}\sigma_1 e^{-isH}, \sigma_1 \right] \Psi_{\lambda_0}, \Psi_{\lambda_0} \right\rangle \\ &= 2\pi g^2 \|\Psi_{\lambda_0}\|^{-2} \int_{0}^{\infty} \mathrm{d}s\, \langle f, W_{-s}\rangle_2 \left\langle \left[e^{-isH}\sigma_1 e^{isH}, \sigma_1 \right] \Psi_{\lambda_0}, \Psi_{\lambda_0} \right\rangle \\ &= ig^2 \|\Psi_{\lambda_0}\|^{-2} \left(T^{(1)} - T^{(2)} \right), \end{aligned} \tag{5.3.73}$$

where we use the abbreviations

$$T^{(j)} := \lim_{q\to 0^+} \lim_{Q\to\infty} T^{(j),q,Q} \tag{5.3.74}$$

for $j = 1, 2$ with

$$\begin{aligned} T^{(1),q,Q} &:= -2\pi i \int_q^Q \mathrm{d}s \int \mathrm{d}^3k\, W(k) f(k) e^{is(|k|+\lambda_0)} \left\langle \sigma_1 \Psi_{\lambda_0}, e^{-isH} \sigma_1 \Psi_{\lambda_0} \right\rangle \\ &= -2\pi i \int_q^Q \mathrm{d}s \int \mathrm{d}r\, G(r) e^{is(r+\lambda_0)} \left\langle \sigma_1 \Psi_{\lambda_0}, e^{-isH} \sigma_1 \Psi_{\lambda_0} \right\rangle \end{aligned} \tag{5.3.75}$$

and

$$T^{(2),q,Q} := -2\pi i \int_q^Q \mathrm{d}s \int \mathrm{d}r\, G(r) e^{is(r-\lambda_0)} \left\langle \sigma_1 \Psi_{\lambda_0}, e^{isH} \sigma_1 \Psi_{\lambda_0} \right\rangle. \tag{5.3.76}$$

Here, we recall (5.1.4):

$$G : \mathbb{R} \to \mathbb{C}, \qquad r \mapsto G(r) := \begin{cases} \int \mathrm{d}\Sigma \mathrm{d}\Sigma'\, r^4 \overline{h(r,\Sigma)} l(r,\Sigma') f(r)^2 & \text{for} \quad r \geq 0 \\ 0 & \text{for} \quad r < 0, \end{cases} \tag{5.3.77}$$

where we write spherical coordinates $k = (r, \Sigma)$ and $k' = (r', \Sigma')$ in (2.2.2) and (2.2.27) recalling the definition of W and that $f(k) \equiv f(|k|)$ only depends on the radial coordinate $r = |k|$. Thanks to (5.3.71), we observe

$$G \in C_c^\infty(\mathbb{R} \setminus \{0\}, \mathbb{C}) \subset \mathcal{S}(\mathbb{R}, \mathbb{C}). \tag{5.3.78}$$

Term $T^{(1),q,Q}$**:** Theorem 4.2.3 guarantees that Ψ_{λ_0}, and therefore, also $\sigma_1 \Psi_{\lambda_0}$ is an analytic vector (see Definition 1.3.4). As pointed out earlier, for the ground state, we can take the set $\mathcal{S}$ to be a neighborhood of 0 which allows us to apply Lemma 5.3.2 and find

$$T^{(1),q,Q} = -\int_q^Q \mathrm{d}s \int \mathrm{d}r\, G(r) e^{is(r+\lambda_0)} \int_{\Gamma(\epsilon_n,R)} \mathrm{d}z\, e^{-isz} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right\rangle. \tag{5.3.79}$$

Here, $\Gamma(\epsilon_n, R) = \Gamma_-(\epsilon_n, R) \cup \Gamma_c(\epsilon_n) \cup \Gamma_d(R)$ is the contour defined in Lemma 5.3.2, i.e., (5.3.2), for sufficiently large $R > 0$ and $n > 2$. We split the term

$$T^{(1),q,Q} = T^{(1),q,Q}_{\epsilon_n,R} + T^{(1),q,Q}_{\epsilon_n} + T^{(1),q,Q}_{R} \tag{5.3.80}$$

according to the different contours parts, see (5.3.2), in the $\mathrm{d}z$-integrals:

$$T^{(1),q,Q}_{\epsilon_n,R} := -\int_q^Q \mathrm{d}s\, J(s) \int_{\Gamma_-(\epsilon_n,R)} \mathrm{d}z\, e^{-isz} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right\rangle, \tag{5.3.81}$$

$$T^{(1),q,Q}_{\epsilon_n} := -\int_q^Q J(s) \int_{\Gamma_c(\epsilon_n)} \mathrm{d}z\, e^{-isz} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right\rangle, \tag{5.3.82}$$

$$T^{(1),q,Q}_{R} := -\int_q^Q \mathrm{d}s\, J(s) \int_{\Gamma_d(R)} \mathrm{d}z\, e^{-isz} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right\rangle, \tag{5.3.83}$$

and we use the definition

$$J : \mathbb{R} \to \mathbb{C}, \qquad s \mapsto J(s) = \int \mathrm{d}r\, G(r) e^{is(r+\lambda_0)}. \tag{5.3.84}$$

We observe that, thanks to (5.3.78), we have $J \in S(\mathbb{R}, \mathbb{C})$ which implies

$$|J(s)| \leq C(1 + |s|^2)^{-1} \tag{5.3.85}$$

for some constant C. Moreover, we have (see (5.2.29))

$$\left| e^{-isz} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle \right| \leq C \|\Psi_{\lambda_0}\|^2 \frac{e^{s \operatorname{Im} z}}{|z - e_1|}, \qquad \forall z \in \Gamma_d(R). \tag{5.3.86}$$

Contribution $T_{\epsilon_n}^{(1),q,Q}$ in (5.3.82)**:** Using (5.3.85), we may start with the bound

$$|T_{\epsilon_n}^{(1),q,Q}| \leq C \sup_{s \in [q,Q]} \left| \int_{\Gamma_c(\epsilon_n)} \mathrm{d}z\, e^{-isz} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle \right|. \tag{5.3.87}$$

It follows from Lemma 5.3.8 together with Definition 5.3.7 that there is a constant C such that, for $s \in [q, Q]$, we have

$$\left| \int_{\Gamma_c(\epsilon_n)} \mathrm{d}z\, e^{-isz} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle \right| \leq C e^{\epsilon_n Q} \frac{\epsilon_n}{\rho_n} \boldsymbol{C}^{n+1} \leq C e^{\epsilon_n Q} \rho_n^{\mu/8}, \tag{5.3.88}$$

where we use (5.2.31). In conclusion, we have for all $0 < q < Q < \infty$

$$\lim_{n \to 0} T_{\epsilon_n}^{(1),q,Q} = 0. \tag{5.3.89}$$

Contribution $T_R^{(1),q,Q}$ in (5.3.83)**:** Using (5.3.85) again, we find

$$|T_R^{(1),q,Q}| \leq C \int_q^Q \mathrm{d}s\, \frac{1}{1 + |s|^2} \left| \int_{\Gamma_d(R)} \mathrm{d}z\, e^{-isz} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle \right|. \tag{5.3.90}$$

For $s \in [q, Q]$, we observe that there is a constant C such that (see (5.2.29))

$$\left| \int_{\Gamma_d(R)} \mathrm{d}z\, e^{-isz} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle \right| \leq \frac{C}{R} \int_0^\infty \mathrm{d}u\, e^{-su \sin(\nu/4)}. \tag{5.3.91}$$

Thereby, as in (5.3.91), we obtain the estimate

$$\begin{aligned} &\lim_{R \to \infty} \int_q^Q \mathrm{d}s\, \frac{1}{1 + |s|^2} \int_{\Gamma_d(R)} \mathrm{d}z\, \left| e^{-isz} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle \right| \\ &\leq \lim_{R \to \infty} \frac{C}{R} \int_q^Q \mathrm{d}s\, \frac{1}{1 + |s|^2} \frac{1}{|s|} = 0. \end{aligned} \tag{5.3.92}$$

Then, we conclude for all $0 < q < Q < \infty$

$$\lim_{R\to\infty} T_R^{(1),q,Q} = 0. \tag{5.3.93}$$

This together with (5.3.89) and (5.3.80) yields that for all $0 < q < Q < \infty$

$$T^{(1),q,Q} = \lim_{n,R\to\infty} T_{\epsilon_n,R}^{(1),q,Q}. \tag{5.3.94}$$

Note that $J \in S(\mathbb{R},\mathbb{C})$. Therefore, we are in the position to apply Lemma 5.3.10 and find

$$T^{(1),q,\infty} := \lim_{Q\to\infty} T^{(1),q,Q} = \lim_{Q\to\infty}\lim_{n,R\to\infty} T_{\epsilon_n,R}^{(1),q,Q} = \lim_{n,R\to\infty} T_{\epsilon_n,R}^{(1),q,\infty}, \tag{5.3.95}$$

where

$$T_{\epsilon_n,R}^{(1),q,\infty} := \lim_{Q\to\infty} T_{\epsilon_n,R}^{(1),q,Q} = -\int_q^\infty \mathrm{d}s\, J(s) \int_{\Gamma_-(\epsilon_n,R)} \mathrm{d}z\, e^{-isz} \left\langle \sigma_1\Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1\Psi_{\lambda_0}^{\theta} \right\rangle. \tag{5.3.96}$$

For fixed n and R, the function $z \mapsto e^{-isz}\left\langle \sigma_1\Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1\Psi_{\lambda_0}^{\theta} \right\rangle$ is bounded in $\Gamma_-(\epsilon_n, R)$. Then, thanks to (5.3.85), we may apply Fubini's theorem and find:

$$\begin{aligned} T_{\epsilon_n,R}^{(1),q,\infty} &= -\int_{\Gamma_-(\epsilon_n,R)} \mathrm{d}z \left\langle \sigma_1\Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1\Psi_{\lambda_0}^{\theta} \right\rangle \int_q^\infty \mathrm{d}s \int \mathrm{d}r\, G(r) e^{is(r+\lambda_0-z)} \\ &= -\int_{\Gamma_-(\epsilon_n,R)} \mathrm{d}z \left\langle \sigma_1\Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - z\right)^{-1} \sigma_1\Psi_{\lambda_0}^{\theta} \right\rangle \int \mathrm{d}s\, \Theta(s-q) \int \mathrm{d}r\, G^{(z)}(r) e^{-isr}. \end{aligned} \tag{5.3.97}$$

In the last step, we use the coordinate transformation $r \to z - \lambda_0 - r$ and the notation

$$G^{(z)} : \mathbb{R} \to \mathbb{C}, \qquad r \mapsto G^{(z)}(r) := G(z - \lambda_0 - r) \qquad z \in \mathbb{R}. \tag{5.3.98}$$

Then, it follows from (5.3.78) together with Definition 5.3.3 that

$$\begin{aligned} \int \mathrm{d}s\, \Theta(s-q) \int \mathrm{d}r\, G^{(z)}(r) e^{-isr} &= \int \mathrm{d}s\, \Theta(s) \int \mathrm{d}r\, G^{(z)}(r) e^{-iqr} e^{-isr} \\ &= \Theta(\mathfrak{F}[G^{(z),q}]) = \mathfrak{F}[\Theta](G^{(z),q}), \end{aligned} \tag{5.3.99}$$

where, for $q > 0$, we define

$$G^{(z),q}(r) := G^{(z)}(r) e^{-iqr}. \tag{5.3.100}$$

Thanks to (5.3.78), we have for $z \in \mathbb{R}$ and $q \geq 0$

$$G^{(z),q} \in C_c^\infty(\mathbb{R} \setminus \{z - \lambda_0\}, \mathbb{C}) \subset \mathcal{S}(\mathbb{R},\mathbb{C}). \tag{5.3.101}$$

It follows from Lemma 5.3.6 that for $z \in \mathbb{R}$

$$\int \mathrm{d}s\, \Theta(s-q) \int \mathrm{d}r\, G^{(z)}(r) e^{-isr} = \pi\delta(G^{(z),q}) - i\,(\mathrm{PV}\,(1/\bullet))\,(G^{(z),q}). \tag{5.3.102}$$

This together with (5.3.97) yields that

$$T^{(1),q,\infty}_{\epsilon_n,R} = T^{(1,1),q,\infty}_{\epsilon_n,R} + T^{(1,2),q,\infty}_{\epsilon_n,R}, \tag{5.3.103}$$

where

$$T^{(1,1),q,\infty}_{\epsilon_n,R} := -\pi \int_{\Gamma_-(\epsilon_n,R)} \mathrm{d}z\, \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi^{\theta}_{\lambda_0} \right\rangle G(z-\lambda_0) \tag{5.3.104}$$

$$T^{(1,2),q,\infty}_{\epsilon_n,R} := i \int_{\Gamma_-(\epsilon_n,R)} \mathrm{d}z\, \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi^{\theta}_{\lambda_0} \right\rangle \lim_{\eta\to 0^+} \int_{\mathbb{R}\setminus[-\eta,\eta]} \mathrm{d}r\, \frac{G(z-\lambda_0-r)e^{-iqr}}{r} \tag{5.3.105}$$

In the following, we shall compute both contributions explicitly.

Contribution $T^{(1,1)}_{\epsilon_n,R}(h,l)$: It follows from (5.3.78) that there are numbers $M > \kappa > 0$ such that supp $G \subset [\kappa, M]$. Recall that everything so far holds for any choice of $n, R > 0$ large enough. For the rest of this proof we will restrict this choice to $R > M$ and $n > 0$ large enough such that $\epsilon_n < \kappa/4$. In this setting, we may turn the dz-integral in an indefinite one, exploiting, the compact support of G and the definition of the contour $\Gamma_-(\epsilon_n, R)$. We thus obtain

$$\begin{aligned} T^{(1,1),q,\infty}_{\epsilon_n,R} &= -\pi \int_{\Gamma_-(\epsilon_n,R)} \mathrm{d}z\, \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi^{\theta}_{\lambda_0} \right\rangle G(z-\lambda_0) \\ &= -\pi \int_{\Gamma_-(\epsilon_n,R)-\lambda_0} \mathrm{d}z\, \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - \lambda_0 - z\right)^{-1} \sigma_1 \Psi^{\theta}_{\lambda_0} \right\rangle G(z) \\ &= -\pi \int_0^\infty \mathrm{d}z\, \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - \lambda_0 - z\right)^{-1} \sigma_1 \Psi^{\theta}_{\lambda_0} \right\rangle G(z) \end{aligned} \tag{5.3.106}$$

Contribution $T^{(1,2)}_{\epsilon_n,R}(h,l)$: In order to calculate $T^{(1,2)}_{\epsilon_n,R}(h,l)$ we can now fall back to Lemma 5.3.9. We recall Definition 5.3.7 and notice that $0 < \epsilon_n < \kappa/4$ for sufficiently large n. Then, as a direct consequence of Lemma 5.3.9, we find (for sufficiently large R)

$$\begin{aligned} \lim_{n,R\to\infty} T^{(1,2),q,\infty}_{\epsilon_n,R} &= i \lim_{n,R\to\infty,\eta\to 0} T_{n,R}(\eta) \\ &= -\pi \int_{\mathbb{R}} \mathrm{d}r\, G(r) e^{-iqr} \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - \lambda_0 - r\right)^{-1} \sigma_1 \Psi^{\theta}_{\lambda_0} \right\rangle \\ &= -\pi \int_0^\infty \mathrm{d}z\, \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - \lambda_0 - z\right)^{-1} \sigma_1 \Psi^{\theta}_{\lambda_0} \right\rangle G(z) e^{-iqz}, \end{aligned} \tag{5.3.107}$$

where $T_{n,R}(\eta)$ is defined in (5.3.39).

Collecting the contributions of (5.3.103), i.e, (5.3.106) and (5.3.107), we establish the identity

$$\begin{aligned}
T^{(1)} &= \lim_{q\to 0^+} \lim_{n,R\to\infty} T^{(1),q,\infty}_{\epsilon_n,R} \qquad (5.3.108)\\
&= -\pi \lim_{q\to 0^+} \int_0^\infty \mathrm{d}z \left\langle \sigma_1\Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - \lambda_0 - z\right)^{-1} \sigma_1\Psi^\theta_{\lambda_0}\right\rangle G(z)(1+e^{-iqz})\\
&= -2\pi \int_0^\infty \mathrm{d}z \left\langle \sigma_1\Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - \lambda_0 - z\right)^{-1} \sigma_1\Psi^\theta_{\lambda_0}\right\rangle G(z)\\
&= -2\pi \int \mathrm{d}^3k\mathrm{d}^3k'\, \overline{h(k)} l(k') f(k) f(k') \delta(|k|-|k'|) \left\langle \sigma_1\Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - \lambda_0 - |k'|\right)^{-1} \sigma_1\Psi^\theta_{\lambda_0}\right\rangle .
\end{aligned}$$

In the third line we applied the dominated convergence theorem which is justified by (5.3.78). Moreover, we have inserted the definition of G using the symbolic notation of the Dirac-delta distribution in the last step.

Term $T^{(2)}$: The second term $T^{(2)}$ can be inferred by repeating the calculation with θ replaced by $\overline{\theta}$ and reflecting the path of integration $\Gamma(\epsilon_n, R)$ on the real axis when applying Lemma 5.3.2. In this case one has to consider the Hamiltonian $H^{\overline{\theta}}$ whose spectrum is given by mirroring the spectrum of H^θ at the real axis. Due to the similarity of the calculation, we omit a proof but only state the result

$$T^{(2)} = 2\pi \int \mathrm{d}^3k\mathrm{d}^3k'\, \overline{h(k)} l(k') f(k) f(k') \delta(|k|-|k'|) \left\langle \sigma_1\Psi^{\theta}_{\lambda_0}, \left(H^{\overline{\theta}} - \lambda_0 + |k'|\right)^{-1} \sigma_1\Psi^{\overline{\theta}}_{\lambda_0}\right\rangle . \qquad (5.3.109)$$

The relative sign in comparison with (5.3.108) is due to the the opposite mathematical orientation of the contour. Inserting (5.3.108) and (5.3.109) in (5.3.73) completes the proof. □

Proof of Theorem 5.1.3. We recall from Theorem 5.1.1 that, for all $h, l \in \mathfrak{h}_0$, we have

$$\begin{aligned}
T(h,l) &= -2\pi i \|\Psi_{\lambda_0}\|^{-2} g^2 \int \mathrm{d}^3k\mathrm{d}^3k'\, \overline{h(k)} f(k) l(k') f(k') \delta(|k|-|k'|) \qquad (5.3.110)\\
&\quad \times \left(\left\langle \sigma_1\Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - \lambda_0 - |k'|\right)^{-1} \sigma_1\Psi^\theta_{\lambda_0}\right\rangle + \left\langle \sigma_1\Psi^{\theta}_{\lambda_0}, \left(H^{\overline{\theta}} - \lambda_0 + |k'|\right)^{-1} \sigma_1\Psi^{\overline{\theta}}_{\lambda_0}\right\rangle \right) .
\end{aligned}$$

It follows from $h, l \in \mathfrak{h}_0$ that there is a constant $\kappa > 0$ such that the support of h and l is contained in the interval $[\kappa, \infty)$. Hence, for the remainder of the proof we only consider $k, k' \in \mathbb{R}^3$ such that $|k|, |k'| \geq \kappa$. Using the identity $P_1^\theta + \overline{P_1^\theta} = 1$, we find

$$\begin{aligned}
&\left\langle \sigma_1\Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - \lambda_0 - |k'|\right)^{-1} \sigma_1\Psi^\theta_{\lambda_0}\right\rangle \qquad (5.3.111)\\
&= \frac{1}{\lambda_1 - \lambda_0 - |k'|} \left\langle \sigma_1\Psi^{\overline{\theta}}_{\lambda_0}, P_1^\theta \sigma_1\Psi^\theta_{\lambda_0}\right\rangle + \left\langle \overline{P_1^{\overline{\theta}}} \sigma_1\Psi^{\overline{\theta}}_{\lambda_0}, \left(H^\theta - \lambda_0 - |k'|\right)^{-1} \overline{P_1^\theta} \sigma_1\Psi^\theta_{\lambda_0}\right\rangle
\end{aligned}$$

Recalling (5.2.25) and the definitions $\Psi^{\theta}_{\lambda_i} = P^{\theta}_i \varphi_i \otimes \Omega$ (for $i = 0, 1$), we observe

$$\left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, P^{\theta}_1 \sigma_1 \Psi^{\theta}_{\lambda_0} \right\rangle = \|\Psi_{\lambda_0}\|^2 (1 + r(g)), \qquad \text{where} \qquad |r(g)| \le Cg \tag{5.3.112}$$

for some constant C (independent of g). Similarly, we find

$$\left\| \overline{P^{\theta}_1} \sigma_1 \Psi^{\theta}_{\lambda_0} \right\| \le \|\Psi_{\lambda_0}\| Cg, \quad \text{and} \quad \left\| \overline{P^{\overline{\theta}}_1} \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0} \right\| \le \|\Psi_{\lambda_0}\| Cg. \tag{5.3.113}$$

Let $\boldsymbol{c}$ be the constant introduced in (5.1.9). Note that the vertex of the cone $\mathcal{C}_m(\lambda_1 - 2\rho_n^{1+\mu/4} e^{-i\nu})$ belongs to the lower (open) half space of the complex plane if

$$-g^2 \boldsymbol{c} + 2\rho_n^{1+\mu/4} \sin(\nu) < 0, \tag{5.3.114}$$

and the condition $\mathcal{C}_m(\lambda_0 - 2\rho_n^{1+\mu/4} e^{-i\nu}) \cap [\kappa + \lambda_0, \infty) = \emptyset$ holds true if

$$-\kappa + 2\rho_n^{1+\mu/4} \cos(\nu) < 0. \tag{5.3.115}$$

We find that there is a fixed number $\tilde{n} \in \mathbb{N}$ (independent of g) such that (5.3.115) is fulfilled for $n \ge \tilde{n}$. Moreover, recalling $\nu \in (0, \pi/16)$, we observe that the conditions (5.3.114) and (5.3.115) are fulfilled for

$$n > \log\Big(\frac{g^2 \boldsymbol{c}}{2\sin(\nu) \rho_0^{1+\mu/4}} \Big) \frac{1}{(1 + \mu/4)\log(\rho)} + \tilde{n}. \tag{5.3.116}$$

We fix $n_0 > 0$ to be the smallest integer number satisfying this inequality. Then, we find

$$n_0 \le \log\Big(\frac{g^2 \boldsymbol{c}}{2\sin(\nu) \rho_0^{1+\mu/4}} \Big) \frac{1}{(1 + \mu/4)\log(\rho)} + \tilde{n} + 1. \tag{5.3.117}$$

For such n_0, the cone $\mathcal{C}_m(\lambda_1 - 2\rho_{n_0}^{1+\mu/4} e^{-i\nu})$ belongs to the lower (open) half space of the complex plane and $\mathcal{C}_m(\lambda_0 - 2\rho_{n_0}^{1+\mu/4} e^{-i\nu}) \cap [\kappa + \lambda_0, \infty) = \emptyset$. Then, for $k' \in \mathbb{R}^3$ with $|k'| \ge \kappa$, we conclude

$$(\lambda_0 + |k'|) \in A \cup \Big(B^{(1)}_0 \setminus \mathcal{C}_m(\lambda_0 - 2\rho_{n_0}^{1+\mu/4} e^{-i\nu}) \Big) \cup \Big(B^{(1)}_1 \setminus \mathcal{C}_m(\lambda_1 - 2\rho_{n_0}^{1+\mu/4} e^{-i\nu}) \Big), \tag{5.3.118}$$

where we recall (5.2.4), (5.2.8) and (5.2.9). Moreover, (5.3.117) implies that

$$\begin{aligned} \boldsymbol{C}^{n_0} &\le \boldsymbol{C}^{\tilde{n}+1} \exp\Big[-\log\Big(\frac{g^2 \boldsymbol{c}}{2\sin(\nu)\rho_0^{1+\mu/4}} \Big) \Big]^{-\frac{\log(\boldsymbol{C})}{(1+\mu/4)\log(\rho)}} \\ &= \boldsymbol{C}^{\tilde{n}+1} \Big(\frac{2\sin(\nu)\rho_0^{1+\mu/4}}{g^2 \boldsymbol{c}} \Big)^{-\frac{\log(\boldsymbol{C})}{(1+\mu/4)\log(\rho)}} \\ &\le C g^{-1}, \end{aligned} \tag{5.3.119}$$

where we use that $\boldsymbol{C}\rho^{\frac{1}{2}(1+\mu/4)} \leq 1$, see (5.2.32). For $k' \in \mathbb{R}^3$ with $|k'| \geq \kappa$ such that $\lambda_0 + |k'| \in \mathbb{R} \cap B_1^{(1)}$, this together with (5.2.37) and (5.3.118) leads us to

$$\left\| \frac{1}{H^\theta - \lambda_0 - |k'|} \right\| \leq Cg^{-1} \frac{1}{\mathrm{dist}(\lambda_0 + |k'|, \mathcal{C}_m(\lambda_1))}. \tag{5.3.120}$$

It is geometrically clear, because of $\mathrm{Im}\lambda_1 < -g^2\boldsymbol{c} < 0$ - see (5.2.13), that there is a constant C (that depends on $\boldsymbol{\nu}$ and m, but not on g) such that, for $k' \in \mathbb{R}^3$ with $|k'| \geq \kappa$ such that $\lambda_0 + |k'| \in \mathbb{R} \cap B_1^{(1)}$, we find

$$|\lambda_0 + |k'| - \lambda_1| \leq C\mathrm{dist}(\lambda_0 + |k'|, \mathcal{C}_m(\lambda_1)). \tag{5.3.121}$$

Then, for $k' \in \mathbb{R}^3$ with $|k'| \geq \kappa$ such that $\lambda_0 + |k'| \in \mathbb{R} \cap B_1^{(1)}$, (5.3.120) and (5.3.121) yield

$$\left\| \frac{1}{H^\theta - \lambda_0 - |k'|} \right\| \leq Cg^{-1} \frac{1}{|\lambda_1 - \lambda_0 - |k'||}. \tag{5.3.122}$$

Similarly, we conclude from (5.2.37) together with (5.3.119) and (5.3.118) that, for $k' \in \mathbb{R}^3$ with $|k'| \geq \kappa$ such that $\lambda_0 + |k'| \in \mathbb{R} \cap B_0^{(1)}$,

$$\left\| \frac{1}{H^\theta - \lambda_0 - |k'|} \right\| \leq Cg^{-1} \frac{1}{\mathrm{dist}(\lambda_0 + |k'|, \mathcal{C}_m(\lambda_0))} \leq Cg^{-1} \frac{1}{\kappa} \leq Cg^{-1} \frac{1}{|\lambda_1 - \lambda_0 - |k'||}, \tag{5.3.123}$$

where the last step follows again from geometrical considerations. Moreover, (5.3.122) and (5.3.123) together with (5.3.118), (5.3.113) and (5.2.29) yield that

$$\begin{aligned} \left| \left\langle \overline{P_1^{\overline{\theta}}} \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - \lambda_0 - |k'|\right)^{-1} \overline{P_1^\theta} \sigma_1 \Psi_{\lambda_0}^\theta \right\rangle \right| &\leq \left\| \overline{P_1^{\overline{\theta}}} \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}} \right\| \left\| \overline{P_1^\theta} \sigma_1 \Psi_{\lambda_0}^\theta \right\| \left\| \left(H^\theta - \lambda_0 - |k'|\right)^{-1} \right\| \\ &\leq \|\Psi_{\lambda_0}\|^2 Cg \frac{1}{\lambda_1 - \lambda_0 - |k'|}, \end{aligned} \tag{5.3.124}$$

for all $k' \in \mathbb{R}$ with $|k'| \geq \kappa > 0$. Consequently, (5.3.111) implies that

$$\|\Psi_{\lambda_0}\|^{-2} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - \lambda_0 - |k'|\right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right\rangle = \frac{1}{\lambda_1 - \lambda_0 - |k'|} \left(1 + R_1(g)\right), \tag{5.3.125}$$

where $|R_1(g)| \leq Cg$ for some constant C (independent of g).

By an analogous computation, we obtain

$$\|\Psi_{\lambda_0}\|^{-2} \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^{\overline{\theta}} - \lambda_0 + |k'|\right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right\rangle = \frac{1}{\overline{\lambda_1} - \lambda_0 + |k'|} \left(1 + R_2(g)\right), \tag{5.3.126}$$

where $|R_2(g)| \leq Cg$ for some constant C (independent of g). It follows from (5.2.28) that there is a constant C (independent of g) such that $|\overline{\lambda_1} - \lambda_0 + |k'||^{-1} \leq C$ for all

$k' \in \mathbb{R}^3$ with $|k'| \geq \kappa$, and hence, it follows from (5.3.125) together with (5.3.126) that, for $k' \in \mathbb{R}^3$ with $|k'| \geq \kappa$, we have

$$\|\Psi_{\lambda_0}\|^{-2} \left(\left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^\theta - \lambda_0 - |k'|\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle + \left\langle \sigma_1 \Psi_{\lambda_0}^{\overline{\theta}}, \left(H^{\overline{\theta}} - \lambda_0 + |k'|\right)^{-1} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle \right)$$
$$= \frac{1}{\lambda_1 - \lambda_0 - |k'|}(1 + R(g)) + \frac{1}{\overline{\lambda_1} - \lambda_0 + |k'|}, \tag{5.3.127}$$

where $|R(g)| \leq Cg$. This together with (5.3.110) implies that

$$T(h,l) = -2\pi i g^2 \int \mathrm{d}^3k \mathrm{d}^3k' \, \overline{h(k)} f(k) l(k') f(k') \delta(|k| - |k'|) \tag{5.3.128}$$
$$\times \left(\frac{1}{\lambda_1 - \lambda_0 - |k'|} + \frac{1}{\overline{\lambda_1} - \lambda_0 + |k'|} + R(g) \frac{1}{\lambda_1 - \lambda_0 - |k'|} \right).$$

Changing to spherical coordinates $k = (r, \Sigma)$ and $k' = (r', \Sigma')$ and recalling the definition of G in (5.1.4) yields

$$T(h,l) = -2\pi i g^2 \int \mathrm{d}r \, G(r) \left(\frac{1}{\lambda_1 - \lambda_0 - r} + \frac{1}{\overline{\lambda_1} - \lambda_0 + r} + \frac{R(g)}{\lambda_1 - \lambda_0 - r} \right)$$
$$= -2\pi i g^2 \int \mathrm{d}r \, G(r) \left(\frac{2(\operatorname{Re}\lambda_1 - \lambda_0)}{(\lambda_1 - \lambda_0 - r)(\overline{\lambda_1} - \lambda_0 + r)} + \frac{R(g)}{\lambda_1 - \lambda_0 - r} \right). \tag{5.3.129}$$

In the following we will show that there is a constant $C(h,l) > 0$ such that

$$\left| \int \mathrm{d}r \, \frac{G(r)}{\lambda_1 - \lambda_0 - r} \right| \leq C(h,l) |\log g|. \tag{5.3.130}$$

Recalling that $|R(g)| \leq Cg$, we then observe that

$$\left| \int \mathrm{d}r \, G(r) \frac{R(g)}{\lambda_1 - \lambda_0 - r} \right| \leq C(h,l) g |\log g|. \tag{5.3.131}$$

Note that the function G does not depend on g. Moreover, we observe from (5.1.4) and $h, l \in \mathfrak{h}_0$ that $G : \mathbb{R} \to \mathbb{C}$ is a smooth function with compact support in the interval $[r_1, r_2]$, where r_1 and r_2 are some real numbers such that $0 < \kappa \leq r_1 \leq r_2$. Hence, there is a constant $C(h,l) > 0$ such that

$$\left| \int \mathrm{d}r \, \frac{G(r)}{\lambda_1 - \lambda_0 - r} \right| \leq C(h,l) \int_{r_1}^{r_2} \mathrm{d}r \, \frac{1}{|\lambda_1 - \lambda_0 - r|}. \tag{5.3.132}$$

In addition, we find

$$\int_{r_1}^{r_2} \mathrm{d}r \, \frac{1}{|\lambda_1 - \lambda_0 - r|} = \int_{r_1}^{r_2} \mathrm{d}r \, \frac{1}{\sqrt{(\operatorname{Re}\lambda_1 - \lambda_0 - r)^2 + (\operatorname{Im}\lambda_1)^2}}$$
$$= \int_{(\operatorname{Re}\lambda_1 - \lambda_0 - r_2)/g^2}^{(\operatorname{Re}\lambda_1 - \lambda_0 - r_1)/g^2} \mathrm{d}u \, \frac{1}{\sqrt{u^2 + g^{-4}(\operatorname{Im}\lambda_1)^2}}, \tag{5.3.133}$$

where we substituted $u = (\operatorname{Re}\lambda_1 - \lambda_0 - r)/g^2$ in the last step. It follows from (5.2.13) that there is a constant $\boldsymbol{c} > 0$ (independent of g) such that $|g^{-2}\operatorname{Im}\lambda_1| \geq \boldsymbol{c}$. This yields

$$\int_{r_1}^{r_2} \mathrm{d}r \, \frac{1}{|\lambda_1 - \lambda_0 - r|} \leq \int_{(\operatorname{Re}\lambda_1-\lambda_0-r_2)/g^2}^{(\operatorname{Re}\lambda_1-\lambda_0-r_1)/g^2} \mathrm{d}u \, \frac{1}{\sqrt{u^2+\boldsymbol{c}^2}} \leq \int_{-\alpha/g^2}^{\alpha/g^2} \mathrm{d}u \, \frac{1}{\sqrt{u^2+\boldsymbol{c}^2}}, \quad (5.3.134)$$

where in the second line, we introduced the notation $\alpha := 1 + \max_{j\in\{0,1\}} |\operatorname{Re}\lambda_1 - \lambda_0 - r_j|$. Note that $\alpha \equiv \alpha(g)$ depends on the the coupling constant g, however, it follows from (5.2.28) that it is bounded from above by constant $\boldsymbol{\alpha}$ (independent of g). Recall that we require g to be sufficiently small. In particular, we have $0 < g < g_0$ for some fixed constant $0 < g_0 < 1$ (see Definition 4.4.3). Then, we find that $\boldsymbol{\alpha}/g^2 > 1$, and consequently, we observe from (5.3.134) that

$$\begin{aligned}\int_{r_1}^{r_2} \mathrm{d}r \, \frac{1}{|\lambda_1 - \lambda_0 - r|} &\leq \int_{-\boldsymbol{\alpha}/g^2}^{\boldsymbol{\alpha}/g^2} \mathrm{d}u \, \frac{1}{\sqrt{u^2+\boldsymbol{c}^2}} \leq \frac{2}{\boldsymbol{c}} + 2\int_{1}^{\boldsymbol{\alpha}/g^2} \mathrm{d}u \, \frac{1}{\sqrt{u^2+\boldsymbol{c}^2}} \\ &\leq \frac{2}{\boldsymbol{c}} + 2\left|\log\left(\frac{\boldsymbol{\alpha}}{g^2}\right)\right| \leq \frac{2}{\boldsymbol{c}} + 2|\log\boldsymbol{\alpha}| + 4|\log g| \\ &\leq C(h,l)|\log(g)|, \end{aligned} \quad (5.3.135)$$

for some constant $C(h,l) > 0$ (independent of g). Here, we recall that $\boldsymbol{\alpha}$ and $\boldsymbol{c}$ do not depend on g. Then, (5.3.135) together with (5.3.132) implies that (5.3.130) holds true. This together with (5.3.129) yields that

$$T(h,l) = -4\pi i g^2 \int \mathrm{d}r \, G(r) \frac{\operatorname{Re}\lambda_1 - \lambda_0}{(\lambda_1 - \lambda_0 - r)(\overline{\lambda_1} - \lambda_0 + r)} + \tilde{R}(h,l),, \quad (5.3.136)$$

and there is a constant $C(h,l)$ such that $|\tilde{R}(h,l)| \leq C(h,l) g^2 g |\log g|$. We observe that this completes the proof of the theorem if we drop the factor $\|\Psi_{\lambda_0}\|^{-2}$ in (5.1.8) (see Remark 5.1.4).

We recall that there is a constant C (independent of g) such that $|\overline{\lambda_1} - \lambda_0 + |k'||^{-1} \leq C$ for all $k' \in \mathbb{R}^3$ with $|k'| \geq \kappa$. Then, we obtain from (5.3.132) that there is a constant $C(h,l)$ such that

$$\left| \int \mathrm{d}r \, G(r) \frac{\operatorname{Re}\lambda_1 - \lambda_0}{(\lambda_1 - \lambda_0 - r)(\overline{\lambda_1} - \lambda_0 + r)} \right| \leq C(h,l)|\log g|. \quad (5.3.137)$$

Moreover, it follows from (5.2.25) that $\|\Psi_{\lambda_0}\|^{-2} = 1 + r(g)$ and $|r(g)| \leq Cg$. This together with (5.3.137) and (5.3.136) implies that

$$T(h,l) = -4\pi i \|\Psi_{\lambda_0}\|^{-2} g^2 \int \mathrm{d}r \, G(r) \frac{\operatorname{Re}\lambda_1 - \lambda_0}{(\lambda_1 - \lambda_0 - r)(\overline{\lambda_1} - \lambda_0 + r)} + R(h,l),, \quad (5.3.138)$$

and there is a constant $C(h,l)$ such that $|R(h,l)| \leq C(h,l) g^2 g |\log g|$. □

6. Scattering formula for the massive Spin-Boson model

In this chapter, we analyze the massive Spin-Boson model introduced in Chapter 1.2, where we set the mass of the scalar field $m > 0$ and the infrared regularization parameter $\mu = 0$. Note that this yields the dispersion relation $\omega(k) = \sqrt{k^2 + m^2}$ and the infrared regularization is not necessary anymore. We point out that our proofs, in this chapter, allow for more general boson form factors f than the one defined in (1.2.3). In particular, f has to be spherical symmetric (in order to simplify our notation), satisfy $f, Df, D^2 f \in L^2(\mathbb{R}^3)$, where D is the generator of dilations introduced in Definition 6.4.1 (ii) below, and the condition

$$f(\sqrt{e_1^2 - m^2}) > 0. \tag{6.0.1}$$

Here, we use a slight abuse notation and identify $f(k) \equiv f(|k|)$. This means that f does not have to be analytic and the infrared singularity is not an issue here ($m = 0$). This being said, for concreteness, we consider the particular choice of f defined in (1.2.3) in the remainder of the chapter and observe that it meets all conditions mentioned above.

In addition, we assume the following throughout the present chapter:

Assumption 6.0.1. *We suppose that* $e_1 - e_0 \notin m\mathbb{N}$*. This implies*

$$\delta := \operatorname{dist}(e_1 - e_0, m\mathbb{N}) > 0, \tag{6.0.2}$$

where the symbol dist *stands for the Euclidean distance. Moreover, we assume the mass of the scalar field to be smaller than the energy level* e_1 *in order to allow for scattering processes.*

Speaking in physical terms, this assumption excludes the possibility that a certain number of bosons with zero momentum are able to flip the atom to the excited state.

6.1. Comparison to previous results

In Chapter 5 (c.f. [23, 22]), we derived a formula revealing the relation between the resonance λ_1 and the integral kernel of the scattering matrix was for the case of a massless scalar field. It was proven that the scattering matrix coefficients of one-boson scattering processes, excluding forward scattering, feature the expected Lorenzian shape in leading order in the neighborhood of the real part of the resonance λ_1. More precisely, it was shown in Theorem 5.1.3 that the leading order in the coupling constant g (for small g) of the integral kernel of the transition matrix T fulfills

$$T(k,k') \sim 4\pi i g^2 \|\Psi_{\lambda_0}\|^{-2} f(k)^2 \delta(|k|-|k'|) \frac{\operatorname{Re}\lambda_1 - \lambda_0}{(|k|+\lambda_0-\lambda_1)(|k|-\lambda_0+\overline{\lambda_1})}. \qquad (6.1.1)$$

Here, Ψ_{λ_0} denotes the (due to the construction, unnormalized) ground state corresponding to λ_0 and δ the Dirac delta distribution. Due to the absence of a spectral gap, a subtle study by means of multiscale perturbation analysis was necessary to construct the ground state and resonance and control the required spectral estimates (see Chapter 4 and [21]). The main tool used to control the time-evolution operator in the scattering regime, and hence, the scattering matrix coefficients, was the Laplace transform representation of the unitary time-evolution operator generated by the corresponding Hamiltonian H, i.e.,

$$\left\langle \phi, e^{-itH}\psi \right\rangle = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathbb{R}+i\epsilon} \mathrm{d}z \, e^{-itz} \left\langle \phi, (H-z)^{-1} \psi \right\rangle. \qquad (6.1.2)$$

In order to justify this identity in a rigorous sense, precise control of the resolvent close to the real axis is needed to infer sufficient decay for the integral to converge. For this purpose, the Hamiltonian was studied with the help of a conveniently chosen complex dilation in which it exhibits a spectrum consisting of the ground-state energy λ_0, a resonance λ_1 having negative imaginary part, and the rest of the spectrum being localized in cones in the lower complex plane attached to λ_0 and λ_1, respectively. Thanks to this fact, a well-defined meaning can be given to (6.1.2) by deforming the integration contour $\mathbb{R}+i\epsilon$ at $-\infty$ and $+\infty$ towards the lower complex plane.

In the case of a scalar field with mass $m>0$ as discussed in this chapter, this strategy fails. The reason is that the spectrum of the corresponding dilated unperturbed Hamiltonian contains the points

$$\{e_0 + km\}_{k\in\mathbb{N}_0} \cup \{e_1 + km\}_{k\in\mathbb{N}_0}, \qquad \text{where} \qquad \mathbb{N}_0 := \mathbb{N}\cup\{0\}. \qquad (6.1.3)$$

This leads to an absence of decay of the corresponding complex dilated resolvent close to the real line, which, in [23], was a crucial ingredient to control the time-evolution operator in the scattering regime. Therefore, compared to Chapter 5 (c.f. [23, 22]), a different strategy to control the time-evolution operator has to be developed which is the content of this chapter. As discussed in Section 6.3 below, we use Mourre theory to obtain the required spectral control. In particular, we combine Mourre theory with perturbation theory and the Feshbach-Schur map. In Section 6.3 we compare this approach to the method of complex dilation which was employed in [23, 57].

We point out to the reader that, in general, Mourre theory has been studied in a variety of models (see, e.g., [6, 5, 28, 44]). We emphasize, however, that our application of this theory is non-standard. In the spirit of [4, 36], we prove a "reduced" limiting absorption principle for the unperturbed Hamiltonian at the excited energy e_1 and apply perturbation theory – see Lemma 6.6.3 and Proposition 6.4.8 (iii) below. One of the main achievements of the present chapter (c.f. [19]) is then to combine the obtained limiting absorption principle with a suitable application of the Feshbach-Schur map. Using in addition Fermi's Golden Rule, we then manage to obtain the required control of the time evolution.

The remainder of this chapter is structured as follows: In Section 6.2, we construct the ground-state of the model and in Section 6.3 we present our main result of this chapter, i.e., Theorem 6.3.2. The remaining sections consist of the main technical ingredient given in Section 6.4 and its proof in Section 6.6, the proof of our main result in Section 6.5. We lay out a roadmap for these sections in the end of Section 6.3.

6.2. Ground-state and standard estimates

The existence of a unique ground-state has already been proven in the more complicated situation of a massless scalar field; see e.g. [51] and [21]. For the massive model at stake, it can be shown using regular perturbation theory. However, for the sake of completeness, we provide a detailed proof in the following.

Proposition 6.2.1 (Ground-state)**.** *For $g > 0$ sufficiently small, H has a unique ground state, i.e., $\lambda_0 = \inf \sigma(H)$ is a simple eigenvalue of H. We have*

$$\lambda_0 = e_0 - g^2\Gamma_0 + R_0(g), \qquad \textit{where} \qquad \Gamma_0 := \|f/(e_1 - e_0 + \omega)\|^2, \tag{6.2.1}$$

and there is a constant $C > 0$ such that $|R_0(g)| \leq Cg^4$. Furthermore, denoting by Ψ_{λ_0} the (unnormalized) ground-state, we have that

$$\|\Psi_{\lambda_0} - \varphi_0 \otimes \Omega\| \leq Cg. \tag{6.2.2}$$

The existence of a ground state can be established for any value of g (see [34]). For the sake of simplicity, we only present the proof for sufficiently small $g > 0$ since our main results require this condition anyways.

First of all, for $0 < r < r' < \infty$ and $w \in \mathbb{C}$, we introduce the notation for the open annulus in the complex plane:

$$D(r, r', w) = \{z \in \mathbb{C} : r < |z - w| < R\}. \tag{6.2.3}$$

Lemma 6.2.2. *Let $g > 0$ be small enough and Assumption 6.0.1 hold true. Then, $H - z$ is invertible for all $z \in \overline{D(m/4, m/2, 0)}$ (defined in (6.2.3)) and*

$$\left\|(H - z)^{-1}\right\| \leq 2\left\|(H_0 - z)^{-1}\right\| \leq 8/m \qquad \forall z \in \overline{D(m/4, m/2, 0)}. \tag{6.2.4}$$

Proof. First of all, note that $\sigma(H_0) = \{0\} \cup [m, \infty)$. This implies that

$$\operatorname{dist}\left(D(m/4, m/2, 0), \sigma(H_0)\right) \geq 4/m, \tag{6.2.5}$$

and hence, $H_0 - z$ is invertible for all $z \in D(m/4, m/2, 0)$, and for those z, we have

$$\left\|(H_0 - z)^{-1}\right\| \leq 4/m. \tag{6.2.6}$$

Moreover, it follows from the standard estimate in Proposition 6.2.5 that

$$\left\|V(H_0 + 1)^{-1}\right\| \leq C, \tag{6.2.7}$$

and hence, we obtain for all $z \in D(m/4, m/2, 0)$

$$\left\|V(H_0 - z)^{-1}\right\| \leq \left\|V(H_0 + 1)^{-1}\right\| \left\|\frac{H_0 + 1}{H_0 - z}\right\| \leq C \sup_{y \geq 0} \left|\frac{y+1}{y-z}\right| \leq C(3 + 4/m). \tag{6.2.8}$$

Consequently, for $g > 0$ sufficiently small, we find

$$\left\|V(H_0 - z)^{-1}\right\| \leq Cg \leq 1/2, \tag{6.2.9}$$

and hence,

$$H - z = (1 + gV(H_0 - z)^{-1})(H_0 - z) \tag{6.2.10}$$

is invertible for all $z \in D(m/4, m/2, 0)$ and the resolvent fulfills

$$\left\|(H - z)^{-1}\right\| \leq 2\left\|(H_0 - z)^{-1}\right\| \leq 8/m. \tag{6.2.11}$$

□

Definition 6.2.3. *We define the contour*

$$\zeta : [0, 2\pi] \to \mathbb{C}, \qquad \varphi \mapsto \zeta(t) := m/4e^{it}. \tag{6.2.12}$$

Furthermore, we define the projections

$$P_{0,at} := (-2\pi i)^{-1} \oint_\zeta \mathrm{d}z\, (H_0 - z)^{-1} = P_{\varphi_0} \otimes P_\Omega \tag{6.2.13}$$

and

$$P_0 := (-2\pi i)^{-1} \oint_\zeta \mathrm{d}z\, (H - z)^{-1}. \tag{6.2.14}$$

Here, P_{φ_0} denotes the projection onto φ_0 and P_Ω the projection onto the vacuum $\Omega \in \mathcal{F}[\mathfrak{h}]$. The equality in (6.2.13) *can be seen by a direct calculation.*

Lemma 6.2.4. *Let $g > 0$ be small enough and Assumption 6.0.1 hold true. Then, we find*

$$\|P_0 - P_{0,at}\| \leq gC < 1. \tag{6.2.15}$$

Proof. It follows from Definition 6.2.3 that

$$\begin{aligned}\|P_0 - P_{0,\mathrm{at}}\| &\leq (2\pi)^{-1} \int_0^{2\pi} \mathrm{d}t \left\|(H - m/4e^{it})^{-1} - (H_0 - m/4e^{it})^{-1}\right\| \\ &\leq g \sup_{t\in[0,2\pi]} \left\|(H - m/4e^{it})^{-1}\right\|\left\|V(H_0 - m/4e^{it})^{-1}\right\|, \qquad (6.2.16)\end{aligned}$$

where we used the resolvent identity in the second step. This together with (6.2.9) and Lemma 6.2.2 completes the proof. □

Proof of Proposition 6.2.1. Clearly, $P_{0,\mathrm{at}} = P_{\varphi_0} \otimes P_\Omega$ is a rank-one projection, and hence, it follows from Lemma 6.2.4 that also P_0 is a rank-one projection. Consequently, the self-adjoint operator H has exactly one eigenvalue in $(-m/4, m/4)$ which we call λ_0 and $\Psi_{\lambda_0} := P_0\varphi_0 \otimes \Omega \in \mathcal{H}$ is non-zero and fulfills $H\Psi_{\lambda_0} = \lambda_0\Psi_{\lambda_0}$.

In the remainder of the proof we compute λ_0 up to second order in g.

$$(\lambda_0 - e_0) \left|\langle \phi_0, P_0\phi_0\rangle\right| = \langle \phi_0, (H - H_0)P_0\phi_0\rangle = g \langle \phi_0, VP_0\phi_0\rangle, \qquad (6.2.17)$$

where we have introduced the notation $\phi_i = \varphi_i \otimes \Omega$ for $i = 0, 1$. Moreover, the resolvent identity yields that

$$\begin{aligned}&\left\langle \phi_0, V(H - z)^{-1}\phi_0\right\rangle = \left\langle \phi_0, V(H_0 - z)^{-1}\phi_0\right\rangle - g\left\langle \phi_0, V(H_0 - z)^{-1}V(H_0 - z)^{-1}\phi_0\right\rangle \\ &+ g^2 \left\langle \phi_0, V(H_0 - z)^{-1}V(H_0 - z)^{-1}V(H_0 - z)^{-1}\phi_0\right\rangle \\ &- g^3 \left\langle \phi_0, V(H - z)^{-1}V(H_0 - z)^{-1}V(H_0 - z)^{-1}V(H_0 - z)^{-1}\phi_0\right\rangle. \qquad (6.2.18)\end{aligned}$$

Note that the even orders of g vanish due to symmetry and recall from (6.2.6) that $\|V(H_0 - z)^{-1}\| \leq C$. This implies that

$$\left\|V(H - z)^{-1}\right\| \leq \left\|(H_0 - z)(H - z)^{-1}\right\|\left\|V(H_0 - z)^{-1}\right\| \leq C(1 + gC). \qquad (6.2.19)$$

Consequently, we obtain

$$\begin{aligned}\left\langle \phi_0, V(H - z)^{-1}\phi_0\right\rangle &= -g(e_0 - z)^{-1} \left\langle a(f)^*\phi_1, (H_0 - z)^{-1}a(f)^*\phi_1\right\rangle + \tilde{R}_0(g) \\ &= -g(e_0 - z)^{-1} \int \mathrm{d}^3k \, |f(k)|^2(e_1 + \omega(k) - z)^{-1} + \tilde{R}_0(g), \qquad (6.2.20)\end{aligned}$$

where $|\tilde{R}_0(g)| \leq Cg^3$. Then, it follows from (6.2.17) together with Definition 6.2.3 that

$$\lambda_0 = e_0 - g^2\Gamma_0 + R_0(g) \qquad (6.2.21)$$

where $R_0(g) = g \left|\langle \phi_0, P_0\phi_0\rangle\right|^{-1} \tilde{R}_0(g)$ and

$$\Gamma_0 := (-2\pi i)^{-1} \oint_\zeta \mathrm{d}z \, (e_0 - z)^{-1} \int \mathrm{d}^3k \, |f(k)|^2(e_1 + \omega(k) - z)^{-1}. \qquad (6.2.22)$$

Fubini's theorem allows for interchanging the order of integration, and hence, we obtain from the Cauchy integral theorem

$$\Gamma_0 = \int \mathrm{d}^3k \, |f(k)|^2 (e_1 - e_0 + \omega(k))^{-1}. \tag{6.2.23}$$

We point out to the reader that this integral is non-singular because of Assumption 6.0.1. This completes the proof of the first part of the proposition. The second part follows from the definition of the ground state:

$$\Psi_{\lambda_0} = P_0 \varphi_0 \otimes \Omega = \varphi_0 \otimes \Omega + \widetilde{\Psi}_{\lambda_0}, \tag{6.2.24}$$

where $\widetilde{\Psi}_{\lambda_0} := (P_0 - P_{0,\mathrm{at}})\varphi_0 \otimes \Omega \in \mathcal{H}$ and Lemma 6.2.4 yields that $\left\| \widetilde{\Psi}_{\lambda_0} \right\| \leq Cg$. Moreover, note that $\varphi_0 \otimes \Omega$ is the unique ground state of H_0, and hence, $P_{0,\mathrm{at}}$ is a rank-one projector. We conclude the uniqueness of Ψ_{λ_0} again from Lemma 6.2.4. □

In the following we recall important properties and estimates for the model at stake. Note that the statements are direct consequences of Proposition 1.3.3, Lemma 1.3.2 and 1.3.1 (see, e.g., [23] and [51], see also [39, Lemma 21]).

Proposition 6.2.5. *For every $h \in \mathfrak{h}$ and $a(h)^\# \in \{a(h)^*, a(h)\}$,*

$$\left\| a(h)^\# (H_f + 1)^{-\frac{1}{2}} \right\| \leq C \|h\|_2, \tag{6.2.25}$$

where C is a positive constant. This implies that gV is infinitesimally bounded with respect to H_0 and, consequently, H is self-adjoint and bounded from below, on the domain

$$\mathcal{D}(H) = D(H_0) = \mathcal{D}(\mathbb{1}_K \otimes H_f), \tag{6.2.26}$$

and the operators

$$H_f (H + i)^{-1}, \qquad H(H_f + 1)^{-1} \tag{6.2.27}$$

are bounded.

6.3. Main result of this chapter – and comparison to results in the massless model

We now come to our main result of this chapter, Theorem 6.3.2 below, which makes precise the relation between the scattering matrix kernel and the resonance in the massive Spin-Boson model.

At first, we recall Definition 5.1.2:

Definition 6.3.1. *Using the notation $d^3x \equiv d\Sigma r^2 dr$ for solid angles Σ and radius r in spherical coordinates, we recall, for all $h, l \in \mathfrak{h}_0$,*

$$G_{h,l} : \mathbb{R} \to \mathbb{C}, \qquad r \mapsto G_{h,l}(r) := \begin{cases} \int d\Sigma d\Sigma' \, r^4 \overline{h(r,\Sigma)} l(r,\Sigma') f(r)^2 & \text{for} \quad r \geq 0 \\ 0 & \text{for} \quad r < 0 \end{cases}. \tag{6.3.1}$$

In the proofs below we will drop the indices h, l and write $G_{h,l} \equiv G$.

Theorem 6.3.2. *Suppose that Assumption 6.0.1 holds. There exists a complex number* Γ_{-0} *with* $\operatorname{Im}\Gamma_{-0} > 0$ *such that for all* $h, l \in \mathfrak{h}_0$ *and* $g > 0$ *sufficiently small, the transition matrix coefficients* (2.1.6) *are given by*

$$T(h,l) = T_P(h,l) + R(h,l), \tag{6.3.2}$$

where

$$T_P(h,l) := 4\pi i g^2 \|\Psi_{\lambda_0}\|^{-2} \int \mathrm{d}r \, \frac{G_{h,l}(r)\,(e_1 - g^2 \operatorname{Re}\Gamma_{-0} - \lambda_0)}{(\omega(r) + \lambda_0 - (e_1 - g^2\Gamma_{-0}))\left(\omega(r) - \lambda_0 + (e_1 - g^2\overline{\Gamma_{-0}})\right)}, \tag{6.3.3}$$

and there is a constant $C(h,l) > 0$ *such that*

$$|R(h,l)| \leq C(h,l) g^2 g^{1/3} |\log(g)|. \tag{6.3.4}$$

In (6.4.50) *below we give an explicit expression of* Γ_{-0}.

$T_P(h,l)$ *is the leading term in terms of powers of* g *for small* g, *and* $R(h,l)$ *is regarded as the error term. This is justified by Remark 5.1.5.*

Not surprisingly, it turns out that

$$\widetilde{\lambda}_1 := e_1 - g^2\Gamma_{-0} \tag{6.3.5}$$

is the leading term of the resonance, up to order g^2. This connection can be made by the standard construction of the resonance by means of complex dilation. This computation is not carried out here since we wanted to focus on the methods of Mourre theory rather than complex dilation; see, e.g., [10] for such a construction for massless fields using the method of complex dilation. Note that, in our situation, the construction is much easier since the dilated Hamiltonian exhibits spectral gaps. For treating resonances within the realm of Mourre theory we refer to [54, 55, 27, 36].

In order to compare this formula with the massless case, see (6.1.1), we may rewrite (6.3.3) in integral kernel form which takes the form

$$T(k,k') \sim 4\pi i g^2 \|\Psi_{\lambda_0}\|^{-2} f(k)^2 \, \frac{|k|\delta(\omega(k) - \omega(k'))}{\omega(k)} \, \frac{\operatorname{Re}\widetilde{\lambda}_1 - \lambda_0}{(|k| + \lambda_0 - \widetilde{\lambda}_1)(|k| - \lambda_0 + \overline{\widetilde{\lambda}_1})}. \tag{6.3.6}$$

There are only two differences in the formulas (6.3.6) and (6.1.1). One is due to the different dispersion relations $\omega(k) = \sqrt{|k|^2 + m^2}$ and $\omega(k) = |k|$ for the massive and massless case, respectively, and the other due to the fact that, in (6.1.1), λ_1 figures the non-perturbative resonance while, in (6.3.6), the entity $\widetilde{\lambda}_1$ is only the second order perturbation in g for small g as explained above. However, the latter difference is not relevant as the rest term $R(h,l)$ in both cases is of order $g^2\, g^{1/3} |\log g|$, and thus, will swallow this difference anyway.

The difference in the order in g of the given estimates of the rest terms $R(h,l)$ between the massive, i.e., $g^2\, g^{1/3}|\log g|$ in Theorem 6.3.2, and the massless case, i.e., $g^2\, g|\log g|$ in Theorem 5.1.3 (c.f. [23, Theorem 2.2]), is solely due to the different techniques which were employed. While in the present chapter the required spectral information is inferred by Mourre theory, in Chapter 5 the method of complex dilation was used. If a fair comparison of both techniques is possible at all, from our experience, it turns out that Mourre theory requires less information about the model, especially, no analyticity properties, to start with, however, gives a little more imprecise estimates of the remainders. In turn, the method of complex dilation is based on these analyticity properties but, given this information, one is able to produce slightly better estimates on the remainders. Since the model features a scalar interaction, the physical perturbation processes only differ for even orders in g. Hence, the different estimates of the remainders inferred by our application of Mourre theory and the method of complex dilations can be expected to be physically insignificant. Furthermore, also technically, there seems to be room for improvement.

Compared to our previous derivation of the transition matrix formula (6.1.1), see Chapter 5 and [23, 22], for the massless Spin-Boson model, there are two main innovations in the strategy of proof. First, as already explained, we do not rely on complex dilations anymore but instead use Mourre theory to infer the required spectral information. And second, as mentioned already in the introduction, we handle the problem caused by the nature of the spectrum of the free dilated Hamiltonian (see (6.1.3)), which is a complication due to non-zero boson mass. In previous works [23] and [21], complex dilations were used both for the construction of the resonance as well as the control of required spectral properties, in particular, the estimates on the relevant resolvents.

The main technical import for the proof of Theorem 6.3.2 is contained in the next Section 6.4.1. There, we provide a central Mourre estimate in Lemma 6.4.7 which implies the limiting absorption principle in Proposition 6.4.8. The latter is employed in Section 6.4.2 in a new way, in a combination with the Feshbach-Schur map, to control the time-evolution operator in the scattering regime, and hence, the transition matrix coefficient under investigation. In Section 6.6 we provide a proof of the limiting absorption principle, i.e., Proposition 6.4.8, which in parts is a self-contained review of results in the literature but also provides a non-standard result, see (6.4.37), which allows to conveniently apply the limiting absorption principle in the context of perturbation theory. In Sections 6.7, 6.8 and 6.9 we collect proofs of several other technical and in most parts well-known auxiliary results which were used for the sake of self-containedness.

Remark 6.3.3. *In the remainder of this chapter we denote by C any generic, positive (indeterminate) constant which may change from line to line in the computations but does not depend on g and the parameters $z, z', \epsilon, \eta, \beta$ introduced below.*

6.4. Technical ingredients

In this section we derive a formula for the leading order term with respect to the coupling constant of a certain matrix element of the time-evolution operator and estimate the error term. We rely on two main ingredients, namely, a limiting absorption principle derived from a Mourre estimate and a Feshbach-Schur map. In the first part, Section 6.4.1, we introduce some notation and prove technical lemmas and a Mourre estimate which allows to derive a limiting absorption principle. The latter is also stated in this section since we use it as a key tool in order to prove our main result. Although some of these results are standard and have been proven for large classes of Hamiltonians, we provide a full detailed proof in Section 6.6. In the second part, Section 6.4.2, we introduce a Feshbach-Schur map and combine it with the limiting absorption principle in order to control a certain matrix element of the time-evolution operator.

6.4.1. Limiting absorption principle

In this section we present the limiting absorption principle based on a Mourre estimate for the model at stake. We follow the construction of [28], see also [51, 36, 44]. We start with introducing some notation.

Definition 6.4.1. *Recall that $\mathfrak{h}_0$ has been defined in* (2.1.1).

(i) For any self-adjoint operator O, we define $\mathrm{d}\Gamma(O)$ as the generator of the unitary one-parameter group $\left\{\Gamma(e^{-itO})\right\}_{t\in\mathbb{R}}$, where

$$\Gamma(e^{-itO}) := \bigoplus_{n=0}^{\infty}(e^{-itO})^{\odot n}, \qquad (e^{-itO})^{\odot 0} := 1. \tag{6.4.1}$$

It follows from Stone's theorem that $\mathrm{d}\Gamma(O)$ is self-adjoint. Note that $H_f = \mathrm{d}\Gamma(\omega)$.

(ii) For $\beta \in \mathbb{R}$, we define the unitary dilation operator

$$u_\beta : \mathfrak{h} \to \mathfrak{h}, \qquad \varphi(k) \mapsto \varphi_\beta(k) := e^{\frac{3}{2}\beta}\varphi(e^\beta k), \qquad \forall k \in \mathbb{R}^3. \tag{6.4.2}$$

We denote by D generator of dilations, which is the generator of the unitary one-parameter group $\{u_\beta\}_{\beta\in\mathbb{R}}$. Note that D is self-adjoint on $\mathcal{D}(D) \subset \mathcal{H}$ due to Stone's theorem.

Moreover, for $\varphi \in \mathfrak{h}_0$ and $\beta \in \mathbb{R}$, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\varphi_\beta(k) = \frac{1}{2}\left(\nabla_k \cdot k + k \cdot \nabla_k\right)\varphi_\beta(k), \qquad k \in \mathbb{R}^3. \tag{6.4.3}$$

This implies that the action of D on $\mathfrak{h}_0$ is given by $\frac{i}{2}(k \cdot \nabla_k + \nabla_k \cdot k)$.

(iii) We introduce the function

$$\xi : \mathbb{R}^3 \to \mathbb{R}, \qquad k \mapsto \xi(k) := k^2/\omega(k). \tag{6.4.4}$$

(iv) *We set*

$$\mathcal{H}_0 := \mathcal{K} \otimes \mathcal{F}_{fin}[\mathfrak{h}_0], \tag{6.4.5}$$

where

$$\mathcal{F}_{fin}[\mathfrak{h}_0] := \Big\{ \Psi = (\psi^{(n)})_{n\in\mathbb{N}_0} \in \mathcal{F}[\mathfrak{h}] \,|\, \exists N \in \mathbb{N}_0 : \psi^{(n)} = 0 \, \forall n \geq N, \\ \forall n \in \mathbb{N} : \psi^{(n)} \in C_c^\infty(\mathbb{R}^{3n} \setminus \{0\}, \mathbb{C}) \Big\}. \tag{6.4.6}$$

(v) *Moreover, for every closed operator A, we denote by*

$$\|\cdot\|_A := \left(\|A\cdot\|^2 + \|\cdot\|^2 \right)^{1/2}, \tag{6.4.7}$$

its graph norm in the domain of A.

Remark 6.4.2. *Note that $\mathcal{H}_0$ and $\mathcal{F}_{fin}[\mathfrak{h}_0]$ are dense subsets of the domains of H and H_f with respect to the graph norm of H and H_f, respectively. In other words, $\mathcal{H}_0$ and $\mathcal{F}_{fin}[\mathfrak{h}_0]$ are cores of H and H_f, respectively.*

The following statement is a collection of general properties of the objects introduced in Definitions 6.4.1, which we will use in the remainder of this chapter.

Lemma 6.4.3. *The following properties hold true:*

(i) $\mathcal{F}_{fin}[\mathfrak{h}_0] \subset \mathcal{D}(H_f) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$.

(ii) $\mathcal{D}(H_f) \subset \mathcal{D}(\Phi(Df))$ *and* $\Phi(Df)(H_f+1)^{-\frac{1}{2}}$ *is bounded (recall the definition of* $\Phi(f)$ *in* (1.2.2)*).*

(iii) $\mathcal{D}(H_f) \subset \mathcal{D}(\mathrm{d}\Gamma(\xi))$ *and* $\mathrm{d}\Gamma(\xi)(H_f+1)^{-1}$ *is bounded.*

(iv) *The operator $[H_f, i\mathrm{d}\Gamma(D)]$ defined as a quadratic form on $\mathcal{D}(H_f) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$ can be uniquely extended to a H-bounded operator on $\mathcal{D}(H) = \mathcal{D}(H_0)$ denoted by $[H_f, i\mathrm{d}\Gamma(D)]^0$. We have the identity:*

$$[H_f, i\mathrm{d}\Gamma(D)]^0 = \mathrm{d}\Gamma(\xi) \tag{6.4.8}$$

on $\mathcal{D}(H_0)$.

(v) *The operator $[\Phi(f), i\mathrm{d}\Gamma(D)]$ defined as a quadratic form on $\mathcal{D}(H_f) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$ can be uniquely extended to a H-bounded operator on $\mathcal{D}(H) = \mathcal{D}(H_0)$ denoted by $[\Phi(f), i\mathrm{d}\Gamma(D)]^0$. We have the identity:*

$$[\Phi(f), i\mathrm{d}\Gamma(D)]^0 = \Phi(Df) \tag{6.4.9}$$

on $\mathcal{D}(H_0)$.

Proof. (i) Clearly, this holds by Definition 6.4.1.

(ii) A direct calculation shows that $Df \in \mathfrak{h}$. We conclude the claim by Proposition 6.2.5.

(iii) Note that, for all $k \in \mathbb{R}^3$, $\xi(k) = \frac{k^2}{\omega(k)} = \omega(k)\frac{k^2}{k^2+m^2} \leq \omega(k)$. This directly implies the desired result.

(iv) Clearly, $[H_f, i\mathrm{d}\Gamma(D)]$ can be defined as a quadratic form on $\mathcal{D}(H_f) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$, and hence, it follows from (i) that, for $\psi \in \mathcal{F}_{\text{fin}}[\mathfrak{h}_0]$, we have

$$\langle \psi, [H_f, i\mathrm{d}\Gamma(D)]\psi\rangle = \langle \psi, [\mathrm{d}\Gamma(\omega), i\mathrm{d}\Gamma(D)]\psi\rangle = \langle \psi, \mathrm{d}\Gamma([\omega, iD])\psi\rangle\,. \tag{6.4.10}$$

Moreover, it follows from a direct calculation that

$$[\omega, iD] = \xi, \tag{6.4.11}$$

on $\mathfrak{h}_0$, and hence,

$$\langle \psi, [H_f, i\mathrm{d}\Gamma(D)]\psi\rangle = \langle \psi, \mathrm{d}\Gamma(\xi)\psi\rangle \qquad \forall \psi \in \mathcal{F}_{\text{fin}}[\mathfrak{h}_0]. \tag{6.4.12}$$

Note that $\mathcal{F}_{\text{fin}}[\mathfrak{h}_0]$ is a core of H_f. This together with (6.4.12) and (iii) implies that $[H_f, i\mathrm{d}\Gamma(D)]$ uniquely extends to an H_0-bounded (and H-bounded) operator on $\mathcal{D}(H) = \mathcal{D}(H_0)$ denoted by $[H_f, i\mathrm{d}\Gamma(D)]^0$.

(v) This statement follows similarly as (iv) while using (ii) instead of (iii) in the last step.

□

For the proof of our main result it suffices to control the time-evolution operator only on a spectral subset close to the excited state. In the following we define a cut-off function with its support localized in such a subset. Recall that $\delta > 0$ has been defined in Assumption 6.0.1.

Definition 6.4.4. *We fix $\chi \in C_c^\infty(\mathbb{R}, [0,1])$ such that supp $\chi \subset (e_1 - 3\delta/4, e_1 + 3\delta/4)$ and $\chi\big|_{[e_1-\delta/2, e_1+\delta/2]} = 1$. Moreover, for $0 < \kappa < 2$ and $g^2 \leq s \leq g^\kappa$, we define χ_s by $\chi_s(r) := \chi(e_1 + (r - e_1)/s)$ for all $r \in \mathbb{R}$.*

The following formula is well-known and can be shown using operator calculus (see [14, 52]). For the sake of completeness, we present a detailed proof in section 6.7 below.

Lemma 6.4.5. *For every $v \in C_c^\infty(\mathbb{R}, [0,1])$, there is a constant $C_v > 0$ such that*

$$\|v(H) - v(H_0)\| \leq gC_v. \tag{6.4.13}$$

For every $s > 0$ in a compact set there is a constant C that depends on this set such that

$$\|\chi_s(H) - \chi_s(H_0)\| \leq Cs^{-1}g. \tag{6.4.14}$$

In the following we derive a positive commutator estimate close to the unperturbed eigenvalue e_1. For this purpose, we set (see (1.3.15))

$$H_{\overline{P}} := \overline{P}H\overline{P}, \quad H_{0,\overline{P}} := \overline{P}H_0\overline{P}, \quad H_{f,\overline{P}} := \overline{P}H_f\overline{P}, \quad V_{\overline{P}} := \overline{P}V\overline{P}, \quad \Phi_{\overline{P}}(f) := \overline{P}\Phi(f)\overline{P}, \tag{6.4.15}$$

where, taking P_{φ_1} and P_Ω the orthogonal projections on the spans of φ_1 and Ω, respectively, we define

$$P := P_{\varphi_1} \otimes P_\Omega, \qquad \overline{P} = \mathbb{1}_{\mathcal{H}} - P. \tag{6.4.16}$$

Remark 6.4.6. *It follows from Lemma 6.4.3 that operator $[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]$, defined as a quadratic form on $\mathcal{D}(H_f) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$, can be uniquely extended to a $H_{\overline{P}}$-bounded operator on $\mathcal{D}(H_{\overline{P}})$. We denote this extension by*

$$[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 = H_{\overline{P}}(\xi, Df). \tag{6.4.17}$$

Lemma 6.4.7 (Mourre estimate)**.** *There is a constant $\alpha > 0$ such that, for sufficiently small $g > 0$,*

$$\chi(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}}) \geq \alpha\chi(H_{\overline{P}})^2, \tag{6.4.18}$$

where we recall Definition 6.4.4.

Proof. We take a fixed function $\upsilon \in C_c^\infty\Big((e_1 - \frac{9}{10}\delta, e_1 + \frac{9}{10}\delta), [0,1]\Big)$ with $\chi\upsilon = \chi$ (since this is fixed, we identify $C \equiv C_\upsilon$ in the constants below).

Note that $\mathrm{d}\Gamma(D)$ commutes with $\overline{P} = \mathbb{1}_{\mathcal{H}} - P$. Then, Lemma 6.4.3 (iv) and (v) yields

$$\upsilon(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\upsilon(H_{\overline{P}}) = \upsilon(H_{\overline{P}})\overline{P}\mathrm{d}\Gamma(\xi)\overline{P}\upsilon(H_{\overline{P}}) + g\upsilon(H_{\overline{P}})\overline{P}\sigma_1 \otimes \Phi(Df)\overline{P}\upsilon(H_{\overline{P}}). \tag{6.4.19}$$

It follows from Lemma 6.4.3 (ii) that $\upsilon(H_{\overline{P}})\overline{P}\sigma_1 \otimes \Phi(Df)(H_{0,\overline{P}}+i)^{-1}\overline{P}(H_{0,\overline{P}}+i)\upsilon(H_{\overline{P}})$ is bounded (notice that $(H_{0,\overline{P}}+i)\upsilon(H_{\overline{P}}) = (H_{0,\overline{P}}+i)(H_{\overline{P}}+i)^{-1}(H_{\overline{P}}+i)\upsilon(H_{\overline{P}})$ is bounded, which follows from our estimates in Lemmas 1.3.1 and 1.3.2). Then, we obtain

$$\Big\| g\upsilon(H_{\overline{P}})\overline{P}\sigma_1 \otimes \Phi(Df)\overline{P}\upsilon(H_{\overline{P}}) \Big\| \leq Cg. \tag{6.4.20}$$

Similarly as above, we argue that $\upsilon(H_{\overline{P}})\overline{P}\mathrm{d}\Gamma(\xi)$ and $\mathrm{d}\Gamma(\xi)\overline{P}\upsilon(H_{\overline{P}})$ are bounded, using Lemma 6.4.3 (iii). Then, Lemma 6.4.5 implies that

$$\upsilon(H_{\overline{P}})\overline{P}\mathrm{d}\Gamma(\xi)\overline{P}\upsilon(H_{\overline{P}}) \geq \upsilon(H_{0,\overline{P}})\overline{P}\mathrm{d}\Gamma(\xi)\overline{P}\upsilon(H_{0,\overline{P}}) - gC. \tag{6.4.21}$$

Plugging (6.4.21) and (6.4.20) into (6.4.19) yields that

$$\upsilon(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\upsilon(H_{\overline{P}}) \geq \upsilon(H_{0,\overline{P}})\overline{P}\mathrm{d}\Gamma(\xi)\overline{P}\upsilon(H_{0,\overline{P}}) - gC. \tag{6.4.22}$$

Set $\ell \in \mathbb{N} \cup \{0\}$ be such that

$$e_1 > \ell m \qquad\qquad e_1 < (\ell+1)m. \tag{6.4.23}$$

Notice that Assumption 6.0.1 implies that

$$|e_1 - \ell m| \geq \delta, \tag{6.4.24}$$

and since $\upsilon \in C_c^\infty\Big((e_1 - \frac{9}{10}\delta, e_1 + \frac{9}{10}\delta), [0,1]\Big)$,

$$\upsilon(H_{0,\overline{P}}) H_{0,\overline{P}} \upsilon(H_{0,\overline{P}}) \geq \Big(\ell m + \frac{1}{10}\delta\Big)\upsilon(H_{0,\overline{P}})^2. \tag{6.4.25}$$

For any self-adjoint operator O, we denote by E_O its resolution of the identity. It follows that

$$E_{H_{0,\overline{P}}}(U) = \begin{cases} \overline{P}E_{H_0}(U), & \text{if } 0 \notin U, \\ P + \overline{P}E_{H_0}(U), & \text{if } 0 \in U. \end{cases} \tag{6.4.26}$$

This is a consequence of the fact that the formula in the right hand side of the equation above defines a resolution of the identity and the integral of the identity function with respect to it equals $H_{0,\overline{P}}$ (notice that P commutes with $E_{H_0}(U)$). Since 0 does not belong to the support of υ, it follows that

$$\upsilon(H_{0,\overline{P}}) = \upsilon(H_0)\overline{P} = \overline{P}\upsilon(H_0)\overline{P}. \tag{6.4.27}$$

Set $\mathcal{N} = \mathrm{d}\Gamma(1)$ the number operator. Since $\omega(k) \geq m$, it follows that $\mathbb{1}_{\mathcal{N} > \ell} H_0 \geq (\ell+1)m$, and therefore (notice that $\mathcal{N}$ commutes with $H_{0,\overline{P}}$ and P and recall (6.4.27)),

$$m\upsilon(H_{0,\overline{P}})^2\mathcal{N} = m\upsilon(H_{0,\overline{P}})^2\mathbb{1}_{\mathcal{N}\leq\ell}\mathcal{N} \leq m\ell\upsilon(H_{0,\overline{P}})^2. \tag{6.4.28}$$

Eqs. (6.4.25) and (6.4.28) imply that

$$\upsilon(H_{0,\overline{P}})\Big(H_{0,\overline{P}} - m\mathcal{N}\Big)\upsilon(H_{0,\overline{P}}) \geq \frac{1}{10}\delta\upsilon(H_{0,\overline{P}})^2. \tag{6.4.29}$$

Since $\xi(k) = \frac{k^2+m^2-m^2}{\omega(k)} = \omega(k) - \frac{m^2}{\omega(k)} \geq \omega(k) - m$, we get that

$$\mathrm{d}\Gamma(\xi) \geq H_{0,\overline{P}} - m\mathcal{N}. \tag{6.4.30}$$

Eqs. (6.4.29) and (6.4.30) imply that

$$\upsilon(H_{0,\overline{P}})\mathrm{d}\Gamma(\xi)\upsilon(H_{0,\overline{P}}) \geq \frac{1}{10}\delta\upsilon(H_{0,\overline{P}})^2. \tag{6.4.31}$$

This together with Lemma 6.4.5 and (6.4.22) lead us to (see also (6.4.27))

$$\upsilon(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\upsilon(H_{\overline{P}}) \geq \frac{1}{10}\delta\upsilon(H_{\overline{P}})^2 - gC. \tag{6.4.32}$$

We multiply by $\chi(H_{\overline{P}})$ from the left and the right and use that $\chi\upsilon = \chi$ to obtain

$$\chi(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}}) \geq \frac{1}{10}\delta\chi(H_{\overline{P}})^2 - gC\chi(H_{\overline{P}})^2. \tag{6.4.33}$$

Our desired result follows from (6.4.33), taking small enough g. □

Proposition 6.4.8 (Limiting absorption principle)**.** *We introduce the notation*

$$\langle d\Gamma(D)\rangle := \left((d\Gamma(D))^2+1\right)^{1/2}. \tag{6.4.34}$$

For sufficiently small $g>0$, $\epsilon\in(0,1)$ *and* $z,z'\in[e_1-\delta/4,e_1+\delta/4]$ *we have*

(i) $\sigma_{pp}(H_{\overline{P}})\cap[e_1-\delta/4,e_1+\delta/4]=\emptyset$, *where* $\sigma_{pp}(H_{\overline{P}})$ *denotes the pure point spectrum of* $H_{\overline{P}}$.

(ii)

$$\left\|\langle d\Gamma(D)\rangle^{-1}(H_{\overline{P}}-z\pm i\epsilon)^{-1}\langle d\Gamma(D)\rangle^{-1}\right\|\le C, \tag{6.4.35}$$

and

$$\left\|\langle d\Gamma(D)\rangle^{-1}(H_{0,\overline{P}}-z\pm i\epsilon)^{-1}\langle d\Gamma(D)\rangle^{-1}\right\|\le C, \tag{6.4.36}$$

(iii)

$$\left\|\langle d\Gamma(D)\rangle^{-1}\left((H_{\overline{P}}-z\pm i\epsilon)^{-1}-(H_{0,\overline{P}}-z'\pm i\epsilon)^{-1}\right)\langle d\Gamma(D)\rangle^{-1}\right\|\le C\left(g^{1/2}+|z-z'|^{1/2}\right). \tag{6.4.37}$$

We recall that the constants above do not depend on ϵ, z, z' *and* g *(c.f. Remark 6.3.3).*

For the convenience of the reader, we provide a proof of statements (ii) and (iii) in Section 6.6 - following [28]. Note that statement (iii) is not standard, similar results are addressed in [36]. Their work also draws from [4]. However, we present no proof for statement (i) since this is not used in the remainder of this chapter and it is a standard result.

6.4.2. Resonance and time-evolution operator

In this section we introduce a Feshbach-Schur map, c.f. [14], in order to derive a formula for the resolvent restricted to a spectral subset. This together with the limiting absorption principle obtained in Proposition 6.4.8 allows then for controlling the leading order term of a certain matrix elements of the time-evolution operator (with respect to the coupling constant) and estimate the error term in Lemma 6.4.14 below.

Definition 6.4.9. *We recall* (6.4.15)–(6.4.16)*. For all* $z\in\mathbb{C}\setminus\sigma(H)$, *we define*

$$F_P(z)\equiv F_P(H-z):=P(H-z)P-g^2PV\overline{P}(H_{\overline{P}}-z)^{-1}\overline{P}VP, \tag{6.4.38}$$

as an operator on the range of P.

The following lemma is an application of the limiting absorption principle derived in Proposition 6.4.8 and allows for the control of certain term of the Feshbach-Schur map introduced in Definition 6.4.9.

Lemma 6.4.10. *For sufficiently small g and every $z \in [e_1-\delta/4, e_1+\delta/4]$ and $\epsilon \in (0,1)$, the following estimates hold true:*

(i)

$$\left\| PV\overline{P}(H_{\overline{P}} - z \pm i\epsilon)^{-1}\overline{P}VP \right\| \leq C. \tag{6.4.39}$$

(ii)

$$\left\| PV\overline{P}(H_{0,\overline{P}} - z \pm i\epsilon)^{-1}\overline{P}VP \right\| \leq C. \tag{6.4.40}$$

(iii) if $|z - e_1| \leq r$,

$$\left\| PV\overline{P}\left((H_{0,\overline{P}} - e_1 \pm i\epsilon)^{-1} - (H_{\overline{P}} - z \pm i\epsilon)^{-1} \right) \overline{P}VP \right\| \leq C(g^{1/2} + r^{1/2}). \tag{6.4.41}$$

We recall that the constants C do not depend on ϵ, z and g (c.f. Remark 6.3.3).

Proof. We take $z \in [e_1 - \delta/4, e_1 + \delta/4]$ and $\epsilon \in (0,1)$. Note that $\mathrm{d}\Gamma(D)$ commutes with P. Then, it follows from Lemma 6.4.3 (v) together with $\mathrm{d}\Gamma(D)P = 0$ that $\mathrm{d}\Gamma(D)\overline{P}VP = i\overline{P}\sigma_1 \otimes a(Df)^*P$, and consequently,

$$\left\| \mathrm{d}\Gamma(D)\overline{P}VP \right\| \leq \|a(Df)^*\Omega\| = \|Df\|. \tag{6.4.42}$$

Moreover, we similarly obtain

$$\left\| \overline{P}VP \right\| \leq C. \tag{6.4.43}$$

We recall the definition of $\langle \mathrm{d}\Gamma(D) \rangle$ in (6.4.34) and observe

$$\left\| \langle \mathrm{d}\Gamma(D) \rangle \overline{P}VP \right\|^2 = \sup_{\Psi \in \mathcal{H}, \|\Psi\|=1} \left\langle \overline{P}VP\Psi, \langle \mathrm{d}\Gamma(D) \rangle^2 \overline{P}VP\Psi \right\rangle \tag{6.4.44}$$
$$= \sup_{\Psi \in \mathcal{H}, \|\Psi\|=1} \left\langle \overline{P}VP\Psi, \left(\mathrm{d}\Gamma(D)^2 + 1 \right) \overline{P}VP\Psi \right\rangle \leq \left\| \mathrm{d}\Gamma(D)\overline{P}VP \right\|^2 + \left\| \overline{P}VP \right\|^2.$$

This together with (6.4.42) and (6.4.43) implies that

$$\left\| \langle \mathrm{d}\Gamma(D) \rangle \overline{P}VP \right\| \leq C, \tag{6.4.45}$$

and hence, $\langle \mathrm{d}\Gamma(D) \rangle \overline{P}VP$ is a bounded operator on $\mathcal{H}$. Then, it follows that also its adjoint is a bounded operator. We obtain that

$$\left\| PV\overline{P}(H_{\overline{P}} - z \pm i\epsilon)^{-1}\overline{P}VP \right\| \leq C \left\| \langle \mathrm{d}\Gamma(D) \rangle^{-1} (H_{\overline{P}} - z \pm i\epsilon)^{-1} \langle \mathrm{d}\Gamma(D) \rangle^{-1} \right\|. \tag{6.4.46}$$

We conclude statement (i) by Proposition 6.4.8 (ii). Statements (ii) and (iii) follow similarly from Proposition 6.4.8 (ii) and (iii). □

Next, we derive an explicit formula for the leading order of the Feshbach-Schur map with respect to the coupling constant. This allows then for an easy approximation of the resolvent restricted on a certain subset in Corollary 6.4.12 below.

Lemma 6.4.11. *For sufficiently small $r, g > 0$, $\epsilon \in (0,1)$ and $z \in \mathbb{R}$ with $|z - e_1| \le r$, we have*

$$F_P(H - z \pm i\epsilon) = (e_1 - z - g^2\Gamma_{\pm\epsilon} \pm i\epsilon)P + R_\epsilon(g, r), \tag{6.4.47}$$

where $\|R_\epsilon(g, r)\| \le Cg^2(g^{1/2} + r^{1/2})$ and

$$\Gamma_{\pm\epsilon} := \int \mathrm{d}^3k \, \frac{f(k)^2}{\omega(k) - e_1 \pm i\epsilon}. \tag{6.4.48}$$

Moreover, recalling $m - e_1 < 0$, we observe that the limits

$$\lim_{\epsilon\to 0} \Gamma_{\pm\epsilon} := \Gamma_{\pm 0} \tag{6.4.49}$$

exist (note that $\Gamma_{\pm\epsilon}$ does not depend on g, r and z) and they are given by

$$\Gamma_{\pm 0} = \mp\pi i\theta(0) + \mathcal{P}\int_{m-e_1}^{\infty} \theta(x)/x dx, \tag{6.4.50}$$

where, for $\tau > m - e_1$, we define

$$\theta(\tau) := 4\pi(e_1 + \tau)((e_1 + \tau)^2 - m^2)^{1/2} f(((e_1 + \tau)^2 - m^2)^{1/2})^2. \tag{6.4.51}$$

Note that $\theta(0) > 0$ and hence (see (6.0.1))

$$\operatorname{Im}\Gamma_{\pm 0} = \mp\pi\theta(0) \neq 0. \tag{6.4.52}$$

Proof. Note that $PVP = 0$ and $PH_0P = e_1P$. We take $\epsilon \in (0,1)$ and $z \in \mathbb{R}$ with $|z - e_1| \le r$. We obtain from Definition 6.4.9 that

$$F_P(H - z \pm i\epsilon) = (e_1 - z \pm i\epsilon)P - g^2\hat{\Gamma}_{\pm\epsilon} + R_\epsilon(g), \tag{6.4.53}$$

where

$$\hat{\Gamma}_{\pm\epsilon}P := PV\overline{P}(H_{0,\overline{P}} - e_1 \pm i\epsilon)^{-1}\overline{P}VP \tag{6.4.54}$$

and

$$R_\epsilon(g) = g^2PV\overline{P}\left((H_{0,\overline{P}} - e_1 \pm i\epsilon)^{-1} - (H_{\overline{P}} - z \pm i\epsilon)^{-1}\right)\overline{P}VP. \tag{6.4.55}$$

For $\kappa > 0$ and sufficiently small $g, r > 0$, Lemma 6.4.10 (iii) implies that $\|R_\epsilon(g)\| \le Cg^2(g^{1/2} + r^{1/2})$. We define $\widetilde{f}_\pm(k) = \frac{f(k)}{e_0 + \omega(k) - e_1 \pm i\epsilon}$ and calculate

$$\begin{aligned}\hat{\Gamma}_{\pm\epsilon}P =& PV\overline{P}(H_{0,\overline{P}} - e_1 \pm i\epsilon)^{-1}\overline{P}\sigma_1 \otimes a(f)^*P = PV\overline{P}(H_{0,\overline{P}} - e_1 \pm i\epsilon)^{-1}\varphi_0 \otimes f \\ =& PV\overline{P}\varphi_0 \otimes \widetilde{f}_\pm = \int \mathrm{d}^3k \, \frac{f(k)^2}{\omega(k) - e_1 \pm i\epsilon} P, \end{aligned} \tag{6.4.56}$$

where we recall $e_0 = 0$. This together with the definition of $\Gamma_{\pm\epsilon}$ in (6.4.48) completes the first part of the proof.

In the following we compute the limits as ϵ tends to zero of $\Gamma_{\pm\epsilon}$. This is actually a consequence of the Sokhotski-Plemelj theorem, we calculate using the changes of variables $s = (r^2 + m^2)^{1/2}$ and $\tau = s - e_1$ (we recall that we identify $f(k) \equiv f(|k|)$ and we do the same with ω):

$$\begin{aligned}\int \mathrm{d}^3k \, \frac{f(k)^2}{\omega(k) - e_1 \pm i\epsilon} &= 4\pi \int_0^\infty \mathrm{d}r \, r^2 f(r)^2 \frac{1}{\omega(r) - e_1 \pm i\epsilon} \\ =&4\pi \int_m^\infty \mathrm{d}s \, s(s^2 - m^2)^{1/2} f((s^2 - m^2)^{1/2})^2 \frac{\epsilon}{(s - e_1) \pm i\epsilon} \\ =&4\pi \int_{m-e_1}^\infty \mathrm{d}\tau \, (e_1 + \tau)((e_1 + \tau)^2 - m^2)^{1/2} f(((e_1 + \tau)^2 - m^2)^{1/2})^2 \frac{1}{\tau \pm i\epsilon}.\end{aligned} \tag{6.4.57}$$

Using (6.4.57) and the Sokhotski-Plemelj theorem, we obtain that

$$\lim_{\epsilon \to 0} \int \mathrm{d}^3k \, \frac{f(k)^2}{\omega(k) - e_1 \pm i\epsilon} = \mp \pi i \theta(0) + \mathcal{P} \int_{m-e_1}^\infty \mathrm{d}x \, \theta(x)/x, \tag{6.4.58}$$

and thereby, we complete the proof. □

Corollary 6.4.12. *For sufficiently small $g, r > 0$, small enough $\epsilon > 0$ (depending on g) and $z \in \mathbb{R}$ with $|z - e_1| \leq r$, the following holds true*

$$P(H - z \pm i\epsilon)^{-1} P = (e_1 - z - g^2 \Gamma_{\pm 0})^{-1} P + \widetilde{R}(\epsilon, g, r), \tag{6.4.59}$$

where

$$\left\| \widetilde{R}(\epsilon, g, r) \right\| \leq C(g^{1/2} + r^{1/2}) \Big| \frac{1}{e_1 - z - g^2 \Gamma_{\pm 0}} \Big|, \tag{6.4.60}$$

and C does not depend on ϵ, g, r and z; c.f. Remark 6.3.3.

Proof. It follows from [14, Eq. (IV.13)] that

$$P(H - z \pm i\epsilon)^{-1} P = F_P(H - z \pm i\epsilon)^{-1}, \tag{6.4.61}$$

which is invertible for small enough ϵ, r and g (this is a consequence of Lemma 6.4.11, we recall that $\operatorname{Im} \Gamma_{\pm 0} \neq 0$). We use Neumann series and Lemma 6.4.11 to get

$$\begin{aligned}\| F_P(H - z \pm i\epsilon)^{-1} - &(e_1 - z - g^2 \Gamma_{\pm 0})^{-1} P \| \\ \leq& \Big| \frac{1}{e_1 - z - g^2 \Gamma_{\pm 0}} \Big| \sum_{n=1}^\infty \Big\| \frac{R_\epsilon(g, r) \pm i\epsilon + g^2 \Gamma_{\pm 0} - g^2 \Gamma_{\pm\epsilon}}{e_1 - z - g^2 \Gamma_{\pm 0}} \Big\|^n \\ \leq& C(g^{1/2} + r^{1/2}) \Big| \frac{1}{e_1 - z - g^2 \Gamma_{\pm 0}} \Big|,\end{aligned} \tag{6.4.62}$$

for small enough g, ϵ and r (we can take, for example, $\epsilon \leq g^{5/2}$ and so small such that $|\Gamma_{\pm 0} - \Gamma_{\pm\epsilon}| \leq g^{1/2}$). □

In addition, we present an easy formula for a certain matrix element of the time-evolution operator restricted to a spectral subset.

Lemma 6.4.13. *We set $\Phi_1 := \varphi_1 \otimes \Omega$. For every $s > 0$, we have*

$$\left\langle \Phi_1, e^{-itH}\chi_s(H)\Phi_1 \right\rangle = \pi^{-1} \lim_{\epsilon\to 0^+} \int_{\mathbb{R}} \mathrm{d}r\, \chi_s(r) e^{-itr}\, \mathrm{Im} \left\langle \Phi_1, (H - r - i\epsilon)^{-1}\Phi_1 \right\rangle. \quad (6.4.63)$$

Proof. The result follows from the spectral theorem and the next calculation

$$\begin{aligned} e^{-it\lambda}\chi_s(\lambda) &= \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{\mathbb{R}} dr e^{-it(\lambda+\epsilon r)} \chi_s(\lambda + \epsilon r) \frac{1}{r^2+1} = \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{\mathbb{R}} dr e^{-itr}\chi_s(r) \frac{\epsilon}{(r-\lambda)^2 + \epsilon^2} \\ &= \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{\mathbb{R}} dr e^{-itr}\chi_s(r)\, \mathrm{Im} \frac{1}{\lambda - r - i\epsilon}. \end{aligned} \quad (6.4.64)$$

□

The following formula strongly relies on the previous results in this section and it is a crucial ingredient for the proof of our main theorem in this chapter.

Lemma 6.4.14. *For sufficiently small $g > 0$, s as in Definition 6.4.4 and Lemma 6.4.5 sufficiently small, and all $t \in \mathbb{R}$ the following holds true*

$$\left\langle \Phi_1, e^{-itH}\Phi_1 \right\rangle = \pi^{-1} \int_{\mathbb{R}} \mathrm{d}z\, e^{-itz}\, \mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1} + r_0(g, s), \quad (6.4.65)$$

where

$$|r_0(g, s)| \le C \left((g^{1/2} + s^{1/2})|\log(g)| + gs^{-1} \right), \quad (6.4.66)$$

and we recall $\Phi_1 = \varphi_1 \otimes \Omega$. The constant C does not depend on g, s and t.

Proof. The spectral calculus implies $\chi(H_0)\Phi_1 = \Phi_1$, and hence, it follows from Lemma 6.4.5 that

$$\left\langle \Phi_1, e^{-itH}\Phi_1 \right\rangle = \left\langle \Phi_1, e^{-itH}\chi_s(H)\Phi_1 \right\rangle + r_1(g, s), \quad \text{where} \quad |r_1(g, s)| \le Cgs^{-1}. \quad (6.4.67)$$

Lemma 6.4.13 yields

$$\left\langle \Phi_1, e^{-itH}\chi_s(H)\Phi_1 \right\rangle = \pi^{-1} \lim_{\epsilon\to 0^+} \int_{\mathbb{R}} \mathrm{d}z\, \chi_s(z) e^{-itz}\, \mathrm{Im} \left\langle \Phi_1, P(H - z - i\epsilon)^{-1}P\Phi_1 \right\rangle. \quad (6.4.68)$$

We calculate:

$$\left\langle \Phi_1, e^{-itH}\chi_s(H)\Phi_1 \right\rangle = \pi^{-1} \lim_{\epsilon\to 0^+} \left(\int_{\mathbb{R}} \mathrm{d}z\, e^{-itz}\, \mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1} + r_2(g, \epsilon, s) + r_3(g, s) \right), \quad (6.4.69)$$

where

$$r_2(g,\epsilon,s) = \pi^{-1}\int_{\mathbb{R}} \mathrm{d}z\, \chi_s(z)e^{-itz}\,\mathrm{Im}\left\langle \Phi_1, \left(P(H-z-i\epsilon)^{-1}P - (e_1 - z - g^2\Gamma_{-0})^{-1}\right)\Phi_1\right\rangle \tag{6.4.70}$$

and

$$r_3(g,s) = \pi^{-1}\int_{\mathbb{R}} \mathrm{d}z\,(1-\chi_s(z))e^{-itz}\,\mathrm{Im}(e_1 - z - g^2\Gamma_{+0})^{-1}. \tag{6.4.71}$$

Now, we use Corollary 6.4.12, for sufficiently small s, to get

$$\begin{aligned}\left|\chi_s(z)e^{-itz}\,\mathrm{Im}\left\langle \Phi_1, \left(P(H-z-i\epsilon)^{-1}P - (e_1 - z - g^2\Gamma_{-0})^{-1}\right)\Phi_1\right\rangle\right| \\ \le C(g^{1/2}+s^{1/2})\left|e_1 - z - g^2\Gamma_{-0}\right|^{-1}\chi_s(z).\end{aligned} \tag{6.4.72}$$

This together with (6.4.70) and Definition 6.4.4 yields then that

$$\begin{aligned}|r_2(g,\epsilon,s)| \le& C(g^{1/2}+s^{1/2})\int \mathrm{d}z\,\chi_s(z)\left|e_1 - z - g^2\Gamma_{-0}\right|^{-1} \\ =& C(g^{1/2}+s^{1/2})\int \mathrm{d}z\,\chi((z-e_1)/s+e_1)\Big((e_1 - z - g^2\,\mathrm{Re}\,\Gamma_{-0})^2 + g^4(\mathrm{Im}\,\Gamma_{-0})^2\Big)^{-1/2} \\ \le& C(g^{1/2}+s^{1/2})\int_{-\frac{3}{4}\delta s - g^2\,\mathrm{Re}\,\Gamma_{-0}}^{\frac{3}{4}\delta s - g^2\,\mathrm{Re}\,\Gamma_{-0}} dr\frac{1}{g^2}\frac{1}{((\frac{r}{g^2})^2 + (\mathrm{Im}\,\Gamma_{-0})^2)^{1/2}} \\ \le& C(g^{1/2}+s^{1/2})\int_{|r|\le csg^{-2}} dr\frac{1}{(r^2 + (\mathrm{Im}\,\Gamma_{-0})^2)^{1/2}},\end{aligned} \tag{6.4.73}$$

where the last step follows for $g>0$ sufficiently small and some constant $c>0$. Here, we recall from Definition 6.4.4 that $g^2 \le s \le g^\kappa$ for some $0<\kappa<2$. Employing that $2\sqrt{x^2+y^2} \ge |x|+|y|$, we find a constant $C>0$ such that

$$|r_2(g,\epsilon,s)| \le C(g^{1/2}+s^{1/2})|\log(g)|. \tag{6.4.74}$$

Moreover, it follows from (6.4.71) together with the definition of χ and $0\le\chi\le 1$ that there is a constant $c>0$ such that

$$\begin{aligned}|r_3(g,s)| \le& \pi^{-1}\int \mathrm{d}z\,(1-\chi_s(z))\left|\mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1}\right| \\ \le& \pi^{-1}g^2\,\mathrm{Im}\,\Gamma_{+0}\int(1-\chi_s(z))\frac{1}{(e_1 - z - g^2\,\mathrm{Re}\,\Gamma_{-0})^2 + g^4\,\mathrm{Im}\,\Gamma_{-0}^2} \\ \le& Cg^2\int_{|r|\ge cs} dr\frac{1}{g^4}\frac{1}{(\frac{r}{g^2})^2 + \mathrm{Im}\,\Gamma_{-0}^2} = C\int_{|x|\ge cs/g^2} dx\frac{1}{x^2 + \mathrm{Im}\,\Gamma_{-0}^2} \le Cg^2 s^{-1}.\end{aligned} \tag{6.4.75}$$

□

6.5. Proof of our main result in this chapter – Theorem 6.3.2

In this section we provide a proof of our main result in this chapter; c.f. Theorem 6.3.2.

Proof of Theorem 6.3.2. Our proof starts from the intermediate scattering formula given in Theorem 2.2.2, i.e., equation (2.2.26). Note that this result was already proven for the massless case in [23, Theorem 4.3] and the proof for the massive case works analogously. Then, for $h, l \in \mathfrak{h}_0$; c.f. (2.1.1), it follows from (2.2.26) together with Lemma 2.2.1 (iv) below that

$$T(h,l) = -2\pi i g \|\Psi_{\lambda_0}\|^{-2} \langle a_-(W)\sigma_1\Psi_{\lambda_0}, \Psi_{\lambda_0}\rangle = -2\pi i g \|\Psi_{\lambda_0}\|^{-2} \langle [a_-(W), \sigma_1]\Psi_{\lambda_0}, \Psi_{\lambda_0}\rangle, \tag{6.5.1}$$

where we recall the definition of W in (2.2.27) and $W \in \mathfrak{h}_0$. It follows from (2.2.2) in Lemma 2.2.1 (ii) that

$$\begin{aligned} T(h,l) &= 2\pi(ig)^2\|\Psi_{\lambda_0}\|^{-2} \int_{-\infty}^0 \mathrm{d}t\, \overline{\langle W_t, f\rangle_2} \left\langle \left[e^{itH}\sigma_1 e^{-itH}, \sigma_1\right]\Psi_{\lambda_0}, \Psi_{\lambda_0}\right\rangle \\ &= 2\pi g^2 \|\Psi_{\lambda_0}\|^{-2} \int_0^\infty \mathrm{d}t\, \langle f, W_{-t}\rangle_2 \left\langle \left[e^{-itH}\sigma_1 e^{itH}, \sigma_1\right]\Psi_{\lambda_0}, \Psi_{\lambda_0}\right\rangle \\ &= 2\pi \|\Psi_{\lambda_0}\|^{-2}\left(T^{(1)} - T^{(2)}\right), \end{aligned} \tag{6.5.2}$$

where we recall the notation $W_s(k) = e^{-is\omega(k)}W(k)$ and use the abbreviations

$$\begin{aligned} T^{(1)} :&= g^2 \int_0^\infty \mathrm{d}t \int \mathrm{d}^3k\, W(k) f(k) e^{it(\omega(k)+\lambda_0)} \left\langle \sigma_1\Psi_{\lambda_0}, e^{-itH}\sigma_1\Psi_{\lambda_0}\right\rangle \qquad (6.5.3) \\ &= g^2 \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}r\, G(r) e^{it(\omega(r)+\lambda_0)} \left\langle \sigma_1\Psi_{\lambda_0}, e^{-itH}\sigma_1\Psi_{\lambda_0}\right\rangle \\ &= g^2 \int_0^\infty \mathrm{d}t\, \widetilde{h}(t) \left\langle \sigma_1\Psi_{\lambda_0}, e^{-itH}\sigma_1\Psi_{\lambda_0}\right\rangle \end{aligned}$$

and

$$T^{(2)} := g^2 \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}r\, G(r) e^{it(\omega(r)-\lambda_0)} \left\langle \sigma_1\Psi_{\lambda_0}, e^{itH}\sigma_1\Psi_{\lambda_0}\right\rangle. \tag{6.5.4}$$

Here, we changed to spherical coordinates $k = (r, \Sigma)$ and take:

$$G(r) = \int \mathrm{d}\Sigma \mathrm{d}\Sigma'\, r^4 \overline{h(r,\Sigma)} l(r,\Sigma') f(r)^2, \qquad \widetilde{h}(t) := \int_0^\infty \mathrm{d}r\, G(r) e^{it(\omega(r)+\lambda_0)}. \tag{6.5.5}$$

Moreover, we observe that $G \in C_c^\infty(\mathbb{R}\setminus\{0\}, \mathbb{C})$. Notice that an integration by parts (using that $e^{i\theta\omega(r)} = \frac{\partial}{\partial r}\Big(e^{i\theta\omega(r)} \frac{1}{i\theta \frac{\partial}{\partial\theta}\omega(r)}\Big) - e^{i\theta\omega(r)} \frac{\partial}{\partial r}\Big(\frac{1}{i\theta\frac{\partial}{\partial\theta}\omega(r)}\Big)$) ensures that

$$\left|\widetilde{h}(t)\right| \le C/(1+t^2), \qquad \forall t \in \mathbb{R}, \tag{6.5.6}$$

which guarantees the existence of the integrals in (6.5.3) and (6.5.4).

Recall $\Phi_1 = \varphi_1 \otimes \Omega$ (see Lemma 6.4.13). It follows from Proposition 6.2.1 that

$$\left\langle \sigma_1 \Psi_{\lambda_0}, e^{-isH} \sigma_1 \Psi_{\lambda_0} \right\rangle = \left\langle \Phi_1, e^{-isH} \Phi_1 \right\rangle + \rho_1(g), \tag{6.5.7}$$

where $|\rho_1(g)| \leq Cg$. Moreover, we recall that Lemma 6.4.14 states that

$$\left\langle \Phi_1, e^{-itH} \Phi_1 \right\rangle = \pi^{-1} \int_{\mathbb{R}} \mathrm{d}z \, e^{-itz} \, \mathrm{Im}(e_1 - z - g^2 \Gamma_{-0})^{-1} + r_0(g,s), \tag{6.5.8}$$

where

$$|r_0(g,s)| \leq C\Big(\Big(g^{1/2} + s^{1/2}\Big) |\log(g)| + gs^{-1} \Big). \tag{6.5.9}$$

Note that [23, Remark 4.8] implies that the first term in (6.5.8) is bounded by a constant as $g \to 0^+$ (this actually follows from computing the integral). Then, (6.5.3) together with (6.5.7) and (6.5.8) yields

$$T^{(1)} = T_0^{(1)} + R_1(g,s), \tag{6.5.10}$$

where

$$T_0^{(1)} := \pi^{-1} g^2 \int_0^\infty \mathrm{d}t \, \widetilde{h}(t) \int_{\mathbb{R}} \mathrm{d}z \, e^{-itz} \, \mathrm{Im}(e_1 - z - g^2 \Gamma_{-0})^{-1}, \tag{6.5.11}$$

and $|R_1(g,s)| \leq Cg^2((g^{1/2} + s^{1/2})|\log(g)| + gs^{-1})$ for some constant $C > 0$. As $\mathrm{Im}\, \Gamma_{-0} > 0$ and $\mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1}$ decays as $|z|^{-2}$ at infinity, (6.5.11) is absolutely integrable, and consequently, Fubini's theorem allows for interchanging the order of integration. Similarly, we argue that we can apply the dominated convergence theorem and conclude

$$T_0^{(1)} = \lim_{\eta \to 0^+} T_0^{(1)}(\eta), \tag{6.5.12}$$

where

$$\begin{aligned} T_0^{(1)}(\eta) &= \pi^{-1} g^2 \int_{\mathbb{R}} \mathrm{d}z \; \mathrm{Im}(e_1 - z - g^2 \Gamma_{-0})^{-1} \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}r \, G(r) e^{it(\omega(r) + \lambda_0 - z + i\eta)} \\ &= \pi^{-1} g^2 \int_{\mathbb{R}} \mathrm{d}z \; \mathrm{Im}(e_1 - z - g^2 \Gamma_{-0})^{-1} \int_0^\infty \mathrm{d}t \widetilde{h}(t) e^{-t\eta} e^{-itz}. \end{aligned} \tag{6.5.13}$$

Again, Fubini's theorem yields for $Q > 0$

$$\int_0^Q \mathrm{d}t \int_0^\infty \mathrm{d}r \, G(r) e^{it(\omega(r) + \lambda_0 - z + i\eta)} = i \int_0^\infty \mathrm{d}r \, \frac{G(r)}{\omega(r) + \lambda_0 - z + i\eta} \Big(1 - e^{iQ(\omega(r) + \lambda_0 - z + i\eta)} \Big). \tag{6.5.14}$$

Moreover, for all $\eta > 0$, we obtain by the integration by parts formula (see above (6.5.6)) together with $G \in C_c^\infty(\mathbb{R} \setminus \{0\}, \mathbb{C})$ that there is a constant $C(\eta, g) > 0$ such that

$$\left| \int_0^\infty \mathrm{d}r \, \frac{G(r)}{\omega(r) + \lambda_0 - z + i\eta} e^{iQ(\omega(r) + \lambda_0 - z + i\eta)} \right| \leq C(\eta, g) Q^{-1}, \tag{6.5.15}$$

and consequently, (6.5.14) implies that

$$\int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}r\, G(r) e^{is(\omega(r)+\lambda_0-z+i\eta)} = i \int_0^\infty \mathrm{d}r\, \frac{G(r)}{\omega(r)+\lambda_0-z+i\eta}. \tag{6.5.16}$$

This together with Fubini's theorem yields that

$$T_0^{(1)}(\eta) = i\pi^{-1} g^2 \int_0^\infty \mathrm{d}r\, G(r) \int_{\mathbb{R}} \mathrm{d}z\ \mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1} \frac{1}{\omega(r)+\lambda_0-z+i\eta}. \tag{6.5.17}$$

For $a > e_1$, we define $\mathcal{Q}_a := [-a,a] \cup \{ae^{-i\varphi} : \varphi \in [0,\pi]\} \subset \overline{\mathbb{C}^-}$ to be a closed contour with mathematical negative orientation. Note that the integrand in (6.5.17) is meromorphic in the lower half of the complex plane and its only pole lies at $z = e_1 - g^2\Gamma_{-0}$, and hence, inside the contour $\mathcal{Q}_a$ by definition. Notice that, for real z, as in (6.5.17),

$$\mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1} = \frac{1}{2i}\Big((e_1 - z - g^2\Gamma_{-0})^{-1} - (e_1 - z - g^2\overline{\Gamma_{-0}})^{-1}\Big), \tag{6.5.18}$$

i.e. we do not conjugate z. We extend the formula above, in a meromorphic way, to the lower half of the complex plane. We obtain, for small enough η, using the residue theorem that

$$\begin{aligned} &\int_{\mathbb{R}} \mathrm{d}z\ \mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1} \frac{1}{\omega(r)+\lambda_0-z+i\eta} \\ &= (2i)^{-1} \lim_{a\to\infty} \int_{\mathcal{Q}_a} \mathrm{d}z\, \frac{1}{\omega(r)+\lambda_0-z+i\eta}\Big((e_1 - z - g^2\Gamma_{-0})^{-1} - (e_1 - z - g^2\overline{\Gamma_{-0}})^{-1}\Big) \\ &= \frac{\pi}{\omega(r)+\lambda_0-(e_1-g^2\Gamma_{-0})+i\eta}. \end{aligned} \tag{6.5.19}$$

This together with (6.5.17) yields that

$$\begin{aligned} \lim_{\eta\to 0^+} T_0^{(1)}(\eta) &= \lim_{\eta\to 0^+} \int_0^\infty \mathrm{d}r\, \frac{ig^2 G(r)}{\omega(r)+\lambda_0-(e_1-g^2\Gamma_{-0})+i\eta} \\ &= \int_0^\infty \mathrm{d}r\, \frac{ig^2 G(r)}{\omega(r)+\lambda_0-(e_1-g^2\Gamma_{-0})}, \end{aligned} \tag{6.5.20}$$

where in the last step we applied the dominated convergence theorem which is justified because $G \in C_c^\infty(\mathbb{R}\setminus\{0\}, \mathbb{C})$. Consequently, it follows from (6.5.10) and (6.5.12) that

$$T^{(1)} = ig^2 \int_0^\infty \mathrm{d}r\, \frac{G(r)}{\omega(r)+\lambda_0-(e_1-g^2\Gamma_{-0})} + R_1(g,s), \tag{6.5.21}$$

where we recall that $|R_1(g,s)| \le Cg^2((s^{1/2}+g^{1/2})|\log(g)| + gs^{-1})$. Analogously, we obtain

$$T^{(2)} = ig^2 \int \mathrm{d}r\, \frac{G(r)}{\omega(r)-\lambda_0+(e_1-g^2\overline{\Gamma_{-0}})} + R_2(g,s), \tag{6.5.22}$$

and $|R_2(g,s)| \leq Cg^2((s^{1/2}+g^{1/2})|\log(g)|+gs^{-1})$ for some constant C. Finally, we conclude from (6.5.21) and (6.5.22) together with (6.5.2) that

$$\begin{aligned} T(h,l) =& 2\pi i g^2 \|\Psi_{\lambda_0}\|^{-2} \int \mathrm{d}r \left(\frac{G(r)}{\omega(r)+\lambda_0-(e_1-g^2\Gamma_{-0})} - \frac{G(r)}{\omega(r)-\lambda_0+(e_1-g^2\overline{\Gamma_{-0}})} \right) \\ &+ R(g,s) \\ =& 4\pi i g^2 \|\Psi_{\lambda_0}\|^{-2} \int \mathrm{d}r \, \frac{G(r)\,(e_1-g^2\,\mathrm{Re}\,\Gamma_{+0}-\lambda_0)}{\left(\omega(r)+\lambda_0-(e_1-g^2\Gamma_{-0})\right)\left(\omega(r)-\lambda_0+(e_1-g^2\overline{\Gamma_{-0}})\right)} \\ &+ R(g,s), \end{aligned} \tag{6.5.23}$$

where $R(g,s) := R_1(g,s)+R_2(g,s)$. Hence, there is a constant $C>0$ such that $|R(g,s)| \leq Cg^2((s^{1/2}+g^{1/2})|\log(g)|+gs^{-1})$. We take $s = g^{2/3}$ and obtain that $|R(g,s)| \leq Cg^2 g^{1/3}|\log(g)|$. This completes the proof. □

6.6. Mourre theory and the limiting absorption principle

In this section we present a proof of Proposition 6.4.8 (ii) and (iii). Although Mourre theory is a standard tool to prove limiting absorption principles, in this section, we do not address the usual procedures because we prove perturbative results in the spirit of [4, 36] (see Proposition 6.4.8 (iii)). Note that in [36] an abstract family of Hamiltonians is studied.

The main result of this section is Proposition 6.4.8 (iii). Despite the fact that Proposition 6.4.8 (ii) is standard, we also prove it because we need it to prove Proposition 6.4.8 (iii). Some other well-known estimates in the context of Mourre theory are not proven in this section – we will give instead proper references (to sections below).

We also mention that we do not employ the original techniques of Mourre to study domain problems and commutators (see [56, 28]). Instead, we directly dilate the operators at stake: our approach is close to the usual one based on the theory of operators of class C^k with respect to a self-adjoint conjugate operator (see [6, 5]), but, in our work, given the explicit form of the operators at stake, we do not need to rely on this theory and we give a more transparent presentation.

In this section we address the limiting absorption principle, i.e. we study the behavior of the resolvent operator $(H_{\overline{P}}-(z\pm i\epsilon))^{-1}$ as $\epsilon>0$ tends to 0 and z belongs to the interval

$$I := [e_1-\delta/4, e_1+\delta/4]. \tag{6.6.1}$$

Of course, the norm of $(H_{\overline{P}}-(z\pm i\epsilon))^{-1}$ tends to infinity as ϵ tends to zero. Then, controlling its behavior requires restricting its domain, and this is achieved by multiplying by the operator

$$\rho := \langle \mathrm{d}\Gamma(D)\rangle^{-1}. \tag{6.6.2}$$

Our goal is to obtain uniform norm-bounds for $\rho(H_{\overline{P}} - (z \pm i\epsilon))^{-1}\rho$ and regularity properties with respect to g (this is what we call above perturbative Mourre theory) and z.

Intuitively, one might consider the operator $H_{\overline{P}} - z$ as a real quantity because it is self-adjoint. One of the clever ideas of Mourre is to add to $H_{\overline{P}} - (z \pm i\epsilon)$ a non-zero imaginary part of size $\eta > 0$ and sign $\pm$ (according to $\pm i\epsilon$). Then, the resulting operator ($H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}$ – see (6.6.4) below) can be intuitively regarded as a real quantity plus $\mp i(\epsilon + \eta)$. It is, therefore, invertible and the norm of its inverse is uniformly bounded with respect to ϵ. Our goal is to study the behavior of the resolvent operator associated to $H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}$ as ϵ and η tend to zero. More precisely, the imaginary part that we refer above is given by the operator $\mp i\eta M^2$, where η is a strictly positive small enough real number and (see Lemma 6.4.7)

$$M^2 := \chi(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}}) \geq \alpha\chi(H_{\overline{P}})^2, \tag{6.6.3}$$

which is a bounded operator (see Remark 6.4.6). We properly select ρ as a function of $\mathrm{d}\Gamma(D)$ because $\rho\mathrm{d}\Gamma(D)$ is bounded. This allows us to control the unbounded operator $\mathrm{d}\Gamma(D)$ in the above commutator. The other operator in this commutator is chosen in order to cancel resolvents (see (6.6.23) and (6.6.25) below for the limiting absorption principle, and (6.6.61) for perturbative results).

We define the operators (for $z \in I$)

$$H_{\overline{P}}^{\pm\eta} := H_{\overline{P}} \mp i\eta M^2, \qquad R^{\pm\eta}(z_{\pm\epsilon}) = \left(H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}\right)^{-1}, \quad z_{\pm\epsilon} := z \pm i\epsilon. \tag{6.6.4}$$

It is a standard result that $H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}$ is invertible (with bounded inverse) – see [28] – and that $R^{\pm\eta}(z_{\pm\epsilon})$ is continuous at $\eta = 0$ and derivable with respect to η, for $\eta > 0$ small enough. Its derivative is given by

$$\mathrm{d}/\mathrm{d}\eta\, R^{\pm\eta}(z_{\pm\epsilon}) = \pm iR^{\pm\eta}(z_{\pm\epsilon})M^2R^{\pm\eta}(z_{\pm\epsilon}), \qquad \forall\eta \in (0, \boldsymbol{\eta}). \tag{6.6.5}$$

For the convenience of the reader we give a proof of this in Section 6.8 below (see also [28]). Moreover, if we multiply $R^{\pm\eta}(z_{\pm\epsilon})$ by an operator that localizes the spectral region of $H_{\overline{P}}$ far away from z, we get a bounded operator which satisfies:

$$\left\|(H_{\overline{P}} + i)\overline{\chi}(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\right\| \leq C, \tag{6.6.6}$$

where $\overline{\chi} = 1 - \chi$. This is proven in Section 6.8 (see also [28]).

As announced above, it follows that the norm of $R^{\pm\eta}(z_{\pm\epsilon})$ can be uniformly bounded (with respect to ϵ). Actually, the following estimate holds:

$$\|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\| \leq C/\eta, \tag{6.6.7}$$

where C does not depend on z, ϵ and g (see [28] and Section 6.8).

Estimate (6.6.7) itself is not enough because we still have the singularity C/η and we need to consider the operator ρ, otherwise we cannot expect to have a limiting absorption principle – this is explained above. For this reason, we define

$$F^{\pm\eta}(z_{\pm\epsilon}) := \rho R^{\pm\eta}(z_{\pm\epsilon})\rho \tag{6.6.8}$$

and get a better estimate which is a key ingredient of Mourre theory. Note that this is the only place where the Mourre estimate (see (6.6.3)) is used:

$$\|(H_{\overline{P}}+i)R^{\pm\eta}(z_{\pm\epsilon})\rho\| \leq C\left(1+\eta^{-1/2}\|F^{\pm\eta}(z_{\pm\epsilon})\|^{1/2}\right). \tag{6.6.9}$$

Eq. (6.6.9) is a standard result (see, e.g., [28]), but we prove it in Section 6.8. Looking at (6.6.7) and (6.6.9), it seams that we get again the unsatisfactory bound

$$\|F^{\pm\eta}(z_{\pm\epsilon})\| \leq C/\eta. \tag{6.6.10}$$

At this point, the line of reasoning becomes more subtle. Actually, in the lines above we never use that M^2 is defined in terms of the commutator $[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0$. The only thing we utilize about M^2 is that it satisfies the Mourre estimate (6.6.3). All the material presented above in this section is standard and it can be directly deduced from the proofs in [28]. Therefore, we do not include proofs of this in the present section. For the convenience of the reader we provide proofs in Section 6.8.

In this section we use all estimates and statements presented above (without proofs) and provide a detailed proof of the limiting absorption principle (Proposition 6.4.8-(ii)) and its perturbative version (Proposition 6.4.8-(iii)). The idea of the proof of Proposition 6.4.8-(ii) (which amounts to bound $\|F^{\pm\eta}(z_{\pm\epsilon})\|$ by a constant) is quite simple, we just write $F^{\pm\eta}(z_{\pm\epsilon})$ as the integral of its derivative. Then, the difficult part is to estimate the referred derivative (Lemma 6.6.2 below). This derivative consists of a sum of several terms and each of them is separately estimated. The most singular term is $Q_{1,1}$ defined in (6.6.23) below. The analysis of $Q_{1,1}$ is the only part of the proof of Proposition 6.4.8-(ii) that requires that M^2 is defined in terms of the commutator $[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0$: we control the unbounded operator $\mathrm{d}\Gamma(D)$ using that $\rho\mathrm{d}\Gamma(D)$ is bounded and $H_{\overline{P}}$ is important to cancel resolvent operators (see (6.6.25) below).

As we mention above, the main result of this section is Proposition 6.4.8-(iii). The proof of it follows the same strategy of the proof of item (ii), but it is substantially more complicated. Again, we study the terms we are interested in using that they are integrals of their derivatives. The difficult part is to estimate the derivatives, which consist on several terms that must be analyzed separately. This is achieved in Lemma 6.6.3 below.

Before we start with the proofs, we state two last results that we use in this section and prove in Appendix 6.9: the operator $R^{\pm\eta}(z_{\pm\epsilon})$ leaves the domain of $\mathrm{d}\Gamma(D)$ invariant. Moreover, there is a bounded operator that we denote by

$$[\mathrm{d}\Gamma(D), M^2]^0 \tag{6.6.11}$$

that represents the quadratic form $[\mathrm{d}\Gamma(D), M^2]$. These results can be proved as in [56, 28] (defining a scale of Hilbert spaces and regularizing the generator of dilations) or [6, 5] (using that the Hamiltonian is of class C^k with respect to the generator of dilations). We provide a more direct proof in Appendix 6.9.

Remark 6.6.1. *The definitions and estimates introduced above in this section are also valid for the case $g = 0$. We distinguish this case by adding everywhere in our notations a subscript 0. For example:*

$$M_0^2 := M^2|_{g=0}, \qquad H_{0,\overline{P}} := H_{\overline{P}}|_{g=0}.$$

Lemma 6.6.2. *For $\geq 0, \boldsymbol{\eta} > 0$ sufficiently small, $\eta \in (0, \boldsymbol{\eta})$, $\epsilon \in (0,1)$, $z, z' \in I$ and $z_{\pm\epsilon} := z \pm i\epsilon$,*

$$\left\| \mathrm{d}/\mathrm{d}\eta \, F^{\pm\eta}(z_{\pm\epsilon}) \right\| \leq C \left(1 + \eta^{-1/2} \left\| F^{\pm\eta}(z_{\pm\epsilon}) \right\|^{1/2} + \left\| F^{\pm\eta}(z_{\pm\epsilon}) \right\| \right). \tag{6.6.12}$$

Proof. It follows from (6.6.3), (6.6.5) and (6.6.8) that

$$\pm i \mathrm{d}/\mathrm{d}\eta \, F^{\pm\eta}(z_{\pm\epsilon}) = Q_1 + Q_2 + Q_3 + Q_4, \tag{6.6.13}$$

where

$$Q_1 := -\rho R^{\pm\eta}(z_{\pm\epsilon})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R^{\pm\eta}(z_{\pm\epsilon})\rho \tag{6.6.14}$$
$$Q_2 := -\rho R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\rho \tag{6.6.15}$$
$$Q_3 := \rho R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R^{\pm\eta}(z_{\pm\epsilon})\rho \tag{6.6.16}$$
$$Q_4 := \rho R^{\pm\eta}(z_{\pm\epsilon})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\rho. \tag{6.6.17}$$

Remark 6.4.6 and (6.6.6) imply that

$$\left\| [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon}) \right\| \leq C. \tag{6.6.18}$$

This yields that

$$\|Q_2\| \leq C \left\| \rho R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}}) \right\| \leq C, \tag{6.6.19}$$

where we use again (6.6.6) (taking the adjoint). Taking the adjoint in (6.6.18), it follows that

$$\|Q_3\| \leq C \left\| R^{\pm\eta}(z_{\pm\epsilon})\rho \right\| \leq C \left(1 + \eta^{-1/2} \left\| F^{\pm\eta}(z_{\pm\epsilon}) \right\|^{1/2} \right), \tag{6.6.20}$$

where we use (6.6.9). Similarly, taking the adjoint in (6.6.9) we obtain that

$$\|Q_4\| \leq C \left(1 + \eta^{-1/2} \left\| F^{\pm\eta}(z_{\pm\epsilon}) \right\|^{1/2} \right). \tag{6.6.21}$$

In the remainder of the proof, we estimate Q_1. For $\phi, \psi \in \mathcal{D}(\mathrm{d}\Gamma(D)) \cap \mathcal{D}(H_{\overline{P}})$, Remark 6.4.6 and the fact that $R^{\pm\eta}(z_{\pm\epsilon})$ leaves the domain of $\mathrm{d}\Gamma(D)$ invariant (see above (6.6.11)) allows us to write

$$\langle \phi, Q_1 \psi \rangle = \langle \phi, Q_{11} \psi \rangle + \langle \phi, Q_{12} \psi \rangle, \tag{6.6.22}$$

where

$$\begin{aligned} \langle \phi, Q_{11} \psi \rangle := & \left\langle \left(H_{\overline{P}} \pm i\eta M^2 - z_{\mp\epsilon} \right) R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D) R^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \\ & - \left\langle \left(-i\mathrm{d}\Gamma(D) \right) R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, \left(H_{\overline{P}} \mp i\eta M^2 - z_{\pm\epsilon} \right) R^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle, \end{aligned} \tag{6.6.23}$$

$$\begin{aligned} \langle \phi, Q_{12} \psi \rangle := \pm i\eta \Big(& \left\langle M^2 R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D) R^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \\ & - \left\langle \left(-i\mathrm{d}\Gamma(D) \right) R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, M^2 R^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \Big). \end{aligned} \tag{6.6.24}$$

Employing that $\|\mathrm{d}\Gamma(D)\rho\| \leq 1$, we find

$$|\langle \phi, Q_{11}\psi\rangle| = \left|\left\langle \left(-i\mathrm{d}\Gamma(D)\right)\rho\phi, R^{\pm\eta}(z_{\pm\epsilon})\rho\psi\right\rangle - \left\langle R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D)\rho\psi\right\rangle\right|$$
$$\leq \|\phi\|\|\psi\| \left(\|R^{\mp\eta}(z_{\mp\epsilon})\rho\| + \|R^{\pm\eta}(z_{\pm\epsilon})\rho\|\right). \tag{6.6.25}$$

It follows again from (6.6.9) that

$$|\langle \phi, Q_{11}\psi\rangle| \leq C\|\phi\|\|\psi\| \left(1 + \eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}\right). \tag{6.6.26}$$

Furthermore, we estimate (using again that $R^{\pm\eta}(z_{\pm\epsilon})$ leaves the domain of $\mathrm{d}\Gamma(D)$ invariant and the text around (6.6.11))

$$|\langle \phi, Q_{12}\psi\rangle| \leq \eta\|\phi\|\|\psi\|\|R^{\mp\eta}(z_{\mp\epsilon})\rho\|\|R^{\pm\eta}(z_{\pm\epsilon})\rho\|\left\|[M^2, \mathrm{d}\Gamma(D)]^0\right\| \tag{6.6.27}$$
$$\leq C\eta\|\phi\|\|\psi\| \left(1 + \eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}\right)^2$$
$$\leq C\|\phi\|\|\psi\| \left(1 + \left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|\right),$$

where we use (6.6.9). It follows from (6.6.26) together with (6.6.27), (6.6.22) and the density of $\mathcal{D}(\mathrm{d}\Gamma(D)) \cap \mathcal{D}(H_{\overline{P}})$ in $\mathcal{H}$ that

$$\|Q_1\| \leq C\left(1 + \eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2} + \left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|\right). \tag{6.6.28}$$

This together with (6.6.13), (6.6.19), (6.6.20) and (6.6.21) completes the proof. □

Proof of Proposition 6.4.8 (ii). Let $\eta \in (0, \boldsymbol{\eta})$ (and $\boldsymbol{\eta}$ is sufficiently small). We use the fundamental theorem of calculus

$$F^{\pm\eta}(z_{\pm\epsilon}) = F^{\pm\boldsymbol{\eta}}(z_{\pm\epsilon}) + \int_{\pm\boldsymbol{\eta}}^{\pm\eta} \mathrm{d}\tilde{\eta}\, \mathrm{d}/\mathrm{d}\tilde{\eta}\, F^{\pm\tilde{\eta}}(z_{\pm\epsilon}), \tag{6.6.29}$$

and (6.6.10) to obtain that there is a constant $C(\boldsymbol{\eta}) > 0$ such that

$$\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\| \leq \left\|F^{\pm\boldsymbol{\eta}}(z_{\pm\epsilon})\right\| + C\left|\int_{\pm\boldsymbol{\eta}}^{\pm\eta} \mathrm{d}\tilde{\eta}/\tilde{\eta}\right| \leq C(\boldsymbol{\eta})\left|\log\eta\right|. \tag{6.6.30}$$

Inserting this in Lemma 6.6.2, we obtain

$$\left\|\mathrm{d}/\mathrm{d}\eta\, F^{\pm\eta}(z_{\pm\epsilon})\right\| \leq C(\boldsymbol{\eta})\eta^{-1/2}\left|\log\eta\right|, \tag{6.6.31}$$

and similarly as above, we find

$$\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\| \leq \left\|F^{\pm\boldsymbol{\eta}}(z_{\pm\epsilon})\right\| + C(\boldsymbol{\eta})\left|\int_{\pm\boldsymbol{\eta}}^{\pm\eta} \mathrm{d}\tilde{\eta}\, \eta^{-1/2}\left|\log\eta\right|\right|. \tag{6.6.32}$$

We conclude that there is a constant $C(\boldsymbol{\eta}) > 0$ such that

$$\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\| \leq C(\boldsymbol{\eta}). \tag{6.6.33}$$

Now we use the text below (6.6.4) and take the limit $\eta \to 0^+$ in (6.6.33). We conclude that (6.4.35) holds true (also (6.4.36), taking $g = 0$). Analogously, we show (6.4.36). □

In the remainder of this section we prove Proposition 6.4.8 (iii). The spirit of the proof is similar to the proof of statement (ii), however, we need additional estimates which are collected in the lemma below.

Lemma 6.6.3. *For $g \geq 0, \boldsymbol{\eta} > 0$ sufficiently small, $\eta \in (0, \boldsymbol{\eta})$, $\epsilon \in (0,1)$, $z, z' \in I$ and $z_{\pm\epsilon} := z \pm i\epsilon$, the following estimates hold true*

(i)

$$\left\| \mathrm{d}/\mathrm{d}\eta \, \left(F^{\pm\eta}(z_{\pm\epsilon}) - F^{\pm\eta}(z'_{\pm\epsilon})\right)\right\| \leq C\eta^{-1/2} \tag{6.6.34}$$

(ii)

$$\left\| \mathrm{d}/\mathrm{d}\eta \, \left(F^{\pm\eta}(z_{\pm\epsilon}) - F^{\pm\eta}(z'_{\pm\epsilon})\right)\right\| \leq C\eta^{-3/2}|z - z'| \tag{6.6.35}$$

(iii)

$$\left\| \mathrm{d}/\mathrm{d}\eta \, \left(F^{\pm\eta}(z_{\pm\epsilon}) - F_0^{\pm\eta}(z_{\pm\epsilon})\right)\right\| \leq C\eta^{-3/2}g, \tag{6.6.36}$$

see Remark 6.6.1.

Proof. (i) It follows from Lemma 6.6.2 and (6.6.33).

(iii) Using the second resolvent identity, Remark 6.6.1, Remark 6.4.6 and (6.6.3), we get

$$\mp i \frac{\mathrm{d}}{\mathrm{d}\eta} \left(F^{\pm\eta}(z_{\pm\epsilon}) - F_0^{\pm\eta}(z_{\pm\epsilon})\right) = \pm i g \frac{\mathrm{d}}{\mathrm{d}\eta} \left(\rho R^{\pm\eta}(z_{\pm\epsilon}) \tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\right), \tag{6.6.37}$$

where (see (6.4.15))

$$\tilde{V}_\eta := \sigma_1 \left(\Phi_{\overline{P}}(f) \mp i\eta\chi(H_{\overline{P}})\Phi_{\overline{P}}(Df)\chi(H_{\overline{P}})\right). \tag{6.6.38}$$

We write

$$\mp i \frac{\mathrm{d}}{\mathrm{d}\eta} \left(F^{\pm\eta}(z_{\pm\epsilon}) - F_0^{\pm\eta}(z_{\pm\epsilon})\right) = g\left(W^{(1)} + W^{(2)} + W^{(3)}\right), \tag{6.6.39}$$

where

$$W^{(1)} := \rho \left(\pm i \frac{\mathrm{d}}{\mathrm{d}\eta} R^{\pm\eta}(z_{\pm\epsilon})\right) \tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho, \tag{6.6.40}$$

$$W^{(2)} := \rho R^{\pm\eta}(z_{\pm\epsilon}) \tilde{V}_\eta \left(\pm i \frac{\mathrm{d}}{\mathrm{d}\eta} R_0^{\pm\eta}(z_{\pm\epsilon})\right)\rho, \tag{6.6.41}$$

$$W^{(3)} := \rho R^{\pm\eta}(z_{\pm\epsilon}) \chi(H_{\overline{P}})\Phi_{\overline{P}}(Df)\chi(H_{\overline{P}}) R_0^{\pm\eta}(z_{\pm\epsilon})\rho. \tag{6.6.42}$$

Eqs. (6.6.5) and (6.6.3) yield that

$$W := W^{(1)} + W^{(2)} = \sum_{i=1}^{4} (W_i^{(1)} + W_i^{(2)}), \tag{6.6.43}$$

where

$$W_1^{(1)} := -\rho R^{\pm\eta}(z_{\pm\epsilon})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{6.6.44}$$

$$W_2^{(1)} := \rho R^{\pm\eta}(z_{\pm\epsilon})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{6.6.45}$$

$$W_3^{(1)} := \rho R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{6.6.46}$$

$$W_4^{(1)} := -\rho R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{6.6.47}$$

$$W_1^{(2)} := -\rho R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})[H_{0,\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{6.6.48}$$

$$W_2^{(2)} := \rho R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})[H_{0,\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{0,\overline{P}}) R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{6.6.49}$$

$$W_3^{(2)} := \rho R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{0,\overline{P}})[H_{0,\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{6.6.50}$$

$$W_4^{(2)} := -\rho R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{0,\overline{P}})[H_{0,\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{0,\overline{P}}) R_0^{\pm\eta}(z_{\pm\epsilon})\rho. \tag{6.6.51}$$

We observe from (6.6.39) that in order to complete the proof of statement (iii) it suffices to show that

$$\|W\| \leq C\eta^{-3/2} \qquad \text{and} \qquad \left\|W^{(3)}\right\| \leq C\eta^{-3/2}. \tag{6.6.52}$$

It follows from Proposition 6.2.5, (6.6.9) and similar estimates that that

$$\begin{aligned}\left\|\tilde{V}_\eta R^{\pm\eta}(z_{\pm\epsilon})\rho\right\| &\leq \left\|\tilde{V}_\eta (H_{f,\overline{P}}+i)^{-1}\right\|\left\|(H_{f,\overline{P}}+i)(H_{\overline{P}}+i)^{-1}\right\|\|(H_{\overline{P}}+i)R^{\pm\eta}(z_{\pm\epsilon})\rho\| \\ &\leq C\left(1+\eta^{-1/2}\|F^{\pm\eta}(z_{\pm\epsilon})\|^{1/2}\right)\end{aligned} \tag{6.6.53}$$

and similarly, using the adjoint operator, we find

$$\left\|\rho R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta\right\| \leq C\left(1+\eta^{-1/2}\|F^{\pm\eta}(z_{\pm\epsilon})\|^{1/2}\right). \tag{6.6.54}$$

This and (6.6.33) imply that

$$\left\|\tilde{V}_\eta R^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \leq C\eta^{-1/2} \qquad \text{and} \qquad \left\|\rho R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta\right\| \leq C\eta^{-1/2}. \tag{6.6.55}$$

Using additionally (6.6.18), we get

$$\left\|W_2^{(1)}\right\| \leq \|\rho R^{\pm\eta}(z_{\pm\epsilon})\|\left\|[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\overline{\chi}(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\right\|\left\|\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \leq C\eta^{-1}. \tag{6.6.56}$$

Eqs. (6.6.55), (6.6.6) and (6.6.7), and the fact that $[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0$ is $H_{\overline{P}}$-bounded (see Remark 6.4.6) imply that

$$\left\|W_3^{(1)}\right\| \leq \|R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}})\|\left\|[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R^{\pm\eta}(z_{\pm\epsilon})\right\|\left\|\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \leq C\eta^{-3/2}. \tag{6.6.57}$$

Moreover, we obtain from (6.6.55), (6.6.6) and (6.6.18) that

$$\left\|W_4^{(1)}\right\| \leq \|\rho R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}})\| \left\|[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\right\| \left\|\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \\ \leq C\eta^{-1/2}. \tag{6.6.58}$$

Analogously, we deduce that

$$\left\|W_2^{(2)}\right\|, \left\|W_3^{(2)}\right\|, \left\|W_4^{(2)}\right\| \leq C\eta^{-3/2}. \tag{6.6.59}$$

Next, we estimate the terms $W_1^{(1)}$ and $W_1^{(2)}$. For $\phi, \psi \in \mathcal{D}(\mathrm{d}\Gamma(D)) \cap \mathcal{D}(H_{\overline{P}})$, we find

$$\left\langle \phi, \left(W_1^{(1)} + W_1^{(2)}\right)\psi \right\rangle = A_1 + A_2 + A_3 + A_4, \tag{6.6.60}$$

where

$$\begin{aligned} A_1 :=& -\left\langle \left(H_{\overline{P}}^{\mp\eta} - z_{\mp\epsilon}\right) R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D) R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \\ &+ \left\langle (-i\mathrm{d}\Gamma(D))\, R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, \left(H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}\right) R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle, \\ A_2 :=& \mp i\eta\Big(\left\langle M^2 R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D) R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \\ &- \left\langle (-i\mathrm{d}\Gamma(D))\, R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, M^2 R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \Big), \\ A_3 :=& -\left\langle \left(H_{0,\overline{P}}^{\mp\eta} - z_{\mp\epsilon}\right) R_0^{\mp\eta}(z_{\mp\epsilon})(\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D) R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \\ &+ \left\langle (-i\mathrm{d}\Gamma(D))\, R_0^{\mp\eta}(z_{\mp\epsilon})(\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, \left(H_{0,\overline{P}}^{\pm\eta} - z_{\pm\epsilon}\right) R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle, \\ A_4 :=& \mp i\eta\Big(\left\langle M^2 R_0^{\mp\eta}(z_{\mp\epsilon})(\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D) R^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \\ &- \left\langle (-i\mathrm{d}\Gamma(D))\, R_0^{\mp\eta}(z_{\mp\epsilon})(\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, M^2 R^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \Big). \end{aligned} \tag{6.6.61}$$

This is possible because ρ maps the Hilbert space $\mathcal{H}$ into the domain of $\mathrm{d}\Gamma(D)$ and - by Lemma 6.9.3 - $R^{\pm}(z_{\pm}\epsilon)$, $(\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})$ and $V_\eta R^{\pm\eta}(z_{\pm\epsilon})$ preserve the domain of $\mathrm{d}\Gamma(D)$ (see above (6.6.11) – this holds true also for $g = 0$, see Remark 6.6.1). We estimate

$$|A_2| \leq \eta\|\phi\|\|\psi\| \left\|\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \|R^{\pm\eta}(z_{\pm\epsilon})\| \|R^{\mp\eta}(z_{\mp\epsilon})\rho\| \left\|[M^2, \mathrm{d}\Gamma(D)]^0\right\|. \tag{6.6.62}$$

Eqs. (6.6.55), (6.6.33), (6.6.9) and (6.6.7) imply that

$$|A_2| \leq C\|\phi\|\|\psi\|\eta^{-1}, \tag{6.6.63}$$

and analogously, we find

$$|A_4| \leq C\|\phi\|\|\psi\|\eta^{-1}. \tag{6.6.64}$$

As we argue above, Lemma 6.9.3 implies that $R^{\pm}(z_{\pm}\epsilon), (\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})$ and $V_\eta R^{\pm\eta}(z_{\pm\epsilon})$ preserve the domain of $\mathrm{d}\Gamma(D)$ (see above (6.6.11) – this holds true also for $g = 0$, see Remark 6.6.1). Moreover, the quadratic form $[i\mathrm{d}\Gamma(D), \tilde{V}_\eta]$ is represented by a $H_{\overline{P}}$-bounded operator that we denote by $[i\mathrm{d}\Gamma(D), \tilde{V}_\eta]^0$ (see Lemma 6.9.5). We obtain that

$$\begin{aligned} A_1 + A_3 = &- \left\langle \left(-i\mathrm{d}\Gamma(D)\right) \rho\phi, R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \\ &+ \left\langle R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, \left[\left(i\mathrm{d}\Gamma(D)\right), \tilde{V}_\eta\right]^0 R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \\ &+ \left\langle R_0^{\mp\eta}(z_{\mp\epsilon})(\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D)\rho\psi \right\rangle. \end{aligned} \tag{6.6.65}$$

It follows from (6.6.9), (6.6.33) and the fact that $[i\mathrm{d}\Gamma(D), \tilde{V}_\eta]^0$ is $H_{\overline{P}}$-bounded (see Lemma 6.9.5) that

$$\left| \left\langle R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, \left[\left(i\mathrm{d}\Gamma(D)\right), \tilde{V}_\eta\right]^0 R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \right| \leq \|\phi\| \|\psi\| \eta^{-1}. \tag{6.6.66}$$

We obtain from (6.6.55) and (6.6.7) that

$$\left| \left\langle \left(-i\mathrm{d}\Gamma(D)\right) \rho\phi, R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \right| \leq C\|\phi\| \|\psi\| \eta^{-3/2}, \tag{6.6.67}$$

$$\left| \left\langle R_0^{\mp\eta}(z_{\mp\epsilon})\tilde{V}_\eta^* R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D)\rho\psi \right\rangle \right| \leq C\|\phi\| \|\psi\| \eta^{-3/2}. \tag{6.6.68}$$

This together with (6.6.65) and (6.6.66) yield that

$$|A_1 + A_3| \leq C\|\phi\| \|\psi\| \eta^{-3/2}. \tag{6.6.69}$$

It follows from (6.6.69), (6.6.63), (6.6.64) and (6.6.60) that

$$\left\| W_1^{(1)} + W_1^{(2)} \right\| \leq C\eta^{-3/2}. \tag{6.6.70}$$

Collecting (6.6.43), (6.6.56), (6.6.57), (6.6.58), (6.6.59) and (6.6.70), we deduce that

$$\|W\| \leq C\eta^{-3/2}. \tag{6.6.71}$$

Eqs. (6.6.33) and (6.6.9) together with the $H_{0,\overline{P}}$-boundedness of $\Phi_{\overline{P}}(Df)$ yield that

$$\left\| W^{(3)} \right\| \leq C\eta^{-1}. \tag{6.6.72}$$

This together with (6.6.71) imply that (6.6.52) holds true and, thereby, we complete the proof of Item (iii).

(ii) The proof of Item (ii) follows the same line of arguments as the proof of Item (iii). In fact, it is simpler since the term $\tilde{V}_\eta$ does not appear.

□

Proof of Proposition 6.4.8 (iii). We estimate, for $z, z' \in I$,

$$\left\| F^0(z'_{\pm\epsilon}) - F^0_0(z_{\pm\epsilon}) \right\| \leq \left\| F^0(z'_{\pm\epsilon}) - F^0(z_{\pm\epsilon}) \right\| + \left\| F^0(z_{\pm\epsilon}) - F^0_0(z_{\pm\epsilon}) \right\|. \tag{6.6.73}$$

Hence, it suffices to show that

$$\left\| F^0(z'_{\pm\epsilon}) - F^0(z_{\pm\epsilon}) \right\| \leq C|z - z'|^{1/2}, \tag{6.6.74}$$

and

$$\left\| F^0(z_{\pm\epsilon}) - F^0_0(z_{\pm\epsilon}) \right\| \leq C g^{1/2}. \tag{6.6.75}$$

In the remainder of the proof we show (6.6.74) and (6.6.75). We start with the first estimate and obtain for $\tilde{\eta} \in (0, \boldsymbol{\eta})$

$$\begin{aligned} F^0(z'_{\pm\epsilon}) - F^0(z_{\pm\epsilon}) = &- \int_0^{\tilde{\eta}} \mathrm{d}\eta \, \frac{d}{d\eta} (F^\eta(z'_{\pm\epsilon}) - F^\eta(z_{\pm\epsilon})) \\ &- \int_{\tilde{\eta}}^{\boldsymbol{\eta}} \mathrm{d}\eta \, \frac{d}{d\eta} (F^\eta(z'_{\pm\epsilon}) - F^\eta(z_{\pm\epsilon})) + F^{\boldsymbol{\eta}}(z'_{\pm\epsilon}) - F^{\boldsymbol{\eta}}(z_{\pm\epsilon}). \end{aligned} \tag{6.6.76}$$

It follows from Lemma 6.6.3 (i) that

$$\left\| \int_0^{\tilde{\eta}} \mathrm{d}\eta \, \frac{d}{d\eta} (F^\eta(z'_{\pm\epsilon}) - F^\eta(z_{\pm\epsilon})) \right\| \leq C \tilde{\eta}^{1/2}. \tag{6.6.77}$$

Moreover, it follows from Lemma 6.6.3 (ii) that

$$\left\| \int_{\tilde{\eta}}^{\boldsymbol{\eta}} \mathrm{d}\eta \, \frac{d}{d\eta} (F^\eta(z'_{\pm\epsilon}) - F^\eta(z_{\pm\epsilon})) \right\| \leq C|z - z'| \tilde{\eta}^{-1/2}, \tag{6.6.78}$$

and it follows from the resolvent identity that there is a constant $C(\boldsymbol{\eta}) > 0$ such that

$$\| F^{\boldsymbol{\eta}}(z'_{\pm\epsilon}) - F^{\boldsymbol{\eta}}(z_{\pm\epsilon}) \| \leq C(\boldsymbol{\eta}) |z - z'|. \tag{6.6.79}$$

Note that, in principle, the constant $C(\boldsymbol{\eta})$ could depend on ϵ and z. However, this is not the case, see (6.6.7) Choosing $\tilde{\eta} = |z - z'|^{1/2}$, we get (6.6.74) from (6.6.76) – (6.6.79). Eq. (6.6.75) can be proven analogously employing item (iii) of Lemma 6.6.3 instead of item (ii). □

6.7. Proof of Lemma 6.4.5

In the following we recall the Helffer-Sjöstrand formula from [29, Section 2.2]. We use it in order to prove Lemma 6.4.5 and certain estimates in the proofs of Section 6.9.

Definition 6.7.1. *For $\upsilon \in C^\infty(\mathbb{R}, \mathbb{C})$, we define its almost analytic extension by*

$$\tilde{\upsilon} : \mathbb{C} \to \mathbb{C}, \qquad \tilde{\upsilon}(z) = \sigma(\operatorname{Re} z, \operatorname{Im} z) \sum_{r=0}^{n} \frac{(i \operatorname{Im} z)^r}{r!} \upsilon^{(r)}(\operatorname{Re} z), \tag{6.7.1}$$

where $n \in \mathbb{N}$, $\upsilon^{(r)}$ denotes the r-th derivative of υ and

$$\sigma(\operatorname{Re} z, \operatorname{Im} z) := \tau \left(\frac{\operatorname{Im} z}{\sqrt{(\operatorname{Re} z)^2 + 1}} \right) \tag{6.7.2}$$

for some $\tau \in C^\infty(\mathbb{R}, \mathbb{C})$ with $\tau(t) = 1$ for all $|t| < 1$ and $\tau(t) = 0$ for all $|t| > 2$. It follows from [29, Section 2.2] that

(i) $\tilde{\upsilon}$ is smooth as a function of $(\operatorname{Re} z, \operatorname{Im} z)$.

(ii) If υ is compactly supported, $|\partial_{\overline{z}} \tilde{\upsilon}(z)| \leq C |\operatorname{Im} z|^n$ (where $\frac{d}{d\overline{z}} = \frac{1}{2}(\frac{d}{dx} + i\frac{d}{dy})$, with $z = x + iy$).

Theorem 6.7.2 (Helffer-Sjöstrand formula)**.** *For every self-adjoint operator and any $\upsilon \in C_0^\infty(\mathbb{R}, \mathbb{C})$, the next formula holds true*

$$\upsilon(O) = \pi^{-1} \int_{\mathbb{C}} \mathrm{d}x \mathrm{d}y \, \partial_{\overline{z}} \tilde{\upsilon}(z)(O - z)^{-1}, \tag{6.7.3}$$

where $z = x + iy$, for $x, y \in \mathbb{R}$. Eq. (6.7.3) *does not depend on n and σ.*

Proof of Lemma 6.4.5. We only prove (6.4.14). Since (6.7.3) does not depend on σ, we choose $n = 2$ and, for $s > 0$, $\sigma_s(\operatorname{Re} z, \operatorname{Im} z) := \tau\left(\frac{1}{s}\frac{\operatorname{Im} z}{\sqrt{(\operatorname{Re} z)^2+1}}\right)$. We denote by $\tilde{\chi}_s$ the corresponding almost analytic extension of χ_s. It follows form (6.7.3) and the resolvent equation that

$$\|\chi_s(H) - \chi_s(H_0)\| = \pi^{-1} \Big\| \int_{\mathbb{C}} \mathrm{d}x \mathrm{d}y \, \partial_{\overline{z}} \tilde{\chi}_s(z)(H - z)^{-1} g V (H_0 - z)^{-1} \Big\|. \tag{6.7.4}$$

We calculate now

$$\partial_{\overline{z}} \tilde{\chi}_s(z) = \frac{1}{2} \sum_{r=0}^{2} \chi_s^{(r)}(x)(iy)^r / r! (\frac{\partial}{x}\sigma_s + i\frac{\partial}{\partial y}\sigma_s) + \frac{1}{2}\chi_s^{(n+1)}(x)(iy)^n / n! \sigma. \tag{6.7.5}$$

Notice that $\|(H-z)^{-1} g V \frac{1}{H_f+1}(H_f+1)(H_0-z)^{-1}\| \leq Cg\frac{1}{|y|^2}$. Moreover, $|\chi_s^{(r)}(x)|\,|y|^r|\frac{\partial}{x}\sigma_s + i\frac{\partial}{\partial y}\sigma_s| \leq C\frac{1}{s}\frac{|y|^r}{s^r}$, $|\chi_s^{(n+1)}(x)||y|^n|\sigma| \leq C\frac{1}{s^3}|y|^2$. This together with (6.7.4) yields

$$\begin{aligned} \|\chi_s(H) - \chi_s(H_0)\| \leq Cg \sum_{r=0}^{2} \int_{\operatorname{supp}(|\chi_s^{(r)}|\,|\frac{\partial}{x}\sigma_s + i\frac{\partial}{\partial y}\sigma_s|)} \mathrm{d}x \mathrm{d}y \, \frac{1}{s}\frac{|y|^{r-2}}{s^r} \\ + Cg \int_{\operatorname{supp}(|\chi_s^{(3)}||\sigma|)} \mathrm{d}x \mathrm{d}y \, \frac{1}{s^3}. \end{aligned} \tag{6.7.6}$$

For $y \in \mathbb{R}$, we observe that the diameter of the support of the functions $\mathbb{R} \ni x \mapsto |\chi_s^{(r)}(x)|\,|\frac{\partial}{x}\sigma_s(x,y) + i\frac{\partial}{\partial y}\sigma_s(x,y)|$ and $\mathbb{R} \ni x \mapsto \operatorname{supp}(|\chi_s^{(3)}(x)||\sigma(x,y)|)$ is of order s. Moreover, for $x \in \mathbb{R}$, we find that the diameter of the support of the function $\mathbb{R} \ni y \mapsto \operatorname{supp}(|\chi_s^{(3)}(x)||\sigma(x,y)|)$ is of order s. We conclude that

$$\|\chi_s(H) - \chi_s(H_0)\| \le C\frac{g}{s}, \tag{6.7.7}$$

which is the desired result. □

6.8. Standard results from Mourre theory

In this section we prove all assertions and estimates described at the beginning of Section 6.6, upto (6.6.9). We adapt the proofs of [28] to our model.

Lemma 6.8.1. *Recall $\chi \in C_c^\infty(\mathbb{R},[0,1])$ from Definition 6.4.4. For $g \ge 0, \boldsymbol{\eta} > 0$ sufficiently small, $\eta \in (0,\boldsymbol{\eta})$, $\epsilon \in (0,1)$, $z \in I$ and $z_{\pm\epsilon} := z \pm i\epsilon$, the following statements hold true:*

(i) The operator $R^{\pm\eta}(z_{\pm\epsilon})$ introduced in (6.6.4) exists and it is in $C^1((0,\boldsymbol{\eta}))$ and $C^0([0,\boldsymbol{\eta}))$ with respect to η. Moreover, the following identity holds true:

$$\mathrm{d}/\mathrm{d}\eta\, R^{\pm\eta}(z_{\pm\epsilon}) = \pm i R^{\pm\eta}(z_{\pm\epsilon})M^2R^{\pm\eta}(z_{\pm\epsilon}), \qquad \forall\eta \in (0,\boldsymbol{\eta}). \tag{6.8.1}$$

(ii)

$$\left\|(H_{\overline{P}} + i)\chi(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\psi\right\| \le C\eta^{-1/2}\left|\langle\psi, R^{\pm\eta}(z_{\pm\epsilon})\psi\rangle\right|^{1/2}. \tag{6.8.2}$$

(iii)

$$\left\|(H_{\overline{P}} + i)\overline{\chi}(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\right\| \le C, \tag{6.8.3}$$

where we recall that $\overline{\chi} = 1 - \chi$.

(iv)

$$\left\|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\right\| \le C/\eta. \tag{6.8.4}$$

(v)

$$\left\|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \le C\left(1 + \eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}\right). \tag{6.8.5}$$

The constants C above do not depend on η, ϵ, z and g, see Remark 6.3.3.

Proof. (i) Recall that $H_{\overline{P}}$ is a closed operator and M^2 is bounded (see Remark 6.4.6). Consequently, $H_{\overline{P}}^{\pm\eta}$ is closed. For $\psi \in \mathcal{D}(H_{\overline{P}})$, we observe that

$$\left\| \left(H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon} \right) \psi \right\|^2 = \left\| \left(H_{\overline{P}}^{\pm\eta} - z \right) \psi \right\|^2 + \epsilon^2 \|\psi\|^2 + 2\eta\epsilon \|M\psi\|^2, \tag{6.8.6}$$

and, thereby, the range of $H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}$ is closed and $H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}$ is injective. It also follows from the equation above that its inverse is bounded. Moreover, $\left(H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon} \right)^*$ fulfills a similar estimate and it is, therefore, injective. This implies that the range of $H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}$ is dense and because it is also closed, $H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}$ is surjective.

In addition, the resolvent identity yields that

$$R^{\pm\eta}(z_{\pm\epsilon}) - R^{\pm\eta_0}(z_{\pm\epsilon}) = \pm i(\eta - \eta_0) R^{\pm\eta}(z_{\pm\epsilon}) M^2 R^{\pm\eta_0}(z_{\pm\epsilon}). \tag{6.8.7}$$

It follows from (6.8.6) that there is a constant $C > 0$ (independent of η) such that $\|R^{\pm\eta}(z_{\pm\epsilon})\| \le C/\epsilon$. This together with (6.8.7) and the fact that M^2 is bounded implies that $R^{\pm\eta}(z_{\pm\epsilon})$ is continuous with respect to η, for $\eta \ge 0$, and differentiable for $\eta > 0$. Moreover, taking $\eta \to 0$ in (6.8.7) we get (6.8.1).

(ii) It follows from Lemma 6.4.7 that there is a constant $\alpha > 0$ such that for $\psi \in \mathcal{H}$

$$\begin{aligned}
&\left\|(H_{\overline{P}} + i)\chi(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\psi\right\|^2 = \left\langle \psi, R^{\pm\eta}(z_{\pm\epsilon})^* (H_{\overline{P}}^2 + 1)\chi^2(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\psi \right\rangle \\
&\le ((e_1+\delta)^2 + 1)\alpha^{-1} \left\langle \psi, R^{\pm\eta}(z_{\pm\epsilon})^* \alpha\chi^2(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\psi \right\rangle \\
&\le ((e_1+\delta)^2 + 1)(2\alpha\eta)^{-1} \left\langle \psi, R^{\pm\eta}(z_{\pm\epsilon})^* (2\eta M^2 + 2\epsilon) R^{\pm\eta}(z_{\pm\epsilon})\psi \right\rangle \\
&= ((e_1+\delta)^2 + 1)(2\alpha\eta)^{-1} \left\langle \psi, i(R^{\pm\eta}(z_{\pm\epsilon})^* - R^{\pm\eta}(z_{\pm\epsilon}))\psi \right\rangle \\
&\le ((e_1+\delta)^2 + 1)(\alpha\eta)^{-1} \left|\left\langle \psi, R^{\pm\eta}(z_{\pm\epsilon})\psi \right\rangle\right| .
\end{aligned} \tag{6.8.8}$$

This implies then statement (ii).

(iii) We calculate

$$\begin{aligned}
\overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon}) &= \overline{\chi}(H_{\overline{P}}) R^0(z_{\pm\epsilon}) \left(H_{\overline{P}} - z_{\pm\epsilon}\right) R^{\pm\eta}(z_{\pm\epsilon}) \\
&= \overline{\chi}(H_{\overline{P}}) R^0(z_{\pm\epsilon}) \left(1 \pm i\eta M^2 R^{\pm\eta}(z_{\pm\epsilon})\right).
\end{aligned} \tag{6.8.9}$$

It follows from Definition 6.4.4 and (6.6.1) that $\|\overline{\chi}(H_{\overline{P}}) R^0(z_{\pm\epsilon})\| \le 4/\delta$. Moreover,

$$\left\| H_{\overline{P}} \overline{\chi}(H_{\overline{P}}) R^0(z_{\pm\epsilon}) \right\| = \left\| \overline{\chi}(H_{\overline{P}}) + z_{\pm\epsilon} \overline{\chi}(H_{\overline{P}}) R^0(z_{\pm\epsilon}) \right\|. \tag{6.8.10}$$

We obtain that

$$\left\| (H_{\overline{P}} + i) \overline{\chi}(H_{\overline{P}}) R^0(z_{\pm\epsilon}) \right\| \le C. \tag{6.8.11}$$

This together with (6.8.9) and the boundedness of M^2 yields that

$$\left\|(H_{\overline{P}} + i)\overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\right\| \le C \left(1 + \eta \|R^{\pm\eta}(z_{\pm\epsilon})\|\right). \tag{6.8.12}$$

Statement (iii) follows then by (iv) which is proven below.

(iv) It follows from (ii) together with (6.8.12) that there are constants $C, \tilde{C} > 0$ such that

$$\begin{aligned} &1 + \left\|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\right\| \\ &\leq 1 + \left\|(H_{\overline{P}} + i)\overline{\chi}(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\right\| + \left\|(H_{\overline{P}} + i)\chi(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\right\| \\ &\leq 1 + \tilde{C}\left(1 + \eta\left\|R^{\pm\eta}(z_{\pm\epsilon})\right\|\right) + C\eta^{-1/2}\left\|R^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}. \end{aligned} \tag{6.8.13}$$

We fix $\boldsymbol{\eta} > 0$ sufficiently small such that $\tilde{C}\boldsymbol{\eta} \leq 1/2$ and $\tilde{C} + 1 \leq C\boldsymbol{\eta}^{-1/2}$. Then, employing $|x| + 1 \leq 2\sqrt{x^2+1}$ for all $x \in \mathbb{R}$, we conclude for $\eta \in (0, \boldsymbol{\eta})$

$$\begin{aligned} &1 + \left\|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\right\| \leq C\eta^{-1/2}\left(1 + \left\|R^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}\right) + \frac{1}{2}\left(1 + \left\|R^{\pm\eta}(z_{\pm\epsilon})\right\|\right) \\ &\leq 2C\eta^{-1/2}\left(1 + \left\|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\right\|\right)^{1/2} + \frac{1}{2}\left(1 + \left\|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\right\|\right). \end{aligned} \tag{6.8.14}$$

This yields then

$$1 + \left\|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\right\| \leq 4C\eta^{-1/2}\left(1 + \left\|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\right\|\right)^{1/2}, \tag{6.8.15}$$

and hence,

$$\left\|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\right\| \leq 16C^2\eta^{-1}, \tag{6.8.16}$$

which implies statement (iv).

(v) For $\psi \in \mathcal{H}$, we apply statement (ii) to the vector $\rho\psi \in \mathcal{H}$ and find that there is a constant $C > 0$ such that

$$\left\|(H_{\overline{P}} + i)\chi(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\rho\psi\right\| \leq C\eta^{-1/2}\left|\langle\psi, F^{\pm\eta}(z_{\pm\epsilon})\psi\rangle\right|^{1/2}, \tag{6.8.17}$$

which implies

$$\left\|(H_{\overline{P}} + i)\chi(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \leq C\eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}. \tag{6.8.18}$$

In addition, it follows from statement (iii) that

$$\left\|(H_{\overline{P}} + i)\overline{\chi}(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \leq C. \tag{6.8.19}$$

This together with (6.8.18) completes the proof of statement (v).

□

6.9. Domain properties and commutator estimates in Mourre theory

6.9.1. Domain properties in Mourre theory

In this section we prove auxiliary technical results that we need in Section 6.6. In particular, we prove that $R^{\pm\eta}(z_{\pm\epsilon})$ (see (6.6.4)) leaves the domain of $\mathrm{d}\Gamma(D)$ invariant –

this (and similar results) might be regarded as the main result of this section, see Lemma 6.9.3. In this work we do not use the standard strategy and we believe that our method is much simpler and more direct than the usual one: A novelty of our presentation is that we do not employ the usual techniques to study domain problems and commutators. The standard presentation of Mourre theory includes a scale of Hilbert spaces and a regularization of the generator of dilations in order to address domain problems (which is a technical and delicate issue – see [28]). In our case, instead of stating scales of Hilbert spaces explicitly and regularizing the generator of dilations, we directly dilate the operators at stake. We point out to the reader that the details of the arguments in this section are rarely found in the literature. A presentation of similar arguments may be found, e.g., in [42].

Definition 6.9.1. *Let B be a closed operator, defined in $\mathcal{H}$. For every $\beta \in \mathbb{R}$, we denote its dilation by*

$$B^{(\beta)} = e^{-i\beta \mathrm{d}\Gamma(D)} B e^{i\beta \mathrm{d}\Gamma(D)}. \tag{6.9.1}$$

For every function $h : \mathbb{R}^3 \to \mathbb{R}$ we denote by $h^{(\beta)}(k) := h(e^{\beta}k)$. A direct calculation shows that (see Definition 6.4.1)

$$H_{\overline{P}}^{(\beta)} = H_{\overline{P}}(\omega^{(\beta)}, u_\beta f), \quad (M^2)^{(\beta)} = \chi(H_{\overline{P}}^{(\beta)}) H_{\overline{P}}(\xi^{(\beta)}, u_\beta Df)\chi(H_{\overline{P}}^{(\beta)}), \tag{6.9.2}$$

see Remark 6.4.6, and (see (6.6.4)*)*

$$(H_{\overline{P}}^{\pm\eta})^{(\beta)} := H_{\overline{P}}^{(\beta)} \mp i\eta (M^2)^{(\beta)}, \qquad (R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)} = \Big((H_{\overline{P}}^{\pm\eta})^{(\beta)} - z_{\pm\epsilon} \Big)^{-1}. \tag{6.9.3}$$

Lemma 6.9.2. *Let B be a bounded operator in $\mathcal{H}$. Assume that the map $\beta \mapsto B^{(\beta)}$ is continuous at* 0 *and, for every $\phi \in \mathcal{D}(\mathrm{d}\Gamma(D))$, the limit*

$$\lim_{\beta \to 0} \frac{1}{\beta}(B^{(\beta)} - B)\phi \tag{6.9.4}$$

exists. Then, $\mathcal{D}(\mathrm{d}\Gamma(D))$ is invariant under B. In particular this holds true if the map $\beta \mapsto B^{(\beta)}$ is differentiable at 0*.*

Proof. We recall that $B\phi \in \mathcal{D}(\mathrm{d}\Gamma(D))$ if and only if the function $\beta \mapsto e^{-i\beta \mathrm{d}\Gamma(D)} B\phi$ is differentiable at 0. Set $\phi \in \mathcal{D}(\mathrm{d}\Gamma(D))$. We notice that the limit

$$\lim_{\beta \to 0} \frac{1}{\beta}(e^{-i\beta \mathrm{d}\Gamma(D)} - 1)B\phi = \lim_{\beta \to 0} \frac{1}{\beta}(B^{(\beta)} - B)\phi + B^{(\beta)}\frac{1}{\beta}(e^{-i\beta \mathrm{d}\Gamma(D)} - 1)\phi \tag{6.9.5}$$

exists because $\phi \in \mathcal{D}(\mathrm{d}\Gamma(D))$ (see (6.9.4) and above). □

Lemma 6.9.3. *The derivatives (recall* (6.6.38)*)*

$$\frac{\partial}{\partial \beta} \frac{1}{H_{\overline{P}}^{(\beta)} - \lambda}\Big|_{\beta=0}, \qquad \frac{d}{d\beta}\chi(H_{\overline{P}}^{(\beta)})\Big|_{\beta=0}, \qquad \frac{\partial}{\partial \beta}(R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}\Big|_{\beta=0}, \tag{6.9.6}$$

$$\frac{\partial}{\partial \beta}((\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon}))^{(\beta)}\Big|_{\beta=0}, \qquad \frac{\partial}{\partial \beta}(\tilde{V}_\eta R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}\Big|_{\beta=0}$$

exist, and therefore, the operators above leave $\mathcal{D}(\mathrm{d}\Gamma(D))$ invariant (see Lemma 6.9.2).

Proof. In this proof, we denote by a dot on the top of a symbol the derivative with respect to β at zero. If it is necessary, we specify below with respect to which norm is the derivative taken. For example, the (point-wise) derivative of $u_\beta f$ with respect to β at *zero* is denoted by $(\dot{u_\beta f})$. In case that the dependence on β is written as a superscript, we sometimes omit the symbol β. For example the (point-wise) derivative of $\xi^{(\beta)}$ at zero is denoted by $\dot{\xi}$.

Moreover, we recall from (1.2.1), (1.2.2) and (1.2.5) that H_0 and H_f depends on ω, V depends on f and H depends on ω and f. Therefore, in the remainder of the proof, we write this dependence explicitly: $H_0 \equiv H_0(\omega)$, $H_f \equiv H_f(\omega)$, $V \equiv V(f)$ and $H \equiv H(\omega, f)$.

A simple calculation shows that

$$\left\|\beta^{-1}(f^{(\beta)} - f) - (\dot{u_\beta f})\right\| \leq C|\beta|, \quad \left|\beta^{-1}\left(\omega^{(\beta)}(k) - \omega(k)\right) - \dot{\omega}(k)\right| \leq C|\beta|\omega(k). \quad (6.9.7)$$

This together with Proposition 6.2.5 (see also (6.9.2) and similar calculations) implies that

$$\left\|\Big(\frac{1}{\beta}(H_{\overline{P}}(\omega, f)^{(\beta)} - H_{\overline{P}}) - H_{\overline{P}}(\dot{\omega}, (\dot{u_\beta f}))\Big)\frac{1}{H_f + 1}\right\| \leq C|\beta|. \quad (6.9.8)$$

Then, the second resolvent identity and Proposition 6.2.5 imply that, for every $\lambda \in \mathbb{C}$ with not vanishing imaginary part (here we proceed as in (6.9.12) below),

$$\frac{\partial}{\partial\beta}\frac{1}{H_{\overline{P}}^{(\beta)} - \lambda}|_{\beta=0} = -\frac{1}{H_{\overline{P}} - \lambda}H_{\overline{P}}(\dot{\omega}, (\dot{u_\beta f}))\frac{1}{H_{\overline{P}} - \lambda}, \quad (6.9.9)$$

and therefore, we obtain that the derivative in the left term of the first line in (6.9.6) exists. Similar proofs (and formulas) hold for $H_f\frac{1}{H^{(\beta)}-\lambda}$ and $\frac{1}{H^{(\beta)}-\lambda}H_f$. Eq. (6.9.9) and the second resolvent equation (used as in (6.9.12) below) allows us to analyze the resolvents in the integrand in the Helffer-Sjöstrand formula ((6.7.3), with $n > 3$) and get (see also Proposition 6.2.5)

$$\frac{d}{d\beta}(H_f + 1)\chi(H_{\overline{P}}^{(\beta)})|_{\beta=0} = \pi^{-1}\int_{\mathbb{C}} \mathrm{d}x\mathrm{d}y\, \partial_{\overline{z}}\,\tilde{\chi}(z)\frac{\partial}{\partial\beta}(H_f + 1)\frac{1}{H_{\overline{P}}^{(\beta)} - \lambda}, \quad (6.9.10)$$

where $z = x + iy$. This implies that the derivative in the middle term of the first line in (6.9.6) exists. Similarly as in (6.9.8), we obtain that

$$\frac{d}{d\beta}H(\xi^{(\beta)}, u_\beta Df)^{(\beta)}\frac{1}{H_f + 1}|_{\beta=0} = H(\dot{\xi}, (\dot{u_\beta Df}))\Big)\frac{1}{H_f + 1}. \quad (6.9.11)$$

Eqs. (6.9.10) and (6.9.11) imply that $(M^2)^{(\beta)}$ is differentiable with respect to β at $\beta = 0$ (see (6.9.2)). This and (6.9.8) imply that $(H_{\overline{P}}^{\pm\eta})^{(\beta)}\frac{1}{H_f+1}$ is differentiable with respect to β at $\beta = 0$. Now we calculate the derivative of $(H_f + 1)(R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}$ at zero using the second resolvent equation:

$$\begin{aligned}
&\frac{1}{\beta}(H_f+1)\Big((R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}-R^{\pm\eta}(z_{\pm\epsilon})\Big) \qquad (6.9.12)\\
&\quad+(H_f+1)R^{\pm\eta}(z_{\pm\epsilon})\Big[\frac{\partial}{\partial\beta}(H^{\pm\eta}_{\overline{P}})^{(\beta)}\frac{1}{H_f+1}|_{\beta=0}\Big](H_f+1)R^{\pm\eta}(z_{\pm\epsilon})\\
&=\Big\{(H_f+1)R^{\pm\eta}(z_{\pm\epsilon})\Big\}\Big\{\Big(\frac{1}{\beta}(H^{\pm\eta}_{\overline{P}}-(H^{\pm\eta}_{\overline{P}})^{(\beta)})\frac{1}{H_f+1}+\Big[\frac{\partial}{\partial\beta}(H^{\pm\eta}_{\overline{P}})^{(\beta)}\frac{1}{H_f+1}|_{\beta=0}\Big)\Big\}\\
&\qquad\qquad\cdot\Big\{(H_f+1)R^{\pm\eta}(z_{\pm\epsilon})\Big\}\\
&\quad+\Big\{(H_f+1)((R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}-R^{\pm\eta}(z_{\pm\epsilon}))\Big\}\Big\{\frac{1}{\beta}(H^{\pm\eta}_{\overline{P}}-(H^{\pm\eta}_{\overline{P}})^{(\beta)})\frac{1}{H_f+1}\Big\}\\
&\qquad\qquad\cdot\Big\{(H_f+1)R^{\pm\eta}(z_{\pm\epsilon})\Big\}.
\end{aligned}$$

It follows from Proposition 6.2.5 and (6.6.7) that $(H_f+1)R^{\pm\eta}(z_{\pm\epsilon})$ is bounded. This and the fact that $(H^{\pm\eta}_{\overline{P}})^{(\beta)}\frac{1}{H_f+1}$ is differentiable with respect to β at $\beta=0$ imply that the first term in the right hand side side of (6.9.12) tends to zero as β goes to zero. The same arguments and the fact that

$$(H_f+1)(R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}=\Big((H_f+1)\frac{}{(H_f+1)^{(\beta)}}\Big)\Big((H_f+1)(R^{\pm\eta}(z_{\pm\epsilon})\Big)^{(\beta)} \qquad (6.9.13)$$

is uniformly bounded for small β (see Proposition 6.2.5 and (6.6.7)) imply that the second term in the right hand side of (6.9.12) is bounded (uniformly with respect to β). Since the second term in the left hand side of (6.9.12) is bounded (see arguments above), it follows that

$$\lim_{\beta\to 0}(H_f+1)\Big((R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}-R^{\pm\eta}(z_{\pm\epsilon})\Big)=0. \qquad (6.9.14)$$

This in turn and the arguments above imply that the second term in the right hand side of (6.9.12) tends to zero as β tends to zero. We conclude that the left hand side of (6.9.12) tends to zero as β tends to zero and, therefore, $(H_f+1)(R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}$ is differentiable at zero. This proves the existence of the derivative in the right term of the first line in (6.9.6). The proof that the derivative of $(\tilde{V}_\eta)^{(\beta)}\frac{1}{H_f+1}$, with respect to β, at zero exists follows exactly the same lines as the corresponding result for $(H^{\pm\eta}_{\overline{P}})^{(\beta)}\frac{1}{H_f+1}$, and therefore, we omit it. Then, using this and that $(H_f+1)(R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}$ is differentiable at zero, we obtain that $\Big(\tilde{V}_\eta\frac{1}{1+H_f}\Big)\Big((1+H_f)R^{\pm\eta}(z_{\pm\epsilon})^{(\beta)}\Big)$ is differentiable at zero. This proves the existence of the derivative in the right term of the second line in (6.9.6). The proof for the left term is analogous. □

6.9.2. Commutator estimates in Mourre theory

Lemma 6.9.4. *Recall that we introduce $\chi\in C_c^\infty(\mathbb{R},[0,1])$ in Definition 6.4.4. The quadratic form $[\chi(H_{\overline{P}}),\mathrm{d}\Gamma(D)]$, defined in the domain of $\mathrm{d}\Gamma(D)$, extends to a bounded* *operator that we denote by* $[\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]^0$. *Additionally,* $(H_{\overline{P}}+i)[\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]^0$ *is bounded.*

Proof. For $\psi, \phi \in \mathcal{D}(\mathrm{d}\Gamma(D)) \cap \mathcal{D}(H_{\overline{P}})$ and $z \in \mathbb{C}\setminus\mathbb{R}$, it follows from Lemma 6.9.3 that

$$\begin{aligned}\left\langle \phi, [(H_{\overline{P}}-z)^{-1}, \mathrm{d}\Gamma(D)]\psi \right\rangle &= \left\langle \mathrm{d}\Gamma(D)(H_{\overline{P}}-\overline{z})^{-1}\phi, (H_{\overline{P}}-z)(H_{\overline{P}}-z)^{-1}\psi \right\rangle \\ &\quad - \left\langle (H_{\overline{P}}-\overline{z})(H_{\overline{P}}-\overline{z})^{-1}\phi, \mathrm{d}\Gamma(D)(H_{\overline{P}}-z)^{-1}\psi \right\rangle \\ &= -\left\langle \phi, (H_{\overline{P}}-\overline{z})^{-1}[H_{\overline{P}}, \mathrm{d}\Gamma(D)]^0(H_{\overline{P}}-z)^{-1}\psi \right\rangle . \end{aligned} \tag{6.9.15}$$

Note that

$$\left\|(H_{\overline{P}}+i)(H_{\overline{P}}-z)^{-1}\right\| \le 1 + \left\|(z+i)(H_{\overline{P}}-z)^{-1}\right\| \le C\left(1+|\operatorname{Re} z||\operatorname{Im} z|^{-1}\right). \tag{6.9.16}$$

Then, we observe from Remark 6.4.6 that

$$\left|\left\langle \phi, [(H_{\overline{P}}-z)^{-1}, \mathrm{d}\Gamma(D)]\psi \right\rangle\right| \le C\|\phi\|\|\psi\||\operatorname{Im} z|^{-1}\left(1+|\operatorname{Re} z||\operatorname{Im} z|^{-1}\right), \tag{6.9.17}$$

and consequently, $[(H_{\overline{P}}-z)^{-1}, \mathrm{d}\Gamma(D)]$ uniquely extends to a bounded operator on $\mathcal{H}$ which we denote by $[(H_{\overline{P}}-z)^{-1}, \mathrm{d}\Gamma(D)]^0$ and

$$[(H_{\overline{P}}-z)^{-1}, \mathrm{d}\Gamma(D)]^0 = -(H_{\overline{P}}-z)^{-1}[H_{\overline{P}}, \mathrm{d}\Gamma(D)]^0(H_{\overline{P}}-z)^{-1}. \tag{6.9.18}$$

This together with Remark 6.4.6, (6.9.16) and the Helffer-Sjöstrand formula (see (6.7.3)) yields

$$\begin{aligned}&\left\|(H_{\overline{P}}+i)[\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]^0\right\| \le \pi^{-1}\int_{\mathbb{C}} \mathrm{d}x\mathrm{d}y\, |\partial_{\overline{z}}\tilde{\chi}(z)| \left\|(H_{\overline{P}}+i)[(H_{\overline{P}}-z)^{-1}, \mathrm{d}\Gamma(D)]^0\right\| \\ &\le \pi^{-1}\int_{\mathbb{C}} \mathrm{d}x\mathrm{d}y\, |\partial_{\overline{z}}\tilde{\chi}(z)| \left\|(H_{\overline{P}}+i)(H_{\overline{P}}-z)^{-1}\right\|^2 \left\|[H_{\overline{P}}, \mathrm{d}\Gamma(D)]^0(H_{\overline{P}}-i)^{-1}\right\| \\ &\le C\int_{\mathbb{C}} \mathrm{d}x\mathrm{d}y\, |\partial_{\overline{z}}\tilde{\chi}(z)| \left(1+|x||y|^{-1}\right)^2, \end{aligned} \tag{6.9.19}$$

where we take $z = x+iy$ for $x, y \in \mathbb{R}$ and $\tilde{\chi}$ is the almost analytic extension of χ (see Definition 6.7.1). In the definition of $\tilde{\chi}$ we choose $n \ge 2$ and, therefore, $|\partial_{\overline{z}}\tilde{\chi}(z)| \le C|\operatorname{Im} z|^2$. Since χ is compactly supported, then $\tilde{\chi}$ is also compactly supported. It follows that

$$\left\|(H_{\overline{P}}+i)[\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]^0\right\| \le C\int_{\mathrm{supp}\,(\tilde{\chi})} dxdy|y|^2\left(1+|x||y|^{-1}\right)^2 \le C. \tag{6.9.20}$$

This completes the proof. □

Lemma 6.9.5. *Recall that we introduce* $\chi \in C_c^\infty(\mathbb{R}, [0,1])$ *in Definition 6.4.4 and* M^2 *in* (6.6.3). *The quadratic form* $[\mathrm{d}\Gamma(D), M^2]$*, defined in the domain of* $\mathrm{d}\Gamma(D)$*, extends to a bounded operator that we denote by* $[\mathrm{d}\Gamma(D), M^2]^0$. *Similarly, the quadratic form* $[i\mathrm{d}\Gamma(D), \tilde{V}_\eta]$ *extends to a* $H_{\overline{P}}$*-bounded operator that we denote by* $[i\mathrm{d}\Gamma(D), \tilde{V}_\eta]^0$.

Proof. For $\phi, \psi \in \mathcal{D}(\mathrm{d}\Gamma(D)) \cap \mathcal{D}(H_{\overline{P}})$, we observe from Lemma 6.9.3 and the $H_{\overline{P}}$-boundedness of $[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0$ that

$$\begin{aligned} &\left\langle \mathrm{d}\Gamma(D)\phi, M^2\psi \right\rangle - \left\langle M^2\phi, \mathrm{d}\Gamma(D)\psi \right\rangle \qquad (6.9.21)\\ &= \left\langle [\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]\phi, [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\psi \right\rangle + \left\langle \mathrm{d}\Gamma(D)\chi(H_{\overline{P}})\phi, [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\psi \right\rangle \\ &- \left\langle [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\phi, [\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]\psi \right\rangle - \left\langle [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\phi, \mathrm{d}\Gamma(D)\chi(H_{\overline{P}})\psi \right\rangle . \end{aligned}$$

It follows from Lemma 6.4.4 and Remark 6.4.6 that

$$\left| \left\langle [\mathrm{d}\Gamma(D), \chi(H_{\overline{P}})]\phi, [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\psi \right\rangle \right| \le C\|\phi\|\|\psi\| \qquad (6.9.22)$$

and

$$\left| \left\langle [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\phi, [\mathrm{d}\Gamma(D), \chi(H_{\overline{P}})]\psi \right\rangle \right| \le C\|\phi\|\|\psi\|. \qquad (6.9.23)$$

Moreover, for $\varphi, \vartheta \in \mathcal{F}_{\mathrm{fin}}[\mathfrak{h}_0]$, we obtain from Lemma 6.4.3 (iv) and (v) that

$$\begin{aligned} &\left\langle \mathrm{d}\Gamma(D)\varphi, [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\vartheta \right\rangle - \left\langle [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\varphi, \mathrm{d}\Gamma(D)\vartheta \right\rangle \\ &= \langle \varphi, [\mathrm{d}\Gamma(D), (\mathrm{d}\Gamma_{\overline{P}}(\xi) + g\sigma_1\Phi_{\overline{P}}(Df))]\vartheta \rangle = \left\langle \varphi, \left(\mathrm{d}\Gamma_{\overline{P}}(\tilde{\xi}) - ig\sigma_1\Phi_{\overline{P}}(D^2f)\right)\vartheta \right\rangle , \qquad (6.9.24) \end{aligned}$$

where $\tilde{\xi} = [D, \xi]$. Direct calculations show that $|\tilde{\xi}| \le C\omega$ and $D^2 f \in \mathfrak{h}$. Proposition 6.2.5 implies that $\left(\mathrm{d}\Gamma_{\overline{P}}(\tilde{\xi}) - ig\sigma_1\Phi_{\overline{P}}(D^2 f)\right)$ is relatively bounded with respect to $H_{\overline{P}}$ and, hence, $[\mathrm{d}\Gamma(D), [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0]$ extends to a $H_{\overline{P}}$-bounded operator which we denote by $[\mathrm{d}\Gamma(D), [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0]^0 = \mathrm{d}\Gamma_{\overline{P}}(\tilde{\xi}) - ig\sigma_1\Phi_{\overline{P}}(D^2 f)$. Employing Lemma 6.9.3, we find a constant $C > 0$ such that

$$\begin{aligned} &\left| \left\langle \mathrm{d}\Gamma(D)\chi(H_{\overline{P}})\phi, [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\psi \right\rangle - \left\langle [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\phi, \mathrm{d}\Gamma(D)\chi(H_{\overline{P}})\psi \right\rangle \right| \\ &= \left| \left\langle \chi(H_{\overline{P}})\phi, \left[\mathrm{d}\Gamma(D), [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\right]^0 \chi(H_{\overline{P}})\psi \right\rangle \right| \le C\|\phi\|\|\psi\|. \qquad (6.9.25) \end{aligned}$$

This together with (6.9.21), (6.9.23) and (6.9.22) implies that there is a constant $C > 0$ such that

$$\left| \left\langle \mathrm{d}\Gamma(D)\phi, M^2\psi \right\rangle - \left\langle M^2\phi, \mathrm{d}\Gamma(D)\psi \right\rangle \right| \le C\|\phi\|\|\psi\|, \qquad (6.9.26)$$

and, thereby, we complete the proof, since $\mathcal{D}(\mathrm{d}\Gamma(D)) \cap \mathcal{D}(H_{\overline{P}})$ is dense in $\mathcal{H}$. The statement concerning $[i\mathrm{d}\Gamma(D), \tilde{V}_\eta]$ is proved following the same lines above. □

7. Outlook

In this last chapter, we conclude with some thoughts and questions that arose during our analysis in the previous chapters that seem worthwhile to be addressed in future research projects. Moreover, we give a brief assessment of the respective difficulties that are to be expected.

Formula for multi-boson scattering processes: One further aim could be to find a formula for multi-boson scattering processes, which are of higher order with respect to the coupling constant. From a physical perspective, it could be interesting to examine which processes are dominating and which are suppressed. This would provide rigorous versions of the well-known exclusion rules. As mentioned above, a major step in this direction has already been achieved in [12], where they provide an expansion of the scattering amplitudes with respect to powers the fine-structure constant for the Pauli-Fierz model. Starting from their results, we expect that this goal can be achieved by the methods presented in this work. However, we encountered difficulties in a potential derivation of an exact version the scattering formula for multi-boson processes. The root cause can already be observed in the intermediate scattering formula, which has been presented in Theorem 2.2.2. In a multi-boson setting, this formula produces a lot of terms, which all require their own careful treatment in view of the long time estimates.

Scattering formulas for n-level systems: Another aim could be to study a generalized version of the Spin-Boson model, where one models an atom with more than two energy levels, i.e., the replacing the atomic part of the Hamiltonian by a n-by-n matrix instead of a 2-by-2 matrix (compare to, e.g., [24]). From a physical point of view, this is an interesting situation since such a model allows for more complicated scattering processes than the model studied in this work. For the latter, heuristically, there are only two possible processes: first, the atom is in its lower energy state, absorbs a boson, and thereby, gets flipped to the excited energy state, and second, the atom is in the excited energy state, emits a boson, and thereby, gets flipped to the lower energy state. In the generalized model with more than two energy levels, additional processes will show up. For example, the processes mentioned in the paragraph above could happen in cascades. It would be interesting to analyze those additional processes and prove that some of them are dominating while others are suppressed as know from physics. Even though the study of a n-level system allows for more processes compared to the 2-level system, we presume that the respective terms look similar. Therefore, we expect that this aim could be achieved with the methods described in this work.

Removing the slight infrared-regularization: Recall that, for the massless case, we studied an infrared-regularized model. In particular, we chose the boson form factor f (defined in (1.2.3)) to have a slightly less singular behavior for small momenta. Consequently, in Chapter 4, we also obtained slightly better key estimates as for the case of no infrared-regularization, which are used in the induction step of the multiscale analysis for the construction of the resonance and the ground-state (see, e.g., Lemma 4.4.11). The corresponding estimates of the dilated resolvent operator are later exploited to control the time-evolution operator in the scattering regime (see, e.g., Lemma 5.3.8). Hence, for the case of no infrared-regularization, these estimates would have to be improved. As shown in [8], there are already techniques to treat this case, e.g., for the construction of the ground-state. The employed multiscale method relies on a certain symmetry of the model which guarantees that the most singular terms vanish. A further research goal could be to extend their method to construct the resonances and control the time-evolution operator such that the formula for the scattering matrix elements (see Theorems 3.0.1 and 3.0.3) can be recovered.

Relaxing Assumption 6.0.1: In Assumption 6.0.1, we suppose that the mass of the scalar field, m, can not be a multiple of the energy gap of the two bound states of the free system $e_1 - e_0$. Heuristically, this excludes situations where an atom being in its lower energy state gets flipped to the excited energy state by a certain number of boson with zero momentum. The question whether this condition can be relaxed is of both physical and mathematical interest. The latter arises since, in the critical case (i.e. $m = n(e_1 - e_0)$ for some $n \in \mathbb{N}$), we find that the dilated free Hamiltonian does not exhibit a spectral gap around e_1. Hence, one has to overcome similar obstacles as in the massless model and we expect that they can be handled by the methods described in this work. From a physical perspective, it would be interesting to know whether this condition is again (only) a technical issue or if the physical properties will change in this case.

Scattering formulas for other models: An obvious research goal is to derive a similar scattering formula, as presented in Theorems 3.0.1 and 3.0.3, in other models of quantum field theory, which provides a deeper physical interpretation than the Spin-Boson model at hand. One could have several such models in mind: for example, the Pauli-Fierz model, the Yukawa model, and ultimately relativistic quantum electrodynamics. One of the crucial ingredients for the proofs of our scattering formulas (Theorems 3.0.1 and 3.0.3) is the good understanding of certain spectral properties of the model since we employ them to control the time-evolution operator in the scattering regime. Regarding this issue, several results and techniques can be found in the literature. For example, in [10], the Pauli-Fierz model is studied and the methods used therein are similar to some of the ones presented in Chapter 4 (and [21]). There is a new obstacle when considering a more complicated model. The proof of the exact scattering formula for the massless model (see Theorem 3.0.3), strongly relies on a special symmetry of the Spin-Boson model, namely, the fact that $\sigma_1 \Psi_{\lambda_0}$ is orthogonal to Ψ_{λ_0}. This symmetry is not (necessarily) present in other models of quantum field theory.

Analyzing the ultraviolet behavior: Maybe one of the most interesting questions in this context is to study also the ultraviolet properties of the model. In particular, a goal could be to construct the ground-state and the resonance for a fixed (but small) coupling constant but for arbitrary large ultraviolet cut-offs. Once this is achieved, one could examine the dependence of the physically relevant entities, such as the energy difference of the excited state and the ground-state and the decay rate of the excited state (given by the imaginary part of the resonance), on the ultraviolet cut-off. Note that Pizzo's multiscale method (see Chapter 4 and [10, 21]), which has been introduced in order to solve the so-called infrared problem, can interestingly also be applied to analyze ultraviolet properties. This has been shown in [33] in order to construct the ground-state of the Yukawa model for an uniformly fixed coupling constant and arbitrary large ultraviolet cut-offs. Unfortunately, it is not obvious how their methods apply to resonances, and therefore, we expect this problem to require substantial efforts.

A. Fourier transform of the Heaviside distribution

In this section we present a detailed proof of the formula for the Fourier transform of the Heaviside distribution presented in Lemma 5.3.6. This section is drawn from [22].

Proof of Lemma 5.3.6. For $\alpha > 0$, we define $g_\alpha \in S'(\mathbb{R}, \mathbb{C})$ by

$$g_\alpha : S(\mathbb{R}, \mathbb{C}) \to \mathbb{C}, \qquad \varphi \mapsto g_\alpha(\varphi) = \int_0^\infty \mathrm{d}x \, e^{-\alpha x} \varphi(x). \tag{A.0.1}$$

It follows from Definition 5.3.3 that for $\varphi \in S(\mathbb{R}, \mathbb{C})$

$$\mathfrak{F}[g_\alpha](\varphi) := g_\alpha\left(\mathfrak{F}[\varphi]\right) = \int_0^\infty \mathrm{d}x \, e^{-\alpha x} \mathfrak{F}[\varphi](x) = \int_0^\infty \mathrm{d}x \, e^{-\alpha x} \int_{\mathbb{R}} \mathrm{d}s \, \varphi(s) e^{-isx}. \tag{A.0.2}$$

The integrand on the right-hand side of (A.0.2) is absolutely integrable because of $\varphi \in S(\mathbb{R}, \mathbb{C})$, and hence, the Fubini-Tonelli theorem yields that

$$\mathfrak{F}[g_\alpha](\varphi) = \int_{\mathbb{R}} \mathrm{d}s \, \varphi(s) \int_0^\infty \mathrm{d}x \, e^{-x(\alpha + is)}. \tag{A.0.3}$$

This together with

$$\int_0^\infty \mathrm{d}x \, e^{-x(\alpha+is)} = \frac{1}{\alpha + is} = \frac{\alpha}{(\alpha^2 + s^2)} - i \frac{s}{(\alpha^2 + s^2)} \tag{A.0.4}$$

implies that

$$\mathfrak{F}[g_\alpha](\varphi) = \int_{\mathbb{R}} \mathrm{d}s \, \frac{\alpha}{(\alpha^2 + s^2)} \varphi(s) - i \int_{\mathbb{R}} \mathrm{d}s \, \frac{s}{(\alpha^2 + s^2)} \varphi(s) =: G_\alpha^{(1)}(\varphi) - i G_\alpha^{(2)}(\varphi). \tag{A.0.5}$$

where

$$G_\alpha^{(1)}(\varphi) = \int_{\mathbb{R}} \mathrm{d}s \, \frac{\alpha}{(\alpha^2 + s^2)} \varphi(s) \tag{A.0.6}$$

and

$$G_\alpha^{(2)}(\varphi) = \int_{\mathbb{R}} \mathrm{d}s \, \frac{s}{(\alpha^2 + s^2)} \varphi(s). \tag{A.0.7}$$

Using coordinate transformation $s \to \alpha s$ yields then that

$$\lim_{\alpha \to 0^+} G_\alpha^{(1)}(\varphi) = \lim_{\alpha \to 0^+} \int_{\mathbb{R}} \mathrm{d}s \, \frac{\varphi(\alpha s)}{1 + s^2} = \varphi(0) \int_{\mathbb{R}} \mathrm{d}s \, \frac{1}{1 + s^2} = \pi \varphi(0) = \pi \delta(\varphi). \tag{A.0.8}$$

The second step follows from the dominated convergence theorem together with the continuity of φ. Moreover, we have

$$G_\alpha^{(2)}(\varphi) = G_\alpha^{(2,1)}(\varphi) + G_\alpha^{(2,2)}(\varphi), \tag{A.0.9}$$

where

$$G_\alpha^{(2,1)}(\varphi) := \int_{\mathbb{R}\setminus[-\alpha^8,\alpha^8]} \mathrm{d}s \, \frac{s}{(\alpha^2+s^2)}\varphi(s) \tag{A.0.10}$$

and

$$G_\alpha^{(2,2)}(\varphi) := \int_{-\alpha^8}^{\alpha^8} \mathrm{d}s \, \frac{s}{(\alpha^2+s^2)}\varphi(s). \tag{A.0.11}$$

We treat these two terms separately. At first, we obtain

$$\begin{aligned} \left|G_\alpha^{(2,2)}(\varphi)\right| &\leq \int_{-\alpha^8}^{\alpha^8} \mathrm{d}s \left|\frac{s}{(\alpha^2+s^2)}\left(\varphi(s)-\varphi(0)\right)\right| + |\varphi(0)| \left|\int_{-\alpha^8}^{\alpha^8} \mathrm{d}s \, \frac{s}{(\alpha^2+s^2)}\right| \\ &\leq 2\alpha^{14} \sup_{s\in[-\alpha^8,\alpha^8]} |\varphi(s)-\varphi(0)| + \frac{|\varphi(0)|}{2}\left|\int_{-\alpha^{16}}^{\alpha^{16}} \mathrm{d}s \, \frac{1}{\alpha^2+s}\right| \end{aligned} \tag{A.0.12}$$

where we have used the coordinate transformation $s' = s^2$ for the second term in the last line. Then, we obtain

$$\left|G_\alpha^{(2,2)}(\varphi)\right| \leq 2\alpha^{14} \sup_{s\in[-\alpha^8,\alpha^8]} |\varphi(s)-\varphi(0)| + \frac{\varphi(0)}{2}\left|\ln\left(1+\alpha^8\right) - \ln\left(1-\alpha^8\right)\right|. \tag{A.0.13}$$

Note that $\ln(\cdot)$ is continuous close to 1 and $\sup_{s\in[-\alpha^8,\alpha^8]} |\varphi(s)-\varphi(0)| < \infty$ since a continuous function has a maximum on a compact set. We conclude

$$\lim_{\alpha\to 0^+} G_\alpha^{(2,2)}(\varphi) = 0. \tag{A.0.14}$$

Finally, for some $R > 0$, we obtain

$$\begin{aligned} G_\alpha^{(2,1)}(\varphi) = &\int_{[-R,R]\setminus[-\alpha^8,\alpha^8]} \mathrm{d}s \, \frac{s}{(\alpha^2+s^2)}\left(\varphi(s)-\varphi(0)\right) + \int_{[-R,R]\setminus[-\alpha^8,\alpha^8]} \mathrm{d}s \, \frac{s}{(\alpha^2+s^2)}\varphi(0) \\ &+ \int_{\mathbb{R}\setminus[-R,R]} \mathrm{d}s \, \frac{s}{(\alpha^2+s^2)}\varphi(s). \end{aligned} \tag{A.0.15}$$

Note that the second term vanishes independently of R because of symmetry, and further, the mean value theorem implies that

$$|\varphi(s)-\varphi(0)| \leq |s|\|\varphi'\|_\infty. \tag{A.0.16}$$

Altogether, this yields that

$$|G_\alpha^{(2,1)}(\varphi)| \leq 2R\|\varphi'\|_\infty + \int_{\mathbb{R}\setminus[-R,R]} \mathrm{d}s \, |\varphi(s)|/|s| < \infty, \tag{A.0.17}$$

This allows us to apply the dominated convergence theorem which yields that

$$\lim_{\alpha\to 0^+} G_\alpha^{(2,1)}(\varphi) = \mathrm{PV}\int_{\mathbb{R}} \mathrm{d}s\, \frac{1}{s}\varphi(s) = \left(\mathrm{PV}\left(\frac{1}{\bullet}\right)\right)(\varphi). \tag{A.0.18}$$

This together with (A.0.14), (A.0.9), (A.0.8) and (A.0.5) implies that

$$\lim_{\alpha\to 0^+} \mathfrak{F}[g_\alpha](\varphi) = \pi\delta(\varphi) - i\left(\mathrm{PV}\left(\frac{1}{\bullet}\right)\right)(\varphi) \qquad \forall\varphi \in S(\mathbb{R},\mathbb{C}). \tag{A.0.19}$$

Moreover, Definition 5.3.3 yields that

$$\lim_{\alpha\to 0^+} \mathfrak{F}[g_\alpha](\varphi) = \lim_{\alpha\to 0^+} g_\alpha(\mathfrak{F}[\varphi]) = \Theta(\mathfrak{F}[\varphi]) = \mathfrak{F}[\Theta](\varphi) \qquad \forall\varphi \in S(\mathbb{R},\mathbb{C}) \tag{A.0.20}$$

and, thereby, we complete the proof. □

B. The principal term $T_p(h, l)$

In this section, we prove that if $G \equiv G(h,l)$ is positive and strictly positive at $\mathrm{Re}\,\lambda_1 - \lambda_0$ then the absolute of the principal term $T_P(h,l)$ can be bounded by a strictly positive constant times g^2. This section is drawn from [23].

Lemma B.0.1. *Suppose that $G \equiv G(h,l)$ is positive and strictly positive at* $\mathrm{Re}\,\lambda_1 - \lambda_0$, *then, for small enough g (depending on G), there is a constant $C(h,l) > 0$ (independent of g) such that*

$$|T_P(h,l)| \geq C(h,l) g^2. \tag{B.0.1}$$

Proof. We set

$$I := \int \mathrm{d}r \frac{G(r)}{(r + \lambda_0 - \mathrm{Re}\,\lambda_1 - ig^2 E_1)(r - \lambda_0 + \overline{\lambda_1})}, \tag{B.0.2}$$

and take small enough g. Recalling (6.3.3), we observe that

$$T_P(h,l) = g^2 E_1 M I. \tag{B.0.3}$$

We recall from the discussion below Definition 5.1.2 that $E_1 = E_I + g^a \Delta$, where $a > 0$, $\Delta \equiv \Delta(g)$ is uniformly bounded and E_I is a strictly negative constant that does not depend on g, see (5.2.11). Additionally, it follows from (5.2.25) together with $\|\varphi_0 \otimes \Omega\| = 1$ that $\|\Psi_{\lambda_0}\| \geq C > 0$, for some constant C that does not depend on g. Moreover, we conclude from (5.2.28) that $\mathrm{Re}\,\lambda_1 - \lambda_0 \geq C > 0$ for some constant C (independent of g). Consequently, (5.1.8) guarantees that there is a constant C (independent of g) such that $|M| \geq C > 0$.

This together with (B.0.3) implies that it suffices to show that there is a constant $C(h,l) > 0$ such that

$$|I| \geq C(h,l), \tag{B.0.4}$$

in order to conclude (B.0.1).

For $\alpha \equiv \alpha_g := \mathrm{Re}\,\lambda_1 - \lambda_0$ and recalling (5.0.2), we observe

$$I = \int \mathrm{d}r \frac{G(r)}{(r - \alpha - ig^2 E_1)(r + \alpha - ig^2 E_1)} = \int \mathrm{d}r \frac{G(r)\left(r^2 - \alpha^2 - g^4 E_1^2 + 2ig^2 E_1 r\right)}{(r^2 - \alpha^2 - g^4 E_1^2)^2 + 4g^4 E_1^2 r^2}. \tag{B.0.5}$$

Let $c > 0$ be such that G is supported in the complement of the ball or radius c and center 0. Then, we have

$$|\operatorname{Im}(I)| \geq |E_1| \int \mathrm{d}r G(r) \frac{2g^2 r}{(r^2 - \alpha^2 - g^4 E_1^2)^2 + 4g^4 E_1^2 c^2}. \tag{B.0.6}$$

Substituting $s = r^2$, yields

$$|\operatorname{Im}(I)| \geq |E_1| \int \mathrm{d}s G(\sqrt{s}) \frac{g^2}{(s - \alpha^2 - g^4 E_1^2)^2 + 4g^4 E_1^2 c^2}. \tag{B.0.7}$$

Since $G(\alpha) \neq 0$, then for small enough g there is a constant r_0, that does not depend on g and a constant $C > 0$ (independent of g) such that $G(\sqrt{s}) \geq C$, for every $s \in [\alpha^2 + g^4 E_1^2 - r_0, -\alpha^2 - g^4 E_1^2 + r_0]$. We apply the change of variables $u = s - \alpha^2 - g^4 E_1^2$ and obtain

$$|\operatorname{Im}(I)| \geq C|E_1| \int_{-r_0}^{r_0} \mathrm{d}s \frac{g^2}{s^2 + 4g^4 E_1^2 c^2}. \tag{B.0.8}$$

Finally, we change to the variable $\tau = s/g^2$ to obtain:

$$|\operatorname{Im}(I)| \geq C|E_1| \int_{-r_0/g^2}^{r_0/g^2} \mathrm{d}\tau \frac{1}{\tau^2 + 4E_1^2 c^2} \geq C|E_1|, \tag{B.0.9}$$

for small enough g (depending on G). □

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

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Name, Vorname

16.07.2019

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Unterschrift Doktorand/in

Formular 3.2