Bachelor's Thesis

Classical Radiation Reaction and Instability of Dynamics

Faculty of Physics
Ludwig-Maximilians-Universität München

Thilo Wick

Munich, June 18th, 2024

Submitted in fulfillment of the requirements for the degree of Bachelor of Science
Supervised by PD Dr. Dirk - André Deckert and Prof. Dr. Harmut Ruhl

Bachelorarbeit

Klassische Strahlungsrückwirkung und Instabilität der Dynamik

Fakultät für Physik
Ludwig-Maximilians-Universität München

Thilo Wick

München, den 18. Juni 2024

Vorgelegt zur Erlangung des akademischen Grades Bachelor of Science
Betreut von PD Dr. Dirk - André Deckert und Prof. Dr. Harmut Ruhl

Abstract

In this thesis, we investigate the consequences of the classical radiation reaction for the stability of dynamics. We present different equations of motion for point charges and extended charges and analyze the problems of runaway solutions and preacceleration. For the semirelativistic Abraham model we review the proof of Lyapunov-instability of the stationary solution by Bauer and Dürr. We close some gaps by rigorously proving conservation of energy and explicitly showing that the initial conditions are ϵ-close to the stationary solution. Considering the mass renormalization for extended charges, we will argue that there is a size limit for rigid charges, below which the dynamics are unstable.

Contents

1 Introduction **1**

2 Electrodynamics of point charges **2**
- 2.1 Runaway solutions . . . 3
- 2.2 Preacceleration . . . 5
- 2.3 The Landau-Lifshitz approximation . . . 6

3 Extended charges **8**
- 3.1 The semirelativistic Abraham model . . . 8
- 3.2 Existence of solutions . . . 10
- 3.3 Renormalization . . . 15
  - 3.3.1 The velocity-dependent mass . . . 15
  - 3.3.2 The charged spherical shell . . . 18
- 3.4 Runaway solutions . . . 19
  - 3.4.1 The full nonlinear problem . . . 19
  - 3.4.2 The Sommerfeld-Page equation . . . 22

4 Conclusion and outlook **24**
- 4.1 Conclusion . . . 24
- 4.2 Outlook . . . 24

A Appendix **V**
- A.1 Heaviside-Lorentz units . . . V
- A.2 Lorentz boost . . . V

1 Introduction

Despite the development of quantum electroctrodynamics, the classical theory of charged particles remains an interesting field of study. One reason is that we expect to obtain a classical theory as a limit from the quantum theory of electrodynamics. Another motivation for studying classical electrodynamics is to better understand the quantum theory.

Over time, a central problem of classical electrodynamics has been the radiation reaction. As described by the Larmor formula, an accelerated charge loses energy due to its interaction with the electromagnetic fields it generates. In some cases, however, this radiation reaction can cause unphysical solutions to the equations of motion, in which the particle accelerates to the speed of light without an external force acting on it, so-called runaway solutions. These were observed by Dirac [8] for point charges, but can also be found for charges of finite size. In this thesis, we will investigate the problem of runaway solutions for point charges as well as extended charges in an attempt to understand the reasons for their occurrence.

This thesis will be structured as follows. In section 2, we will discuss the coupled Maxwell-Lorentz equations for point charges. Then we will present the general solutions of the Maxwell equations for point charges, which are given by the Liénard-Wiechert fields, as well as their implications for the Lorentz force. Subsequently, we will present the Lorentz-Abraham-Dirac equation of motion for point charges and illustrate how runaway solutions occur. A condition on the solutions will be discussed, which eliminates runaway solutions but turns out to introduce the problem of preacceleration. Finally, we will present the Landau-Lifshitz approximation of the Lorentz-Abraham-Dirac equation, which attempts to solve the problems of runaways and preacceleration.

In section 3, we will investigate the stability of the coupled dynamics for extended charges. The first step will be to introduce the semirelativistic Abraham model of rigid charges. We will review the results of Bauer and Dürr [4] for global existence and uniqueness of solutions for the coupled Maxwell-Lorentz equations. We will prove conservation of energy, even in the case of negative bare mass, which was omitted by Bauer and Dürr. Next, we will examine the velocity-dependent inertial mass of charges in the Abraham model and see that there is a size limit below which extended charges must have negative bare mass. For a particle with negative bare mass, we expect the radiation damping to switch signs and cause an accelerated particle to accelerate even further, causing a runaway solution. We will attempt to substantiate this expectation by reviewing the proof by Bauer and Dürr of Lyapunov-instability for the Abraham model in the case of negative bare mass and a quadratic potential with positive curvature. In this proof, we will close some gaps by showing that the initial conditions are ϵ-close to the stationary solution and clarifying the runaway-argument. We will further support our expectation of runaway solutions in the case of negative bare mass by briefly discussing the nonrelativistic Sommerfeld-Page equation of motion for extended charges and its relativistic generalization, the Caldirola equation.

We will use Heaviside-Lorentz units throughout this thesis and provide a useful translation recipe in the appendix. We will use the metric tensor with the signature $(-,+,+,+)$.

2 Electrodynamics of point charges

There are several reasons for wanting to derive an equation of motion for point charges. Since the results would not depend on the choice of charge distribution, they would be more fundamental than for a theory of extended charges [4]. Additionally, one does not have to consider the cohesive forces which are required to hold an extended charge together [11]. Lastly, a theory of point charges would be naturally relativistic invariant [4], since there are no transformations of the charge distribution to consider.

When studying the dynamics of a charged particle, one conventionally sets out knowing either the world line of the particle or the electromagnetic fields. In the case of the prescribed world line, the electromagnetic fields are determined through Maxwell's equations; if the fields are given, the motion is governed by the Lorentz force. An intuitive approach to studying the dynamics of point particle interacting with its own fields would be to have the particle act as the source in Maxwell's equations and insert the resulting fields into the Lorentz equation, which governs the motion of the particle according to the Newton force law. We set the particle's charge to e and $c = 1$. If required, c can be reintroduced later. This yields the following equations of motion [25, ch. 2.3]:

$$\partial_t B(t,x) = -\nabla \times E(t,x)\,, \tag{1}$$
$$\partial_t E(t,x) = \nabla \times B(t,x) - e\delta(x-q(t))v(t)\,, \tag{2}$$

$$\nabla \cdot E(t,x) = e\delta(x-q(t))\,, \quad \nabla \cdot B(t,x) = 0\,, \tag{3}$$

$$\frac{d}{dt}\left(m_{\mathrm{b}}\gamma v(t)\right) = e\left(E_{\mathrm{in}}(q(t)) + E(q(t),t) + v(t) \times \left(B_{\mathrm{in}}(q(t)) + B(q(t),t)\right)\right)\,. \tag{4}$$

Here, m_{b} is the inertial mass of the particle without considering the interaction with its fields, also called the bare mass, and $\gamma = (1-v^2)^{-1/2}$. E_{in} and B_{in} are the incoming external fields. The well-known solutions to Maxwell's equations for a point charge in motion are the Liénard-Wiechert fields [14, 25]:

$$\begin{aligned} E^{LW,v}(t,x) &= \frac{e}{4\pi}\left[\frac{(1-v^2)(\hat{n}-v)}{(1-v\cdot\hat{n})^3|x-q|^2} + \frac{\hat{n}\times[(\hat{n}-v)\times\dot{v}]}{(1-v\cdot\hat{n})^3|x-q|}\right]_{t=t_r}\,, \\ B^{LW,v}(t,x) &= \hat{n}\times E^{LW,v}(t,x)\,, \\ \hat{n} &= \frac{x-q(t)}{|x-q(t)|}\,, \\ t_r &= t - |x-q(t_r)|\,. \end{aligned} \tag{5}$$

We would now want to insert the Liénard-Wiechert fields into (4). However, they diverge as $|x-q(t)|^{-2}$ at $x = q(t)$. Therefore, E and B are singular at the very point where they are to be evaluated for the Lorentz force, meaning the Lorentz force is not well-defined for a point charge [4].

Dirac, who wanted to obtain an equation of motion for the electron, argued that an electron was too simple to have any structure [8]. In order to circumvent the singularity of the fields of a point charge, he used conservation of energy and momentum for a finite sized charge. By taking the limit in the size of the charge, Dirac obtained an equation of motion for point particles. The resulting equation [8, 18, 25]

$$(m_\mathrm{b} + m_\mathrm{f})\dot{u}^\mu = m\dot{u}^\mu = eF_\mathrm{in}^{\mu\nu}u_\nu + \frac{e^2}{6\pi}\left[\ddot{u}^\mu - \dot{u}^\nu\dot{u}_\nu u^\mu\right] \tag{6}$$

is commonly known as the Lorentz-Dirac or Lorentz-Abraham-Dirac equation. Here $F_\mathrm{in}^{\mu\nu}$ describes the fields from external sources acting on the particle, which Dirac called the incident field. We denote the four-velocity by $u^\mu = \dot{q}^\mu$, where the dot denotes differentiation by the proper time s. We will from now on refer to (6) as the LAD equation.

Since the Lorentz force is not used in the derivation of this equation, it is also valid for non-electromagnetic external forces [20, ch. 4.3]. These can be taken into account by adding the four-vector of the external force F_ext^μ to the right-hand side [22, ch. 6.5]. Since m is fixed to the experimentally determined inertial mass, the infinity of the field energy m_f for a point charge (cf. sec. 3.3.2) has to be compensated by the bare mass m_b being negative and infinite. This is known as the classical mass renormalization. Ignoring the second term on the right hand side, (6) looks like a typical equation of motion of the form $F = ma$. Since the second term is due to the particle's interaction with its own fields [8], it is often called the *radiation reaction* [18].

2.1 Runaway solutions

By not using the Lorentz force in its derivation, the LAD equation solves the problem of evaluating the fields generated by a point charge at their singularity. We will see, however, that (6) allows solutions for which the velocity of the particle approaches the speed of light without an external force acting upon it. These clearly go against our physical expectations.

We will illustrate this point by presenting the analysis for the case of one-dimensional motion following the same procedure as Dirac [8]. In the absence of external fields, (6) can be rewritten as

$$a\dot{u}^\mu - \ddot{u}^\mu + \dot{u}^\nu\dot{u}_\nu u^\mu = 0\,, \quad a = \frac{6\pi m}{e^2}\,. \tag{7}$$

We now write q^μ in coordinates, $q^\mu = (t, x, y, z)$, and choose initial conditions with the initial velocity and acceleration four-vectors lying entirely in the $x{-}t{-}$plane of space-time.

$$u^\mu(0) = (\dot{t}_0, \dot{x}_0, 0, 0) \tag{8}$$

$$\dot{u}^\mu(0) = (\ddot{t}_0, \ddot{x}_0, 0, 0) \tag{9}$$

Due to the symmetry of the problem, the motion remains entirely in the $x - t -$ plane. Therefore it suffices to consider the x and t components of (7),

$$a\ddot{x} - \dddot{x} + (-\ddot{t}^2 + \ddot{x}^2)\dot{x} = 0\,, \tag{10}$$

$$a\ddot{t} - \dddot{t} + (-\ddot{t}^2 + \ddot{x}^2)\dot{t} = 0\,. \tag{11}$$

Since $\dot{t} = \mathrm{d}t/\mathrm{d}s = \gamma$ and $\dot{q}^i = \gamma v^i,\ i = 1, 2, 3$, we see that

$$u^\mu u_\mu = -1\,. \tag{12}$$

By derivating (12) twice, we obtain the following:

$$u^\mu \dot{u}_\mu = 0\,, \tag{13}$$
$$u^\mu \ddot{u}_\mu + \dot{u}^\mu \dot{u}_\mu = 0\,. \tag{14}$$

In our case of one-dimensional motion, these read

$$-\dot{t}^2 + \dot{x}^2 = -1\,, \tag{15}$$
$$-\dot{t}\ddot{t} + \dot{x}\ddot{x} = 0\,, \tag{16}$$
$$-\dot{t}\dddot{t} + \dot{x}\dddot{x} - \ddot{t}^2 + \ddot{x}^2 = 0\,. \tag{17}$$

By applying these properties, we can see that $\dot{x} \times (10) - \dot{t} \times (11)$ vanishes identically. Therefore equations (11) and (10) are equivalent and it is sufficient to consider (10) to obtain the solutions. Solving (15) for $\dot{t}$ yields

$$\dot{t} = \sqrt{1 + \dot{x}^2}\,, \tag{18}$$

which we can insert into (16) to obtain

$$\ddot{t} = \ddot{x}\frac{\dot{x}}{\sqrt{1 + \dot{x}^2}}\,. \tag{19}$$

We now insert this into (10), which gives

$$a\ddot{x} - \dddot{x} + \frac{\dot{x}\ddot{x}^2}{1 + \dot{x}^2} = 0\,. \tag{20}$$

If $\ddot{x} \neq 0$, we can write (20) as

$$a - \frac{\dddot{x}}{\ddot{x}} + \frac{\dot{x}\ddot{x}}{1 + \dot{x}^2} = 0\,. \tag{21}$$

Integration of (21) by proper time yields

$$as - \ln(\ddot{x}) + \frac{1}{2}\ln(1 + \dot{x}^2) = l\,, \tag{22}$$

where l is a constant of integration. We choose the origin, from which the proper time s is measured, such that $l = -\ln(a)$. Taking the exponential of (22) therefore gives

$$\frac{\ddot{x}}{\sqrt{1 + \dot{x}^2}} = ae^{as}\,, \tag{23}$$

which we integrate once more to find

$$\operatorname{arcsinh}(\dot{x}) = e^{as} + b\,, \tag{24}$$

with another constant of integration b. Solving this for $\dot{x}$ and inserting into (18) now yields the result

$$\dot{x} = \sinh\left(e^{as} + b\right)\,, \tag{25}$$
$$\dot{t} = \cosh\left(e^{as} + b\right)\,. \tag{26}$$

Let us now examine the asymptotic behaviour of this solution. For $s \to -\infty$ the velocity tends to a constant,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}s}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^{-1} = \frac{\dot{x}}{\dot{t}} \to \tanh(b)\,. \tag{27}$$

However, as s increases from $-\infty$, the velocity increases steadily. This goes so far that as $s \to +\infty$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} \to 1\,. \tag{28}$$

The velocity tends to the velocity of light for large proper times s. Since this result does not agree with experimental observations or reasonable expectations, Dirac proposed to eliminate these kinds of solutions by requiring that the velocity be constant after the particle is left alone [8]. We will see however, that this condition leads to particles in a way anticipating external forces, a phenomenon called *preacceleration*.

2.2 Preacceleration

By imposing the condition that the velocity of the particle has to be constant after the interaction with external forces, we have eliminated runaway solutions to the LAD equation. This additional condition, however, allows the particle to experience preacceleration. We will demonstrate this following Dirac's example of an electron at rest which encounters a pulse of radiation [8],

$$E(t,x) = (E_x, E_y, E_z) = (k\delta(t-y), 0, 0)\,. \tag{29}$$

If magnetic effects, such as the magnetic field induced by the time-dependent electric field, are neglected, the motion will remain entirely in the $x-t-$ plane of space-time. After inserting (29), (6) takes the form

$$a\ddot{x} - \dddot{x} + (-\ddot{t}^2 + \ddot{x}^2)\dot{x} = \kappa\delta(t)\dot{t}\,, \tag{30}$$

with a as in (7) and κ defined by $\kappa = 6\pi k/e$. As before, the equations containing $\dddot{x}$ and $\dddot{t}$ are equivalent and we will only have to consider one of the two. We will choose k small enough for the particle's velocity to be so small against the velocity of light that relativistic effects can be neglected. In the nonrelativistic approximation (30) becomes

$$a\ddot{x} - \dddot{x} = \kappa\delta(t)\,, \tag{31}$$

where the dots now denote differentiation with respect to t. The nonrelativistic version of the Lorentz-Abraham-Dirac equation is also known as the Abraham-Lorentz equation. Integration of (31) gives

$$\ddot{x}(t) = a\dot{x}(t) - \kappa\theta(t) + b\,, \tag{32}$$

where b is a constant of integration. From this we can deduce that at time $t = 0$, $\ddot{x}$ increases discontinuously by $-\kappa$. Also, according to the condition stated at the end of the previous section, $\dot{x}$ must be constant as soon as there is no external force acting on it, which is the case for $t > 0$. Therefore $\ddot{x}(t > 0) = 0$ which implies $\ddot{x}(0) = \kappa$. For times $t \neq 0$, (31) becomes

$$a\ddot{x} = \dddot{x}\,, \tag{33}$$

the general solution for which is

$$\dot{x} = l_1 e^{at} + l_2\,, \tag{34}$$

l_1 and l_2 being constants of integration. To fix these constants, we take into account that the particle is initially at rest, $\dot{x} \to 0$ for $t \to -\infty$. Earlier, we also found that $\ddot{x}(0) = \kappa$. With these two conditions we determine $l_1 = \kappa/a$ and $l_2 = 0$. Due to the form of $\ddot{x}$, cf. (32), we expect $\dot{x}$ to be continuous at $t = 0$. This yields the equation of motion

$$\dot{x} = \begin{cases} \frac{\kappa}{a} e^{at} & \text{if } t \leq 0\,, \\ \frac{\kappa}{a} & \text{if } t > 0\,. \end{cases} \tag{35}$$

From this it can be seen that the particle starts accelerating before it is acted upon by an external force; it exhibits the so-called preacceleration. This disagrees with the expectation that a particle should remain in uniform motion until acted upon by a force. According to Dirac [8], this preacceleration is not problematic, since the timescales are too short to be observed. A possible justification for the existence of preacceleration would be that for an extended charge the force hits one end of the charge distribution before it hits the center of mass [8]. However, the equation of motion from which Dirac set out was obtained by taking the point particle limit. This would also enable faster-than-light information transfer, since the particle accelerates and thus radiates before the pulse reaches the center of the particle. Dirac therefore assumed that the interior of the electron was “a region of failure [. . .] of some of the elementary properties of space-time” [8].

2.3 The Landau-Lifshitz approximation

In the course of deriving equation (6), Dirac [8] made some assumptions that should be kept in mind. For one, the derivation involves integrating the electromagnetic energy-momentum tensor over a three dimensional hypersurface in space-time. This tube is obtained by taking a sphere around the position of the particle and moving it along in time, thus creating a three-dimensional tube around the world line. The tube is usually capped at the ends with spacelike hypersurfaces. The integral of the energy-momentum tensor over these caps diverges, but this can be taken care of by absorbing the divergent terms into the renormalized mass [20, ch. 4.3]. Also, the energy-momentum tensor is valid only for continuous charge distributions, not for point charges [8]. Additionally, the conservation of four-momentum is only guaranteed if $\ddot{q}(-\infty) = \ddot{q}(+\infty)$ [20, ch. 5.5]. This condition is not satisfied by the types of runaway solutions discussed in section 2.1. The external force is also required to be an analytic function of time [27, ch. 2.1], which is not the case for the delta force considered in section 2.2. However, preacceleration is not a result of fast-changing forces and also occurs for forces that are analytic functions of time [15, 26]. Finally, it should be kept in mind that the bracketed term in (6) is small compared to the external force [22, 25]. This is, of course, neglected by setting the external force to zero.

Using this assumption of smallness of the radiation reaction, one can obtain the Landau-Lifshitz approximation of the Lorentz-Abraham-Dirac equation, as described by

Rohrlich [22]. The first step is to rewrite (6) as

$$m\dot{u}^\mu = F^\mu + P^{\mu\nu}\tau_0 \frac{\mathrm{d}(m\dot{u}_\nu)}{\mathrm{d}s}\,, \tag{36}$$

where F^μ is the total force acting on the particle, $\tau_0 = \frac{e^2}{6\pi m}$ and P is defined by $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ with the metric tensor $g^{\mu\nu}$. Since τ_0 is small – the largest value for any particle is $\tau_0 = 0.62 \times 10^{-23}\,\mathrm{s}$ for the electron [22] – we view the second term in (36) as a small perturbation and $m\dot{u}_\nu = F_\nu$ as the unperturbed equation. We therefore replace $m\dot{u}_\nu$ by F_ν to obtain

$$m\dot{u}^\mu = F^\mu + \tau_0 P^{\mu\nu}\frac{\mathrm{d}F_\nu}{\mathrm{d}s} = F^\mu + \tau_0\left(\frac{\mathrm{d}F^\mu}{\mathrm{d}s} + u^\mu u^\alpha \frac{\mathrm{d}F_\alpha}{\mathrm{d}s}\right). \tag{37}$$

This equation is valid as long as the assumption made in the derivation,

$$\left|\tau_0 P^{\mu\nu}\frac{\mathrm{d}F_\nu}{\mathrm{d}s}\right| \ll |F^\mu|\,, \tag{38}$$

is satisfied. This is not the same equation as given by Landau and Lifshitz [17, sec. 76]. However, it is significantly less complicated and both provide physically acceptable dynamics [22].

It can be seen immediately that, unlike the LAD equation, (37) does not allow runaway solutions for free particles. This is because for a force which vanishes during a finite time interval, meaning that both the force and its derivative are zero, the acceleration also vanishes. To examine the problem of preacceleration, we will introduce the convenient coordinates $u^\mu = (\cosh w, \hat{n}\sinh w)$ with an arbitrary unit vector $\hat{n}$ and $f^\mu = m^{-1}F^\mu$. Therefore, in the case of one-dimensional motion, (37) can be written as

$$\dot{w} = f + \tau_0 \dot{f}\,, \tag{39}$$

and the condition (38) becomes

$$|\tau_0 \dot{f}| \ll |f|\,. \tag{40}$$

By this condition, delta forces like in section 2.2 are not allowed. Thus at least the most obvious case of preacceleration is excluded from the domain of validity of the equation of motion.

The Landau-Lifshitz equation is an approximation of the LAD equation, but Rohrlich [22] argues that the equation is accurate within the bounds of classical physics since further corrections would lead into the domain of quantum mechanics [22, ch. 9]. From a physical standpoint, this is satisfying if one aims to obtain accurate dynamics reflecting the results of real experiments. Considering the theory itself, however, it would be desirable to have a way of obtaining stable dynamics without validity cutoffs like (38), even if the results do not agree with empirical evidence.

3 Extended charges

To avoid the singularity of the fields generated by a point charge at its location, one can consider an extended charge distribution [25, ch. 2.3]. There are two approaches that have been investigated so far. The *Abraham model* assumes a charge distribution which is fixed in the laboratory frame and independent of velocity, hence this model is also called the nonrelativistically rigid model. One should keep in mind that this model is not very realistic, since the particle's length along its axis of movement in its proper frame of reference would tend to infinity as the velocity relative to the laboratory frame tends to the velocity of light [27, ch. 3]. A more realistic model would be relativistically rigid, where the charge distribution is fixed in its proper frame of reference and thus appears Lorentz contracted in the laboratory frame. This model is called the *Lorentz model*, for an in-depth treatment of which we refer to Appel and Kiessling [2].

3.1 The semirelativistic Abraham model

Due to its greater simplicity, we will from now on focus on the Abraham model in the setting of Bauer and Dürr [4]. The charge distribution is set to be $e\varphi$, where φ is smooth and of compact support, $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$. We shall also require that $\int \mathrm{d}x\varphi(x) = 1$ [25, ch. 2.4], since this fixes the particles charge to e and will make notation a little simpler. Note that this is not strictly necessary. The charge distribution is rigidly carried along with a mass point for which the space coordinate q is located in its center, and the charge is considered not to be rotating. Unlike in section 2.1, we will consider the case with an external harmonic potential, $V(q) = \kappa q^2/2$. Finally, we will set the velocity of light $c = 1$ and the particle's bare mass $m_\mathrm{b} = \alpha = \pm 1$. The motion is then governed by the following system of differential equations, which shall be called the Maxwell-Lorentz system of a rigid charge [4, 25]:

$$
\begin{aligned}
\dot{q}(t) &= \frac{\alpha p(t)}{\sqrt{1+p(t)^2}}\,, \quad p(t) = \alpha\gamma\dot{q}(t)\,,\\
\dot{p}(t) &= -\kappa q(t) + e\left(E_\varphi(q(t),t) + \frac{\alpha p(t)}{\sqrt{1+p(t)^2}} \times B_\varphi(q(t),t)\right)\,,\\
\dot{E}(t,x) &= \nabla \times B(t,x) - \frac{\alpha p(t)}{\sqrt{1+p(t)^2}} e\varphi\left(x - q(t)\right)\,,\\
\dot{B}(t,x) &= -\nabla \times E(t,x)\,,\\
\nabla \cdot E(t,x) &= e\varphi\left(x - q(t)\right)\,,\\
\nabla \cdot B(t,x) &= 0\,.
\end{aligned} \tag{41}
$$

For readability, we denote the convolution $\varphi * E$ by E_φ and from now on use the notation $\varphi_q = \varphi(\cdot - q)$ for $q \in \mathbb{R}^3$. Since point charges are not allowed by the conditions imposed on the charge distribution $e\varphi$, the Lorentz force is well defined. The model is called semirelativistic, because the relativistic momentum and kinetic energy are used but the charge distribution is nonrelativistically rigid, as noted earlier.

In order to obtain the existence of solutions for the Maxwell-Lorentz system, it will be advantageous to write (41) as an evolution equation. The first step is to set up the Hilbert space,

$$\mathcal{H} = \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \left(L^2\right)^3 \oplus \left(L^2\right)^3 . \tag{42}$$

$L^2 := L^2(\mathbb{R}^3)$ is the Hilbert space of real-valued, measurable and square-integrable functions on $\mathbb{R}^3$, and $\left(L^2\right)^3$ denotes the Hilbert space of 3-tuples from L^2. An element of $\mathcal{H}$ will be called a state vector $\psi = (q, p, E, B)$. The canonical scalar product is denoted by $x \cdot y$ for two vectors $x, y \in \mathbb{R}^3$ and by $\langle f, g\rangle$ for two functions $f, g \in (L^2)^3$. We will use $|\cdot|$ for the norm on $\mathbb{R}^3$ and $\|\cdot\|$ for the norm on $\left(L^2\right)^3$. The norm $\|\cdot\|_{\mathcal{H}}$ on $\mathcal{H}$ is induced by the scalar product $\langle\cdot\,,\cdot\rangle_{\mathcal{H}}$, which is defined as follows:

$$\langle \psi\,, \psi'\rangle_{\mathcal{H}} = q \cdot q' + p \cdot p' + \langle E, E'\rangle + \langle B\,, B'\rangle\,. \tag{43}$$

We can now rewrite the first four lines of (41) as an evolution equation for the $\mathcal{H}$-valued map on $\mathbb{R}$, $\psi(\cdot) : t \mapsto (q(t), p(t), E(t), B(t))$, which represents the evolution of the physical state ψ in time,

$$\dot{\psi}(t) = A\psi(t) + J\left(\psi(t)\right) . \tag{44}$$

The linear operator A in (44) is defined by

$$A\psi = (0, 0, \nabla \times B, -\nabla \times E)\,, \tag{45}$$

and the nonlinear operator J by

$$J(\psi) = \left(\frac{\alpha p}{\sqrt{1+p^2}}\,, -\kappa q + e\left(E_\varphi(q) + \frac{\alpha p}{\sqrt{1+p^2}} \times B_\varphi(q)\right)\,, \frac{-\alpha p}{\sqrt{1+p^2}} e\varphi_q\,, 0\right) . \tag{46}$$

The domain of A is defined by

$$\mathcal{D}(A) = \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus W^{\mathrm{curl}} \oplus W^{\mathrm{curl}}\,, \tag{47}$$

with

$$W^{\mathrm{curl}} = \left\{E \in \left(L^2\right)^3 \mid \nabla \times E \in \left(L^2\right)^3\right\}\,, \tag{48}$$

where the partial derivatives in the curl are distributional derivatives. For $n \in \mathbb{N}, n \geq 2$ we define the domain of the n-th power A^n of A by

$$\mathcal{D}(A^n) = \left\{\psi \in \mathcal{D}(A) \mid A^j\psi \in \mathcal{D}(A),\ j = 1, \ldots, n-1\right\} . \tag{49}$$

We will use the notation $\mathcal{D}(A^1) = \mathcal{D}(A)$. A strongly continuous one-parameter group is defined as follows [24, ch. 12].

Definition 1 (Strongly continuous one-parameter group)**.** A family $\{U_t\}_{t\in\mathbb{R}}$ of operators on $\mathcal{H}$ is called a *strongly continuous one-parameter group of unitary operators* if

1. U_t is unitary for any $t \in \mathbb{R}$,
2. $U_{t+s} = U_t U_s$ for all $t, s \in \mathbb{R}$,

3. the map $\mathbb{R} \ni t \mapsto U_t\psi \in \mathcal{H}$ is continuous for any $\psi \in \mathcal{H}$.

We find that A generates a strongly continuous one-parameter group of unitary operators on $\mathcal{H}$.

Lemma 1. *There is a strongly continuous family $\{U_t\}_{t\in\mathbb{R}}$ of unitary operators $U_t : \mathcal{H} \to \mathcal{H}$ such that*

$$\lim_{h\to 0} \frac{U_h\psi - \psi}{h} = A\psi\,, \quad U_t\psi \in \mathcal{D}(A)\,, \quad AU_t\psi = U_tA\psi \tag{50}$$

for each $\psi \in \mathcal{D}(A)$ and $t \in \mathbb{R}$.

Proof. Refer to Bauer and Dürr [4, lemma 1]. □

As for J, we may choose $\mathcal{D}(A)$ as a domain of J as defined in (46) [4, lemma 2],

$$J : \mathcal{D}(A) \to \bigcap_{n=0}^{\infty} \mathcal{D}(A^n)\,. \tag{51}$$

3.2 Existence of solutions

Bauer and Dürr [4] proved global existence of solutions for the Maxwell-Lorentz system (41). It is important for our objective of investigating the stability of the dynamics to have a rigorous proof of global solutions and the conditions under which they exist. Since the details of this proof go beyond the scope of this thesis, we will present the results of Bauer and Dürr and briefly sketch the ideas behind the proofs. For the full proofs, we refer to [4].

To prove global existence and uniqueness of solutions to (44), the first step is to prove local existence and uniqueness. The local existence will then be extended to global existence by showing that the interval on which local solutions exist can be extended.

Theorem 1 (Local existence and uniqueness)**.** *For each $\psi_0 \in \mathcal{D}(A^n), n \geq 1$, there is a $T > 0$ and a map $\psi(\cdot) : [0, T) \to \mathcal{D}(A^n)$ with the following properties:*

1. *$\psi(\cdot)$ is a n-times strongly continuously differentiable solution of (44) with initial value $\psi(0) = \psi_0$ and $\mathrm{d}^j/\mathrm{d}t^j\psi(t) \in \mathcal{D}(A^{n-j})$ for all $t \in [0, T)$ and $j = 0, \ldots, n$.*

2. *If $\tilde{\psi}(\cdot) : [0, \widetilde{T}) \to \mathcal{D}(A)$ is a strongly continuously differentiable solution of (44) with initial value $\tilde{\psi}(0) = \psi_0$, then $\tilde{\psi}(t) = \psi(t)$ for $t \in [0, \widetilde{T}) \cap [0, T)$.*

Proof. Refer to Bauer and Dürr [4, proposition 1]. □

The proof is conducted by using the contraction mapping principle to prove the existence and uniqueness of local solutions to the integral equation

$$\psi(t) = U_t\psi_0 + \int_0^t U_{t-s}J(\psi(s))\mathrm{d}s\,, \tag{52}$$

and showing that a unique solution to (52) is a unique solution to (44) with initial value $\psi(0) = \psi_0$ [4].

The next step on the way to global existence is to consider the last two lines of (41), which were disregarded up to now. To do this, we will impose the following constraint on the initial conditions:

$$\nabla \cdot E_0 = \rho(\cdot - q_0)\,, \;\; \nabla \cdot B_0 = 0\,. \tag{53}$$

Theorem 2. *Suppose $\psi_0 = (q_0, p_0, E_0, B_0) \in \mathcal{D}(A^n), n \geq 1$. Suppose further that q_0, E_0 and B_0 satisfy* (53). *Then the solution $\psi(\cdot) = (q(\cdot), p(\cdot), E(\cdot), B(\cdot)) : [0, T) \to \mathcal{D}(A^n)$ of* (44) *with initial value $\psi(0) = \psi_0$, cf. theorem 1, has the following properties:*

1. *For all $t \in [0, T)$, the functions $q(\cdot), E(\cdot)$ and $B(\cdot)$ satisfy*
$$\nabla \cdot E(t) = \rho(\cdot - q(t))\,, \quad \nabla \cdot B(t) = 0\,, \tag{54}$$
with distributional derivatives in the divergence.

2. *If $n \geq 2\lambda + 2$ for some $\lambda \in \mathbb{N}$, then, for every $t \in [0, T)$, each of $E(t)$ and $B(t)$ is equal almost everywhere in $\mathbb{R}^3$ to a function in $\mathcal{C}^{2\lambda}(\mathbb{R}^3)^3$. If $\psi_0 \in \bigcap\limits_{n=0}^{\infty} \mathcal{D}(A^n)$, then, for every $t \in [0, T)$, each of $E(t)$ and $B(t)$ is equal almost everywhere in $\mathbb{R}^3$ to a function in $\mathcal{C}^{\infty}(\mathbb{R}^3)^3$.*

Proof. The proof can be found in [4, proposition 2]. □

In order to prove the first point, one can use the fact that $t \mapsto E(t)$ is continuously differentiable from theorem 1 to write

$$E(t) = E_0 + \int\limits_0^t \mathrm{d}s \dot{E}(s)$$

and use (41) to calculate $\nabla \cdot E(t)$. The same procedure can be applied to $B(t)$. For the second point one uses the fact that

$$\nabla \times (\nabla \times E) = \nabla\left(\nabla \cdot E\right) - \Delta E\,,$$

where Δ denotes the Laplacian, and $\widehat{\Delta^l E}(k) = (-1)^l |k|^{2l} \hat{E}(k)$. Using Plancherel's theorem [23], one obtains that $\partial^l / \partial x_j^l E(t) \in (L^2)^3$ for $0 \leq l \leq 2\lambda + 2$ and the same property for $B(t)$. By these properties and Sobolev's lemma [23], $E(t)$ and $B(t)$ are equal almost everywhere in $\mathbb{R}^3$ to a function in $\mathcal{C}^{2\lambda}(\mathbb{R}^3)^3$. Refer to Bauer and Dürr [4] for the full proof.

Now only global existence remains. We will define $\mathcal{M}_n$ as the subset of states in $\mathcal{D}(A^n)$ satisfying (54),

$$\mathcal{M}_n = \left\{\psi \in \mathcal{D}(A^n) | \psi \text{ satisfies (54)}\right\}\,. \tag{55}$$

Theorem 3. *Let $\psi_0 = (q_0, p_0, E_0, B_0) \in \mathcal{D}(A^n), n \geq 1$. Suppose further that q_0, E_0 and B_0 satisfy* (53). *Then the following is true:*

1. *(Global existence) There is a function $\psi(\cdot) = (q(\cdot), p(\cdot), E(\cdot), B(\cdot)) : \mathbb{R} \to \mathcal{M}_n$ with the following properties: $\psi(\cdot)$ is a n times strongly continuously differentiable solution of* (44) *with initial value $\psi(0) = \psi_0$ and $\mathrm{d}^j/\mathrm{d}t^j \psi(t) \in \mathcal{D}(A^{n-j})$ for all $t \in \mathbb{R}$ and $j = 0, \ldots, n$. The functions $q(\cdot), E(\cdot)$ and $B(\cdot)$ satisfy* (54) *for all $t \in \mathbb{R}$. Thus, $\psi(\cdot)$ is a global solution of the Maxwell-Lorentz system, with curl and divergence in the distributional sense.*

2. *(Uniqueness) Suppose $\Lambda \in \mathbb{R}$ is an interval and $T_0 \in \Lambda$. If $\tilde{\psi}(\cdot) : \Lambda \to \mathcal{D}(A)$ is a continuously differentiable solution of* (44) *with initial value $\tilde{\psi}(T_0) = \psi(T_0)$, then $\tilde{\psi}(t) = \psi(t)$ for all $t \in \Lambda$.*

3. *(Regularity) If $n \geq 2\lambda + 2$ for some $\lambda \in \mathbb{N}$, then for every $t \in \mathbb{R}$ each of $E(t)$ and $B(t)$ is equal almost everywhere in $\mathbb{R}^3$ to a function in $\mathcal{C}^{2\lambda}(\mathbb{R}^3)^3$. If $\psi_0 \in \bigcap_{n=0}^{\infty} \mathcal{D}(A^n)$, then for every $t \in \mathbb{R}$ each of $E(t)$ and $B(t)$ is equal almost everywhere in $\mathbb{R}^3$ to a function in $\mathcal{C}^\infty(\mathbb{R}^3)^3$.*

Proof. Refer to the proof of theorem 1 in [4]. □

The proof of this theorem is approached as follows [4]. Assume $\bar{T}$ is the supremum of all $T > 0$ for which there exists a solution of (44) according to theorem 1. Global existence is proven by showing that the map $t \mapsto \|\psi(t)\|_{\mathcal{H}}$ is bounded in $[0, \bar{T})$. The solution can then be continued beyond $\bar{T}$, since the length of the interval over which the contraction mapping principle can be used only depends on this bound. Therefore $\bar{T}$ cannot be the supremum and there exists a global solution. Uniqueness and regularity are already included in theorems 1 and 2 respectively, since their proofs do not exclude global solutions.

It should be noted that this proof does not require energy conservation and therefore includes the case of negative bare mass, for which the energy is no longer bounded from below.

For the case of positive bare mass and an external potential which is bounded from below, global existence was also proved by Komech and Spohn [16]. A summary can be found in [25, ch. 2.4]. In this case, the velocity is bounded and under the additional conditions of $\hat{\varphi}(k) \neq 0$ and smooth external potentials, even $\lim_{t\to\infty} \ddot{q}(t) = 0$ [16, 25].

Later we will require conservation of energy. Bauer and Dürr [4] claimed that the energy of the Maxwell-Lorentz system in conserved, however they did not provide a proof. To prove conservation of energy, we require the following result.

Lemma 2. *Let $E(t,x), B(t,x)$ be a solution to the Maxwell equations with the sources ρ and j given by*

$$\rho(t,x) = \rho(x - q(t)) \,, \quad j(t,x) = \rho(x - q(t))\dot{q}(t) \,, \tag{56}$$

where $\rho(x)$ is smooth and of compact support, $\operatorname{supp}\rho \subset B_R(0)$ *for some $R > 0$ and $q(t)$ is a strictly time-like trajectory. Let the initial fields at $t_0 = 0$ be given by $E_0, B_0 \in \mathcal{C}^\infty(\mathbb{R}^3)^3$ satisfying* (53) *as well as*

$$|E_0(x)| + |B_0(x)| + |x| \sum_{i=1}^{3} \left(|\partial_{x_i} E_0(x)| + |\partial_{x_i} B_0(x)|\right) = \mathop{\mathcal{O}}_{|x|\to\infty} \left(|x|^{-(1+\epsilon)}\right) \tag{57}$$

for some $\epsilon > 0$. Then the fields at time t are of order $|x|^{-(1+\epsilon)}$,

$$|E(t,x)| + |B(t,x)| = \mathop{\mathcal{O}}_{|x|\to\infty} \left(|x|^{-(1+\epsilon)}\right) \tag{58}$$

Proof. The solution to the Maxwell equation with sources $\rho(t,x), j(t,x)$ and initial conditions E_0, B_0 is given by [6, 25]

$$\begin{pmatrix} E(t,x) \\ B(t,x) \end{pmatrix} = \begin{pmatrix} -\partial_t & \nabla\times \\ -\nabla\times & \partial_t \end{pmatrix} K_t * \begin{pmatrix} E_0(x) \\ B_0(x) \end{pmatrix} + \int_0^t \mathrm{d}s \begin{pmatrix} -\nabla & \partial_s \\ 0 & \nabla\times \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho(s,x) \\ j(s,x) \end{pmatrix}, \quad (59)$$

where $K_t = K_t^- - K_t^+$ and

$$K_t^\pm(x) = \frac{\delta(|x| \pm t)}{4\pi|x|}\,. \quad (60)$$

For any smooth function $f \in \mathcal{C}^\infty$, the convolution with K_t is given by

$$K_t * f(x) = t \fint_{\partial B_{|t|}(0)} \mathrm{d}\sigma(y) f(x-y) := t \int_{\partial B_{|t|}(0)} \mathrm{d}\sigma(y) \frac{f(x-y)}{4\pi t^2}\,. \quad (61)$$

First, we will examine the asymptotic behavior of the second term in (59). Let $t \in \mathbb{R}$. By (61),

$$K_{t-s} * \rho(s,x) = (t-s) \fint_{\partial B_{|t-s|}(0)} \mathrm{d}\sigma(y) \rho(x-y-q(s))\,. \quad (62)$$

By the properties of the support of ρ, this term vanishes if $\partial B_{|t-s|}(0) \cap B_R(x-q(s)) = \emptyset$. Since $j(t,x) = \rho(t,x)\dot{q}(t)$, the same holds for the convolution $K_{t-s} * j(s,x)$. Now, R is fixed, $q(s)$ is strictly time-like and $|t-s| \in [0,t]$ with $t \in \mathbb{R}$. Therefore, there is an $x' \in \mathbb{R}$ with $0 < x' < \infty$, such that the second term in (59) vanishes for $|x| > x'$.

Let us now consider the first term in (59). The condition (57) ensures that there is a constant $1 \le C < \infty$ such that for $|x|$ large enough,

$$\left(|E_0(x)| + |B_0(x)| + |x| \sum_{i=1}^3 \left(|\partial_{x_i} E_0(x)| + |\partial_{x_i} B_0(x)| \right) \right) |x|^{1+\epsilon} \le C\,. \quad (63)$$

Using this and (61), for $t \in \mathbb{R}$ and $|x|$ large enough, we find the estimates [6, proof of theorem 4.18]

$$\begin{aligned} |\partial_{x_i} K_t * E_0(x)| &\le |t| \fint_{\partial B_{|t|}(0)} \mathrm{d}\sigma(y) \frac{|\partial_{x_i} E_0(x-y)||x-y|^{1+\epsilon}}{|x-y|^{1+\epsilon}} \\ &\le |t| \fint_{\partial B_{|t|}(0)} \mathrm{d}\sigma(y) \frac{C}{(|x|-|y|)^{1+\epsilon}} \\ &\le \frac{C|t|}{(|x|-|t|)^{1+\epsilon}} = \underset{|x|\to\infty}{\mathcal{O}} \left(|x|^{-(1+\epsilon)} \right), \end{aligned} \quad (64)$$

and

$$
\begin{aligned}
|\partial_t K_t * E_0(x)| &\leq \fint_{\partial B_{|t|}(0)} \mathrm{d}\sigma(y)|E_0(x-y)| + |t| \fint_{\partial B_1(0)} \mathrm{d}\sigma(y)|(y\cdot\nabla)E_0(x-|t|y)| \\
&\leq \fint_{\partial B_{|t|}(0)} \mathrm{d}\sigma(y)\frac{|E_0(x-y)||x-y|^{1+\epsilon}}{|x-y|^{1+\epsilon}} \\
&\quad + |t| \fint_{\partial B_1(0)} \mathrm{d}\sigma(y)\frac{\sum_{i=1}^{3}|\partial_{x_i}E_0(x-|t|y)||x-|t|y|^{2+\epsilon}}{|x-|t|y|^{2+\epsilon}} \\
&\leq \frac{C}{(|x|-|t|)^{1+\epsilon}} + \frac{C|t|}{(|x|-|t|)^{2+\epsilon}} \\
&= \underset{|x|\to\infty}{\mathcal{O}}\left(|x|^{-(1+\epsilon)}\right) ,
\end{aligned}
\tag{65}
$$

where we used that $|x-y|^{1+\epsilon} \geq (|x|-|y|)^{1+\epsilon}$. Since we required that $|x|$ be large enough, we can choose $|x| > |t|$ in which case the denominator does not vanish. We obtain the same estimates for $B(t,x)$ by replacing E_0 with B_0. Therefore, by (59), the fields at time t meet the decay condition $|E(t,x)| + |B(t,x)| = \underset{|x|\to\infty}{\mathcal{O}}(|x|^{-(1+\epsilon)})$. □

Lemma 3 (Conservation of energy)**.** *Let $\psi(\cdot) = (q(\cdot), p(\cdot), E(\cdot), B(\cdot)) : \mathbb{R} \to \mathcal{M}_n$, $n \geq 1$, be a solution of* (41) *where the initial fields at time $t_0 = 0$ satisfy the decay condition* (57) *for some $\epsilon > 0$ and $E_0, B_0 \in \mathcal{C}^\infty(\mathbb{R}^3)^3$. Then the quantity H,*

$$
H = \alpha\sqrt{1+p(t)^2} + \frac{1}{2}\kappa q(t)^2 + \frac{1}{2}\int \mathrm{d}x \left(E(t,x)^2 + B(t,x)^2\right) \tag{66}
$$

is independent of t. We call H the total energy of the Maxwell-Lorentz system.

Proof. By theorem 3, $\psi(\cdot)$ is n-times strongly continuously differentiable. Therefore, we calculate the time derivative of H. We find

$$
\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\alpha p(t)}{\sqrt{1+p(t)^2}}\left[-\kappa q(t) + e\left(E_\varphi(q(t),t) + \frac{\alpha p(t)}{\sqrt{1+p(t)^2}} \times B_\varphi(q(t),t)\right)\right] \\
&\quad + \kappa q(t)\frac{\alpha p(t)}{\sqrt{1+p(t)^2}} + \int_{\mathbb{R}^3} \mathrm{d}x\big[E(t,x)\left(\nabla\times B(t,x)\right) - B(t,x)\left(\nabla\times E(t,x)\right)\big] \\
&\quad - \frac{\alpha p(t)}{\sqrt{1+p(t)^2}} e E_\varphi(q(t)) \\
&= -\int_{\mathbb{R}^3} \mathrm{d}x \nabla\cdot\left(E(t,x)\times B(t,x)\right) \\
&= -\lim_{R\to\infty} \int_{\partial B_R(0)} \mathrm{d}\sigma(x)\cdot\left(E(t,x)\times B(t,x)\right) \\
&= -\lim_{R\to\infty} \fint_{\partial B_R(0)} \mathrm{d}\sigma(x)\cdot 4\pi|x|^2\left(E(t,x)\times B(t,x)\right) ,
\end{aligned}
\tag{67}
$$

where $d\sigma$ is the orientated surface element and we used the divergence theorem in the last step.

By theorem 3, $\rho = e\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$. Also, $\psi(t)$ is continuous and therefore is bounded on any compact real interval $[T_1, T_2]$. Therefore, since $v(t) < 1$, there exists a $v_{\max} < 1$ such that $v(t) \le v_{\max}$ $\forall t \in [T_1, T_2]$. The trajectory is strictly time-like on any compact interval $[T_1, T_2] \subset \mathbb{R}$. Due to this and the condition on E_0 and B_0, we may apply lemma 2. Therefore, for $t \in \mathbb{R}$, $E(t,x)$ and $B(t,x)$ drop of sufficiently,

$$|E(t,x)| + |B(t,x)| = \mathop{\mathcal{O}}_{|x|\to\infty}\left(|x|^{-(1+\epsilon)}\right) , \tag{68}$$

and the surface term vanishes and $\frac{\mathrm{d}H}{\mathrm{d}t} = 0$. □

It is justified to call H the total energy of the Maxwell-Lorentz system, since it can be obtained from the Abraham model's Lagrangian [cf. 25, ch. 2.4]. To prove conservation of energy we imposed strong conditions on the initial fields, namely $E_0, B_0 \in \mathcal{C}^\infty$. This might not be strictly necessary for the proof, but it is sufficient for our purposes.

3.3 Renormalization

3.3.1 The velocity-dependent mass

From Newtonian mechanics, we know the inertial mass as the proportionality constant connecting the force and the acceleration or equivalently as the result of derivating a particle's momentum with respect to its velocity. Akin to the mass renormalization in (6), we will see that a particle's inertial mass is made up of two terms, one of which is due to the particle's interaction with its fields. This term is often called the electromagnetic or electrodynamic mass. In his book, Abraham [1, ch. 20] argued that for charged particles, one can only expect the acceleration to be proportional to the applied force if the motion is rectilinear in the absence of external forces. Abraham also stated that the concept of electromagnetic mass is only justified for quasistatic motion, where the velocity changes little in the amount of times it takes for light to traverse the particle. Since this applies to any observable motion of electrons [1, ch. 24], we will assume quasistatic motion for the purpose of obtaining the velocity-dependent mass. In this case, we can assume that the fields of the particle in motion are the fields obtained for rectilinear motion at the same velocity [1].

A solution to (44) moving at constant velocity v is called a soliton, and the state vector is given by $\psi(t) = S_{q+vt,v}$ [25, ch. 4.1] with

$$S_{q,v} = \left(q, \frac{\alpha v}{\sqrt{1-v^2}}, E^v(x-q), B^v(x-q)\right) . \tag{69}$$

The fields E^v and B^v are given by their Fourier transforms [25]

$$\begin{aligned} \hat{E}^v(k) &= i\left[k^2 - (v\cdot k)^2\right]^{-1} e\hat{\varphi}(k)\left(-k + v(v\cdot k)\right) , \\ \hat{B}^v(k) &= -i\left[k^2 - (v\cdot k)^2\right]^{-1} e\hat{\varphi}(k)\left(v \times k\right) , \end{aligned} \tag{70}$$

where $v \in \mathbb{V} = \{v | \, |v| < 1\}$ and

$$\hat{f}(k) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{d}x e^{-ik\cdot x} f(x) \,,$$
$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{d}x e^{ik\cdot x} \hat{f}(k) \,.$$

We can now obtain the velocity-dependent mass from the energy-momentum relation as described by Spohn [25, ch. 4.1]. It is assumed throughout that there are no external potentials. The total momentum of the particle is given by

$$\mathcal{P} = m_{\mathrm{b}} \gamma v + \int \mathrm{d}^3 x \left(E(x) \times B(x) \right) \,, \tag{71}$$

which is conserved by the coupled Maxwell-Lorentz equations. $\mathcal{P}$ corresponds physically to the total momentum since it is the conserved quantity linked by Noether's theorem to the Lagrangian of the Abraham model. By inserting (70), we find the momentum of a soliton,

$$\begin{aligned} P_{\mathrm{s}}(v) &= m_{\mathrm{b}} \gamma v + e^2 \int \mathrm{d}^3 k |\hat{\varphi}(k|^2 \Big(\left[k^2 - (v \cdot k)^2 \right]^{-1} v \\ &\quad - \gamma^{-2} \left[k^2 - (v \cdot k)^2 \right]^{-2} (v \cdot k) k \Big) \\ &= v \left(m_{\mathrm{b}} \gamma + m_{\mathrm{f}} |v|^{-3} \left[-|v| + (1 + v^2) \operatorname{arctanh}(|v|) \right] \right) \,, \end{aligned} \tag{72}$$

where we used Plancherel's theorem [23] to replace $\int \mathrm{d}^3 x \left(E(x) \times B(x) \right) = \int \mathrm{d}^3 k (\hat{E}(k) \times \hat{B}(k))$. By m_{f} we denote the electrostatic energy of the charge distribution $e\varphi$,

$$m_{\mathrm{f}} = \frac{1}{2} e^2 \int \mathrm{d}^3 x \mathrm{d}^3 x' \frac{\varphi(x) \varphi(x')}{4\pi |x - x'|} = \frac{1}{2} \int \mathrm{d}^3 k |\hat{E}(k)|^2 \,, \tag{73}$$

where we integrated by parts and used Plancherel's theorem [23] in the second step. In order to calculate the energy of a soliton, we insert (70) into (66) and set $\kappa = 0$,

$$\begin{aligned} H_{\mathrm{s}}(v) &= m_{\mathrm{b}} \gamma + \frac{1}{2} e^2 \int \mathrm{d}^3 k |\hat{\varphi}(k)|^2 \frac{(1 + v^2) k^2 - (3 - v^2)(v \cdot k)^2}{\left(k^2 - (v \cdot k)^2 \right)^2} \\ &= m_{\mathrm{b}} \gamma + m_{\mathrm{f}} |v|^{-1} \left[-|v| + 2 \operatorname{arctanh}(|v|) \right] \,. \end{aligned} \tag{74}$$

The map $\mathbb{V} \ni v \mapsto P_{\mathrm{s}}(v) \in \mathbb{R}^3$ is bijective and therefore invertible. We will write the velocity as a function of total momentum $v(P_{\mathrm{s}})$, which is the inverse to $P_{\mathrm{s}}(v)$. This way we obtain the energy-momentum relation,

$$H_{\mathrm{eff}}(p) = H_{\mathrm{s}}(v(p)) \,, \tag{75}$$

which depends on the charge distribution only through its electrostatic energy m_{f}. We see that

$$H_{\mathrm{s}}(v) = P_{\mathrm{s}}(v) \cdot v - T(v) \,, \tag{76}$$

with

$$T(v) = -m_{\mathrm{b}}\gamma^{-1} - m_{\mathrm{f}}|v|^{-1}(1-|v|^2)\,\mathrm{arctanh}(|v|)\,. \tag{77}$$

Remembering that the Hamiltonian of a system can be obtained from the Lagrangian by $H(p,q) = p\cdot\dot{q} - L(q,\dot{q})$, this suggests that H_{s} will play the role of an effective Hamiltonian and T the role of the inertial term in an effective Lagrangian. Note that we are considering the case without an external potential. Hamilton's equations are given by

$$\dot{q} = \nabla_p H\,, \tag{78}$$
$$\dot{p} = -\nabla_q H\,, \tag{79}$$

where $p = \nabla_v H$ is equivalent to (78). By using $P_{\mathrm{s}}(v) = \nabla_v T(v)$, we can calculate $\nabla_v H_{\mathrm{s}}(v)$,

$$\frac{\mathrm{d}P_{\mathrm{s}}(v)}{\mathrm{d}v}v = \nabla_v H_{\mathrm{s}}(v)\,. \tag{80}$$

This suggests that the 3×3 matrix $dP_{\mathrm{s}}(v)/\mathrm{d}v$, the components of which are given by $(\mathrm{d}P_{\mathrm{s}}(v)/\mathrm{d}v)_{ij} = \mathrm{d}P_{\mathrm{s}}(v)_j/\mathrm{d}v_i$, is to be considered as the velocity-dependent mass. We can immediately see that this mass will be made up of two components, one containing the bare mass m_{b} and one containing the electrostatic energy m_{f}. We will write the field contribution to the velocity dependent mass as $m_{\mathrm{f}}(v)$ and split it up into a longitudinal and a transversal component,

$$m_{\mathrm{f}}(v) = \frac{\mathrm{d}\left(P_{\mathrm{s}}(v) - m_{\mathrm{b}}\gamma v\right)}{\mathrm{d}v} = m_{\mathrm{l}}(|v|)\hat{v}\otimes\hat{v} + m_{\mathrm{t}}(|v|)(\mathbb{1} - \hat{v}\otimes\hat{v})\,. \tag{81}$$

Here, $\hat{v}$ is a unit vector in the direction of v and $\hat{v}\otimes\hat{v}$ is given by $(\hat{v}\otimes\hat{v})_{ij} = \hat{v}_i\hat{v}_j$. By inserting (72) we calculate

$$\begin{aligned}\mathrm{d}P_{\mathrm{s}}(v)_j/\mathrm{d}v_i = m_{\mathrm{f}}&\left(\frac{v_iv_j}{|v|^2}|v|^{-3}\right)\left(\frac{2|v|}{1-|v|^2} - 2\,\mathrm{arctanh}\left(|v|\right)\right)\\ &+ m_{\mathrm{f}}\left(\delta_{ij} - \frac{v_iv_j}{|v|^2}\right)|v|^{-3}\left(-|v| + (1+|v|^2)\,\mathrm{arctanh}\left(|v|\right)\right)\,,\end{aligned} \tag{82}$$

from which we can ascertain

$$m_{\mathrm{l}}(|v|) = m_{\mathrm{f}}|v|^{-3}\left(2|v|(1-|v|^2)^{-1} - 2\,\mathrm{arctanh}(|v|)\right)\,, \tag{83}$$
$$m_{\mathrm{t}}(|v|) = m_{\mathrm{f}}|v|^{-3}\left(-|v| + (1+|v|^2)\,\mathrm{arctanh}(|v|)\right)\,, \tag{84}$$

for the longitudinal and transversal components. We can expand these in $|v|$, which gives

$$m_{\mathrm{l}}(v) = \frac{4}{3}m_{\mathrm{f}}\left(1 + \frac{6}{5}|v|^2 + \cdots\right)\,, \tag{85}$$
$$m_{\mathrm{t}}(v) = \frac{4}{3}m_{\mathrm{f}}\left(1 + \frac{2}{5}|v|^2 + \cdots\right)\,. \tag{86}$$

We also expand $E_{\mathrm{s}}(v)$ in v, using

$$\gamma = 1 + \frac{1}{2}v^2 + \cdots\,, \quad |v|^{-1}\left(-|v| + 2\,\mathrm{arctanh}(|v|)\right) = 1 + \frac{2}{3}v^2 + \cdots\,. \tag{87}$$

Therefore, in the nonrelativistic approximation, we have

$$H_\mathrm{s}(v) - H_\mathrm{s}(0) \approx \frac{1}{2}\left(m_\mathrm{b} + \frac{4}{3}m_\mathrm{f}\right)v^2\,, \quad P_\mathrm{s}(v) \approx \left(m_\mathrm{b} + \frac{4}{3}m_\mathrm{f}\right)v\,. \tag{88}$$

The effective mass is given by

$$m_\mathrm{eff} = m_\mathrm{b} + \frac{4}{3}m_\mathrm{f}\,. \tag{89}$$

This effective mass is the inertial mass that can actually be determined experimentally. The name renormalization comes from the process of adjusting m_b to the calculated electrostatic energy m_f, so that m_eff matches the empirical rest mass.

For a more geometric approach to calculating the velocity-dependent mass, refer to Abraham [1, ch. 20].

3.3.2 The charged spherical shell

To illustrate the implications of this renormalization, we will calculate m_f for the easiest possible charge distribution, a homogeneously charged spherical shell. This model is commonly used for the electron. The charge distribution $e\varphi$ is defined by

$$\varphi(x) = \frac{1}{4\pi R^2}\delta(|x| - R)\,. \tag{90}$$

We determine the electric field by applying Gauss's law, according to which the field inside of the charged surface vanishes.

$$|E(x)| = \begin{cases} 0 & \text{if } |x| < R\,, \\ \frac{e}{4\pi|x|^2} & \text{if } |x| \geq R\,. \end{cases} \tag{91}$$

We will calculate the electrostatic energy as follows:

$$m_\mathrm{f} = \frac{1}{2}\int \mathrm{d}^3x |E(x)|^2\,. \tag{92}$$

This is equivalent to (73) by Plancherel's theorem [23]. Inserting (91) we obtain the electrostatic energy

$$m_\mathrm{f} = \frac{e^2}{8\pi R}\,, \tag{93}$$

which varies with the radius R of the charged sphere like $1/R$. Since the effective mass $m_\mathrm{eff} = m_\mathrm{b} + \frac{4}{3}m_\mathrm{f}$ is fixed to the experimentally determined inertial mass, the bare mass must be negative for charges whose radius is smaller than

$$R' = \frac{e^2}{6\pi m_\mathrm{exp}}\,. \tag{94}$$

Reintroducing c, this is 2/3 times the classical electron radius [9],

$$r_\mathrm{cl} = \frac{e^2}{4\pi m_\mathrm{exp} c^2} \approx 2.82 \times 10^{-13}\,\mathrm{cm}\,. \tag{95}$$

For a charged ball, the prefactor would be 4/5. The consequences of having a negative bare mass will be investigated in the next section.

3.4 Runaway solutions

3.4.1 The full nonlinear problem

For φ spherically symmetric, the evolution equation (44) has a stationary solution ψ_{s} [4],

$$\psi_{\mathrm{s}} = (0, 0, E_{\mathrm{C}}, 0)\,, \tag{96}$$

$$E_{\mathrm{C}}(x) = \frac{e}{4\pi}\int \mathrm{d}x'\varphi(x')\frac{x - x'}{|x - x'|^3}\,. \tag{97}$$

Let us first check that this is indeed a stationary solution of the Maxwell-Lorentz system (41). It is obvious that $\nabla \cdot B = 0$. To calculate $\nabla \cdot E_{\mathrm{C}}(x)$, we use the well-known solution of Poisson's equation

$$\Delta\frac{1}{|x - x'|} = -4\pi\delta(x - x') \tag{98}$$

and find that $\nabla \cdot E_{\mathrm{C}}(x) = e\varphi(x)$. Therefore, by theorem 3 there exists a solution $\psi(t)$ of (41) with initial value $\psi(0) = \psi_{\mathrm{s}}$. Since $\varphi(x)$ is spherically symmetric, it is an even function of x, while $E_{\mathrm{C}}(x)$ is an odd function of x. Therefore $\dot{p}(0) = 0$. Also, since $p(0) = 0$, $\dot{q}(0) = 0$. Define

$$C(x) = \frac{e}{4\pi}\frac{x}{|x|^3} = -\frac{e}{4\pi}\nabla\frac{1}{|x|}\,. \tag{99}$$

We observe that $E_{\mathrm{C}}(x) = \varphi * C(x)$ and therefore $\nabla \times E_{\mathrm{C}}(x) = 0$ since $\partial_i(f * g)(x) = (f * \partial_i g)(x)$ and any gradient field is free of rotation. This means that $\dot{B}(0, x) = 0$. Finally, since $p(0) = 0$ and $B(0, x) = 0$, $\dot{E}(0, x) = 0$. We conclude that $\dot{\psi}(0) = 0$ and therefore $\psi(t) = \psi_{\mathrm{s}}\ \forall t > 0$.

Bauer and Dürr [4] proved that ψ_{s} is Lyapunov-unstable in the case of negative bare mass ($\alpha = -1$) and an attractive potential ($\kappa > 0$). That proof has some gaps which we will attempt to fill in the following.

Theorem 4. *Let $\alpha = -1$, $\kappa > 0$ and let $\rho = e\varphi$ be spherically symmetric. Then the stationary solution ψ_{s} of the Maxwell-Lorentz system* (41) *is unstable in the sense of Lyapunov: For each $\epsilon > 0$ there is a solution $t \mapsto \psi(t) = (q(t), p(t), E(t), B(t))$ of* (41) *with $\|\psi_{\mathrm{s}} - \psi(0)\|_{\mathcal{H}} < \epsilon$ and*

$$\lim_{t\to\infty} |q(t)| = \infty\,, \quad \lim_{t\to\infty} |p(t)| = \infty\,, \quad \textit{i.e.}\ \lim_{t\to\infty} \|\psi_{\mathrm{s}} - \psi(t)\|_{\mathcal{H}} = \infty$$

Proof. Let u be some unit vector, $u \in \mathbb{R}^3$ and $|u| = 1$. We choose the following initial data:

$$\begin{aligned}
q_0 &= x_0 u\,, \quad x_0 > 0\,,\\
\dot{q}_0 &= \frac{-p_0}{\sqrt{1 + p_0^2}} = v_0 u\,, \quad v_0 > 0\,,\\
E_0(x) &= \frac{e}{4\pi}\int \mathrm{d}x'\varphi(x')\frac{x - q_0 - x'}{|x - q_0 - x'|^3} = E_{\mathrm{C}}(x - q_0)\,,\\
B_0(x) &= 0\,,
\end{aligned}$$

where we will require

$$v_0 > \sqrt{1 - \left(1 + \frac{1}{2}\kappa x_0^2\right)^{-2}}\,. \tag{100}$$

These initial data satisfy (53) and can be written in the form of a state-vector as follows:

$$\psi_0 = (q_0, p_0, E_0, B_0) = \left(q_0, \frac{-\dot{q}_0}{\sqrt{1-\dot{q}_0^2}}, E_\mathrm{C}(\cdot - q_0), 0\right).$$

Notice that $\psi_0 \in \bigcap_{n=0}^{\infty} \mathcal{D}(A^n)$, since $E_\mathrm{C} \in \mathcal{C}^\infty(\mathbb{R}^3)^3$, so by theorem 3 there is a global solution $t \mapsto \psi(t) = (q(t), p(t), E(t), B(t))$ of the Maxwell-Lorentz system (41) with $\psi(0) = \psi_0$. We still have to check that the initial data satisfy $\|\psi_\mathrm{s} - \psi_0\|_\mathcal{H} < \epsilon$. It will be easier to use the square of the norm,

$$\|\psi_\mathrm{s} - \psi_0\|_\mathcal{H}^2 = x_0^2 + \frac{v_0^2}{1-v_0^2} + \int \mathrm{d}x\, |E_\mathrm{C}(x) - E_\mathrm{C}(x-q_0)|^2\,. \tag{101}$$

Let us first check the first two terms. We set

$$v_0^2 = 1 - \left(1 + \frac{1}{2}\kappa x_0^2\right)^{-2} + \Delta = \frac{\left(\frac{1}{2}\kappa x_0^2\right)^2 + 2\left(\frac{1}{2}\kappa x_0^2\right)}{\left(1 + \left(\frac{1}{2}\kappa x_0^2\right)\right)^2} + \Delta\,, \tag{102}$$

for some $\Delta > 0$. Since the first term is a quotient of polynomials in x_0^2 and the denominator is never zero, v_0^2 depends continuously on x_0^2. We also notice that the first term vanishes in the case $x_0^2 = 0$. Likewise, the second term in (101) depends continuously on v_0^2 and is zero for $v_0^2 = 0$. Therefore, we can make the first two terms arbitrarily small by choosing x_0 and Δ small enough.

The behaviour of the third term remains to be established. We will define the auxillary function $L : \mathbb{R}^3 \to \mathbb{R}$,

$$L(x) = \int \mathrm{d}y |E_\mathrm{C}(y) - E_\mathrm{C}(y-x)|^2\,.$$

Since $E_\mathrm{C} \in \mathcal{C}^\infty(\mathbb{R}^3)^3$, $L(x)$ is at least once continuously differentiable, $L(x) \in \mathcal{C}^1(\mathbb{R}^3)$. Therefore L is locally Lipschitz-continuous [10, ch. 12]. Since L is locally Lipschitz, there exists a neighbourhood U of $0 \in \mathbb{R}^3$, such that $\forall x \in U\ \exists M \in \mathbb{R}_0^+ : |L(x) - L(0)| \leq M|x|$. Noticing that $L(0) = 0$ we can therefore find a q_0 and M, for which $|L(q_0)| \leq M|q_0| = Mx_0$. This means that $\|\psi_\mathrm{s} - \psi_0\|_\mathcal{H} < \epsilon$ is indeed satisfied by choosing x_0, v_0 sufficiently small.

Without proof we make the same assumption as Bauer and Dürr, which is that, due to $q_0, p_0 \parallel u$ and the rotational symmetry of E_0, B_0 around the axis through u, the motion of q will remain one-dimensional,

$$q(t) = x(t)u \quad (t \in \mathbb{R})\,, \tag{103}$$

with a real-valued function $t \mapsto x(t)$ satisfying $x(0) = x_0$ and $\dot{x}(0) = v_0$. Since the initial fields satisfy the decay condition (57), we can now make use of energy conservation, cf.

lemma 3,

$$
\begin{aligned}
H &= \frac{-1}{\sqrt{1-\dot{x}(t)^2}} + \frac{1}{2}\kappa x(t)^2 + \frac{1}{2}\int \mathrm{d}x \left(E(t,x)^2 + B(t,x)^2\right) \\
&= \frac{-1}{\sqrt{1-v_0^2}} + \frac{1}{2}\kappa x_0^2 + \frac{1}{2}\int \mathrm{d}x E_{\mathrm{C}}(x)^2
\end{aligned}
\tag{104}
$$

for all $t \in \mathbb{R}$. We decompose the electric field into a longitudinal and a transversal part [cf. 3, ch. 1.16], $E = E_{\parallel} + E_{\perp}$, where $\nabla \cdot E_{\perp} = 0$ and $\nabla \times E_{\parallel} = 0$. The Fourier modes of the longitudinal and transversal components are given by $\hat{E}_{\parallel}(k) = \hat{k}(\hat{k}\cdot\hat{E}(k))$ and $\hat{E}_{\perp} = \hat{E} - \hat{E}_{\parallel}$ [cf. 25, ch. 6.3], where $\hat{k}$ denotes the unit vector in direction of k, and $\hat{E}$ denotes the Fourier mode of E. By theorem 3, $E \in (L^2)^3$ and we can apply Parseval's formula [23, thm. 7.9], $\int \mathrm{d}x E_{\parallel}\cdot E_{\perp} = \int \mathrm{d}k \hat{E}_{\parallel}\cdot\hat{E}_{\perp}$. The longitudinal and transversal components are orthogonal in Fourier space,

$$
\hat{E}_{\parallel}\cdot\hat{E}_{\perp} = \hat{E}_{\parallel}\cdot(\hat{E}-\hat{E}_{\parallel}) = \hat{k}(\hat{k}\cdot\hat{E})\cdot\hat{E} - (\hat{k}\cdot\hat{E})^2 = 0\,,
$$

so $\int \mathrm{d}x E_{\mathrm{C}}\cdot E_{\mathrm{t}} = 0$. We also notice that

$$
\nabla\cdot E_{\parallel}(t,x) = \nabla\cdot E(t,x) = e\varphi(x-q(t))\,. \tag{105}
$$

Taking into account that $\nabla\times E_{\parallel} = 0$,we can solve (105) using a Green's function to obtain $E_{\parallel}(t,x) = E_{\mathrm{C}}(x-q(t))$ [3, ch. 9.7]. Therefore

$$
\int \mathrm{d}x\left(E^2+B^2\right) \geq \int \mathrm{d}x E^2 = \int \mathrm{d}x\left(E_{\mathrm{C}}^2 + 2E_{\mathrm{C}}\cdot E_{\perp} + E_{\perp}^2\right) \geq \int \mathrm{d}x E_{\mathrm{C}}^2\,. \tag{106}
$$

From (104) we thus obtain

$$
\frac{1}{\sqrt{1-\dot{x}(t)^2}} \geq \frac{1}{2}\kappa\left(x(t)^2 - x_0^2\right) + \frac{1}{\sqrt{1-v_0^2}} > 1 \tag{107}
$$

for all $t\in\mathbb{R}$, where we used the condition (100) in the second step. Therefore, and since $x_0, v_0 > 0$, there exists a constant $b > 0$ such that $\forall t \geq 0 : \dot{x}(t) > b$. By integration we find that $\forall t \geq 0 : x(t) \geq bt + a$, where a is another constant. Again using the condition (100), we find

$$
\frac{1}{\sqrt{1-\dot{x}(t)^2}} \geq \frac{1}{2}\kappa\left(bt+a\right)^2 \tag{108}
$$

for all $t \geq 0$ and therefore

$$
\lim_{t\to\infty}|x(t)| = \infty\,, \quad \lim_{t\to\infty}|\dot{x}(t)| = 1\,. \tag{109}
$$

Because of (103) this is what we had to show. □

We slightly improved on the proof by Bauer and Dürr [4] by explicitly showing that the initial data are ϵ-close to ψ_{s} and clarifying the final argument in (108).

This proof does not remain without some weak spots. While the assumption of one-dimensional motion (103) seems reasonable, it still has to be proven rigorously. Another point of concern is the attractive potential. Setting $\kappa > 0$ is required to make the conservation of energy argument work without allowing solutions with constant velocity. In combination with the negative bare mass this means, however, that even a particle with zero charge would run away, according to Newton's law $m\ddot{q}(t) = F = -\kappa q(t)$.

The mechanism described in theorem 4 cannot be understood as the particle's energy loss to radiation causing it to accelerate despite a binding potential. Rather it is to be understood as an acceleration caused by external forces which is not being impeded by the particle's interaction with its own fields.

While the result of theorem 4 is certainly a much less powerful statement, we still expect runaways to occur even without an external potential as long as the bare mass is negative. To substantiate this claim, we will conduct a brief thought experiment. The fact that an accelerated particle loses energy, as described phenomenologically by the Larmor formula [12], suggests that the interaction of the particle with its own radiation acts as a damping force countering the acceleration caused by external forces. By setting the bare mass negative, the sign of the acceleration caused by this force would change as well and start acting in the same direction as the particle's existing acceleration, thus causing the particle to accelerate indefinitely. We will pursue this thought in the next section.

3.4.2 The Sommerfeld-Page equation

To support our expectation of runaway solutions in the absence of an external potential, we will briefly discuss the Sommerfeld-Page equation, which is a linearized nonrelativistic equation of motion for extended charges. It is derived under the assumptions that the particle is instantaneously at rest and that the charge distribution is rigid and spherically symmetric, thus agreeing with the Abraham model used in the previous section. One sets out from the Lorentz force and expresses the electric field using the four-potential. In the four-potential, the four-current is evaluated at the retarded time. Expanding the four-current at the time t, one obtains the expression [14, 19]

$$m_{\mathrm{b}}\ddot{q}(t) = F_{\mathrm{ext}}(t) - \frac{1}{6\pi}\sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n\frac{\mathrm{d}^{n+2}q(t)}{\mathrm{d}t^{n+2}}\gamma_n\,, \tag{110}$$

where we reintroduced the bare mass m_{b} and the coefficients γ_n are defined through the charge distribution $\rho = e\varphi$,

$$\gamma_n = e^2\int \mathrm{d}x \int \mathrm{d}x'\varphi(t,x)\varphi(t,x')|x-x'|^{n-1}\,. \tag{111}$$

The detailed derivation of (110) can be found in [14, ch. 16.3]. For a charged sphere with a charge distribution as defined in (90), the coefficients can be calculated explicitly,

$$\gamma_n = \frac{2e^2(2R)^{n-1}}{n+1}\,, \tag{112}$$

and the series in (110) converges, yielding

$$m_{\mathrm{b}}\ddot{q}(t) = F_{\mathrm{ext}} + \frac{e^2}{12\pi R^2}\left(\exp\left(-2R\frac{\mathrm{d}}{\mathrm{d}t}\right) - 1\right)\dot{q}(t)\,. \tag{113}$$

Here, $\exp\left(-2R\frac{\mathrm{d}}{\mathrm{d}t}\right)$ is a time delay operator. Thus, if $m_\mathrm{b} \neq 0$, we obtain [15, 19]

$$\ddot{q}(t) = \frac{F_\mathrm{ext}(t)}{m_\mathrm{b}} + \frac{e^2}{12\pi m_b R^2}\left[\dot{q}(t-2R) - \dot{q}(t)\right] , \tag{114}$$

which is the Sommerfeld-Page equation.

We will now again consider the case without external forces. The general solution to (114) with $F_\mathrm{ext} = 0$ is given by

$$q(t) = ae^{\omega t} , \tag{115}$$

with a constant vector a. For a runaway solution in the sense of section 3.4.1 it is required for the real part of ω to be positive. Inserting (115) into (114) it can be seen that, aside from the trivial solution $\omega = 0$, ω is given by

$$\omega = \frac{e^2}{12\pi m_\mathrm{b} R^2}\left(e^{-2\omega R} - 1\right) . \tag{116}$$

Therefore the real part of ω can only be positive if $m_\mathrm{b} < 0$. By writing (114) as an integral equation with a Green's function, one can also see that preacceleration occurs only in the case of negative bare mass [15, 19].

The nonrelativistic approximation seems suboptimal for the purpose of investigating runaway solutions, in which the velocity will quickly approach the speed of light. The relativistic generalization of the Sommerfeld-Page equation is the Caldirola equation [5, 13, 15, 21, 27],

$$m_\mathrm{b}\frac{\mathrm{d}u^\mu}{\mathrm{d}s} = F^\mu_\mathrm{ext} + \frac{e^2}{12\pi R^2}\left[u^\mu(s-2R) - u^\mu(s)u_\nu(s)u^\nu(s-2R)\right] . \tag{117}$$

Like the Sommerfeld-Page equation, the Caldirola equation does not allow runaway solutions or preacceleration in the case of positive bare mass [21]. It is however not obvious that there should not be any runaway solutions in the case of negative bare mass. In the limit $R \to 0$, the Caldirola equation reduces to the LAD equation [13, 21], which exhibits runaway solutions and preacceleration as discussed in sections 2.1 and 2.2. Remembering the relationship between a particle's size and its bare mass, as discussed in section 3.3.2, this further suggests that the classical dynamics of charged particles are only stable for particles larger than some finite size.

4 Conclusion and outlook

4.1 Conclusion

We discussed the coupled Maxwell-Lorentz dynamics for point charges and found that the solutions of the Maxwell equations for a point charge in motion cause the Lorentz force to be ill-defined. We analyzed the Lorentz-Abraham-Dirac equation of motion for point charges, which circumvents this problem by using conservation of energy and momentum instead of the Lorentz force in its derivation, and found that it allows runaway solutions. We also found that by imposing an additional condition on the acceleration, which eliminates runaway solutions, one causes the solutions to exhibit preacceleration, where the particle starts to accelerate even before being acted upon by an external force. Lastly for point charges, we presented the Landau-Lifshitz approximation of the LAD equation, which eliminates runaway solutions in the case of a free particle and does not allow a delta force for which we demonstrated the problem of preacceleration with the LAD equation.

We also examined the alternative approach to the problem of the divergent fields of a point charge, which is to introduce an extended charge distribution. We introduced the semirelativistic Abraham model of a rigid charge and set up the dynamics as an evolution equation. We recounted the results of Bauer and Dürr [4] for global existence of solutions to this evolution equation and explicitly proved conservation of energy. By explicitly calculating the velocity-dependent mass of a charged particle, we showed that for particles below a certain size, the bare mass has to be negative if their inertia is to match the empirically determined value. We closed some gaps in the proof by Bauer and Dürr [4], which showed Lyapunov-instability of the stationary solution to the dynamics in the case of negative bare mass, by explicitly checking that the initial conditions are ϵ-close to the stationary solution and slightly modifying them in order to make the runaway argument clearer. Thus we proved the existence of runaway solutions in the case of negative bare mass and a quadratic potential with positive curvature. However, it turned out that this mechanism is not the same we would expect physically, which is that the damping force caused by the radiation reaction causes the particle to accelerate further due to its negative bare mass, regardless of an external potential. To support this expectation, we discussed the Sommerfeld-Page and Caldirola equations of motion for extended charges, which are only free of runaway solutions in the case of positive bare mass.

In conclusion, we found that the classical dynamics of a nonrelativistically rigid charged particle interacting with its own fields seem to be accurate only for particles with non-negative bare mass. This turned out to be the case for charge distributions larger than a certain size, depending on their shape. This matches our physical intuition, as we would expect a particle with negative inertia to react to a damping force by accelerating further.

4.2 Outlook

For further investigations, it would be interesting to find out more about the mechanism behind runaway solutions. It still remains to be shown rigorously that the motion remains one-dimensional for the initial conditions given in the proof of theorem 4. A next step would be to improve on the initial conditions used in the proof of theorem 4. We used a translated Coulomb field in the initial conditions and while this worked out to give us a global solution according to theorem 3, it is not what we would expect from a physical standpoint. In the following, we will present two ideas for incorporating the physically expected fields of a charge in rectilinear motion into the initial conditions in the proof of theorem 4 as a starting point for further investigations.

For an extended charge in motion, we would expect the fields to be the Liénard-Wiechert fields of a point charge convoluted with the charge distribution,

$$\begin{aligned} E^v_\varphi(t,x) &= \varphi * E^{LW,v}(t, x - q_0)\,, \\ B^v_\varphi(t,x) &= \varphi * B^{LW,v}(t, x - q_0) = \hat{n} \times E^v_\varphi(t,x)\,, \end{aligned} \tag{118}$$

with $E^{LW,v}, B^{LW,v}$ defined in (5). We will assume that q_0 and v point along the x_1-axis. Let us first gain an overview over the frames of reference involved in our problem. The coordinates of the laboratory frame L shall be denoted by $x^\mu = (t, x_1, x_2, x_3)$. For simplicity we will introduce another frame L', which is related to L by translation, $x'^\mu = (t', x'_1, x'_2, x'_3) = (t, x_1 - x_0, x_2, x_3)$. Therefore $E^v_\varphi(t,x) = \varphi * E^{LW,v}(t,x')$. We will denote by $\tilde{x}^\mu$ the coordinates in the momentary rest frame of the particle (RF), which is related to L and L' by a Lorentz transformation,

$$\tilde{x}^i = \Lambda(v)^i_\nu \tilde{x}'^\nu = \begin{pmatrix} \gamma(x'_1 - vt) \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \gamma(x_1 - x_0 - vt) \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \gamma R_1 \\ R_2 \\ R_3 \end{pmatrix}. \tag{119}$$

In the case of rectilinear motion, $q(t) = q_0 + vt$, R is the vector pointing from the momentary location $q(t)$ of the particle to the location x of the observer in the laboratory frame.

We will now examine the transformation of the fields. If RF is moving at velocity v relative to L', the transformation of the fields is given by (133),

$$E(t,x') = \begin{pmatrix} \tilde{E}_1(\tilde{t},\tilde{x}) \\ \gamma(\tilde{E}_2(\tilde{t},\tilde{x}) + v\tilde{B}_3(\tilde{t},\tilde{x})) \\ \gamma(\tilde{E}_3(\tilde{t},\tilde{x}) - v\tilde{B}_2(\tilde{~},\tilde{x})) \end{pmatrix}\,, \quad B(t,x') = \begin{pmatrix} \tilde{B}_1(\tilde{t},\tilde{x}) \\ \gamma(\tilde{B}_2(\tilde{t},\tilde{x}) - v\tilde{E}_3(\tilde{t},\tilde{x})) \\ \gamma(\tilde{B}_3(\tilde{t},\tilde{x}) + v\tilde{E}_2(\tilde{t},\tilde{x})) \end{pmatrix}. \tag{120}$$

In the rest frame, we assume that the electric field is given by a Coulomb field and the magnetic field vanishes,

$$\tilde{E}(\tilde{x}) = \tilde{\varphi} * C(\tilde{x})\,, \quad \tilde{B}(\tilde{x}) = 0\,, \tag{121}$$

with $C(x)$ defined in (99) and $\tilde{\varphi}$ an unknown charge distribution.

The next step will be to determine the relationship between φ and $\tilde{\varphi}$. For rectilinear motion the Liénard-Wiechert electric field can be expressed as [12, ch. 12.3.2]

$$E^{LW,v}(t,x') = \frac{e}{4\pi} \frac{\gamma R}{((\gamma R_1)^2 + R_2^2 + R_3^2)^{3/2}}\,. \tag{122}$$

According to (120) and (121), the electric field transforms like

$$E^v_\varphi(t,x) = \varphi * E^{LW,v}(t,x') = \operatorname{diag}(1,\gamma,\gamma)\tilde{\varphi} * E_{\mathrm{C}}(\tilde{x})\,. \tag{123}$$

Using (122) and (123) and setting $\tilde{y} = (\gamma y_1, y_2, y_3)$, we obtain

$$
\begin{aligned}
E^v_\varphi(t,x) &= \mathrm{diag}\,(1,\gamma,\gamma)\int \mathrm{d}\tilde{y}\tilde{\varphi}(\tilde{y})\frac{e}{4\pi}\frac{\tilde{x}-\tilde{y}}{|\tilde{x}-\tilde{y}|^3} \\
&= \gamma\frac{e}{4\pi}\int \mathrm{d}y\tilde{\varphi}(\tilde{y})\frac{\gamma(R-y)}{((\gamma(R_1-y_1))^2+(R_2-y_2)^2+(R_3-y_3)^2)^{3/2}} \\
&= \tilde{\varphi}' * E^{LW,v}(t,x')
\end{aligned}
\tag{124}
$$

This implies

$$
\varphi(y) = \tilde{\varphi}'(y) = \gamma\tilde{\varphi}(\gamma y_1, y_2, y_3)\,. \tag{125}
$$

We will also check Gauss' law for our initial field $E^v_\varphi(t,x)$. To do this, we first establish

$$
\nabla' = \left(\frac{\partial}{\partial x_1'}, \frac{\partial}{\partial x_2'}, \frac{\partial}{\partial x_3'}\right) = \left(\gamma\frac{\partial}{\partial \tilde{x}_1}, \frac{\partial}{\partial \tilde{x}_2}, \frac{\partial}{\partial \tilde{x}_3}\right), \tag{126}
$$

with $\partial/\partial x_i = \partial/\partial x_i'$ and use the notation $\tilde{\nabla} = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$. Using this and (126) we can calculate

$$
\begin{aligned}
\nabla\cdot E^v_\varphi(t,x) &= \mathrm{diag}\,(1,\gamma,\gamma)\nabla'\cdot(\tilde{\varphi} * C)(\tilde{x}) \\
&= \gamma\frac{e}{4\pi}\int \mathrm{d}\tilde{y}\tilde{\varphi}(\tilde{y})\tilde{\nabla}\cdot\frac{\tilde{x}-\tilde{y}}{|\tilde{x}-\tilde{y}|^3} \\
&= e\gamma\tilde{\varphi}(\tilde{x}) \\
&= e\varphi(x-q(t))\,.
\end{aligned}
\tag{127}
$$

Thus Gauss' law as stated in (41) is satisfied.

One could now attempt to find a similar estimate as in (106) but using the Liénard-Wiechert fields instead. For that, it would be advantageous to have the field part of the energy in the form

$$
\int \mathrm{d}x\left(E^v_\varphi(t,x)^2 + B^v_\varphi(t,x)^2\right) = a(v)\int \mathrm{d}xF(t,x)^2\,, \tag{128}
$$

where $F(t,x)$ is some expression that does not depend on the velocity. Using (120), we can write

$$
\int \mathrm{d}x\left(E^v_\varphi(t,x)^2 + B^v_\varphi(t,x)^2\right) = \int \mathrm{d}x\left((\tilde{\varphi} * C(\tilde{x}))^2 + 2v^2\gamma^2\left[(\tilde{\varphi} * C(\tilde{x}))_2^2 + (\tilde{\varphi} * C(\tilde{x}))_3^2\right]\right)\,. \tag{129}
$$

Since we know from (125) that $\tilde{\varphi}(x)$ is symmetric with regard to rotations along the x_1-axis, the two terms in the square brackets are equal to each other. However, to obtain an expression of the type (128), we would need to know the relationship between the terms $\int \mathrm{d}x(\tilde{\varphi} * C(\tilde{x}))_1^2$ and $\int \mathrm{d}x(\tilde{\varphi} * C(\tilde{x}))_{2,3}^2$.

An alternative approach would be to write the fields at time t as the sum of the velocity and acceleration terms of the Liénard-Wiechert fields, $E^v + E^a, B^v + B^a$. The field part of the energy corresponding to E^v, B^v was calculated in (74) and is strictly increasing with $|v|$. To carry out the proof by the same method as in section 4, it would remain to be shown that $\int \mathrm{d}xE^v\cdot E^a = 0$ so that $\int \mathrm{d}x(E^2+B^2) \geq \int \mathrm{d}x((E^v)^2+(B^v)^2)$. For rectilinear motion this follows immediately from (5) in the case of a point charge, but for an extended charge it is not obvious.

In both cases it would have to be shown that the initial conditions containing the Liénard-Wiechert fields are also ϵ-close to the stationary solution (96) in the $\mathcal{H}$-norm.

A Appendix

A.1 Heaviside-Lorentz units

Any expressions given in Heaviside-Lorentz units can easily be translated to Gaussian-cgs or SI units. To do this, replace the quantities in column 1 of table 1 by the the quantities in column 2 or 3 respectively [7, 14]. For an exhaustive list of relations refer to [7].

Heaviside-Lorentz	Gaussian-cgs	SI
(E^{HL}, ϕ^{HL})	$\frac{1}{\sqrt{4\pi}}(E^G, \phi^G)$	$\sqrt{\varepsilon_0}(E^{SI}, \phi^{SI})$
$(e^{HL}, \rho^{HL}, j^{HL})$	$\sqrt{4\pi}(e^G, \rho^G, j^G)$	$\frac{1}{\sqrt{\varepsilon_0}}(e^{SI}, \rho^{SI}, j^{SI})$
(B^{HL}, A^{HL})	$\frac{1}{\sqrt{4\pi}}(B^G, A^G)$	$\frac{1}{\sqrt{\mu_0}}(B^{SI}, A^{SI})$

Table 1: Translation between H.-L., Gaussian-cgs and SI units of measurement

A.2 Lorentz boost

Recall the four-potential $A_\mu = (-\phi, A)$. The field strength tensor is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We will use the metric tensor with the signature $(-,+,+,+)$.

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} . \tag{130}$$

The indices can be raised by employing the metric tensor g, $F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$. $F^{\mu\nu}$ is an order two tensor and thus transforms like

$$F^{\mu\nu}(x) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \tilde{F}^{\alpha\beta}(\tilde{x}) \tag{131}$$

using $\mathrm{x}^\mu = \Lambda^\mu{}_\nu \tilde{\mathrm{x}}^\nu$ for the transformation of the position four-vector $\mathrm{x} = (ct, x)$. Now, if the frame described by x is moving at velocity v in the x_1 direction relative to the frame described by $\tilde{\mathrm{x}}$, we set $\Lambda = \Lambda(v)$,

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{132}$$

with $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$, and obtain the transformed fields

$$E(t,x) = \begin{pmatrix} \tilde{E}_1(\tilde{t},\tilde{x}) \\ \gamma(\tilde{E}_2(\tilde{t},\tilde{x}) - \beta\tilde{B}_3(\tilde{t},\tilde{x}) \\ \gamma(\tilde{E}_3(\tilde{t},\tilde{x}) + \beta\tilde{B}_2(\tilde{t},\tilde{x}) \end{pmatrix} , \quad B(t,x) = \begin{pmatrix} \tilde{B}_1(\tilde{t},\tilde{x}) \\ \gamma(\tilde{B}_2(\tilde{t},\tilde{x}) + \beta\tilde{E}_3(\tilde{t},\tilde{x})) \\ \gamma(\tilde{B}_3(\tilde{t},\tilde{x}) - \beta\tilde{E}_2(\tilde{t},\tilde{x}) \end{pmatrix} . \tag{133}$$

References

[1] M. Abraham. *Theorie der Elektrizität 2: Elektromagnetische Theorie der Strahlung.* Teubner, 1905.

[2] W. Appel and M. K.-H. Kiessling. Mass and spin renormalization in Lorentz electrodynamics. *Ann. Phys. (N. Y.)*, 289(1):24–83, 2001. doi: 10.1006/aphy.2000.6119.

[3] G. B. Arfken and H. J. Weber. *Mathematical Methods for Physicists.* Elsevier, 6th edition, 2006.

[4] G. Bauer and D. Dürr. The Maxwell-Lorentz system of a rigid charge. *Ann. Henri Poincaré*, 2:179–196, 2001. doi: 10.1007/PL00001030.

[5] P. Caldirola. A new model of classical electron. *Nuovo Cimento*, 3(suppl. 2): 297–343, 1956. doi: 10.1007/BF02743686.

[6] D.-A. Deckert. *Electrodynamic Absorber Theory.* PhD thesis, LMU Munich, 2009.

[7] E. A. Desloge. Relation between equations in the international, electrostatic, electromagnetic, Gaussian, and Heaviside–Lorentz systems. *Am. J. Phys.*, 62(7):601–609, 1994. doi: 10.1119/1.17534.

[8] P. A. M. Dirac. Classical theory of radiating electrons. *Proc. R. Soc. Lond. A*, 167 (929):148–169, 1938. doi: 10.1098/rspa.1938.0124.

[9] R. P. Feynman, R. B. Leighton, and M. Sands. *The Feynman Lectures on Physics, Volume II: Mainly Electromagnetism and Matter.* Basic Books, 2010.

[10] O. Forster. *Analysis 2.* Springer, 11th edition, 2017.

[11] J. Frenkel. Zur Elektrodynamik punktförmiger Elektronen. *Z. Phys.*, 32:518–534, 1925. doi: 10.1007/BF01331692.

[12] D. J. Griffiths. *Introduction to Electrodynamics.* Cambridge, 5th edition, 2024.

[13] K. Iqbal. *Radiation effects on relativistic electrons in strong external fields.* PhD thesis, LMU Munich, 2012.

[14] J. D. Jackson. *Classical Electrodynamics.* Wiley, 3rd edition, 1998.

[15] K.-J. Kim. The equation of motion of an electron: a debate in classical and quantum physics. *Nucl. Instrum. Methods Phys. Res. A*, 429(1-3):1–8, 1999. doi: 10.1016/S0168-9002(99)00047-9.

[16] A. Komech and H. Spohn. Long-time asymptotics for the coupled Maxwell-Lorentz equations. *Comm. Partial Differential Equations*, 25(3-4):559–584, 2000. doi: 10.1080/03605300008821524.

[17] L. D. Landau and E. M. Lifshitz. *Course of Theoretical Physics. Volume 2, The Classical Theory of Fields*. Butterworth-Heinemann, fourth revised english edition, 1975.

[18] K. T. McDonald. On the history of the radiation reaction, 2020. URL http://kirkmcd.princeton.edu/examples/selfforce.pdf. last accessed May 30th 2024.

[19] E. J. Moniz and D. Sharp. Radiation reaction in nonrelativistic quantum electrodynamics. *Phys. Rev. D*, 15(10):2850–2865, 1977. doi: 10.1103/PhysRevD.15.2850.

[20] S. Parrott. *Relativistic Electrodynamics and Differential Geometry*. Springer, 1987.

[21] F. Rohrlich. The dynamics of a charged sphere and the electron. *Am. J. Phys.*, 65 (11):1051–1056, 1997. doi: 10.1119/1.18719.

[22] F. Rohrlich. *Classical Charged Particles*. World Scientific, 3rd edition, 2007.

[23] W. Rudin. *Functional Analysis*. McGraw-Hill, 2nd edition, 1991.

[24] P. Sołtan. *A Primer on Hilbert Space Operators*. Birkhäuser, 2018.

[25] H. Spohn. *Dynamics of Charged Particles and Their Radiation Field*. Cambridge University Press, 2004.

[26] D. Villarroel. Preacceleration in classical electrodynamics. *Phys. Rev. E*, 66(4): 046624, 2002. doi: 10.1103/PhysRevE.66.046624.

[27] A. Yaghjian. *Relativistic Dynamics of a Charged Sphere: Updating the Lorentz-Abraham Model*. Springer International Publishing, 2022.

Eigenständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig verfasst zu haben und keine anderen als die in der Arbeit angegebenen Quellen und Hilfsmittel benutzt zu haben.

München, den 18.06.2024

Name