Bachelor's Thesis

A Mathematical Approach to the Dirac Equation: Representation Theory of the Lorentz Group

Ein Mathematischer Zugang zur Dirac-Gleichung:
Darstellungstheorie der Lorentz-Gruppe

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1 Introduction

The theories of relativity and quantum mechanics are among the most successful in physics. While a full unification of general relativity and quantum mechanics has yet to be achieved, such a unification has been partially accomplished for special relativity. A key step in this direction is the Dirac equation, the relativistic extension of the Schrödinger equation. This equation describes particles with spin $\frac{1}{2}$, such as electrons, in a way that is consistent with the principles of special relativity.

In this thesis, we will approach the Dirac equation from a mathematical perspective. Beginning with the symmetries of spacetime, we will explore which equations are appropriate for formulating a quantum theory that is relativistically admissible. This leads to the Klein-Gordon equation and the Dirac equation as natural candidates.

During the early development of quantum mechanics, it was recognized that the mathematical structures employed in this theory are intimately related to those studied in representation theory in mathematics. From this perspective, the framework of quantum mechanics appears to align naturally with the mathematical language of representation theory.

Although there are many excellent books on the Dirac equation and representation theory, the language used is often abbreviated and not always rigorously presented. As a result, significant time is required to become familiar with the subject matter. The goal of this thesis is to describe the mathematical framework in a consistent and precise terminology, drawn from mathematical representation theory, which will be introduced gradually alongside the necessary mathematical definitions.

In Chapter 2, we focus on the physical space of special relativity, the Minkowski space, and investigate its symmetries. This leads us to the Lorentz transformations, which form a Lie group, a differentiable manifold endowed with a group structure. In discussing its group properties, we encounter the mathematical concept of group representations, which is closely tied to the description of the spin of a particle. We will show that the connected component of the Lorentz group with the neutral element can be decomposed into rotations and Lorentz boosts. Since the manifold of rotations is not simply connected, we are led to the mathematical concept of the universal covering group.

In Chapter 3, we determine the universal covering groups. We find that the rotation group is covered by $SU(2)$, and for the Lorentz group, the universal covering group is $SL(2, \mathbb{C})$. The latter proves to be a suitable foundation for constructing further representations. Through this approach, we can systematically derive relativistically invariant field equations, including the covariant Dirac equation.

In Chapter 4, we take a different perspective by delving into the fact that the Lorentz group forms a Lie group. This leads to a specific structure known as the Lie algebra. We begin with a general introduction to the Lie algebra of matrix Lie groups, and then apply these concepts to the Lorentz group, yielding the Lorentz algebra. The basis elements of this algebra, which are the infinitesimal generators of the Lorentz group, are closely related to the symmetries of spacetime. In particular, the generators of rotations correspond to the components of angular momentum, while the generators of boosts are associated with transformations between different inertial frames, linking time and space coordinates.

To conclude the discussion, we will provide an outlook on further extensions of this theory by considering the Poincaré group, which includes both spatial and temporal translations. This allows for a more complete description of the system's dynamics, including its time evolution.

2 Lorentz Group

2.1 Minkowski Space

Events in spacetime are described with respect to a reference frame by coordinates $x = (x^0, \vec{x})^T \in \mathbb{R}^4$, where $\vec{x} = (x^1, x^2, x^3)^T \in \mathbb{R}^3$ represents the spatial coordinates, and $x^0 = ct$ is the temporal coordinate. A transformation of the reference frame is described by the affine transformation

$$x \mapsto x' = \Lambda \cdot x + a \tag{1}$$

where $\Lambda \in \mathbb{R}^{4\times 4}$ is a matrix representing the linear part of the transformation and $a \in \mathbb{R}^4$ is a vector representing the translations. $\cdot$ is the standard matrix multiplication.

The aim is to determine which of these transformations are consistent with the principles of special relativity and, therefore, represent transformations between all possible inertial frames.
The principles of relativity can be stated as follows:

- All inertial frames are equivalent for the description of physical laws.
- The speed of light c is constant in all inertial frames.

Light rays originating from the origin propagate along the light cone described by

$$c^2t^2 - |\vec{x}|^2 = 0 \tag{2}$$

where $|\cdot|$ denotes the Euclidean norm. It consists of the past light cone ($t < 0$) and the future light cone ($t > 0$). Translations in space and time a can be disregarded.
Given this, the above principles can be rewritten as

$$(x^0)^2 - \vec{x}^2 = 0 \iff (x'^0)^2 - \vec{x'}^2 = 0 \tag{3}$$

where we use the notation $|\vec{x}|^2 = \vec{x}^2$.
It is useful to introduce the following quadratic forms:

$$Q(x) := x^T g x \tag{4}$$
$$Q'(x) := Q(x') = x^T \Lambda^T g \Lambda x \tag{5}$$

where

$$g = (g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6}$$

is the metric tensor, for which we chosen a signature $(1, -1, -1, -1)$. We use the standard index notation with covariant and contravariant indices along with the Einstein summation convention as explained in (Thaller, 1992, Chapter 2.1)

For a fixed $\vec{x}$, the quadratic form (4) vanishes when $x^0 = \pm|\vec{x}|$. From the condition in (3), it follows that the quadratic form in (5) must also vanish. Thus, it can be generally expressed as

$$Q'(x) = \lambda(x^0 - |\vec{x}|)(x^0 + |\vec{x}|) = \lambda\left((x^0)^2 - \vec{x}^2\right) = \lambda Q(x) \tag{7}$$

The case $\lambda \neq 1$ corresponds to a change of units and is not considered further. Consequently, we obtain $Q'(x) = Q(x)$, implying $\forall x \in \mathbb{R}^4 : x^T \Lambda^T g \Lambda x = x^T g x$. This condition is equivalent to

$$\Lambda^T g \Lambda = g \quad \text{or} \quad \Lambda^\mu{}_\rho g_{\mu\nu} \Lambda^\nu{}_\tau = g_{\rho\tau} \tag{8}$$

This leads to the introduction of the bilinear form

$$\langle x, y \rangle = x^T g y \tag{9}$$

which can be shown to be a indefinite scalar product (Scharf, 2014, Chapter 0.1), and remains invariant under the aforementioned transformations:

$$\langle \Lambda x, \Lambda y \rangle = (\Lambda x)^T g (\Lambda y) = x^T \Lambda^T g \Lambda y = x^T g y = \langle x, y \rangle \tag{10}$$

This is how constancy of the speed of light in all inertial frames, disregarding changes of units, is mathematically reflected as the relativistic invariance of the scalar product. The space $(\mathbb{R}^4, \langle\cdot,\cdot\rangle)$ is referred to as Minkowski space $\mathbb{M}$.

2.2 Lorentz Transformations

Definition 2.1 (Lorentz transformation)**.** *A Lorentz transformation is a linear map* $\Lambda : \mathbb{R}^4 \to \mathbb{R}^4$ *such that* $\forall x, y \in \mathbb{R}^4 : \langle \Lambda x, \Lambda y \rangle = \langle x, y \rangle$.

As demonstrated, Lorentz transformations can also be characterized by Equation (8). We now proceed to show that the set of Lorentz transformations forms a group.

Definition 2.2 (Group)**.** *A group is a pair* $(G, *)$*, consisting of a set* G *and a binary operation* $* : G \times G \to G$*,* $(a, b) \mapsto a * b$*, which satisfies the following conditions:*

1. *Associativity:* $\forall a, b, c \in G : (a * b) * c = a * (b * c)$
2. *Existence of identity element:* $\exists e \in G \; \forall g \in G : g * e = g = e * g$
3. *Existence of inverse element:* $\forall g \in G \; \exists g^{-1} \in G : g * g^{-1} = e = g^{-1} * g$

In this context, the group operation is the standard matrix multiplication. First, we demonstrate that the composition of two Lorentz transformations results in another Lorentz transformation:

$$(\Lambda_1 \Lambda_2)^T g \Lambda_1 \Lambda_2 = \Lambda_2^T \Lambda_1^T g \Lambda_1 \Lambda_2 = \Lambda_2^T g \Lambda_2 = g \tag{11}$$

Conditions 1 and 2 follow directly from the properties of matrix multiplication. It remains to verify condition 3:

$$1 = -\det(g) = -\det(\Lambda^T g \Lambda) = -\underbrace{\det(\Lambda^T)}_{=\det(\Lambda)} \underbrace{\det(g)}_{=-1} \det(\Lambda) = \det(\Lambda)^2 \implies \det(\Lambda) = \pm 1 \neq 0 \tag{12}$$

This implies that Λ is invertible and Λ^{-1} is itself a Lorentz transformation:

$$(\Lambda^{-1})^T g \Lambda^{-1} = (\Lambda^{-1})^T (\Lambda^T g \Lambda) \Lambda^{-1} = ((\Lambda^{-1})^T \Lambda^T) g (\Lambda \Lambda^{-1}) = g \tag{13}$$

Consequently, it has been established that the Lorentz transformations indeed form a group, which shall be denoted by $\mathcal{L} = O(1,3)$ from now on.

Next, we explore how a group acts on a set, which will lead us to the concept of group representations. We define:

Definition 2.3 (Group Action). *A group action of a group G on a set M is a map*

$$\alpha : G \times M \to M,\ (g, x) \mapsto g \cdot x \tag{14}$$

that assigns to each pair (g, x) of a group element $g \in G$ and an element $x \in M$ another element $g \cdot x \in M$, such that

$$\forall g_1, g_2 \in G\ \forall x \in M : g_1 \cdot (g_2 \cdot x) = (g_1 * g_2) \cdot x \tag{15}$$

It follows that

$$\forall x \in M \text{ with } e \text{ the identity element of } G : e \cdot x = x \tag{16}$$

since

$$\forall g \in G\ \forall x \in M : g \cdot (e \cdot x) = (g * e) \cdot x = g \cdot x \tag{17}$$

In general, M can be an arbitrary set with a complex structure.
To simplify the problem, we consider the linearization of the set M. Let W be a vector space over a field $\mathbb{K}$. Define $F(M) = \{f \mid f : M \to W\} \subseteq W^M$ as the set of functions on the set M. This approach linearizes the problem, as $F(M)$ forms a vector space:

$$\forall x \in M\ \forall f_1, f_2 \in F(M) : (f_1 + f_2)(x) := f_1(x) + f_2(x) \tag{18}$$
$$\forall \lambda \in \mathbb{K}\ \forall x \in M\ \forall f \in F(M) : (\lambda f)(x) := \lambda f(x) \tag{19}$$

Given a group action α of G on M, one can then define an "action of G on $F(M)$".

Definition 2.4 (Group Representation). *A representation of a group G is a pair (V, π), consisting of a vector space V and a group homomorphism*

$$\pi : G \to GL(V),\ g \mapsto \pi(g) \tag{20}$$

where $GL(V)$ denotes the group of invertible linear maps $V \to V$.

The group homomorphism property of π requires that $\forall g_1, g_2 \in G$:

$$\pi(g_1)\pi(g_2) = \pi(g_1 * g_2) \tag{21}$$

It ensures that a "group action on $F(M)$" is well-defined. From here on, we will just use the term "homomorphism", always referring to group homomorphisms unless stated otherwise. The general idea is to linearize the group action by linearizing both the set M

and the group G.

If V is a finite-dimensional vector space, it is well-known (see Axler (2024)) that there exists an isomorphism $GL(V) \cong GL(n, \mathbb{C})$ between the group of invertible linear maps on V and the group of invertible $n \times n$ matrices with complex entries, where n denotes the dimension of V.

The above definition of a Lorentz transformation is based on the standard representation of the Lorentz group, which corresponds to the 4×4 matrices with matrix multiplication as group action. This representation is also known as the defining representation. Other representations can also be chosen, and we will explore them later. However, the next chapters will work within the standard representation.

A 4×4 matrix is characterized by $4 \cdot 4 = 16$ parameters. Due to the symmetry of Equation (8) in the indices ρ and τ, there are $\sum_{i=1}^{4} i = 10$ constraints. Hence, a Lorentz transformation has $16 - 10 = 6$ degrees of freedom.
In addition to its group structure, the set of Lorentz transformations possesses a rich geometric structure, forming a six-dimensional manifold. Moreover, it can be shown that these structures are compatible, and the Lorentz transformations constitute a Lie group. In the following, these concepts will first be defined, and the corresponding statements will then be proven.

Definition 2.5 (Topological Manifold)**.** *A topological manifold M is a topological space (M, τ), which satisfies the following conditions:*

1. *Locally Euclidean: Each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.*

2. *Hausdorff: For every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.*

3. *Second Countable: There exists a countable basis for the topology of M.*

With the introduction of charts, atlases, and transition maps on the manifold (Lee, 2012, Chapters 1 and 2), we are able define the concept of smoothness for manifolds:

Definition 2.6 (Smooth Manifold)**.** *A smooth manifold is a topological manifold equipped with an atlas $\mathcal{A}$ such that all transition maps are smooth.*

Definition 2.7 (Smooth Map)**.** *Let M and N be smooth manifolds. A map $f : M \to N$ is smooth if for every point $p \in M$, there exists a chart (U, ϕ) with $p \in U$ and a chart (V, ψ) for N with $f(p) \in V$, such that the composition $\psi \circ f \circ \phi^{-1}$ is smooth.*

Definition 2.8 (Lie Group)**.** *A Lie group G is a smooth manifold with a group structure*

$$m : G \times G \to G, (g_1, g_2) \mapsto g_1 * g_2 \tag{22}$$

such that both the multiplication map m and the inversion map $i : G \to G$, $g \mapsto g^{-1}$, are smooth.

We now aim to prove that the set of Lorentz transformations indeed forms a Lie group. The strategy involves using the fact that the general linear group over the real numbers $GL(n, \mathbb{R})$, which is the group of all invertible $n \times n$ matrices with real entries, is a Lie group (Lee, 2012, Chapter 7), and then applying the closed subgroup theorem (Lee, 2012, Chapter 20).

Theorem 2.1 (Closed Subgroup Theorem)**.** *Any closed subgroup H of a Lie group G is a Lie subgroup (and thus a submanifold) of G.*

We have already established that $\mathcal{L}$ forms a matrix group and thus is a subgroup of $GL(4, \mathbb{R})$. It is also a closed subset of $GL(4, \mathbb{R}) \cong \mathbb{R}^{16}$, as can be seen from the following considerations:

- The condition (8) yields a set of polynomial equations in the entries of Λ.
- These equations can be rewritten in the form $P(\Lambda^{\mu}{}_{\nu}) = 0$, where $P(\Lambda^{\mu}{}_{\nu})$ are polynomials and thus continuous functions of $\Lambda^{\mu}{}_{\nu}$.
- Since 0 is a closed subset of $\mathbb{R}^{16}$, it follows from the continuity of $P(\Lambda^{\mu}{}_{\nu})$ that the preimage $\Lambda^{\mu}{}_{\nu}$ is also closed in $\mathbb{R}^{16}$.

By applying the theorem, we conclude that the Lorentz transformations form a Lie group.

2.3 Components of the Lorentz Group

Next, we will examine the connectedness of the set of Lorentz transformations. This concept will become important later when we study the Lie algebra of the Lorentz group, which can be seen as a kind of "derivative" of the Lie group. To transition from the algebra back to the group, integration along a path is required. Thus, the notion of path-connectedness is particularly relevant.
We consider the case where $\rho = \tau = 0$ in Equation (8):

$$(\Lambda^{0}{}_{0})^2 - (\Lambda^{1}{}_{0})^2 - (\Lambda^{2}{}_{0})^2 - (\Lambda^{3}{}_{0})^2 = 1 \tag{23}$$

It follows that $(\Lambda^{0}{}_{0})^2 \geq 1$, which implies

$$\Lambda^{0}{}_{0} \geq 1 \quad \text{or} \quad \Lambda^{0}{}_{0} \leq -1 \tag{24}$$

where we use the notation $\geq$ for "greater than or equal to" and $\leq$ for "less than or equal to". From Equation (12), we already know that

$$\det(\Lambda) = 1 \quad \text{or} \quad \det(\Lambda) = -1 \tag{25}$$

As a result, there are four components of the Lorentz group:

$$\mathcal{L}_{+}^{\uparrow} \equiv \{\Lambda \in \mathcal{L} : \Lambda^{0}{}_{0} \geq +1, \det(\Lambda) = +1\} \tag{26}$$
$$\mathcal{L}_{-}^{\uparrow} \equiv \{\Lambda \in \mathcal{L} : \Lambda^{0}{}_{0} \geq +1, \det(\Lambda) = -1\} \tag{27}$$
$$\mathcal{L}_{+}^{\downarrow} \equiv \{\Lambda \in \mathcal{L} : \Lambda^{0}{}_{0} \leq -1, \det(\Lambda) = +1\} \tag{28}$$
$$\mathcal{L}_{-}^{\downarrow} \equiv \{\Lambda \in \mathcal{L} : \Lambda^{0}{}_{0} \leq -1, \det(\Lambda) = -1\} \tag{29}$$

As we will see later, all four components are path-connected, meaning all components have been identified.

Of particular interest is the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow = SO(1,3)^+$, which is the only one that forms a subgroup, as we will demonstrate.
First, we show that $\mathcal{L}^\uparrow = \mathcal{L}_+^\uparrow \cup \mathcal{L}_-^\uparrow = O(1,3)^+$ is a subgroup. To prove closure under composition, we need to show that ${(\tilde{\Lambda}\Lambda)^0}_0 \geq 1$ for ${\Lambda^0}_0 \geq 1, {\tilde{\Lambda}^0}_0 \geq 1$.

$$ {(\tilde{\Lambda}\Lambda)^0}_0 = {\tilde{\Lambda}^0}_0 {\Lambda^0}_0 + \sum_{i=1}^{3} {\tilde{\Lambda}^0}_i {\Lambda^i}_0 = {\tilde{\Lambda}^0}_0 {\Lambda^0}_0 - \sum_{i=1}^{3} {\tilde{\Lambda}^i}_0 {\Lambda^i}_0 \tag{30} $$

Using the 00-component of Equation (8)

$$ ({\Lambda^0}_0)^2 - ({\Lambda^1}_0)^2 - ({\Lambda^2}_0)^2 - ({\Lambda^3}_0)^2 = 1 \tag{31} $$

we obtain for the 3-norm

$$ \|({\Lambda^1}_0, {\Lambda^2}_0, {\Lambda^3}_0)^T\| = \sqrt{({\Lambda^0}_0)^2 - 1} \tag{32} $$

Using the Cauchy-Schwarz inequality

$$ |\langle({\tilde{\Lambda}^1}_0, {\tilde{\Lambda}^2}_0, {\tilde{\Lambda}^3}_0)^T, ({\Lambda^1}_0, {\Lambda^2}_0, {\Lambda^3}_0)^T\rangle| \leq \|({\tilde{\Lambda}^1}_0, {\tilde{\Lambda}^2}_0, {\tilde{\Lambda}^3}_0)^T\| \|({\Lambda^1}_0, {\Lambda^2}_0, {\Lambda^3}_0)^T\| \tag{33} $$

$$ \sum_{i=1}^{3} {\tilde{\Lambda}^i}_0 {\Lambda^i}_0 \leq \sqrt{({\tilde{\Lambda}^0}_0)^2 - 1}\sqrt{({\Lambda^0}_0)^2 - 1} \tag{34} $$

where $\langle\cdot,\cdot\rangle$ here is the Euclidean scalar product, we get

$$ {(\tilde{\Lambda}\Lambda)^0}_0 \geq {\tilde{\Lambda}^0}_0 {\Lambda^0}_0 - \sqrt{({\tilde{\Lambda}^0}_0)^2 - 1}\sqrt{({\Lambda^0}_0)^2 - 1}, \tag{35} $$

$$ = {\tilde{\Lambda}^0}_0 {\Lambda^0}_0 - \sqrt{({\tilde{\Lambda}^0}_0 {\Lambda^0}_0)^2 - \underbrace{({\tilde{\Lambda}^0}_0)^2}_{\geq 1} - \underbrace{({\Lambda^0}_0)^2}_{\geq 1} + 1}, \tag{36} $$

$$ \geq {\tilde{\Lambda}^0}_0 {\Lambda^0}_0 - \sqrt{({\tilde{\Lambda}^0}_0 {\Lambda^0}_0)^2 - 1} \tag{37} $$

Next, we use the following norm property:

$$ \forall a, b \in \mathbb{R}, a > b : \|a\| - \|b\| \leq \|a - b\| \tag{38} $$

$$ \iff \sqrt{a^2} - \sqrt{b^2} \leq \sqrt{a^2 - b^2} \tag{39} $$

and finally arrive at

$$ {(\tilde{\Lambda}\Lambda)^0}_0 \geq {\tilde{\Lambda}^0}_0 {\Lambda^0}_0 - \left(\sqrt{({\tilde{\Lambda}^0}_0 {\Lambda^0}_0)^2} - \sqrt{1^2}\right) = 1 \tag{40} $$

The identity element is contained in $\mathcal{L}^\uparrow$ because it is in $\mathcal{L}_+^\uparrow$. The same is true for the inverse:

$$ \Lambda^T g \Lambda = g \iff \Lambda^T g = g\Lambda^{-1} \iff \Lambda^{-1} = g\Lambda^T g \tag{41} $$

$$ \implies {(\Lambda^{-1})^0}_0 = {\Lambda^0}_0 \geq 1 \tag{42} $$

This completes the proof.

In a similar manner, it can be shown that $\mathcal{L}_+ = \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\downarrow = SO(1,3)$ also forms a subgroup:

- Closure: For $\det(\Lambda) = 1$ and $\det(\tilde{\Lambda}) = 1$, we have $\det(\tilde{\Lambda}\Lambda) = \det(\tilde{\Lambda})\det(\Lambda) = 1$.
- Neutral element: The identity element is contained in $\mathcal{L}_+^\uparrow$.
- Inverse element: From $\det(\Lambda) = 1$, it follows that $\det(\Lambda^{-1}) = \frac{1}{\det(\Lambda)} = 1$.

The proofs for $\mathcal{L}^\uparrow$ and $\mathcal{L}_+$ are independent of each other and can be performed simultaneously. Thus, we find that $\mathcal{L}_+^\uparrow$ is indeed a subgroup of $\mathcal{L}$.
The other connected components can be obtained through discrete transformations of the proper orthochronous Lorentz group:

$$\mathcal{L}_-^\uparrow = P\mathcal{L}_+^\uparrow, \quad \mathcal{L}_+^\downarrow = T\mathcal{L}_+^\uparrow, \quad \mathcal{L}_-^\downarrow = PT\mathcal{L}_+^\uparrow \tag{43}$$

with

- the parity transformation $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} : (x_0, \vec{x})^T \mapsto (x_0, -\vec{x})^T$
- the time-reversal transformation $T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : (x_0, \vec{x})^T \mapsto (-x_0, \vec{x})^T$
- the space-time-inversion $PT = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} : (x_0, \vec{x})^T \mapsto (-x_0, -\vec{x})^T$

In short, we can express this as

$$\mathcal{L}/\mathcal{L}_+^\uparrow = \{\mathbb{1}_4, P, T, PT\} \tag{44}$$

where $\mathcal{L}/\mathcal{L}_+^\uparrow$ denotes the factor set (Thaller, 1992, Chapter 3.1).

2.4 Rotations and Boosts

The next goal is to find the general form of the proper orthochronous Lorentz group and to decompose it into its components.
We start with the following ansatz:

$$\Lambda = \begin{pmatrix} a_{1\times1} & b_{1\times3}^T \\ c_{3\times1} & D_{3\times3} \end{pmatrix} \tag{45}$$

$$\Lambda^T g \Lambda = \begin{pmatrix} a & c^T \\ b & D \end{pmatrix} \begin{pmatrix} 1 & 0^T \\ 0 & -\mathbb{1}_3 \end{pmatrix} \begin{pmatrix} a & b^T \\ c & D \end{pmatrix} = \begin{pmatrix} a^2 - c^T c & ab^T - (Dc)^T \\ ab - Dc & bb^T - DD^T \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 & 0^T \\ 0 & -\mathbb{1}_3 \end{pmatrix} = g \tag{46}$$

This leads to the conditions

$$a^2 - c^T c = 1, \quad ab = Dc, \quad bb^T - DD^T = -\mathbb{1}_3 \tag{47}$$

First, we consider transformations that leave the x^0 component, i.e., the time component, unchanged. To achieve this, we set $a = 1$. The first condition then gives $c = 0$, and the second condition implies $b = 0$. From the third condition, we obtain $DD^T = \mathbb{1}_3$, meaning that D belongs to the special orthogonal group $SO(3)$, which is the group of all orthogonal 3×3 matrices with determinant 1, and corresponds to a rotation. In four dimensions, this results in

$$\Lambda(\vec{\varphi}) = \begin{pmatrix} 1 & 0^T \\ 0 & R \end{pmatrix} \tag{48}$$

where $R \in SO(3)$. R is characterized by the rotation vector $\vec{\varphi} = \varphi\vec{n}$, which specifies the rotation axis $\vec{n}$ and the rotation angle $\varphi \in [0, \pi)$.
If we choose the rotation axis to be the $\vec{e}_3$ axis, the familiar formula is given by

$$R(\vec{e}_3, \varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{49}$$

and

$$\Lambda(\vec{e}_3, \varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{50}$$

The notation is intended to show that $\Lambda(\vec{\varphi})$ represents a rotation about an arbitrary axis, while $\Lambda(\vec{e}_3)$ denotes a rotation specifically about the $\vec{e}_3$-axis. To describe a rotation about an arbitrary axis, we can decompose the vector $\vec{x}$ to be transformed into a component parallel and a component perpendicular to the rotation axis:

$$\vec{x}_{\parallel} = \frac{\vec{\varphi} \cdot \vec{x}}{\varphi^2}\vec{\varphi}, \quad \vec{x}_{\perp} = \vec{x} - \vec{x}_{\parallel} \tag{51}$$

The parallel and perpendicular components of the rotated vector are given by

$$\vec{x'}_{\parallel} = \vec{x}_{\parallel}, \quad \vec{x'}_{\perp} = \cos\varphi\vec{x}_{\perp} + \sin\varphi\frac{\vec{\varphi}}{\varphi} \times \vec{x}_{\perp} \tag{52}$$

Thus, for the rotated vector we get

$$\begin{aligned} \vec{x'} = \vec{x'}_{\parallel} + \vec{x'}_{\perp} &= \vec{x}_{\parallel} + \cos\varphi\vec{x}_{\perp} + \sin\varphi\frac{\vec{\varphi}}{\varphi} \times \vec{x}_{\perp} = \vec{x}_{\parallel} + \cos\varphi(\vec{x} - \vec{x}_{\parallel}) + \sin\varphi\frac{\vec{\varphi}}{\varphi} \times (\vec{x} - \vec{x}_{\parallel}) \\ &= \cos\varphi\vec{x} + (1 - \cos\varphi)\vec{x}_{\parallel} + \sin\varphi\frac{\vec{\varphi}}{\varphi} \times \vec{x} = \cos\varphi\vec{x} + (1 - \cos\varphi)\frac{\vec{\varphi} \cdot \vec{x}}{\varphi^2}\vec{\varphi} + \sin\varphi\frac{\vec{\varphi}}{\varphi} \times \vec{x} \end{aligned} \tag{53}$$

In component form, this reads as

$$x'^i = \cos\varphi x^i + (1 - \cos\varphi)\frac{\varphi^i\varphi_j}{\varphi^2}x^j + \sin\varphi\epsilon^i{}_{jk}\frac{\varphi^j}{\varphi}x^k = \cos\varphi x^i + (1 - \cos\varphi)\frac{\varphi^i\varphi_j}{\varphi^2}x^j - \sin\varphi\epsilon^i{}_{jk}\frac{\varphi^k}{\varphi}x^j \tag{54}$$

which gives

$$\Lambda^i{}_j = \cos\varphi\delta^i{}_j + (1-\cos\varphi)\frac{\varphi^i\varphi_j}{\varphi^2} - \sin\varphi\epsilon^i{}_{jk}\frac{\varphi^k}{\varphi} \tag{55}$$

In a similar manner to (50), a boost in the direction of $\vec{e}_3$ can be defined as

$$\Lambda(\vec{e}_3,\chi) := \begin{pmatrix} \cosh\chi & 0 & 0 & \sinh\chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\chi & 0 & 0 & \cosh\chi \end{pmatrix} \tag{56}$$

where $\chi \in [0,\infty)$ is the rapidity. It can be shown that $\Lambda(\vec{e}_3,\chi) \in \mathcal{L}_+^\uparrow$. For brevity, we will write $\Lambda(\chi)$.

$$\begin{aligned} \Lambda(\chi)^T g \Lambda(\chi) &= \begin{pmatrix} \cosh^2\chi - \sinh^2\chi & 0 & 0 & \cosh\chi\sinh\chi - \sinh\chi\cosh\chi \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sinh\chi\cosh\chi - \cosh\chi\sinh\chi & 0 & 0 & \sinh^2\chi - \cosh^2\chi \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g \end{aligned} \tag{57}$$

and $\det(\Lambda(\chi)) = \cosh^2\chi - \sinh^2\chi = 1, \quad (\Lambda(\chi))^0{}_0 = \cosh\chi \geq 1.$

$\Lambda(\chi)$ even forms a one-parameter subgroup of $\mathcal{L}_+^\uparrow$:

$$\begin{aligned} \Lambda(\chi_1)\Lambda(\chi_2) &= \begin{pmatrix} \cosh\chi_1\cosh\chi_2 + \sinh\chi_1\sinh\chi_2 & 0 & 0 & \cosh\chi_1\sinh\chi_2 + \sinh\chi_1\cosh\chi_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sinh\chi_1\cosh\chi_2 + \cosh\chi_1\sinh\chi_2 & 0 & 0 & \sinh\chi_1\sinh\chi_2 + \cosh\chi_1\cosh\chi_2 \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\chi_1+\chi_2) & 0 & 0 & \sinh(\chi_1+\chi_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\chi_1+\chi_2) & 0 & 0 & \cosh(\chi_1+\chi_2) \end{pmatrix} = \Lambda(\chi_1+\chi_2) \end{aligned} \tag{58}$$

which demonstrates closure. The identity element is obtained for $\chi = 0$ and the inverse is given by

$$\Lambda^{-1}(\chi) = \begin{pmatrix} \cosh\chi & 0 & 0 & -\sinh\chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\chi & 0 & 0 & \cosh\chi \end{pmatrix} \tag{59}$$

Next, we will demonstrate that this Lorentz transformation can indeed be interpreted as a boost. Consider a free, classical particle that is at rest in the inertial frame I. At two time points t_1 and t_2, it is described by the coordinates $(ct_1,\vec{x})^T$ and $(ct_2,\vec{x})^T$. When transformed into another inertial frame I', the coordinates are given by $(ct'_1, \vec{x'_1})^T$ and $(ct'_2, \vec{x'_2})^T$.

$$\Lambda(\chi)(ct_1, \vec{x})^T = \begin{pmatrix} \cosh(\chi)ct_1 + \sinh(\chi)x^3 \\ x^1 \\ x^2 \\ \cosh(\chi)ct_1 + \sinh(\chi)x^3 \end{pmatrix} = (t'_1, \vec{x'_1})^T \tag{60}$$

$$\Lambda(\chi)(ct_2, \vec{x})^T = \begin{pmatrix} \cosh(\chi)ct_2 + \sinh(\chi)x^3 \\ x^1 \\ x^2 \\ \cosh(\chi)ct_2 + \sinh(\chi)x^3 \end{pmatrix} = (t'_2, \vec{x'_2})^T \tag{61}$$

The relative velocity of I' with respect to I is given by

$$\begin{aligned} \beta = \frac{v}{c} = \frac{\|\vec{x'_2} - \vec{x'_1}\|}{t'_2 - t'_1} &= \frac{\sinh(\chi)ct_2 + \cosh(\chi)x^3 - (\sinh(\chi)ct_1 + \cosh(\chi)x^3)}{\cosh(\chi)ct_2 + \sinh(\chi)x^3 - (\cosh(\chi)ct_1 + \sinh(\chi)x^3)} \\ = \frac{\sinh(\chi)(ct_2 - ct_1)}{\cosh(\chi)(ct_2 - ct_1)} &= \tanh\chi \end{aligned} \tag{62}$$

With $\cosh\chi = \frac{1}{\sqrt{1-\tanh^2\chi}}$, $\cosh\chi = \frac{\tanh\chi}{\sqrt{1-\tanh^2\chi}}$, and $\gamma := \frac{1}{\sqrt{1-\vec{\beta}^2}}$, the boost can be expressed in terms of $\vec{\beta}$:

$$\Lambda(\chi) = \begin{pmatrix} \gamma & 0 & 0 & \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta & 0 & 0 & \gamma \end{pmatrix} \tag{63}$$

Using these components, it can be demonstrated that any proper orthochronous Lorentz transformation can be decomposed into a rotation, a boost in a coordinate direction, and another rotation:

Theorem 2.2. *Every proper orthochronous Lorentz transformation $\Lambda \in \mathcal{L}_+^\uparrow$ can be expressed in the form $\Lambda = \Lambda(\vec{\varphi}_1)\Lambda(\chi)\Lambda(\vec{\varphi}_2)$.*

Proof. Construct the three-vector $\vec{\Lambda_0} = (\Lambda^1{}_0, \Lambda^2{}_0, \Lambda^3{}_0)^T \neq 0$ from the matrix $\Lambda^\mu{}_\nu$ and normalize it:

$$\vec{c} = \lambda\vec{\Lambda_0} = (c_1, c_2, c_3)^T, \quad (c_1)^2 + (c_2)^2 + (c_3)^2 = 1 \tag{64}$$

Choose two normalized three-vectors $\vec{a} = (a_1, a_2, a_3)^T$ and $\vec{b} = (b_1, b_2, b_3)^T$ such that $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a positively oriented basis of the three-space.
Then the matrix

$$\Lambda^{-1}(\vec{\varphi}_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \\ 0 & c_1 & c_2 & c_3 \end{pmatrix} \tag{65}$$

is a rotation $\in SO(3)$. The positive orientation ensures that $\det(\Lambda^{-1}(\vec{\varphi}_1)) = +1$.
Therefore,

$$\Lambda^{-1}(\vec{\varphi}_1)\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ 0 & d_{11} & d_{12} & d_{13} \\ 0 & d_{21} & d_{22} & d_{23} \\ d_{30} & d_{31} & d_{32} & d_{33} \end{pmatrix} \tag{66}$$

is in $\mathcal{L}_+^\uparrow$ because both Λ and $\Lambda^{-1}(\vec{\varphi_1})$ belong to $\mathcal{L}_+^\uparrow$, so their composition does as well. The two zeros in the first column arise from the fact that $\vec{a}$, $\vec{b}$, and $\vec{c}$ form an orthonormal basis.
Now, consider the two three-vectors $\vec{d_1} = (d_{11}, d_{12}, d_{13})^T$ and $\vec{d_2} = (d_{21}, d_{22}, d_{23})^T$, which are orthonormal. This follows from a transformation of (8):

$$\Lambda^T g \Lambda = g \iff \Lambda^T g = g\Lambda^{-1} \iff \Lambda^T = g\Lambda^{-1} g \iff g\Lambda^T = \Lambda^{-1} g \iff \Lambda g \Lambda^T = g \tag{67}$$

$$\Lambda^\mu{}_\rho g^{\rho\tau} \Lambda^\nu{}_\tau = g^{\mu\nu} \tag{68}$$

$$\Lambda^\mu{}_0 \Lambda^\nu{}_0 - \sum_{i=1}^{3} \Lambda^\mu{}_i \Lambda^\nu{}_i = \begin{cases} 0 & \text{for } \mu \neq \nu \\ 1 & \text{for } \mu = \nu = 0 \\ -1 & \text{for } \mu = \nu \neq 0 \end{cases} \tag{69}$$

Add a third three-vector $\vec{e} = (e_1, e_2, e_3)^T$ to again form an orthonormal basis with positive orientation.
The matrix

$$\Lambda^{-1}(\vec{\varphi_2}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d_{11} & d_{21} & e_1 \\ 0 & d_{12} & d_{22} & e_2 \\ 0 & d_{13} & d_{23} & e_3 \end{pmatrix} \tag{70}$$

is again a rotation $\in SO(3)$, and the product

$$\Lambda^{-1}(\vec{\varphi_1})\Lambda\Lambda^{-1}(\vec{\varphi_2}) = \begin{pmatrix} \Lambda^0{}_0 & 0 & 0 & f_{03} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ f_{30} & 0 & 0 & f_{33} \end{pmatrix} \tag{71}$$

is once again a Lorentz transformation. The entries in the second and third columns, as well as the two zeros in the fourth column, result from the fact that the rows of $\Lambda^{-1}(\vec{\varphi_1})$ are orthonormal (see Equation (69)) and that the multiplication by $\Lambda^{-1}(\vec{\varphi_2})$ corresponds to multiplication by the corresponding rows. We now show that the matrix corresponds to a boost in the $\vec{e_3}$ direction. To do this, we consider the conditions in (47):

$$(\Lambda^0{}_0)^2 - (f_{30})^2 = 1, \quad \Lambda^0{}_0 f_{03} = f_{33} f_{30}, \quad (f_{03})^2 - (f_{33})^2 = -1 \tag{72}$$

This system of equations is solved by

$$\Lambda^0{}_0 = \cosh\chi, \quad f_{03} = \sinh\chi, \quad f_{30} = \sinh\chi, \quad f_{33} = \cosh\chi \tag{73}$$

and we obtain $\Lambda^{-1}(\vec{\varphi_1})\Lambda\Lambda^{-1}(\vec{\varphi_2}) = \Lambda(\chi)$, which implies $\Lambda(\vec{\varphi_1})\Lambda(\chi)\Lambda(\vec{\varphi_2}) = \Lambda$.

It remains to consider the case where $(\Lambda^1{}_0, \Lambda^2{}_0, \Lambda^3{}_0)^T = 0$. In this case, the first condition in (47) gives $\Lambda^0{}_0 = 1$, and the second condition yields $(\Lambda^0{}_1, \Lambda^0{}_2, \Lambda^0{}_3)^T = 0$. Thus, Λ is a rotation, and the theorem is trivially satisfied.

□

The decomposition in the theorem is not unique, which can be seen from the construction in the proof.
For the rotation matrix $\Lambda^{-1}(\vec{\varphi_1})$, the first normalized vector was given. The other two vectors can be chosen orthonormally with positive orientation in the plane orthogonal to it. This ultimately corresponds to an arbitrary rotation of the two new vectors around the axis of the given vector.
In Equation (66), one could have swapped the order of $\Lambda^{-1}(\vec{\varphi_1})$ and Λ. The proof would then proceed almost identically, but the freedom of choice would apply to the other rotation.
This raises the question of how to interpret this freedom of choice. For a boost in the coordinate direction, the orientation perpendicular to the boost axis does not matter. Therefore, one can perform rotations before and after the boost which cancel each other out.

Theorem 2.2 can be used to show that $\mathcal{L}_+^\uparrow$ is path-connected, a mathematical concept, which we introduce next:

Definition 2.9 (Path-connectedness)**.** *A subset A of a topological space M is called path-connected if, for any two points $x, y \in A$, there exists a path with initial point x and terminal point y that lies entirely within A. That is, for $a, b \in \mathbb{R}$, there exists a continuous map*

$$\gamma : [a,b] \to A \quad \textit{with } \gamma(a) = x,\ \gamma(b) = y \tag{74}$$

We recall the fact that for a metric space X a map $f : X \to \mathbb{R}^{n\times n}$ is continuous if and only if all components $f^\mu{}_\nu,\ \mu,\nu = 1,\dots,n$ are continuous.

Lemma 2.3. *$\mathcal{L}_+^\uparrow$ is path-connected.*

Proof. Let Λ_1 and Λ_2 be two arbitrary elements of $\mathcal{L}_+^\uparrow$, and let id denote the identity element of $\mathcal{L}_+^\uparrow$. It suffices to show that there exists a path with initial point Λ_1 and terminal point id, and a path with initial point id and terminal point Λ_2, i.e., continuous maps

$$\gamma_1 : [t_1, 0] \to \mathcal{L}_+^\uparrow \quad \text{with } \gamma_1(t_1) = \Lambda_1 \text{ and } \gamma_1(0) = \text{id} \tag{75}$$

$$\gamma_2 : [0, t_2] \to \mathcal{L}_+^\uparrow \quad \text{with } \gamma_2(0) = \text{id} \text{ and } \gamma_2(t_2) = \Lambda_2 \tag{76}$$

$\Lambda(\vec{\varphi_1})$, $\Lambda(\vec{\varphi_2})$, and $\Lambda(\chi)$ are continuous maps and approach the identity as $\vec{\varphi_1}, \vec{\varphi_2}, \chi \to 0$. Since the composition of continuous maps is again continuous, it follows from the above theorem 2.2 that $\Lambda = \Lambda(\vec{\varphi_1})\Lambda(\chi)\Lambda(\vec{\varphi_2})$ can be continuously deformed into the identity. □

Due to Equation (43), the other three components $\mathcal{L}_-^\uparrow$, $\mathcal{L}_+^\downarrow$, and $\mathcal{L}_-^\downarrow$ are also path-connected.

With $\Lambda = \Lambda(\vec{\varphi_1})\Lambda(\chi)\Lambda(\vec{\varphi_2})$ and definition of the new rotation $\Lambda(\vec{\tilde{\varphi}}) := \Lambda(\vec{\varphi_1})\Lambda(\vec{\varphi_2})$, we obtain

$$\Lambda = \Lambda(\vec{\varphi_1})\Lambda(\chi)\Lambda^{-1}(\vec{\varphi_1})\Lambda(\vec{\tilde{\varphi}}) = \Lambda(\vec{\chi_1})\Lambda(\vec{\tilde{\varphi}}) \tag{77}$$

where $\Lambda(\vec{\chi_1}) = \Lambda(\vec{\varphi_1})\Lambda(\chi)\Lambda^{-1}(\vec{\varphi_1})$ is a boost in an arbitrary direction. Similarly, we also obtain

$$\Lambda = \Lambda(\vec{\tilde{\varphi}})\Lambda^{-1}(\vec{\varphi_2})\Lambda(\chi)\Lambda(\vec{\varphi_2}) = \Lambda(\vec{\tilde{\varphi}})\Lambda(\vec{\chi_2}) \tag{78}$$

where $\Lambda(\vec{\chi_2}) = \Lambda^{-1}(\vec{\varphi_2})\Lambda(\chi)\Lambda(\vec{\varphi_2})$.

Here, $\vec{\chi} = \chi\vec{n}$ is the boost vector, where $\chi \in [0, \infty)$ is the rapidity, and $\vec{n}$ is the boost axis.

The ambiguity in the decomposition from the above theorem disappears in Equations (77) and (78). The part that could be chosen arbitrarily was precisely the one perpendicular to the boost axis, and it is not affected by $\Lambda(\chi)$. Therefore, this part of the rotation commutes with $\Lambda(\chi)$ and cancels out with the inverse rotation.

Next, we want to take a closer look at boosts in an arbitrary direction. Instead of explicitly calculating $\Lambda(\vec{\chi}) = \Lambda(\vec{\varphi})\Lambda(\chi)\Lambda^{-1}(\vec{\varphi})$, we can avoid this lengthy computation by taking a similar approach to the one used at the beginning of the chapter for rotations. The vector to be transformed is decomposed into components parallel and perpendicular to the boost axis:

$$\vec{x}_\parallel = \frac{\vec{\chi}\cdot\vec{x}}{\chi^2}\vec{\chi}, \quad \vec{x}_\perp = \vec{x} - \vec{x}_\parallel \tag{79}$$

Using Equation (56), the components of the boosted vector are obtained as

$$x'^0 = \cosh\chi x^0 + \sinh\chi\frac{\vec{\chi}}{\chi}\cdot\vec{x}, \quad \vec{x'}_\parallel = \cosh\chi\vec{x}_\parallel + \sinh\chi\frac{\vec{\chi}}{\chi}x^0, \quad \vec{x'}_\perp = \vec{x}_\perp \tag{80}$$

The boosted vector is then given by

$$\begin{aligned}\vec{x'} &= \vec{x'}_\parallel + \vec{x'}_\perp = \cosh\chi\vec{x}_\parallel + \sinh\chi\frac{\vec{\chi}}{\chi}x^0 + \vec{x}_\perp = \cosh\chi\frac{\vec{\chi}\cdot\vec{x}}{\chi^2}\vec{\chi} + \sinh\chi\frac{\vec{\chi}}{\chi}x^0 + \vec{x} - \frac{\vec{\chi}\cdot\vec{x}}{\chi^2}\vec{\chi}\\ &= \vec{x} + (\cosh\chi - 1)\frac{\vec{\chi}\cdot\vec{x}}{\chi^2}\vec{\chi} + \sinh\chi\frac{\vec{\chi}}{\chi}x^0\end{aligned} \tag{81}$$

In component form, this reads as

$$x'^i = (\delta^i{}_j + (\cosh\chi - 1)\frac{\chi^i\chi_j}{\chi^2})x^j + \sinh\chi\frac{\chi^i}{\chi}x^0, \quad x'^0 = \cosh\chi x^0 + \sinh\chi\frac{\chi_i}{\chi}x^i \tag{82}$$

and we obtain

$$\Lambda^0{}_0 = \cosh\chi, \quad \Lambda^0{}_i = \sinh\chi\frac{\chi_i}{\chi}, \quad \Lambda^i{}_0 = \sinh\chi\frac{\chi^i}{\chi}, \quad \Lambda^i{}_j = \delta^i{}_j + (\cosh\chi - 1)\frac{\chi^i\chi_j}{\chi^2} \tag{83}$$

$$\Lambda(\vec{\chi}) = \begin{pmatrix}\cosh\chi & \sinh\chi\vec{n}^T\\ \sinh\chi\vec{n} & \mathbb{1}_3 + (\cosh\chi - 1)\vec{n}\vec{n}^T\end{pmatrix} \tag{84}$$

Using the relation from Equation (62), this can also be written as

$$\Lambda(\vec{\beta}) = \begin{pmatrix}\gamma & \gamma\beta\vec{n}^T\\ \gamma\beta\vec{n} & \mathbb{1}_3 + (\gamma - 1)\vec{n}\vec{n}^T\end{pmatrix} \tag{85}$$

2.5 Composition of Boosts

We want to examine the composition of two arbitrary boosts. To enhance clarity, we use the following notation in this chapter: B denotes a boost in an arbitrary direction, R represents an arbitrary 4-dimensional rotation matrix, and $\mathcal{R}$ signifies an arbitrary 3-dimensional rotation matrix. In general, boosts do not form a subgroup of the Lorentz group, meaning the composition of two boosts does not necessarily result in another boost. As a counterexample, consider:

$$B_x = \begin{pmatrix} \gamma_1 & \gamma_1\beta_1 & 0 & 0 \\ \gamma_1\beta_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{86}$$

$$B_y = \begin{pmatrix} \gamma_2 & 0 & \gamma_2\beta_2 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_2\beta_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{87}$$

$$B_xB_y = \begin{pmatrix} \gamma_1 & \gamma_1\beta_1 & 0 & 0 \\ \gamma_1\beta_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_2 & 0 & \gamma_2\beta_2 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_2\beta_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma_1\gamma_2 & \gamma_1\beta_1 & \gamma_1\gamma_2\beta_2 & 0 \\ \gamma_1\beta_1\gamma_2 & \gamma_1 & \gamma_1\beta_1\gamma_2\beta_2 & 0 \\ \gamma_2\beta_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{88}$$

Clearly, this result only assumes the form (85) when $\gamma_1 = \gamma_2 = 1$ and $\beta_1 = \beta_2 = 0$.
In general, B_2B_1 is not a boost, but $B_2B_1 \in \mathcal{L}_+^\uparrow$. Therefore, by Theorem 2.2 we can express it as $B_2B_1 = RB_{21}$, where B_{21} is a boost and $R \in SO(3)$.

$$\begin{aligned} B_2B_1 &= \begin{pmatrix} \gamma_2 & \gamma_2\vec{\beta_2}^T \\ \gamma_2\vec{\beta_2} & \mathbb{1}_3 + (\gamma_2 - 1)\vec{n_2}\vec{n_2}^T \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_1\vec{\beta_1}^T \\ \gamma_1\vec{\beta_1} & \mathbb{1}_3 + (\gamma_1 - 1)\vec{n_1}\vec{n_1}^T \end{pmatrix} \\ &= \begin{pmatrix} \gamma_2\gamma_1(1 + \vec{\beta_2}\cdot\vec{\beta_1}) & \gamma_2\gamma_1\vec{\beta_1}^T + \gamma_2\vec{\beta_2}^T(\mathbb{1}_3 + (\gamma_1 - 1)\vec{n_1}\vec{n_1}^T) \\ \gamma_2\gamma_1\vec{\beta_2} + \gamma_1(\mathbb{1}_3 + (\gamma_2 - 1)\vec{n_2}\vec{n_2}^T)\vec{\beta_1} & \gamma_2\gamma_1\vec{\beta_2}\vec{\beta_1}^T + (\mathbb{1}_3 + (\gamma_2 - 1)\vec{n_2}\vec{n_2}^T)(\mathbb{1}_3 + (\gamma_1 - 1)\vec{n_1}\vec{n_1}^T) \end{pmatrix} \end{aligned} \tag{89}$$

$$RB_{21} = \begin{pmatrix} 1 & 0^T \\ 0 & \mathcal{R} \end{pmatrix} \begin{pmatrix} \gamma_{21} & \gamma_{21}\vec{\beta_{21}}^T \\ \gamma_{21}\vec{\beta_{21}} & \mathbb{1}_3 + (\gamma_{21} - 1)\vec{n_{21}}\vec{n_{21}}^T \end{pmatrix} = \begin{pmatrix} \gamma_{21} & \gamma_{21}\vec{\beta_{21}}^T \\ \gamma_{21}\mathcal{R}\vec{\beta_{21}} & \mathcal{R}(\mathbb{1}_3 + (\gamma_{21} - 1)\vec{n_{21}}\vec{n_{21}}^T) \end{pmatrix} \tag{90}$$

By comparing the first row and column, we obtain:

- $\gamma_{21} = \gamma_2\gamma_1(1 + \vec{\beta_2}\cdot\vec{\beta_1})$
- $\gamma_{21}\vec{\beta_{21}}^T = \gamma_2\gamma_1\vec{\beta_1}^T + \gamma_2\vec{\beta_2}^T(\mathbb{1}_3 + (\gamma_1 - 1)\vec{n_1}\vec{n_1}^T)$
- $\gamma_{21}\mathcal{R}\vec{\beta_{21}} = \gamma_2\gamma_1\vec{\beta_2} + \gamma_1(\mathbb{1}_3 + (\gamma_2 - 1)\vec{n_2}\vec{n_2}^T)\vec{\beta_1}$

The second bullet point yields the relativistic addition of velocities:

$$\vec{\beta_{21}} = \frac{\vec{\beta_1} + \gamma_1^{-1}(\mathbb{1}_3 + (\gamma_1 - 1)\vec{n_1}\vec{n_1}^T)\vec{\beta_2}}{1 + \vec{\beta_2} \cdot \vec{\beta_1}} = \frac{\vec{\beta_1} + \gamma_1^{-1}\vec{\beta}_{2\perp 1} - \vec{\beta}_{2\parallel 1}}{1 + \vec{\beta_2} \cdot \vec{\beta_1}} \tag{91}$$

Now consider the effect of switching the order of B_1 and B_2. We know that $B_2B_1 = RB_{21}$, $R^T = R^{-1}$, and from (85) $B^T = B$:

$$B_1B_2 = (B_2^T B_1^T)^T = (B_2B_1)^T = (RB_{21})^T = B_{21}^T R^T = B_{21}R^{-1} \tag{92}$$

To express this in the form "first boost, then rotation", we proceed as follows:

$$B_1B_2 = R^{-1}(RB_{21}R^{-1}) \tag{93}$$

$$(RB_{21}R^{-1})^T = (R^{-1})^T B_{21}^T R^T = RB_{21}R^{-1} \tag{94}$$

Thus, $RB_{21}R^{-1}$ is a pure boost, and we define $B_{12} := RB_{21}R^{-1}$, leading to

$$B_1B_2 = R^{-1}B_{12} \tag{95}$$

When the boosts are swapped, the inertial frame is rotated in the opposite direction after the boost.

The relationship between B_{21} and B_{12} can be determined by performing an analogous calculation for B_1B_2 as for B_2B_1:

- $\gamma_{12} = \gamma_1\gamma_2(1 + \vec{\beta_1} \cdot \vec{\beta_2})$
- $\gamma_{12}\vec{\beta_{12}}^T = \gamma_1\gamma_2\vec{\beta_2}^T + \gamma_1\vec{\beta_1}^T(\mathbb{1}_3 + (\gamma_2 - 1)\vec{n_2}\vec{n_2}^T)$
- $\gamma_{12}\mathcal{R}\vec{\beta_{12}} = \gamma_1\gamma_2\vec{\beta_1} + \gamma_2(\mathbb{1}_3 + (\gamma_1 - 1)\vec{n_1}\vec{n_1}^T)\vec{\beta_2}$

A comparison of the third bullet point from B_2B_1 and the second bullet point from B_1B_2 transposed gives

$$\gamma_{21}\mathcal{R}\vec{\beta_{21}} = \gamma_{12}\vec{\beta_{12}} \overset{\gamma_{21}=\gamma_{12}}{\Longleftrightarrow} \mathcal{R}\vec{\beta_{21}} = \vec{\beta_{12}} \tag{96}$$

Thus, the boost vectors are rotated by the same angle as the inertial frames after the boost.

Inverting B_2B_1 results in

$$(B_2B_1)^{-1} = (RB_{21})^{-1} = B_{21}^{-1}R^{-1} = (R^{-1}B_{12}R)^{-1}R^{-1} = R^{-1}B_{12}^{-1} \tag{97}$$

This implies a boost in the $-\vec{\beta_{12}}$ direction.

2.6 Rotation and Boost Manifold

In the previous sections, we examined the group structure of rotations and boosts. Now, we turn to their topological structure.

Boosts:
Boosts are parameterized by the boost vector $\vec{\chi} = \chi\vec{n},\ \chi \in [0, \infty)$. This shows that the set of all boosts is diffeomorphic to $\mathbb{R}^3$. As a result, the manifold is simply connected and non-compact.

We can define the so-called mass-shell

$$M_{pos} \equiv \{p \in \mathbb{R}^4 \mid \langle p, p\rangle = m^2,\ p^0 > 0\} = \{(\sqrt{\vec{p}^2 + m^2}, \vec{p})^T \in \mathbb{R}^4 \mid \vec{p} \in \mathbb{R}^3\}, \quad m > 0 \tag{98}$$

and show that the boost manifold is diffeomorphic to it. M_{pos} contains the point $q = (m, 0, 0, 0)^T$ and for any $p \in M_{pos}$, there exists exactly one boost Λ_p such that $\Lambda_p q = p$ as the following calculation shows:

$$\Lambda_p q = \begin{pmatrix} \gamma m \\ \gamma\vec{\beta}m \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} \sqrt{\vec{p}^2 + m^2} \\ \vec{p} \end{pmatrix} = p \tag{99}$$

We choose $\vec{\beta} = \frac{\vec{p}}{\sqrt{\vec{p}^2+m^2}}$ and verify

$$\gamma m = \frac{m}{\sqrt{1 - \vec{\beta}^2}} = \frac{m}{\sqrt{1 - \frac{\vec{p}^2}{\vec{p}^2+m^2}}} = \frac{m}{\sqrt{\frac{m^2}{\vec{p}^2+m^2}}} = \sqrt{\vec{p}^2 + m^2} \quad \checkmark \tag{100}$$

$$\gamma\vec{\beta}m = \sqrt{\vec{p}^2 + m^2}\frac{\vec{p}}{\sqrt{\vec{p}^2 + m^2}} = \vec{p} \quad \checkmark \tag{101}$$

The map is also differeantiable, which concludes the proof of the claim.

Rotations:
Rotations preserve the origin, Euclidean distance, and orientation. They satisfy $\langle Rx, Ry\rangle = \langle x, R^T Ry\rangle \stackrel{!}{=} \langle x, y\rangle$ and $\det(R) = 1$, implying $RR^T = \mathbb{1}$ and $\det(R) = 1$. This shows that rotations are isomorphic to the special orthogonal group $SO(3)$.

Rotations are parameterized by the rotation vector $\vec{\varphi} = \varphi\vec{n},\ \varphi \in [0, \pi)$. Thus, the set of all rotations is diffeomorphic to the ball B with radius π in $\mathbb{R}^3$, where we must identify antipodal points on the surface of B because a rotation of π in $\vec{n}$-direction is the same as a rotation of π in $-\vec{n}$-direction. The manifold has no boundary.
A rotation about a fixed axis can be visualized by moving radially outward from the origin. Upon reaching the boundary of the sphere (corresponding to a half-turn of π), the motion continues inward from the opposite side of the sphere, representing a full rotation of 2π.

The manifold is also diffeomorphic to the 3-dimensional real projective space $\mathbb{RP}^3$, which is formed by identifying antipodal points of the 3-sphere. This diffeomorphism corresponds to the parametrization of the rotation group by Euler angles.

As a consequence, the rotation manifold is compact and path-connected, but it is not simply connected, a mathematical concept, which we introduce next:

Definition 2.10 (Free Homotopy of Paths)**.** *Let X be a topological space. Two closed paths $\gamma_0, \gamma_1 : [a, b] \to X$ are called freely homotopic in X if there exists a continuous map $H : [a, b] \times [0, 1] \to X$ such that*

1. $H(t, 0) = \gamma_0(t), \quad H(t, 1) = \gamma_1(t) \quad$ *for all* $t \in [a, b]$
2. $H(a, s) = H(b, s) \quad$ *for all* $s \in [0, 1]$

Definition 2.11 (Null-homotopic Path)**.** *A continuous closed path $\gamma : [0, 1] \to X$ with base point $x = \gamma(0) = \gamma(1)$ in a topological space X is called null-homotopic if it is homotopic to the constant path at x.*

Definition 2.12 (Simply Connected Space)**.** *A topological space X is called simply connected if it is path-connected and if every continuous closed path in X is null-homotopic.*

For example, the diameter through two antipodal points is a closed path, but it cannot be continuously shrunk to a point. This shows that the rotation manifold is not simply connected.

However, it would be desirable to have a simply connected Lie group to work with. We will briefly justify this here. In a quantum theory, due to the axioms taken, projective representations are of importance (see A). However, working with these representations is more difficult, which is why we prefer to switch to (regular) representations. Under certain conditions, all projective representations of a connected Lie group are "covered" by the unitary representations of the universal covering group (see Thaller (1992) Theorems 2.18, 2.21, 2.22, 2.23).

The universal covering group can be understood as the "smallest" Lie group that covers G and is simply connected (Thaller, 1992, Chapter 2.3).

Definition 2.13. *Let X be a topological space. A covering of X is a topological space $\tilde{X}$ and a continuous map*

$$\Phi : \tilde{X} \to X \tag{102}$$

such that for every $x \in X$ there exists an open neighborhood U_x of x, a discrete space D_x and for every $d \in D_x$ an open set V_d such that

$$\Phi^{-1}(U_x) = \bigsqcup_{d \in D_x} V_d \tag{103}$$

and

$$\Phi|_{V_d} : V_d \to U_x \tag{104}$$

is a homeomorphism for every $d \in D_x$.

Definition 2.14. *A covering group of a topological group G is a covering space $\tilde{G}$ of G such that $\tilde{G}$ is a topological group and the covering map $\Phi : \tilde{G} \to G$ is a continuous group homomorphism, i.e.*

$$\Phi(\tilde{g})\Phi(\tilde{h}) = \Phi(\tilde{g}\tilde{h}) \tag{105}$$

One can now consider what the covering group of the rotation group might look like. The manifold is simply connected and locally resembles the rotation manifold (e.g., it has no boundary).

As previously described, the rotation manifold is isomorphic to the ball B in $\mathbb{R}^3$ with radius π and identified antipodal points. When transitioning to a simply connected set without boundary, we obtain two balls, B_1 and B_2, in $\mathbb{R}^3$ with radius π and mutually identified points.

Similarly, one could start from the 3-sphere $\mathcal{S}^3$ with identified antipodal points, or equivalently $\mathbb{RP}^3$, which is also isomorphic to the rotation manifold. The simply connected covering without boundary in this case is the 3-sphere $\mathcal{S}^3$.

These two perspectives are connected as follows: the 3-sphere can be projected onto two balls in $\mathbb{R}^3$ with identified boundary points.

Thus, topologically, the covering appears like the 3-sphere. To obtain the universal covering group, the covering map must be chosen as a continuous group homomorphism.

3 Representations

3.1 Double Cover of the Rotation and the Lorentz Group

We aim to find the universal covering group of $SO(3)$. We propose that $SU(2)$, the group of all 2×2 unitary matrices with complex entries and determinant 1, is isomorphic to the universal covering group of $SO(3)$.

It has the correct topology:

$$\begin{aligned} U \in SU(2) \implies U &= \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1, \quad a, b \in \mathbb{C} \\ &= \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}, \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1, \quad x_1, x_2, x_3, x_4 \in \mathbb{R} \end{aligned} \tag{106}$$

This shows that $SU(2)$ is isomorphic to the 3-sphere, and thus it is a covering of $SO(3)$.

Next, we need to find a continuous group homomorphism.
The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{107}$$

These form a basis of the 3-dimensional real vector space of complex, traceless, Hermitian 2×2 matrices, denoted TH(2). Their properties are given by

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}_2, \quad [\sigma_i, \sigma_j] = 2i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k, \quad \mathrm{tr}(\sigma_i) = 0 \tag{108}$$

An element $\hat{x}$ in this vector space is defined as

$$\hat{x} = \sum_{i=1}^{3} x^i\sigma_i = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}, \quad \vec{x} \in \mathbb{R}^3 \tag{109}$$

This defines an isomorphism between $\mathbb{R}^3$ and TH(2). This is useful because $SO(3)$ matrices act on 3-dimensional objects, while $SU(2)$ matrices act on 2-dimensional objects.
Moreover, we have

$$\det(\hat{x}) = -(x^3)^2 - (x^1)^2 - (x^2)^2 = -\langle \vec{x}, \vec{x} \rangle \tag{110}$$

Next, we investigate how $\hat{x}$ could transform. The most general transformation that preserves hermiticity is

$$(\hat{x}')^\dagger = (A\hat{x}B)^\dagger = B^\dagger\hat{x}^\dagger A^\dagger = B^\dagger\hat{x}A^\dagger \overset{!}{=} A\hat{x}B = \hat{x}' \implies A^\dagger = B \tag{111}$$

The most general transformation that preserves the trace is

$$\mathrm{tr}(\hat{x}') = \mathrm{tr}(A\hat{x}B) \overset{\mathrm{tr}(ABC)=\mathrm{tr}(BCA)=\mathrm{tr}(CAB)}{=} \mathrm{tr}(\hat{x}BA) \overset{!}{=} \mathrm{tr}(\hat{x}) \implies A^{-1} = B \tag{112}$$

Thus, we require $A^\dagger = A^{-1}$, which holds for $SU(2)$ matrices.

This transformation also preserves the scalar product:

$$\begin{aligned}\langle \vec{x'}, \vec{x'} \rangle &= -\det(\hat{x}') = -\det(U\hat{x}U^\dagger) = -\det(U\hat{x}U^{-1}) \\ &= -\det(U)\det(\hat{x})\det(U^{-1}) = -\det(U)\det(\hat{x})\frac{1}{\det(U)} = -\det(\hat{x}) = \langle \vec{x}, \vec{x} \rangle\end{aligned} \tag{113}$$

Therefore, $\hat{x}' = U\hat{x}U^\dagger$ seems like a plausible candidate for the transformation.

Let us verify this. We need some additional properties:

$$\sigma_i\sigma_j = \frac{1}{2}\left(\{\sigma_i, \sigma_j\} + [\sigma_i, \sigma_j]\right) = \frac{1}{2}\left(2\delta_{ij}\mathbb{1}_2 + 2i\epsilon_{ijk}\sigma_k\right) = \delta_{ij}\mathbb{1}_2 + i\epsilon_{ijk}\sigma_k \tag{114}$$

$$\begin{aligned}\hat{x}\hat{y} = (\vec{x}\cdot\vec{\sigma})(\vec{y}\cdot\vec{\sigma}) &= \left(\sum_{i=1}^{3} x^i\sigma_i\right)\left(\sum_{j=1}^{3} y^j\sigma_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} x^i\sigma_i y^j\sigma_j \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} x^i y^j(\delta_{ij} + i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k) = \sum_{i=1}^{3} x^i y_i + \sum_{k=1}^{3}\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\epsilon_{ijk}x^i y^j\right)\sigma_k \\ &= \vec{x}\cdot\vec{y} + i\vec{\sigma}\cdot(\vec{x}\wedge\vec{y})\end{aligned} \tag{115}$$

Lemma 3.1. *For two $n \times n$ matrices U, H over the complex numbers with $U := e^{itH} = \sum_{n\in\mathbb{N}_0}\frac{(itH)^n}{n!}$, the following holds: U is unitary if and only if H is Hermitian.*

Proof. Let U, H be $n\times n$ matrices over the complex numbers. The expression $U := e^{itH} = \sum_{n\in\mathbb{N}_0}\frac{(itH)^n}{n!}$ is well-defiend since it converges for any H (Hall, 2015, Chapter 2.1).

$$\begin{aligned}U \text{ unitary} \iff U^\dagger U = \mathbb{1} \iff & e^{-itH^\dagger}e^{itH} = 1 \quad \Big| \frac{d}{dt}_{|t=0} \\ \implies & -iH^\dagger + iH = 0 \\ \iff & H^\dagger = H \iff H \text{ Hermitian}\end{aligned} \tag{116}$$

$$H \text{ Hermitian} \implies U^\dagger U = e^{-itH^\dagger}e^{itH} = e^{-itH}e^{itH} = \mathbb{1} \iff U \text{ unitary} \tag{117}$$

□

Lemma 3.2. *For any $n \times n$ complex matrix A, it holds that* $\det(e^A) = e^{\mathrm{tr}(A)}$.

Proof. Let A be a complex $n \times n$ matrix.
For any square $n \times n$ matrix A over the complex numbers $\mathbb{C}$, there exists a Jordan decomposition

$$A = PJP^{-1} \tag{118}$$

where P is an invertible matrix, and

$$J = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{pmatrix} \quad \text{with} \quad J_i = \begin{pmatrix} \lambda_j & 1 & & & 0 \\ & \lambda_j & 1 & & 0 \\ & & \ddots & \ddots & \\ & & & \lambda_j & 1 \\ 0 & & & & \lambda_j \end{pmatrix}, \ \lambda_j \in \mathbb{C} \tag{119}$$

$$\begin{aligned}\det(\exp(A)) &= \det(\exp(PJP^{-1})) = \det(P\exp(J)P^{-1}) = \det(P)\det(\exp(J))\det(P^{-1})\\ &= \det(\exp(J)) = \prod_{i=1}^{n}\exp(\lambda_i) = \exp(\sum_{i=1}^{n}\lambda_i) = \exp(\mathrm{tr}(J))\\ &= \exp(\mathrm{tr}(PJP^{-1})) = \exp(\mathrm{tr}(A))\end{aligned} \tag{120}$$

□

The general form of an element $U(\vec{\varphi}) \in SU(2)$ can be expressed as

$$\begin{aligned}U(\vec{\varphi}) &= \exp\left(-\frac{i}{2}\vec{\varphi}\cdot\vec{\sigma}\right) = \cos\left(\frac{\varphi}{2}\right)\mathbb{1}_2 - i\sin\left(\frac{\varphi}{2}\right)\vec{n}\cdot\vec{\sigma}\\ &= \begin{pmatrix}\cos\left(\frac{\varphi}{2}\right) - i\sin\left(\frac{\varphi}{2}\right)n^3 & i\sin\left(\frac{\varphi}{2}\right)(-n^1+in^2)\\ i\sin\left(\frac{\varphi}{2}\right)(-n^1-in^2) & \cos\left(\frac{\varphi}{2}\right)+i\sin\left(\frac{\varphi}{2}\right)n^3\end{pmatrix}\end{aligned} \tag{121}$$

The first expression shows that $U(\vec{\varphi})$ is unitary (117) and that $\det(U(\vec{\varphi})) = 1$ (120), since $\mathrm{tr}(\sigma_i) = 0$.
The factor $\frac{1}{2}$ is deliberately chosen so that it cancels out in the subsequent calculation using addition theorems.

$$U^\dagger(\vec{\varphi}) = \exp\left(\frac{i}{2}\vec{\varphi}\cdot\vec{\sigma}\right) = \cos\left(\frac{\varphi}{2}\right)\mathbb{1}_2 + i\sin\left(\frac{\varphi}{2}\right)\vec{n}\cdot\vec{\sigma} \tag{122}$$

$$\begin{aligned}U(\vec{\varphi})\ \hat{x}\ U^\dagger(\vec{\varphi}) &= U(\vec{\varphi})\ \vec{x}\cdot\vec{\sigma}\ U^\dagger(\vec{\varphi})\\ &= \cos^2\left(\frac{\varphi}{2}\right)\vec{x}\cdot\vec{\sigma} + \cos\left(\frac{\varphi}{2}\right)\vec{x}\cdot\vec{\sigma}\ i\sin\left(\frac{\varphi}{2}\right)\vec{n}\cdot\vec{\sigma} - i\sin\left(\frac{\varphi}{2}\right)\vec{n}\cdot\vec{\sigma}\ \vec{x}\cdot\vec{\sigma}\cos\left(\frac{\varphi}{2}\right)\\ &\quad + \sin^2\left(\frac{\varphi}{2}\right)\vec{n}\cdot\vec{\sigma}\ \vec{x}\cdot\vec{\sigma}\ \vec{n}\cdot\vec{\sigma}\\ &\overset{(115)}{=} \cos^2\left(\frac{\varphi}{2}\right)\vec{x}\cdot\vec{\sigma} - 2\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right)\vec{\sigma}\cdot(\vec{x}\wedge\vec{n})\\ &\quad + \sin^2\left(\frac{\varphi}{2}\right)(\vec{n}\cdot\vec{\sigma}(\vec{x}\cdot\vec{n} + i\vec{\sigma}\cdot(\vec{x}\wedge\vec{n}))\\ &\overset{(115)}{=} \cos^2\left(\frac{\varphi}{2}\right)\vec{x}\cdot\vec{\sigma} - 2\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right)\vec{\sigma}\cdot(\vec{x}\wedge\vec{n})\\ &\quad + \sin^2\left(\frac{\varphi}{2}\right)(\vec{n}\cdot\vec{x}\ \vec{n}\cdot\vec{\sigma} + \underbrace{i\vec{n}\cdot(\vec{x}\wedge\vec{n})}_{=0} - \vec{\sigma}\cdot\underbrace{(\vec{n}\wedge(\vec{x}\wedge\vec{n}))}_{=(\vec{n}\cdot\vec{n})\vec{x}-(\vec{n}\cdot\vec{x})\vec{n}})\\ &= \left(\cos^2\left(\frac{\varphi}{2}\right)\vec{x} - 2\underbrace{\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right)}_{=\frac{1}{2}\sin\varphi}\vec{x}\wedge\vec{n} + \sin^2\left(\frac{\varphi}{2}\right)((\vec{n}\cdot\vec{x})\vec{n} - \vec{x} + (\vec{n}\cdot\vec{x})\vec{n})\right)\cdot\vec{\sigma}\\ &= \left(\frac{1}{2}(1+\cos(\varphi))\vec{x} - \sin(\varphi)\vec{x}\wedge\vec{n} + \frac{1}{2}(1-\cos(\varphi))(2(\vec{n}\cdot\vec{x})\vec{n} - \vec{x})\right)\cdot\vec{\sigma}\\ &= (\cos(\varphi)\vec{x} + \sin(\varphi)\vec{n}\wedge\vec{x} + (1-\cos(\varphi))(\vec{n}\cdot\vec{x})\vec{n})\cdot\vec{\sigma}\\ &\overset{(55)}{=} (R(\vec{\varphi})\vec{x})\cdot\vec{\sigma} = R(\hat{\vec{\varphi}})\vec{x}\end{aligned} \tag{123}$$

In the sixth step, we used the identities $\cos^2(\frac{\varphi}{2}) = \frac{1}{2}(1 + \cos(\varphi))$ and $\sin^2(\frac{\varphi}{2}) = \frac{1}{2}(1 - \cos(\varphi))$.

This equation establishes a relationship between $U \in SU(2)$ and $R \in SO(3)$:

$$(R(U)\vec{x}) \cdot \vec{\sigma} = U\ \vec{x} \cdot \vec{\sigma}\ U^\dagger \tag{124}$$

$$(R(U_1U_2)\vec{x}) \cdot \vec{\sigma} = U_1U_2\ \vec{x} \cdot \vec{\sigma}\ U_2^\dagger U_1^\dagger = U_1\ (R(U_2)\vec{x}) \cdot \vec{\sigma}\ U_1^\dagger = (R(U_1)R(U_2)\vec{x}) \cdot \vec{\sigma} \tag{125}$$

$$\implies R(U_1U_2) = R(U_1)R(U_2) \tag{126}$$

Thus, we have indeed found a group homomorphism. This homomorphism is also continuous, since all operations in $R(\vec{\varphi})$ and $U(\vec{\varphi})$ are continuous.
Since $SU(2)$ is isomorphic to the 3-sphere, it is also simply connected. This confirms that $SU(2)$ is the universal covering group of $SO(3)$.

It is evident that $U(\vec{\varphi})$ and $-U(\vec{\varphi}) \in SU(2)$ correspond to the same rotation $R(\vec{\varphi}) \in SO(3)$, as the two negative signs cancel each other out in Equation (123). Therefore, $SU(2)$ is a double cover of $SO(3)$.

We now transition from the case of rotations that preserve the scalar product and orientation in 3-dimensional Euclidean space to proper orthochronous Lorentz transformations, which preserve the scalar product, orientation, and time direction in 4-dimensional Minkowski space.
Therefore, we add the identity matrix to the Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{127}$$

These matrices form a basis of the 4-dimensional real vector space of complex, Hermitian 2×2 matrices H(2), which do not have to be traceless.
An element $\hat{x}$ in this space is given by

$$\hat{x} = \sum_{\mu=0}^{3} x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \quad x \in \mathbb{R}^4 \tag{128}$$

This defines an isomorphism between $\mathbb{R}^4$ and H(2).
It holds that

$$\det(\hat{x}) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = \langle x, x \rangle \tag{129}$$

The most general transformation that preserves Hermiticity is given by

$$\hat{x}' = A\hat{x}A^\dagger \tag{130}$$

To preserve the scalar product, we require $\det(A) = 1$:

$$\langle x', x' \rangle = \det(\hat{x}') = \det(A\hat{x}A^\dagger) = \det(A)\det(\hat{x})\det(A^\dagger) = \det(\hat{x}) = \langle x, x \rangle \tag{131}$$

Thus, $SL(2, \mathbb{C})$, the group of 2×2 complex matrices with determinant 1, with $\hat{x}' = A\hat{x}A^\dagger$ could be a good candidate for the universal covering group of the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$.

To verify this, we require the following theorem (Hall, 2015, Chapter 2.5):

Theorem 3.3 (Polar Decomposition)**.** *Any invertible matrix $A \in \mathbb{C}^{n\times n}$ can be uniquely written as*

$$A = Ue^H \tag{132}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary and $H \in \mathbb{C}^{n\times n}$ is Hermitian.

We decompose $A \in SL(2,\mathbb{C})$ as $A = Ue^H$ with $U \in SU(2)$ and $H \in \mathrm{SH}(2) = \{X \in \mathbb{C}^{2\times 2} \mid X = X^\dagger, \det(X) = 1\}$.

$$\begin{aligned}
e^{H(\vec{\chi})}\ \hat{x}\ (e^{H(\vec{\chi})})^\dagger &= e^{H(\vec{\chi})}\ x\cdot\sigma\ (e^{H^\dagger(\vec{\chi})}) = e^{\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}}\ (x^0\sigma_0 + \vec{x}\cdot\vec{\sigma})\ e^{\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}} \\
&= \left(\cosh\left(\frac{\chi}{2}\right)\sigma_0 + \sinh\left(\frac{\chi}{2}\right)\vec{n}\cdot\vec{\sigma}\right)\ (x^0\sigma_0 + \vec{x}\cdot\vec{\sigma})\ \left(\cosh\left(\frac{\chi}{2}\right)\sigma_0 + \sinh\left(\frac{\chi}{2}\right)\vec{n}\cdot\vec{\sigma}\right) \\
&= x^0\left(\cosh^2\left(\frac{\chi}{2}\right)\sigma_0 + 2\cosh\left(\frac{\chi}{2}\right)\sinh\left(\frac{\chi}{2}\right)\vec{n}\cdot\vec{\sigma} + \sinh^2\left(\frac{\chi}{2}\right)\vec{n}\cdot\vec{\sigma}\ \vec{n}\cdot\vec{\sigma}\right) \\
&\quad + \cosh^2\left(\frac{\chi}{2}\right)\vec{x}\cdot\vec{\sigma} + \cosh\left(\frac{\chi}{2}\right)\vec{x}\cdot\vec{\sigma}\ \sinh\left(\frac{\chi}{2}\right)\vec{n}\cdot\vec{\sigma} + \sinh\left(\frac{\chi}{2}\right)\vec{n}\cdot\vec{\sigma}\ \vec{x}\cdot\vec{\sigma}\cosh\left(\frac{\varphi}{2}\right) \\
&\quad + \sinh^2\left(\frac{\chi}{2}\right)\vec{n}\cdot\vec{\sigma}\ \vec{x}\cdot\vec{\sigma}\ \vec{n}\cdot\vec{\sigma} \\
&\overset{(115)}{=} x^0\left(\cosh^2\left(\frac{\chi}{2}\right)\sigma_0 + 2\cosh\left(\frac{\chi}{2}\right)\sinh\left(\frac{\chi}{2}\right)\vec{n}\cdot\vec{\sigma} + \sinh^2\left(\frac{\chi}{2}\right)\sigma_0\right) \\
&\quad + \cosh^2\left(\frac{\chi}{2}\right)\vec{x}\cdot\vec{\sigma} + 2\cosh\left(\frac{\chi}{2}\right)\sinh\left(\frac{\chi}{2}\right)\vec{n}\cdot\vec{x}\ \sigma_0 \\
&\quad + \sinh^2\left(\frac{\chi}{2}\right)(\vec{n}\cdot\vec{\sigma}(\vec{x}\cdot\vec{n} + i\vec{\sigma}\cdot(\vec{x}\wedge\vec{n})) \\
&\overset{(115)}{=} x^0\left(\cosh^2\left(\frac{\chi}{2}\right)\sigma_0 + 2\cosh\left(\frac{\chi}{2}\right)\sinh\left(\frac{\chi}{2}\right)\vec{n}\cdot\vec{\sigma} + \sinh^2\left(\frac{\chi}{2}\right)\sigma_0\right) \\
&\quad + \cosh^2\left(\frac{\varphi}{2}\right)\vec{x}\cdot\vec{\sigma} + 2\cosh\left(\frac{\varphi}{2}\right)\sinh\left(\frac{\varphi}{2}\right)\vec{n}\cdot\vec{x}\ \sigma_0 \\
&\quad + \sinh^2\left(\frac{\chi}{2}\right)(\vec{n}\cdot\vec{x}\ \vec{n}\cdot\vec{\sigma} + \underbrace{i\vec{n}\cdot(\vec{x}\wedge\vec{n})}_{=0} - \vec{\sigma}\cdot\underbrace{(\vec{n}\wedge(\vec{x}\wedge\vec{n}))}_{=(\vec{n}\cdot\vec{n})\vec{x}-(\vec{n}\cdot\vec{x})\vec{n}}) \\
&= \left(\cosh^2\left(\frac{\chi}{2}\right)x^0 + \sinh^2\left(\frac{\chi}{2}\right)x^0 + 2\cosh\left(\frac{\chi}{2}\right)\sinh\left(\frac{\chi}{2}\right)\vec{n}\cdot\vec{x}\right)\sigma_0 \\
&\quad + (2\cosh\left(\frac{\chi}{2}\right)\sinh\left(\frac{\chi}{2}\right)x^0\vec{n} + \cosh^2\left(\frac{\chi}{2}\right)\vec{x} \\
&\quad + \sinh^2\left(\frac{\chi}{2}\right)((\vec{n}\cdot\vec{x})\vec{n} - \vec{x} + (\vec{n}\cdot\vec{x})\vec{n}))\cdot\vec{\sigma} \\
&= \left(\frac{1}{2}(\cosh(\chi)+1)x^0 + \frac{1}{2}(\cosh(\chi)-1)x^0 + \sinh(\chi)\vec{n}\cdot\vec{x}\right)\sigma_0 \\
&\quad + \left(\sinh(\chi)x^0\vec{n} + \frac{1}{2}(\cosh(\chi)+1)\vec{x} + \frac{1}{2}(\cosh(\chi)-1)((\vec{n}\cdot\vec{x})\vec{n} - \vec{x} + (\vec{n}\cdot\vec{x})\vec{n})\right)\cdot\vec{\sigma} \\
&= \left(\cosh(\chi)x^0 + \sinh(\chi)\vec{n}\cdot\vec{x}\right)\sigma_0 \\
&\quad + \left(\sinh(\chi)x^0\vec{n} + \vec{x} + (\cosh(\chi)-1)(\vec{n}\cdot\vec{x})\vec{n}\right)\cdot\vec{\sigma} \\
&\overset{(83)}{=} (B(\vec{\chi})x)\cdot\sigma = \hat{B}(\vec{\chi})x
\end{aligned} \tag{133}$$

For $U \in SU(2)$, we have essentially performed the calculation in the previous section on rotations:

$$\begin{aligned} U(\vec{\varphi})\ \hat{x}\ U^\dagger(\vec{\varphi}) &= U(\vec{\varphi})\ (x^0\sigma_0 + \vec{x}\cdot\vec{\sigma})\ U^\dagger(\vec{\varphi}) \\ &= x^0 U(\vec{\varphi})U^\dagger(\vec{\varphi}) + U(\vec{\varphi})\ \vec{x}\cdot\vec{\sigma}\ U^\dagger(\vec{\varphi}) \\ &\overset{(123)}{=} x^0 + (\mathcal{R}(\vec{\varphi})\vec{x})\cdot\vec{\sigma} \\ &= (R(\vec{\varphi})x)\cdot\sigma = \hat{R}(\vec{\varphi})x \end{aligned} \tag{134}$$

Using Theorem 2.2 and Theorem 3.3, we have thus found the following relationship between $A \in SL(2,\mathbb{C})$ and $\Lambda \in \mathcal{L}_+^\uparrow$:

$$(\Lambda(A)x)\cdot\sigma = A\ x\cdot\sigma\ A^\dagger \tag{135}$$

We can show that the mapping defined above is a group homomorphism:

$$(\Lambda(A_1A_2)x)\cdot\sigma = A_1A_2\ x\cdot\sigma\ A_2^\dagger A_1^\dagger = A_1\ (\Lambda(A_2)x)\cdot\sigma\ A_1^\dagger = (\Lambda(A_1)\Lambda(A_2)x)\cdot\sigma \tag{136}$$

$$\implies \Lambda(A_1A_2) = \Lambda(A_1)\Lambda(A_2) \tag{137}$$

Next, we want to explicitly specify the group homomorphism.
Define

$$\begin{aligned} \sigma^\mu &= (\sigma_0, -\sigma_i), \quad \sigma_\mu = (\sigma_0, \sigma_i) \\ \tilde{\sigma}^\mu &= (\sigma_0, \sigma_i), \quad \tilde{\sigma}_\mu = (\sigma_0, -\sigma_i) \end{aligned} \tag{138}$$

From Equation (114), we have

$$\sigma_i\sigma_j = \delta_{ij}\sigma_0 + i\epsilon_{ijk}\sigma_k \implies \operatorname{tr}(\sigma_i\sigma_j) = \delta_{ij}\operatorname{tr}(\sigma_0) + i\epsilon_{ijk}\operatorname{tr}(\sigma_k) = 2\delta_{ij} \tag{139}$$

$$\sigma_0\sigma_\mu = \sigma_\mu\sigma_0 = \sigma_\mu \implies \operatorname{tr}(\sigma_0\sigma_\mu) = \operatorname{tr}(\sigma_\mu\sigma_0) = \operatorname{tr}(\sigma_\mu) = 2\delta_{0\mu} \tag{140}$$

$$\implies \operatorname{tr}(\sigma_\mu\sigma_\nu) = 2\delta_{\mu\nu} \tag{141}$$

The inverse mapping of the isomorphism (128) is given by

$$\frac{1}{2}\operatorname{tr}(\hat{x}\tilde{\sigma}^\mu) = \frac{1}{2}\operatorname{tr}(x^\nu\sigma_\nu\tilde{\sigma}^\mu) = \frac{1}{2}x^\nu\operatorname{tr}(\sigma_\nu\tilde{\sigma}^\mu) = \frac{1}{2}x^\nu 2\delta_\nu{}^\mu = x^\mu \tag{142}$$

For the transformation properties of x and $\hat{x}$, we have

$$(x')^\mu = \frac{1}{2}\operatorname{tr}(\hat{x}'\tilde{\sigma}^\mu) = \frac{1}{2}\operatorname{tr}(A\hat{x}A^\dagger\tilde{\sigma}^\mu) = \frac{1}{2}\operatorname{tr}(Ax^\nu\sigma_\nu A^\dagger\tilde{\sigma}^\mu) = \frac{1}{2}\operatorname{tr}(A\sigma_\nu A^\dagger\tilde{\sigma}^\mu)x^\nu \overset{!}{=} \Lambda^\mu{}_\nu x^\nu \tag{143}$$

$$\implies \Lambda^\mu{}_\nu = \frac{1}{2}\operatorname{tr}(A\sigma_\nu A^\dagger\tilde{\sigma}^\mu) \overset{\operatorname{tr}(XY)=\operatorname{tr}(YX)}{=} \frac{1}{2}\operatorname{tr}(A^\dagger\tilde{\sigma}^\mu A\sigma_\nu) \tag{144}$$

Here again, we see that A and $-A \in SL(2,\mathbb{C})$ correspond to the same Lorentz transformation $\Lambda \in \mathcal{L}_+^\uparrow$.

Theorem 3.4. *The kernel $Z = \Phi^{-1}(\{\mathbb{1}_4\})$ of $\Phi : A \to \Lambda(A)$ is given by $\{\mathbb{1}_2, -\mathbb{1}_2\}$, and $\mathcal{L}_+^\uparrow = SL(2,\mathbb{C})/\{\mathbb{1}_2, -\mathbb{1}_2\}$.*

Proof.

$$Z = \Phi^{-1}(\{\mathbb{1}_4\}) = \{A \in SL(2,\mathbb{C}) \mid A\sigma_\mu A^\dagger = \sigma_\mu\} \tag{145}$$

Taking $\sigma_\mu = \sigma_0$, we find that $A \in Z$ is unitary. It follows that $A\sigma_\mu = \sigma_\mu A$ for $A \in Z$, meaning A must be a multiple of $\mathbb{1}_2$. Since A must also be unitary, we obtain $A = +\mathbb{1}_2$ or $A = -\mathbb{1}_2$.
Therefore, $A = \pm B$ if and only if $\Lambda(A) = \Lambda(B)$. □

Thus, we have proven the following theorem:

Theorem 3.5. *$SL(2,\mathbb{C})$ is the double cover of $\mathcal{L}_+^\uparrow$, and the covering map*

$$\Phi : SL(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow,\ A \mapsto \Lambda(A) = \frac{1}{2}\operatorname{tr}(A^\dagger \tilde{\sigma}^\mu A \sigma_\nu) \tag{146}$$

is a group homomorphism.

This homomorphism is also continuous.

Because of Theorem 2.2 and Theorem 3.3, we can decompose $A \in SL(2,\mathbb{C})$ into $U \in SU(2)$ and e^H with $H \in \mathrm{SH}(2)$, which correspond to a rotation and a boost. Therefore, $SL(2,\mathbb{C})$ is diffeomorphic to $\mathcal{S}^3 \times \mathbb{R}^3$, both of which are simply connected. Hence, $SL(2,\mathbb{C})$ is simply connected.

In conclusion, this shows that $SL(2,\mathbb{C})$ is isomorphic to the universal covering group of $\mathcal{L}_+^\uparrow$.

3.2 Spinor Representation of the Lorentz Group

Equation (135) shows that the tuple $(\mathrm{H}(2), \mathcal{A})$, with

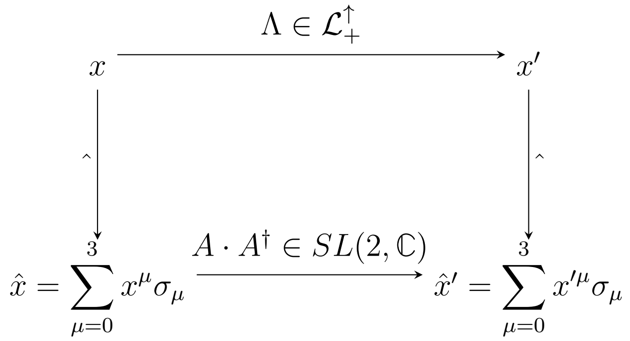
$$\begin{aligned} \mathcal{A} : \mathcal{L}_+^\uparrow &\longrightarrow SL(\mathrm{H}(2)) \\ \Lambda &\longmapsto (\hat{x} \mapsto A_\Lambda \hat{x} A_\Lambda^\dagger), \quad A_\Lambda \in SL(2,\mathbb{C}) \end{aligned} \tag{147}$$

is a representation of $\mathcal{L}_+^\uparrow$:

$$A_\Lambda\ x \cdot \sigma\ A_\Lambda^\dagger = (\Lambda x) \cdot \sigma \tag{148}$$

$$\begin{aligned} A_{\Lambda_1\Lambda_2}\ x \cdot \sigma\ A_{\Lambda_1\Lambda_2}^\dagger &= (\Lambda_1\Lambda_2 x) \cdot \sigma = A_{\Lambda_1}\ (\Lambda_2 x) \cdot \sigma\ A_{\Lambda_1}^\dagger \\ &= A_{\Lambda_1} A_{\Lambda_2}\ x \cdot \sigma\ A_{\Lambda_2}^\dagger A_{\Lambda_1}^\dagger \end{aligned} \tag{149}$$

$\mathcal{A}$ is also well-defined because $\hat{x} \mapsto A_\Lambda \hat{x} A_\Lambda^\dagger$ is linear and invertible.



The diagram illustrates the representation $(\mathrm{H}(2), \mathcal{A})$.
$\hat{\ }: \mathbb{R}^4 \to \mathrm{H}(2)$ denotes the isomorphism defined in (128) and the symbol $\cdot$ in $A \cdot A^\dagger$ serves as a placeholder.

As a next step one might try to define a representation $(\mathbb{C}^2, \tilde{D}^{(\frac{1}{2},0)})$, with

$$\begin{aligned} \tilde{D}^{(\frac{1}{2},0)} : \mathcal{L}_+^\uparrow &\longrightarrow SL(2,\mathbb{C}) \\ \Lambda &\longmapsto (u \mapsto A_\Lambda u) \end{aligned} \tag{150}$$

However, this representation would not be well-defined, as it is $1:2$. An element $\Lambda \in \mathcal{L}_+^\uparrow$ could be mapped to both $A_\Lambda \in SL(2,\mathbb{C})$ and $-A_\Lambda \in SL(2,\mathbb{C})$.
This issue does not arise with $(\mathrm{H}(2), \mathcal{A})$, because the sign of A_Λ cancels out in this case.

We can instead use $SL(2,\mathbb{C})$ rather than $\mathcal{L}_+^\uparrow$ and define a representation $(\mathbb{C}^2, D^{(\frac{1}{2},0)})$, with

$$\begin{aligned} D^{(\frac{1}{2},0)} : SL(2,\mathbb{C}) &\longrightarrow SL(2,\mathbb{C}) \\ \Lambda &\longmapsto (u \mapsto A_\Lambda u) \end{aligned} \tag{151}$$

It follows from Equation (149) that $D^{(\frac{1}{2},0)}$ is a homomorphism: $A_{\Lambda_1\Lambda_2} = A_{\Lambda_1}A_{\Lambda_2}$.
Therefore, we are working with representations of the universal covering group. This is done here to obtain a well-defined mapping. But it fits perfectly with our discussion at the end of Chapter 2, where we mentioned that it makes sense to use representations of the universal covering group.
The vectors $u \in \mathbb{C}^2$, which the representation map acts on, are called spinors.

A_Λ acts in a natural way in both cases $(\mathrm{H}(2), \mathcal{A})$ and $(\mathbb{C}^2, D^{(\frac{1}{2},0)})$:
On $\mathrm{H}(2)$, A_Λ acts, as is usual for matrices, from both the left and the right:

$$\mathcal{A}(\Lambda) : \mathrm{H}(2) \to \mathrm{H}(2), \hat{x} \mapsto \hat{x}' = A_\Lambda \hat{x} A_\Lambda^\dagger \tag{152}$$

On $\mathbb{C}^2$, A_Λ acts on the vector from the left:

$$D^{(\frac{1}{2},0)}(\Lambda) : \mathbb{C}^2 \to \mathbb{C}^2, u \mapsto u' = A_\Lambda u \tag{153}$$

Inspired by Equation (129), we define the symplectic bilinear form

$$\langle \cdot, \cdot \rangle : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}, \quad u \times v \mapsto \langle u, v \rangle = u_1 v_2 - u_2 v_1 = \det(u, v) \tag{154}$$

which is invariant under $SL(2,\mathbb{C})$:

$$\begin{aligned} \langle u', v' \rangle &= \det((Au, Av)) = \det \begin{pmatrix} A^1{}_1 u^1 + A^1{}_2 u^2 & A^1{}_1 v^1 + A^1{}_2 v^2 \\ A^2{}_1 u^1 + A^2{}_2 u^2 & A^2{}_1 v^1 + A^2{}_2 v^2 \end{pmatrix} = \det(A(u,v)) \\ &= \underbrace{\det(A)}_{=1} \det((u,v)) = \det((u,v)) = \langle u, v \rangle \end{aligned} \tag{155}$$

We now introduce a suitable index notation:

$$\mathbb{C}^2 \ni u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (u_\alpha) \tag{156}$$

$$D^{(\frac{1}{2},0)}(\Lambda) : \mathbb{C}^2 \to \mathbb{C}^2, u_\beta \mapsto u'_\alpha = A_\alpha{}^\beta u_\beta \tag{157}$$

With the definition $\epsilon = (\epsilon^{\alpha\beta}) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we can write

$$\langle u, v\rangle = u_\alpha \ \epsilon^{\alpha\beta} \ v_\beta \tag{158}$$

We can define $u^\alpha := \epsilon^{\alpha\beta} u_\beta$, and consequently $u_\alpha = \epsilon_{\alpha\beta} u^\beta$ with $(\epsilon_{\alpha\beta}) = \epsilon^{-1} = \epsilon^T$. Thus, we obtain

$$\langle u, v\rangle = u_\alpha v^\alpha \tag{159}$$

This shows that a summation over an upper and lower index is invariant under $SL(2, \mathbb{C})$. ϵ can be understood as the metric tensor for spinors.

For the representation of $SL(2, \mathbb{C})$ we can introduce a second representation space $\overline{\mathbb{C}}^2$. The complex conjugated vector space $(\overline{\mathbb{C}}, \bar{+}, \bar{\cdot})$ has the same elements and additive group structure as $(\mathbb{C}, +, \cdot)$ but the scalar multiplication involves complex conjugation of the scalars

$$\bar{\cdot} : \mathbb{C} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}, (\lambda, u) \mapsto \lambda \bar{\cdot} u = \overline{\lambda} \cdot u \tag{160}$$

This gives a representation $(\overline{\mathbb{C}}^2, D^{(0,\frac{1}{2})})$ with

$$\begin{aligned} D^{(0,\frac{1}{2})} : SL(2, \mathbb{C}) &\longrightarrow SL(2, \overline{\mathbb{C}}) \\ \Lambda &\longmapsto (\overline{u} \mapsto \overline{A}_\Lambda \overline{u}) \end{aligned} \tag{161}$$

where $\overline{A}_\Lambda$ is the complex conjugate matrix.
That this is a representation follows from Equation (149):

$$\overline{A}_{\Lambda_1\Lambda_2} = \overline{A_{\Lambda_1} A_{\Lambda_2}} = \overline{A}_{\Lambda_1} \overline{A}_{\Lambda_2} \tag{162}$$

The two representations $(\mathbb{C}^2, D^{(\frac{1}{2},0)})$ and $(\overline{\mathbb{C}}^2, D^{(0,\frac{1}{2})})$ are not equivalent in the following sense:

Definition 3.1 (Equivalent Representations)**.** *Two representations* (V_1, π_1) *and* (V_2, π_2) *with* $\pi_1 : G \to GL(V_1)$ *and* $\pi_2 : G \to GL(V_2)$ *are called equivalent if there is an isomorphism* $T : V_1 \to V_2$ *such that* $\forall g \in G$*:*

$$T\pi_1(g)T^{-1} = \pi_2(g) \tag{163}$$

This definition is meaningful, as demonstrated by the following heuristic:

$$v' = \pi_2(g)v = T\pi_1(g)T^{-1}Tu = T\pi_1(g)u = Tu' = v' \tag{164}$$

Now, we show that the representations are not equivalent:
For equivalent representations $\pi_1(g)$ and $\pi_2(g) = T\pi_1(g)T^{-1}$, it holds that $\forall g \in G$:

$$\mathrm{tr}(\pi_2(g)) = \mathrm{tr}(T\pi_1(g)T^{-1}) = \mathrm{tr}(\pi_1(g)T^{-1}T) = \mathrm{tr}(\pi_1(g)) \tag{165}$$

However, for $\pi_1(\Lambda) = D^{(\frac{1}{2},0)}(\Lambda) = A_\Lambda$ and $\pi_2(\Lambda) = D^{(0,\frac{1}{2})}(\Lambda) = \overline{A}_\Lambda$, this condition is not satisfied for

$$A_\Lambda = \begin{pmatrix} 2i & 0 \\ 0 & -\frac{i}{2} \end{pmatrix} \in SL(2, \mathbb{C}) \tag{166}$$

because $\mathrm{tr}(A) = \frac{3}{2}i$ and $\mathrm{tr}(\overline{A}) = \begin{pmatrix} -2i & 0 \\ 0 & \frac{i}{2} \end{pmatrix} = -\frac{3}{2}i$.

It follows that $(\mathbb{C}^2, D^{(\frac{1}{2},0)})$ and $(\overline{\mathbb{C}}^2, D^{(0,\frac{1}{2})})$ are two non-equivalent representations of $\mathcal{L}_+^\uparrow$.

We can explore more deeply what the representation $(\overline{\mathbb{C}}^2, D^{(0,\frac{1}{2})})$ means: We propose that it has something to do with the parity transformation. To realize a parity transformation, we define in contrast to ((128)):

$$\tilde{\hat{x}} = \sum_{\mu=0}^{3} x^\mu \tilde{\sigma}_\mu = x^0\sigma_0 - \sum_{i=1}^{3} x^i \sigma_i \tag{167}$$

We have

$$\epsilon A \epsilon^{-1} = \begin{pmatrix} A^2{}_2 & -A^2{}_1 \\ -A^1{}_2 & A^1{}_1 \end{pmatrix} \tag{168}$$

It follows

$$\epsilon\overline{\sigma_0}\epsilon^{-1} = \sigma_0, \quad \epsilon\overline{\sigma_i}\epsilon^{-1} = -\sigma_i \quad \Longrightarrow \epsilon\overline{\sigma_\mu}\epsilon^{-1} = \tilde{\sigma}_\mu \tag{169}$$

and we obtain

$$\tilde{\hat{x}} = \epsilon\bar{\hat{x}}\epsilon^{-1} \tag{170}$$

Thus, for the transformation in the reflected space one gets

$$\tilde{\hat{x}}' = \epsilon\overline{\hat{x}'}\epsilon^{-1} = \epsilon\overline{A\hat{x}A^\dagger}\epsilon^{-1} = \epsilon\overline{A}(\epsilon^{-1}\tilde{\hat{x}}\epsilon)\overline{A^\dagger}\epsilon^{-1} = (\epsilon\overline{A}\epsilon^{-1})\tilde{\hat{x}}(\epsilon\overline{A^\dagger}\epsilon^{-1}) = (A^\dagger)^{-1}\tilde{\hat{x}}A^{-1} \tag{171}$$

The last equality follows from

$$A\epsilon A^T = \begin{pmatrix} -A^1{}_2 & A^1{}_1 \\ -A^2{}_2 & A^2{}_1 \end{pmatrix}\begin{pmatrix} A^1{}_1 & A^2{}_1 \\ A^1{}_2 & A^2{}_2 \end{pmatrix} \overset{\det(A)=1}{=} \epsilon \iff \epsilon A^T\epsilon^{-1} = A^{-1}$$
$$\iff (\epsilon^{-1})^\dagger (A^T)^\dagger \epsilon^\dagger = (A^{-1})^\dagger \iff \epsilon\overline{A}\epsilon^{-1} = (A^\dagger)^{-1} \tag{172}$$

The transformation law

$$\tilde{\hat{x}}' = (\epsilon\overline{A}\epsilon^{-1})\tilde{\hat{x}}(\epsilon\overline{A^\dagger}\epsilon^{-1}) \tag{173}$$

defines a representation, which with the isomorphism $\epsilon : \overline{\mathbb{C}}^2 \to \overline{\mathbb{C}}^2$ is equivalent to $(\overline{\mathbb{C}}^2, D^{(0,\frac{1}{2})})$.

Thus, $(\overline{\mathbb{C}}^2, D^{(0,\frac{1}{2})})$ is required to realise equalities which consider parity transformations.

As mentioned, one obtains equivalent representations by choosing an isomorphic vector space and appropriately adjusting the homomorphism of the representation. Since these representations do not add anything new, it makes sense to group all equivalent representations together:

$(\frac{1}{2}, 0)$ are the equivalent representations of $(\mathbb{C}^2, D^{(\frac{1}{2},0)})$ and $(0, \frac{1}{2})$ are the equivalent representations of $(\overline{\mathbb{C}}^2, D^{(0,\frac{1}{2})})$.

$(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are called spinor representations and are, in a sense, the fundamental representations of $\mathcal{L}_+^\uparrow$. All other representations can be constructed from these two fundamental representations using spinor calculus (van der Waerden (1974)). The fact that these two representations are so fundamental is related to the fact that $SL(2, \mathbb{C})$ is the universal covering group of $\mathcal{L}_+^\uparrow$.

3.3 Invariant Field Equations

So far, we have only considered finite-dimensional representations, where the vector space is finite-dimensional. In quantum mechanics, however, we deal with functions defined on Minkowski space

$$\psi : \mathbb{M} \to \mathbb{C}^d, x \mapsto \langle x|\psi\rangle \tag{174}$$

$$\psi \in \mathcal{H} = L^2(\mathbb{R}^3)^d \tag{175}$$

This setting fits well with our introduction of representations, where we linearized M by taking the space of functions $F(M) = \{f \mid f : M \to W\} \subseteq W^M$ as the vector space of the representation. This vector space is infinite-dimensional if M contains infinitely many elements. This is the case for Minkowski space, and thus we obtain infinite-dimensional representations (see also Maciejko (2020)).

So, we consider the space of functions as the vector space of the representation. The set M mentioned above in our case is Minkowski space $\mathbb{M}$, which corresponds to the vector space $\mathbb{R}^4$. We want to examine the general transformation behavior in this context:
Let G be a group and V, W be vector spaces.
We have representation mappings

$$\Lambda : G \to GL(V), \quad S : G \to GL(W) \tag{176}$$

and we are interested in a function

$$\psi : V \to W \tag{177}$$

i.e. representations on the vector space W^V. That means we consider the representation mapping

$$\begin{aligned} T : G &\longrightarrow GL(W^V) \\ A &\longmapsto (T_A : \psi \mapsto S_A \circ \psi \circ \Lambda_A^{-1}) \end{aligned} \tag{178}$$

This mapping is chosen such that $\forall g, h \in G$:

$$T_{g*h} = T_g \circ T_h \tag{179}$$

as shown by the following calculation:

$$T_g \circ T_h \circ \psi = S_g \circ S_h \circ \psi \circ \Lambda_h^{-1} \circ \Lambda_g^{-1} = S_g \circ S_h \circ \psi \circ (\Lambda_g \circ \Lambda_h)^{-1} = S_{g*h} \circ \psi \circ \Lambda_{g*h}^{-1} = T_{g*h} \circ \psi \tag{180}$$

In the third step, we used the fact that S and Λ are representation mappings, and thus group homomorphisms.
Therefore, we generally obtain

$$\psi(x) \overset{A}{\longmapsto} \psi'(x') = (T_A\psi)(x') = S_A\psi(\Lambda_A^{-1} \cdot x') = S_A\psi(x) \tag{181}$$

As Equations (174) and (175) already suggest, the wave function ψ can have a d-dimensional range. At this point, the Dirac theory differs from the Schrödinger theory, where the wave function is one-dimensional. The dimension of the wave function ψ characterizes different representations, which, as we will see later, correspond to a particular property of a particle: the spin.

Representations with the lowest dimension of the wave function's range:
$(0,0)$: For a scalar field $\psi : \mathbb{M} \to \mathbb{C}, x \mapsto \langle x|\psi\rangle$, $D^{(0,0)}(\Lambda)$ is a one-valued map, and the requirement that $D^{(0,0)}$ is a group homomorphism gives us

$$\lambda = D^{(0,0)}(\Lambda_1\Lambda_2) = D^{(0,0)}(\Lambda_1)D^{(0,0)}(\Lambda_2) = \lambda^2 \implies \lambda = 1 \implies D^{(0,0)}(\Lambda) = 1 \tag{182}$$

From Equation (181), it follows

$$\psi'(x') \overset{(181)}{=} D^{(0,0)}(\Lambda)\psi(x) = \psi(\Lambda^{-1} \cdot x') \tag{183}$$

$(\frac{1}{2}, 0)$: This spinor representation transforms the two-valued field $\psi_\alpha : \mathbb{M} \to \mathbb{C}^2, x \mapsto \langle x|\psi_\alpha\rangle$ in the following way:

$$\psi'_\alpha(x') \overset{(181)}{=} A_\alpha{}^\beta \psi_\beta(x) = A_\alpha{}^\beta \psi_\beta(\Lambda^{-1} \cdot x') \tag{184}$$

$(0, \frac{1}{2})$: This spinor representation transforms the two-valued field $\psi_{\overline{\alpha}} : \mathbb{M} \to \overline{\mathbb{C}}^2, x \mapsto \langle x|\psi_{\overline{\alpha}}\rangle$ as follows:

$$\psi'_{\overline{\alpha}}(x') \overset{(181)}{=} \overline{A}_{\overline{\alpha}}{}^{\overline{\beta}} \psi_{\overline{\beta}}(x) = \overline{A}_{\overline{\alpha}}{}^{\overline{\beta}} \psi_{\overline{\beta}}(\Lambda^{-1} \cdot x') \tag{185}$$

Now, the goal is to find invariant differential equations for various fields. Here, we closely follow Scharf (2014). We need a representation of the differential operator $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Therefore, we must have a 4-dimensional irreducible representation.

Definition 3.2 (Irreducible Representation)**.** *A representation (V, π) of a group G with representation map*

$$\pi : G \to GL(V), g \mapsto \pi(g) \tag{186}$$

is called irreducible if there are no non-trivial G-invariant subspaces, i.e., $W_1 = \{0\}$ and $W_2 = V$ are the only G-invariant subspaces.
A subspace $W \subseteq V$ is called G-invariant if for all $w \in W$ and all $g \in G$: $\pi(g)w \in W$.

The irreducibility of the representation is crucial because such representations cannot be decomposed further, and as a consequence they describe fundamental properties of the one-particle quantum system of interest.

All finite-dimensional irreducible representations of $\mathcal{L}_+^\uparrow$ can be obtained in the following way (Scharf, 2014, Chapter 1.1):
We consider the vector space

$$\mathrm{sym}(\mathbb{C}^2)^{\otimes n} \times \mathrm{sym}(\overline{\mathbb{C}}^2)^{\otimes m} \ni u_{\alpha_1 \dots \alpha_n \overline{\beta}_1 \dots \overline{\beta}_m} \tag{187}$$

where $\mathrm{sym}(\mathbb{C}^2)^{\otimes n}$ is the symmetric part of the n-fold tensor product of $\mathbb{C}^2$. For a detailed definition see (Lee, 2012, Chapter 12). Then we get the following irreducible representation:

$$D^{(\frac{n}{2},\frac{m}{2})} : SL(2,\mathbb{C}) \to SL(\mathrm{sym}(\mathbb{C}^2)^{\otimes n} \times \mathrm{sym}(\overline{\mathbb{C}}^2)^{\otimes m}), \Lambda \mapsto D^{(\frac{n}{2},\frac{m}{2})}(\Lambda) \tag{188}$$

with

$$\begin{aligned} D^{(\frac{n}{2},\frac{m}{2})}(\Lambda) : \mathrm{sym}(\mathbb{C}^2)^{\otimes n} \times \mathrm{sym}(\overline{\mathbb{C}}^2)^{\otimes m} &\to \mathrm{sym}(\mathbb{C}^2)^{\otimes n} \times \mathrm{sym}(\overline{\mathbb{C}}^2)^{\otimes m}, \\ u_{\gamma_1 \dots \gamma_n \overline{\delta}_1 \dots \overline{\delta}_m} \mapsto u'_{\alpha_1 \dots \alpha_n \overline{\beta}_1 \dots \overline{\beta}_m} &= A_{\alpha_1}^{\ \gamma_1} \dots A_{\alpha_n}^{\ \gamma_n} \, \overline{A}_{\overline{\beta}_1}^{\ \overline{\delta}_1} \dots \overline{A}_{\overline{\beta}_m}^{\ \overline{\delta}_m} \, u_{\gamma_1 \dots \gamma_n \overline{\delta}_1 \dots \overline{\delta}_m} \end{aligned} \tag{189}$$

To determine the dimension of $\mathrm{sym}(\mathbb{C}^2)^{\otimes n}$, we start by considering the basis of $\mathbb{C}^2$, denoted as e_1, e_2. The basis of $\mathrm{sym}(\mathbb{C}^2)^{\otimes n}$ is then given by:

- $e_1 \otimes e_1 \otimes \dots \otimes e_1 \quad$ (n times e_1)
- $e_2 \otimes e_1 \otimes \dots \otimes e_1 + e_1 \otimes e_2 \otimes \dots \otimes e_1 + \dots + e_1 \otimes e_1 \otimes \dots \otimes e_2 \quad$ ($(n-1)$ times e_1)
- $\dots$
- $e_1 \otimes e_2 \otimes \dots \otimes e_2 + e_2 \otimes e_1 \otimes \dots \otimes e_2 + \dots + e_2 \otimes e_2 \otimes \dots \otimes e_1 \quad$ ($(n-1)$ times e_2)
- $e_2 \otimes e_2 \otimes \dots \otimes e_2 \quad$ (n times e_2)

Thus, the number of basis elements is $n+1$ and we have $\dim(\mathrm{sym}(\mathbb{C}^2)^{\otimes n}) = n+1$.

It follows that $\dim(\mathrm{sym}(\mathbb{C}^2)^{\otimes n} \times \mathrm{sym}(\overline{\mathbb{C}}^2)^{\otimes m}) = (n+1)(m+1)$.

The desired 4-dimensional representation is therefore obtained for $n = 1$ and $m = 1$, i.e., $(\frac{1}{2}, \frac{1}{2})$:

$$D^{(\frac{1}{2},\frac{1}{2})} : SL(2,\mathbb{C}) \to SL(\mathrm{sym}(\mathbb{C}^2) \times \mathrm{sym}(\overline{\mathbb{C}}^2)), \Lambda \mapsto D^{(\frac{1}{2},\frac{1}{2})}(\Lambda) \tag{190}$$

with

$$D^{(\frac{1}{2},\frac{1}{2})}(\Lambda) : \mathrm{sym}(\mathbb{C}^2) \times \mathrm{sym}(\overline{\mathbb{C}}^2) \to \mathrm{sym}(\mathbb{C}^2) \times \mathrm{sym}(\overline{\mathbb{C}}^2), u_{\gamma\overline{\delta}} \mapsto u'_{\alpha\overline{\beta}} = A_{\alpha}^{\ \gamma} \, \overline{A}_{\overline{\beta}}^{\ \overline{\delta}} \, u_{\gamma\overline{\delta}} \tag{191}$$

The matrices $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ form a basis not only for the real vector space of Hermitian 2×2 matrices but also for the complex vector space of 2×2 matrices.
Define the isomorphism

$$\hat{\psi}_{\alpha\overline{\beta}} = \sum_{\mu=0}^{3} \psi^\mu (\sigma_\mu)_{\alpha\overline{\beta}} \tag{192}$$

which links the spinor $\hat{\psi}_{\alpha\overline{\beta}}$ and the four-vector ψ^μ, analogous to Equation (128). We obtain the correct transformation behavior, as shown by Equation (149):

$$\hat{\psi}'_{\alpha\overline{\beta}} = A_\alpha{}^\gamma \hat{\psi}_{\gamma\overline{\delta}} \overline{A}_{\overline{\beta}}^{\overline{\delta}} = (A\hat{\psi}A^\dagger)_{\alpha\overline{\beta}} \overset{(149)}{=} (\hat{\Lambda\psi})_{\alpha\overline{\beta}} \tag{193}$$

Equation (149) was demonstrated for $x \in \mathbb{R}^4$, but the calculation is entirely analogous for $\mathbb{C}^4$.

For ∂_μ and $\partial^\mu = g^{\mu\nu}\partial_\nu$, we have

$$\hat{\partial} = (\partial_{\alpha\overline{\beta}}) = \sigma_\mu \partial^\mu = \sigma_0\partial_0 - \vec{\sigma}\cdot\vec{\nabla} \tag{194}$$

and

$$\partial^{\alpha\overline{\beta}} = \epsilon^{\alpha\gamma}\epsilon^{\overline{\beta}\overline{\delta}}\partial_{\gamma\overline{\delta}} = (\epsilon\hat{\partial}\epsilon^T)^{\alpha\overline{\beta}} \tag{195}$$

From Equation (168) and $\epsilon^{-1} = \epsilon^T$, we obtain

$$\epsilon\sigma_0\epsilon^T = \sigma_0, \quad \epsilon\sigma_i\epsilon^T = -\sigma_i^T \tag{196}$$

Thus, it follows

$$\partial^{\alpha\overline{\beta}} = (\sigma_0\partial_0 + \vec{\sigma}^T\cdot\vec{\nabla})^{\alpha\overline{\beta}} \tag{197}$$

Now, we can write down differential equations for various fields.
Scalar field:
A scalar differential operator has no free indices, implying $\partial_{\alpha\overline{\beta}}\partial^{\alpha\overline{\beta}}$. This leads to the equation

$$\partial_{\alpha\overline{\beta}}\partial^{\alpha\overline{\beta}}\psi(x) + a\psi(x) = 0 \tag{198}$$

where $a \in \mathbb{C}$ is a constant. The second-order differential operator $\partial_{\alpha\overline{\beta}}\partial^{\alpha\overline{\beta}}$ can be expressed as follows:

$$\begin{aligned}\partial_{\alpha\overline{\beta}}\partial^{\gamma\overline{\beta}} &= (\sigma_0\partial_0 - \vec{\sigma}\cdot\vec{\nabla})_{\alpha\overline{\beta}}(\sigma_0\partial_0 + \vec{\sigma}^T\cdot\vec{\nabla})^{\gamma\overline{\beta}} = (\sigma_0\partial_0 - \vec{\sigma}\cdot\vec{\nabla})_{\alpha\overline{\beta}}(\sigma_0\partial_0 + \vec{\sigma}\cdot\vec{\nabla})^{\overline{\beta}\gamma} \\ &= (\sigma_0\sigma_0\partial_0^2 - \vec{\sigma}\cdot\vec{\sigma}\vec{\nabla}^2)_\alpha{}^\gamma \overset{(114)}{=} (\mathbb{1}_2\partial_0^2 - \mathbb{1}_2\vec{\nabla}^2)_\alpha{}^\gamma = (\partial_0^2 - \vec{\nabla}^2)\delta_\alpha{}^\gamma\end{aligned} \tag{199}$$

This implies that

$$\partial_{\alpha\overline{\beta}}\partial^{\alpha\overline{\beta}} = \partial_0^2 - \vec{\nabla}^2 = \Box \tag{200}$$

where $\Box$ is the wave operator. Consequently, we obtain for the invariant scalar field equation

$$(\Box + a)\psi(x) = 0 \tag{201}$$

Here, the constant a has the dimension $\frac{1}{\text{length}^2}$. To determine the physical meaning of a, we construct it using the mass of the particle and fundamental constants from relativity and quantum mechanics. This leads to the Compton wavelength: $\lambda = \frac{\hbar}{mc}$. Thus, we arrive at the Klein-Gordon equation

$$\boxed{\Box\psi(x) + \left(\frac{mc}{\hbar}\right)^2\psi(x) = 0} \tag{202}$$

For the massless case $m = 0$, the equation reduces to the wave equation

$$\Box\psi(x) = 0 \tag{203}$$

Spinor field:
For a spinor field with one free index, we have the following building blocks available:

$$\partial^{\alpha\overline{\beta}}\varphi_\alpha \iff (\sigma_0\partial_0 + \vec{\sigma}^T \cdot \vec{\nabla})^T\varphi = (\sigma_0\partial_0 + \vec{\sigma} \cdot \vec{\nabla})\varphi \tag{204}$$

$$\partial_{\alpha\overline{\beta}}\varphi^\alpha \iff (\sigma_0\partial_0 - \vec{\sigma} \cdot \vec{\nabla})^T\varphi = (\sigma_0\partial_0 - \vec{\sigma}^T \cdot \vec{\nabla})\varphi \tag{205}$$

$$\partial^{\alpha\overline{\beta}}\chi_{\overline{\beta}} \iff (\sigma_0\partial_0 + \vec{\sigma}^T \cdot \vec{\nabla})\chi \tag{206}$$

$$\partial_{\alpha\overline{\beta}}\chi^{\overline{\beta}} \iff (\sigma_0\partial_0 - \vec{\sigma} \cdot \vec{\nabla})\chi \tag{207}$$

Here, $\varphi \in \mathbb{C}^2$ is called a right-handed Weyl spinor, and $\chi \in \overline{\mathbb{C}}^2$ is called a left-handed Weyl spinor. The first two and the last two terms can be transformed into one another using the spinor metric. While φ and χ do not represent the rotation angle or rapidity, this notation is commonly used. These expressions lead to first-order differential equations, with the simplest ones being:
the right-handed Weyl equation

$$\partial^{\alpha\overline{\beta}}\varphi_\alpha(x) = 0 \iff (\sigma_0\partial_0 + \vec{\sigma} \cdot \vec{\nabla})\varphi(x) = 0 \tag{208}$$

and the left-handed Weyl equation

$$\partial_{\alpha\overline{\beta}}\chi^{\overline{\beta}}(x) = 0 \iff (\sigma_0\partial_0 - \vec{\sigma} \cdot \vec{\nabla})\chi(x) = 0 \tag{209}$$

It is not possible to add mass terms like $a\varphi_\alpha$ or $b\chi^{\overline{\beta}}$ to these equations because the indices do not match. Therefore, these equations describe massless particles. Another important point is that the equations are not invariant under parity transformations $P : \vec{x} \mapsto -\vec{x}$. For the right-handed Weyl equation, the transformed equation is

$$\tilde{\varphi}(x) = \varphi(x^0, -\vec{x}) \qquad (\sigma_0\partial_0 - \vec{\sigma} \cdot \vec{\nabla})\tilde{\varphi}(x) = 0 \tag{210}$$

For the left-handed Weyl equation, the transformed equation is

$$\tilde{\chi}(x) = \chi(x^0, -\vec{x}) \qquad (\sigma_0\partial_0 + \vec{\sigma} \cdot \vec{\nabla})\tilde{\chi}(x) = 0 \tag{211}$$

Equations (208) and (210) and Equations (209) and (211) would only be equivalent if there existed a linear transformation T such that

$$T(\sigma_0\partial_0 - \vec{\sigma} \cdot \vec{\nabla})T^{-1}\tilde{\varphi}(x) = (\sigma_0\partial_0 + \vec{\sigma} \cdot \vec{\nabla})\tilde{\varphi}(x) \tag{212}$$

and similarly

$$T(\sigma_0\partial_0 + \vec{\sigma} \cdot \vec{\nabla})T^{-1}\tilde{\chi}(x) = (\sigma_0\partial_0 - \vec{\sigma} \cdot \vec{\nabla})\tilde{\chi}(x) \tag{213}$$

which implies $T\sigma_i = -\sigma_i T$. Explicit calculation shows that this is only valid for $T = 0$. Therefore, the Weyl equations are not invariant under spatial reflection.

We now seek a parity-invariant equation that allows for a mass term. Observing that the right-handed Weyl equation is equivalent to the transformed left-handed Weyl equation and vice versa, we can construct coupled equations

$$\partial^{\alpha\overline{\beta}}\varphi_\alpha(x) = a\chi^{\overline{\beta}}(x) \iff (\sigma_0\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\varphi(x) = a\chi(x) \tag{214}$$

$$\partial_{\alpha\overline{\beta}}\chi^{\overline{\beta}}(x) = b\varphi_\alpha(x) \iff (\sigma_0\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\chi(x) = b\varphi(x) \tag{215}$$

The transformed equations

$$\partial^{\alpha\overline{\beta}}\tilde{\varphi}_\alpha(x) = a\tilde{\chi}^{\overline{\beta}}(x) \iff (\sigma_0\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\tilde{\varphi}(x) = a\tilde{\chi}(x) \tag{216}$$

$$\partial_{\alpha\overline{\beta}}\tilde{\chi}^{\overline{\beta}}(x) = b\tilde{\varphi}_\alpha(x) \iff (\sigma_0\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\tilde{\chi}(x) = b\tilde{\varphi}(x) \tag{217}$$

are identical to the original equations if one sets

$$a = b, \quad \tilde{\varphi}(x) = k\chi(x), \quad \tilde{\chi}(x) = k\varphi(x) \tag{218}$$

where $k \in \mathbb{C}$ is a constant. Additionally, we have

$$\varphi(x) = \tilde{\tilde{\varphi}}(x) = k\tilde{\chi}(x) = k^2\varphi(x) \implies |k| = 1 \tag{219}$$

By choosing $k = 1$ and $a = -i\frac{mc}{\hbar}$, we obtain the Dirac equation for a particle with mass m:

$$i\hbar(\sigma_0\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\varphi(x) = mc\chi(x) \tag{220}$$

$$i\hbar(\sigma_0\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\chi(x) = mc\varphi(x) \tag{221}$$

The factor of i is introduced to ensure unitary time evolution.
The two coupled two-component equations can be written as a single four-component equation by introducing a Dirac spinor

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{222}$$

and the gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \iff \gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{223}$$

Equation (223) represents the Weyl representation of the gamma matrices. Using these, we arrive at the Dirac equation in its standard covariant form:

$$\boxed{i\hbar\gamma^\mu\partial_\mu\psi(x) = mc\psi(x)} \tag{224}$$

From our previous construction, we already know how $\psi(x)$ transforms under a Lorentz transformation:

$$\psi(x) = \begin{pmatrix} \varphi_\alpha(x) \\ \chi^{\overline{\beta}}(x) \end{pmatrix} \longmapsto \psi'(x') = S_\Lambda\psi(x) \tag{225}$$

where

$$S_\Lambda = \begin{pmatrix} A_\Lambda & 0 \\ 0 & \epsilon^{-1}\overline{A}_\Lambda \epsilon \end{pmatrix} \overset{\text{similar to (172)}}{=} \begin{pmatrix} A_\Lambda & 0 \\ 0 & (A_\Lambda^\dagger)^{-1} \end{pmatrix}, \qquad A_\Lambda \in SL(2,\mathbb{C}) \tag{226}$$

The two ϵ factors in S_Λ arise because the index $\overline{\beta}$ in $\chi^{\overline{\beta}}$ is an upper index. It must first be lowered so that the known transformation behavior holds and then raised again afterward.

We can now explicitly check whether the Dirac equation is invariant under Lorentz transformations:

$$i\hbar\gamma^\mu \partial'_\mu \psi'(x') = mc\psi'(x') \tag{227}$$
$$i\hbar\gamma^\mu \partial'_\mu S_\Lambda \psi(x) = mcS_\Lambda \psi(x) \tag{228}$$
$$i\hbar\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu S_\Lambda \psi(x) = mcS_\Lambda \psi(x) \tag{229}$$
$$i\hbar S_\Lambda^{-1}\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu S_\Lambda \psi(x) = mc\psi(x) \tag{230}$$
$$i\hbar S_\Lambda^{-1}\gamma^\mu S_\Lambda \Lambda_\mu{}^\nu \partial_\nu \psi(x) = mc\psi(x) \tag{231}$$

Therefore, it must hold that

$$\boxed{S_\Lambda^{-1}\gamma^\mu S_\Lambda \Lambda_\mu{}^\nu = \gamma^\nu} \tag{232}$$
$$S_\Lambda^{-1}\gamma^\mu S_\Lambda \Lambda_\mu{}^\nu \Lambda^\alpha{}_\nu = \Lambda^\alpha{}_\nu \gamma^\nu \tag{233}$$
$$S_\Lambda^{-1}\gamma_\mu S_\Lambda \Lambda^{\mu\nu} \Lambda^\alpha{}_\nu = \Lambda^\alpha{}_\nu \gamma^\nu \tag{234}$$
$$S_\Lambda^{-1}\gamma^\mu S_\Lambda g_{\mu\rho} \Lambda^\rho{}_\tau g^{\tau\nu} \Lambda^\alpha{}_\nu = \Lambda^\alpha{}_\nu \gamma^\nu \tag{235}$$

Using the condition from (8):

$$g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\tau = g_{\rho\tau} \iff g^{\mu\nu} g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\tau g^{\rho\tau} = g^{\mu\nu} g_{\rho\tau} g^{\rho\tau} \iff \Lambda^\mu{}_\rho \Lambda^\nu{}_\tau g^{\rho\tau} = g^{\mu\nu} \tag{236}$$

we get

$$S_\Lambda^{-1}\gamma^\mu S_\Lambda g_{\mu\rho} \underbrace{\Lambda^\rho{}_\tau g^{\tau\nu} \Lambda^\alpha{}_\nu}_{g^{\rho\alpha}} = \Lambda^\alpha{}_\nu \gamma^\nu \tag{237}$$
$$S_\Lambda^{-1}\gamma^\mu S_\Lambda g_{\mu\rho} g^{\rho\alpha} = \Lambda^\alpha{}_\nu \gamma^\nu \tag{238}$$
$$S_\Lambda^{-1}\gamma^\mu S_\Lambda \delta_\mu{}^\alpha = \Lambda^\alpha{}_\nu \gamma^\nu \tag{239}$$
$$S_\Lambda^{-1}\gamma^\alpha S_\Lambda = \Lambda^\alpha{}_\nu \gamma^\nu \tag{240}$$
$$\boxed{S_\Lambda^{-1}\gamma^\mu S_\Lambda = \Lambda^\mu{}_\nu \gamma^\nu} \tag{241}$$

This is the invariance condition for the Dirac equation. We can observe that the matrices $S_\Lambda^{-1}\gamma^\mu S_\Lambda$ transform like a 4-vector.
Following this condition, we would recover the transformation matrix S_Λ as described earlier in (226).

Next, let us examine what kind of particles are described by the Dirac equation. For this, we analyze the behavior of the Dirac spinor under rotations $\Lambda = R(\vec{\varphi})$. Since the elements of the covering group $A_{R(\vec{\varphi})} = U(\vec{\varphi}) \in SU(2)$ are unitary, the transformation behavior is given by

$$\psi'(x') = \begin{pmatrix} U(\vec{\varphi}) & 0 \\ 0 & U(\vec{\varphi}) \end{pmatrix} \psi(x) = \begin{pmatrix} U(\vec{\varphi}) & 0 \\ 0 & U(\vec{\varphi}) \end{pmatrix} \psi(R(\vec{\varphi})^{-1} x') \tag{242}$$

This corresponds to the spinor representation $D^{\frac{1}{2}} \oplus D^{\frac{1}{2}}$ of the rotation group $SO(3)$, where the underlying vector space is $\mathbb{C}^2 \oplus \mathbb{C}^2$. Therefore, the Dirac equation describes particles with spin $\frac{1}{2}$.

We also observe that for rotations, a two-dimensional spinor is sufficient to obtain a parity-invariant equation, as the transformation matrix for both the right-handed and left-handed Weyl spinors is the same. However, the inclusion of time and the resulting boosts make a four-dimensional Dirac spinor necessary. This is because the Hermitian boost matrices satisfy $(A^{\dagger}_{\lambda(\vec{\chi})})^{-1} = A^{-1}_{\lambda(\vec{\chi})} \neq A_{\lambda(\vec{\chi})}$.
This can be understood by noting that for rotations, a spatial reflection does not make a difference. However, for boosts, a parity transformation leads to a boost in the opposite direction, which corresponds to the inverse boost.

4 Lie Algebra

4.1 Introduction to Lie Algebra

We now place our previous discussions in a broader context. As mentioned earlier, the Lorentz group is a Lie group, which gives it a special structure. Specifically, it is a differentiable manifold that is compatible with its group structure.
This manifold structure allows us to introduce local coordinates. Due to its differentiable nature, we can differentiate a group element with respect to a coordinate and obtain an infinitesimal generator of the group. For general Lie groups, this process provides the tangent space of the Lie group at the identity element. This vector space inherits a special structure from the group itself. Together with this structure the vector space is known as the Lie algebra.

However, we will not explore the topic in full generality here. Instead, we take advantage of the fact that the Lorentz group is a matrix group, i.e., $G \subset GL(n, \mathbb{C})$. Thus, we define the Lie algebra as follows:

Definition 4.1 (Lie Algebra)**.** *For G a Lie group of $n \times n$ invertible matrices, the Lie algebra $\mathfrak{g}$ of G is the space of $n \times n$ matrices X such that $e^{tX} \in G$ for all $t \in \mathbb{R}$.*

The exponential of a matrix is given by the usual series

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \mathbb{1} + A + \frac{1}{2}A^2 + \cdots + \frac{1}{n!}A^n + \dots \tag{243}$$

which converges, e.g., for any finite dimensional matrix A (Hall, 2015, Chapter 2.1).

This more concrete approach simplifies the discussion but comes at the cost of losing some important properties of the Lie algebra. For example, we do not immediately see that the Lie algebra is the tangent space at the identity element of the group. However, we do have the following:

$$\frac{d}{dt} e^{tX} = X e^{tX} \tag{244}$$

which implies that

$$X = \left(\frac{d}{dt} e^{tX} \right)_{t=0} \tag{245}$$

While the map

$$\exp : \mathbb{R} \times \mathrm{Mat}(n, \mathbb{C}) \to GL(n, \mathbb{C}), (t, X) \mapsto e^{tX} \tag{246}$$

where $\mathrm{Mat}(n, \mathbb{C})$ is the space of $n \times n$ matrices over the complex numbers, is not surjective in general, meaning not every group element can be written as e^{tX}, it is surjective near the identity element. Therefore, the calculation above gives us the Lie algebra as the tangent space at the identity element.

For now, we aim to establish a weaker result, namely, that the Lie algebra is at least a vector space for $G = GL(n, \mathbb{C})$. Let $X \in \mathrm{Mat}(n, \mathbb{C})$. The exponential e^{tX} is an invertible matrix for all $t \in \mathbb{R}$, with inverse e^{-tX}, so $e^{tX} \in GL(n, \mathbb{C})$. Thus, we conclude that $\mathrm{Mat}(n, \mathbb{C})$ is the Lie algebra of $GL(n, \mathbb{C})$, and $\mathrm{Mat}(n, \mathbb{C})$ is a vector space.

The same can be shown for any subgroup $G \subset GL(n, \mathbb{C})$.

The special structure inherited by the Lie algebra from the group structure is the Lie bracket:

Definition 4.2 (Lie Bracket)**.** *The Lie bracket operation on $\mathfrak{g}$ is a bilinear, anti-symmetric map given by the commutator of matrices*

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \; (X, Y) \mapsto [X, Y] = XY - YX \tag{247}$$

We must still show that this is well-defined, meaning that $X, Y \in \mathfrak{g}$ implies $[X, Y] = XY - YX \in \mathfrak{g}$. For a detailed proof, see (Woit, 2024, Chapter 5.1).

Since the Lie algebra, unlike the Lie group, is a vector space, we can introduce a basis $X_1, \ldots, X_n$. The Lie bracket can then be written as

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k \tag{248}$$

where c_{ijk} are the structure constants.

For the Lie algebra $\mathfrak{so}(3)$ of the group $SO(3)$, we obtain

$$J_1 = i\frac{d}{d\varphi_1} R_1(\varphi_1)_{|\varphi_1=0} = -i\frac{d}{d\varphi_1}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\varphi_1 & -\sin\varphi_1\\ 0 & \sin\varphi_1 & \cos\varphi_1\end{pmatrix} = i\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix} \tag{249}$$

$$J_2 = i\frac{d}{d\varphi_2} R_1(\varphi_2)_{|\varphi_2=0} = -i\frac{d}{d\varphi_0}\begin{pmatrix}\cos\varphi_2 & 0 & \sin\varphi_2\\ 0 & 1 & 0\\ -\sin\varphi_2 & 0 & \cos\varphi_2\end{pmatrix} = i\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix} \tag{250}$$

$$J_3 = i\frac{d}{d\varphi_3} R_1(\varphi_3)_{|\varphi_3=0} = -i\frac{d}{d\varphi_0}\begin{pmatrix}\cos\varphi_3 & -\sin\varphi_3 & 0\\ \sin\varphi_3 & \cos\varphi_3 & 0\\ 0 & 0 & 1\end{pmatrix} = i\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{251}$$

i.e., by explicit calculation

$$[J_i, J_k] = i\epsilon_{ijk} J_k \tag{252}$$

In physics, based on Noether's theorem, the group elements are identified as symmetry transformations, and the elements of the Lie algebra, which generate these transformations, are interpreted as observables. Therefore, it is customary to introduce a factor of i to the generators J so that they become Hermitian.

For $\mathfrak{su}(2)$, we obtain

$$J_1 = i\frac{d}{d\varphi_1}U_1(\varphi_1)\big|_{\varphi_1=0} = i\frac{d}{d\varphi_1}e^{-\frac{i}{2}\sigma_1} = \frac{1}{2}\sigma_1 \tag{253}$$

$$J_2 = i\frac{d}{d\varphi_2}U_2(\varphi_2)\big|_{\varphi_2=0} = i\frac{d}{d\varphi_2}e^{-\frac{i}{2}\sigma_2} = \frac{1}{2}\sigma_2 \tag{254}$$

$$J_3 = i\frac{d}{d\varphi_3}U_3(\varphi_3)\big|_{\varphi_3=0} = i\frac{d}{d\varphi_3}e^{-\frac{i}{2}\sigma_3} = \frac{1}{2}\sigma_3 \tag{255}$$

Thus, using the commutator for Pauli matrices, we also find

$$[J_i, J_k] = i\epsilon_{ijk}J_k \tag{256}$$

We observe that different groups can share the same Lie algebra. While the Lie algebra of a group can be uniquely determined by differentiation, the reverse is not true. According to a theorem by Lie, commonly referred to as "Lie's Third Theorem", every finite-dimensional Lie algebra corresponds to a unique simply connected Lie group (Hall, 2015, Chapter 5.10).
In our example, the Lie algebra with the commutator $[J_i, J_k] = i\epsilon_{ijk}J_k$ generates the group $SU(2)$.

The central idea behind Lie algebras is to linearize the Lie group at the identity element, allowing for easier calculations due to the vector space structure. This corresponds to breaking down a group element into many small linear transformations that are applied consecutively. For infinitesimal transformations, this leads to the exponential map, $\exp(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$. This justifies the initial definition of the Lie algebra.

Similar to groups, one can also define a representation for Lie algebras:

Definition 4.3 (Lie Algebra Representation)**.** *A Lie algebra representation (V, ϕ) of a Lie algebra $\mathfrak{g}$ on an n-dimensional complex vector space V is given by a linear map*

$$\phi : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C}) = \mathrm{Mat}(n, \mathbb{C}), X \mapsto \phi(X) \tag{257}$$

satisfying

$$\phi([X, Y]) = [\phi(X), \phi(Y)] \tag{258}$$

Implicitly, we have already chosen a representation when defining the generators of $SO(3)$ and $SU(2)$, specifically the defining representation.

Next, we want to connect this with a Lie group and its representation. As mentioned earlier, an element of the Lie group can be described by its behavior near the identity element. When this is applied to the group representation and the Lie algebra representation by the homomorphism property, we find that the group representation $\pi : G \to GL(n, \mathbb{C})$ is largely determined by its behavior at the identity element and, thus, by its derivative π'.
This result is captured in the following theorem:

Theorem 4.1. *If* $\pi : G \to GL(n, \mathbb{C})$ *is a group representation, then*

$$\pi' : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C}) = \mathrm{Mat}(n, \mathbb{C}), X \mapsto \pi'(X) = \frac{d}{dt}(\pi(e^{tX}))\big|_{t=0} \tag{259}$$

satisfies the following properties:

1. $\pi(e^{tX}) = e^{t\pi'(X)}$
2. $\pi'([X, Y]) = [\pi'(X), \pi'(Y)]$

For the proof, see (Woit, 2024, Chapter 5.4).

This theorem shows that we can study Lie group representations (V, π) by analyzing the corresponding Lie algebra representation (V, π'). This is generally much easier since π' is a linear map.

4.2 Lorentz Algebra

Now, we wish to apply this to the Lorentz group, which, as previously mentioned, is a Lie group. We follow the discussion in Pelster (2021). Near the identity, a Lorentz transformation can be written as

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{260}$$

where ω is infinitesimal. Using Equation (8), we obtain for the infinitesimal Lorentz transformation

$$g_{\mu\nu}(\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\tau} + \omega^{\nu}{}_{\tau}) = g_{\rho\tau} + \omega_{\rho\tau} + \omega_{\tau\rho} + \mathcal{O}(\omega^2) \stackrel{!}{=} g_{\rho\tau} \tag{261}$$

From this, it follows that

$$\omega_{\rho\tau} = -\omega_{\tau\rho} \tag{262}$$

This implies that $\omega_{\alpha\beta}$ is antisymmetric, reducing the 16 components to 6 independent parameters. An element of the Lorentz algebra can be written as

$$\omega^{\mu}{}_{\nu} = g^{\alpha\mu}\delta^{\beta}{}_{\nu}\omega_{\alpha\beta} \tag{263}$$

Due to the antisymmetry of $\omega_{\alpha\beta}$, we also have

$$\omega^{\mu}{}_{\nu} = \frac{1}{2}(g^{\alpha\mu}\delta^{\beta}{}_{\nu} - g^{\beta\mu}\delta^{\alpha}{}_{\nu})\omega_{\alpha\beta} \tag{264}$$

Thus, we have made the expression inside the brackets antisymmetric as well. We can now define the basis elements of the Lorentz algebra as

$$(M^{\alpha\beta})^{\mu}{}_{\nu} = i(g^{\alpha\mu}\delta^{\beta}{}_{\nu} - g^{\beta\mu}\delta^{\alpha}{}_{\nu}) \tag{265}$$

and express the Lie algebra in terms of this basis:

$$\omega^{\mu}{}_{\nu} = -\frac{i}{2}(M^{\alpha\beta})^{\mu}{}_{\nu}\omega_{\alpha\beta} \tag{266}$$

Here, we have used the standard representation (the 4×4 representation) of the Lorentz algebra.
Next, we can calculate the Lie bracket, which will allow us to determine a representation-independent property of the Lie algebra:

$$\begin{aligned}
[M^{\alpha\beta}, M^{\gamma\delta}]^{\mu}{}_{\nu} &= (M^{\alpha\beta})^{\mu}{}_{\rho}(M^{\gamma\delta})^{\rho}{}_{\nu} - (M^{\gamma\delta})^{\mu}{}_{\rho}(M^{\alpha\beta})^{\rho}{}_{\nu} \\
&= i^2 \left(g^{\alpha\mu}\delta^{\beta}{}_{\rho} - g^{\beta\mu}\delta^{\alpha}{}_{\rho}\right)\left(g^{\gamma\rho}\delta^{\delta}{}_{\nu} - g^{\delta\rho}\delta^{\gamma}{}_{\nu}\right) - i^2\left(g^{\gamma\mu}\delta^{\delta}{}_{\rho} - g^{\delta\mu}\delta^{\gamma}{}_{\rho}\right)\left(g^{\alpha\rho}\delta^{\beta}{}_{\nu} - g^{\beta\rho}\delta^{\alpha}{}_{\nu}\right) \\
&= -\left(g^{\alpha\mu}\delta^{\beta}{}_{\rho}g^{\gamma\rho}\delta^{\delta}{}_{\nu} - g^{\alpha\mu}\delta^{\beta}{}_{\rho}g^{\delta\rho}\delta^{\gamma}{}_{\nu} - g^{\beta\mu}\delta^{\alpha}{}_{\rho}g^{\gamma\rho}\delta^{\delta}{}_{\nu} + g^{\beta\mu}\delta^{\alpha}{}_{\rho}g^{\delta\rho}\delta^{\gamma}{}_{\nu}\right) \\
&\quad + \left(g^{\gamma\mu}\delta^{\delta}{}_{\rho}g^{\alpha\rho}\delta^{\beta}{}_{\nu} - g^{\gamma\mu}\delta^{\delta}{}_{\rho}g^{\beta\rho}\delta^{\alpha}{}_{\nu} - g^{\delta\mu}\delta^{\gamma}{}_{\rho}g^{\alpha\rho}\delta^{\beta}{}_{\nu} + g^{\delta\mu}\delta^{\gamma}{}_{\rho}g^{\beta\rho}\delta^{\alpha}{}_{\nu}\right) \\
&= -\left(g^{\alpha\mu}g^{\beta\gamma}\delta^{\delta}{}_{\nu} - g^{\alpha\mu}g^{\beta\delta}\delta^{\gamma}{}_{\nu} - g^{\beta\mu}g^{\alpha\gamma}\delta^{\delta}{}_{\nu} + g^{\beta\mu}g^{\alpha\delta}\delta^{\gamma}{}_{\nu}\right) \\
&\quad + \left(g^{\gamma\mu}g^{\alpha\delta}\delta^{\beta}{}_{\nu} - g^{\gamma\mu}g^{\beta\delta}\delta^{\alpha}{}_{\nu} - g^{\delta\mu}g^{\alpha\gamma}\delta^{\beta}{}_{\nu} + g^{\delta\mu}g^{\beta\gamma}\delta^{\alpha}{}_{\nu}\right) \\
&= -g^{\beta\gamma}(g^{\alpha\mu}\delta^{\delta}{}_{\nu} - g^{\delta\mu}\delta^{\alpha}{}_{\nu}) + g^{\beta\delta}(g^{\alpha\mu}\delta^{\gamma}{}_{\nu} - g^{\gamma\mu}\delta^{\alpha}{}_{\nu}) \\
&\quad + g^{\alpha\gamma}(g^{\beta\mu}\delta^{\delta}{}_{\nu} - g^{\delta\mu}\delta^{\beta}{}_{\nu}) - g^{\alpha\delta}(g^{\beta\mu}\delta^{\gamma}{}_{\nu} - g^{\gamma\mu}\delta^{\beta}{}_{\nu}) \\
&= ig^{\beta\gamma}(M^{\alpha\delta})^{\mu}{}_{\nu} - ig^{\beta\delta}(M^{\alpha\gamma})^{\mu}{}_{\nu} - ig^{\alpha\gamma}(M^{\beta\delta})^{\mu}{}_{\nu} + ig^{\alpha\delta}(M^{\beta\gamma})^{\mu}{}_{\nu}
\end{aligned} \tag{267}$$

This leads to the following relation:

$$\boxed{i[M^{\alpha\beta}, M^{\gamma\delta}] = g^{\alpha\gamma}M^{\beta\delta} + g^{\beta\delta}M^{\alpha\gamma} - g^{\alpha\delta}M^{\beta\gamma} - g^{\beta\gamma}M^{\alpha\delta}} \tag{268}$$

To simplify this expression, we analyze the cases for temporal and spatial indices separately:

$$[M^{0i}, M^{0j}], \quad [M^{0i}, M^{jk}], \quad [M^{ij}, M^{kl}] \tag{269}$$

We now define the rotation generators J_i and the boost generators K_i as follows:

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk} \iff M^{ij} = \epsilon_{ijk}J_k, \qquad K_i = M^{0i} \tag{270}$$

The "if and only if"-symbol is justified by the following calculation:

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk} = \frac{1}{2}\epsilon_{ijk}\epsilon_{jkl}J_l = \frac{1}{2}\cdot 2\delta_{il}J_l = J_i \tag{271}$$

Thus, we obtain

$$[K_i, K_j] = [M^{0i}, M^{0j}] = i(-M^{ij} - (-M^{00})) \overset{M \text{ antisym.}}{=} -iM^{ij} = -i\epsilon_{ijk}J_k \tag{272}$$

$$\begin{aligned}
[J_i, K_j] &= \frac{1}{2}\epsilon_{ikl}[M^{kl}, M^{0j}] = \frac{1}{2}\epsilon_{ikl}i(-M^{l0} - (-M^{k0})) \overset{M \text{ antisym.}}{=} \frac{1}{2}i\epsilon_{ikl}(M^{0l} - M^{0k}) \\
&= i\epsilon_{ikl}M^{0l} = i\epsilon_{ikl}K_l
\end{aligned} \tag{273}$$

$$
\begin{aligned}
[J_i, J_j] &= \frac{1}{4}\epsilon_{ikl}\epsilon_{jmn}[M^{kl}, M^{mn}] \\
&= \frac{1}{4}i(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{im}\delta_{kn}\delta_{jl} + \delta_{in}\delta_{jk}\delta_{lm} - \delta_{im}\delta_{jk}\delta_{ln} - \delta_{ij}\delta_{kn}\delta_{lm} - \delta_{in}\delta_{km}\delta_{jl}) \\
&\quad (-\delta^{kn}M^{lm} - \delta^{lm}M^{kn} - (-\delta^{km}M^{ln}) - (-\delta^{ln}M^{km})) \\
&= \frac{1}{4}i(-\delta_{ij}\delta_{lm}M^{lm} - 3\delta_{im}\delta_{jl}M^{lm} - \delta_{ij}\delta_{lm}M^{lm} + \delta_{im}\delta_{jl}M^{lm} + 3\delta_{ij}\delta_{lm}M^{lm} + \delta_{im}\delta_{jl}M^{lm} \\
&\quad - \delta_{ij}\delta_{kn}M^{kn} - \delta_{ij}\delta_{kn}M^{kn} - 3\delta_{in}\delta_{jk}M^{kn} + \delta_{in}\delta_{jk}M^{kn} + 3\delta_{ij}\delta_{kn}M^{kn} + \delta_{in}\delta_{jk}M^{kn} \\
&\quad + 3\delta_{ij}\delta_{ln}M^{ln} + \delta_{in}\delta_{jl}M^{ln} + \delta_{in}\delta_{jl}M^{ln} - \delta_{ij}\delta_{ln}M^{ln} - \delta_{ij}\delta_{ln}M^{ln} - 3\delta_{in}\delta_{jl}M^{ln} \\
&\quad + 3\delta_{ij}\delta_{km}M^{km} + \delta_{im}\delta_{jk}M^{km} + \delta_{im}\delta_{jk}M^{km} - 3\delta_{im}\delta_{jk}M^{km} - \delta_{ij}\delta_{km}M^{km} - \delta_{ij}\delta_{km}M^{km}) \\
&\overset{M\text{ antisym.}}{=} \frac{1}{4}i(-3\delta_{im}\delta_{jl}M^{lm} + \delta_{im}\delta_{jl}M^{lm} + \delta_{im}\delta_{jl}M^{lm} \\
&\quad - 3\delta_{in}\delta_{jk}M^{kn} + \delta_{in}\delta_{jk}M^{kn} + \delta_{in}\delta_{jk}M^{kn} \\
&\quad + \delta_{in}\delta_{jl}M^{ln} + \delta_{in}\delta_{jl}M^{ln} - 3\delta_{in}\delta_{jl}M^{ln} \\
&\quad + \delta_{im}\delta_{jk}M^{km} + \delta_{im}\delta_{jk}M^{km} - 3\delta_{im}\delta_{jk}M^{km}) \\
&= \frac{1}{4}i(-\delta_{im}\delta_{jl}M^{lm} - \delta_{in}\delta_{jk}M^{kn} - \delta_{in}\delta_{jl}M^{ln} - \delta_{im}\delta_{jk}M^{km}) = -\frac{1}{4}i\delta_{im}\delta_{jl}M^{lm} \\
&\overset{M\text{ antisym.}}{=} -\frac{1}{2}i\delta_{im}\delta_{jl}(M^{lm} - M^{ml}) = \frac{1}{2}i(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})M^{lm} = \frac{1}{2}i\epsilon_{ijk}\epsilon_{klm}M^{lm} \\
&= i\epsilon_{ijk}J_k
\end{aligned}
\tag{274}
$$

In summary, we have

$$
\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk}J_k \\
[J_i, K_j] &= i\epsilon_{ikl}K_l \\
[K_i, K_j] &= -i\epsilon_{ijk}J_k
\end{aligned}
\tag{275}
$$

To move towards the simply connected covering group of the Lorentz group, we use the exponential map. From Equation (266), we obtain

$$\Lambda = e^{-\frac{i}{2}M^{\alpha\beta}\omega_{\alpha\beta}} \tag{276}$$

We define, analogous to Equation (270):

$$\Phi^i = \frac{1}{2}\epsilon_{ijk}\omega_{jk} \iff \omega_{ij} = \epsilon_{ijk}\Phi^k, \qquad \mathrm{X}^i = \omega_{0i} \tag{277}$$

and thus find

$$\Lambda = e^{-i\vec{\Phi}\cdot\vec{J} - i\vec{\mathrm{X}}\cdot\vec{K}} \tag{278}$$

Here, $\vec{\Phi}$ and $\vec{\mathrm{X}}$ should not be mixed up with the rotation vector $\vec{\varphi}$ and the boost vector $\vec{\chi}$, since in general $e^{-i\vec{\Phi}\cdot\vec{J} - i\vec{\mathrm{X}}\cdot\vec{K}} \neq e^{-i\vec{\Phi}\cdot\vec{J}}e^{-i\vec{\mathrm{X}}\cdot\vec{K}}$. However, $\vec{\Phi}$ and $\vec{\mathrm{X}}$ are linear combinations of $\vec{\varphi}$ and $\vec{\chi}$.

Here, J_i and K_i are general basis elements of the Lorentz algebra. By choosing a representation of the Lorentz algebra and using the first statement of Theorem 4.1, we obtain a representation of the simply connected covering group of the Lorentz group. For example, the Lie brackets (275) can be solved by $\pi'(J_i) = \frac{1}{2}\sigma_i$ and $\pi'(K_i) = \frac{i}{2}\sigma_i$. This provides a representation of $SL(2,\mathbb{C})$, which is expected since $SL(2,\mathbb{C})$ is the universal covering group of $\mathcal{L}_+^\uparrow$.

Next, we move on to the generators of Lorentz transformations acting on spacetime-dependent objects, specifically fields, drawing inspiration from Eichmann (2020). We start with the simplest case: a scalar field. We have

$$\begin{aligned} \psi(\Lambda^{-1}x) &\overset{(183)}{=} \psi'(x) = e^{-\frac{i}{2}\omega_{\mu\nu}M_L^{\mu\nu}}\psi(x) \\ &= \left(1 - \frac{i}{2}\omega_{\mu\nu}M_L^{\mu\nu}\right)\psi(x) \end{aligned} \tag{279}$$

where we consider the transformation near the identity and have linearized the exponential map. Rearranging the terms, we get

$$\psi(x) - \psi(\Lambda^{-1}x) = \frac{i}{2}\omega_{\mu\nu}M_L^{\mu\nu}\psi(x) \tag{280}$$

To find $M^{\mu\nu}$, we must rewrite the left-hand side in the appropriate form:

$$\psi(x) - \psi(\Lambda^{-1}x) = \underbrace{\delta x_\mu}_{\omega_{\mu\nu}x^\nu}\partial^\mu\psi(x) \tag{281}$$

where δx_μ is an infinitesimal Lorentz transformation, and by Equation (260), we have $\delta x_\mu = \omega_{\mu\nu}x^\nu$. Thus, we obtain

$$\begin{aligned} \psi(x) - \psi(\Lambda^{-1}x) &= \omega_{\mu\nu}x^\nu\partial^\mu\psi(x) \overset{\omega \text{ antisym.}}{=} \frac{1}{2}(\omega_{\mu\nu} - \omega_{\nu\mu})x^\nu\partial^\mu\psi(x) \\ &= \frac{1}{2}\omega_{\mu\nu}(x^\nu\partial^\mu - x^\mu\partial^\nu)\psi(x) \end{aligned} \tag{282}$$

Therefore, one gets

$$\boxed{M_L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)} \tag{283}$$

To distinguish these generators from the previous ones, we denote them with the subscript L and refer to the earlier generators with the subscript S.

We can now decompose this generator into rotation and boost generators using Equation (270):

$$(J_L)_i = \frac{1}{2}\epsilon_{ijk}M_L^{jk} = \frac{1}{2}i\epsilon_{ijk}(x^j\partial^k - x^k\partial^j) = i\epsilon_{ijk}x^j\partial^k \overset{P^\mu = i\partial^\mu}{=} \epsilon_{ijk}x^jP^k = \vec{x}\times\vec{P} \tag{284}$$

$$(K_L)_i = M_L^{0i} = i(x^0\partial^i - x^i\partial^0) \overset{P^\mu = i\partial^\mu}{=} x^0P^i - x^iP^0 \tag{285}$$

J_L represents the classical orbital angular momentum of the system, while K_L is its counterpart for boosts. This generator connects the time coordinate with the momentum (position generator) and the spatial coordinate with the energy (time generator).

For a general field, the generator of Lorentz transformations is given by

$$M^{\mu\nu} = M_S^{\mu\nu} + M_L^{\mu\nu} \tag{286}$$

The generator consists of an intrinsic part and an orbital part, both for rotations and boosts. The orbital part was already known, while the intrinsic part arises from the higher-dimensional spinor representation.

5 Outlook

Until now, we have focused exclusively on Lorentz transformations, which preserve the scalar product. However, our initial requirement was the constancy of the speed of light. It can be shown that this condition also holds when spacetime translations are included. In fact, this is the most general transformation that satisfies the principles of relativity. This broader set of transformations is known as the Poincaré transformations.

Definition 5.1 (Poincaré Transformation)**.** *A Poincaré transformation* $\Pi = (\Lambda, a)$*, where* $\Lambda \in \mathcal{L}$ *and* $a \in \mathbb{R}^4$ *is a mapping from* $\mathbb{R}^4$ *to* $\mathbb{R}^4$ *which is defiend as*

$$\Pi(x) = \Lambda x + a \tag{287}$$

The set of all Poincaré transformations forms a group, known as the Poincaré group, denoted by $\mathcal{P}$.

- Closure:
  Let $(\Lambda_1, a_1), (\Lambda_2, a_2) \in \mathcal{P}$.

$$x''^{\mu} = (\Lambda_2)^{\mu}{}_{\rho} x'^{\rho} + a_2^{\mu} = (\Lambda_2)^{\mu}{}_{\rho}((\Lambda_1)^{\rho}{}_{\nu} x^{\nu} + a_1^{\rho}) + a_2^{\mu} = (\Lambda_2)^{\mu}{}_{\rho}(\Lambda_1)^{\rho}{}_{\nu} x^{\nu} + (\Lambda_2)^{\mu}{}_{\rho} a_1^{\rho} + a_2^{\mu} \tag{288}$$

$$\implies \Lambda^{\mu}{}_{\nu} = (\Lambda_2)^{\mu}{}_{\rho}(\Lambda_1)^{\rho}{}_{\nu}, \quad a^{\mu} = (\Lambda_2)^{\mu}{}_{\nu} a_1^{\nu} + a_2^{\mu} \tag{289}$$

  Therefore:

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \tag{290}$$

  Since $\Lambda_2\Lambda_1 \in \mathcal{L}$ and $\Lambda_2 a_1 + a_2 \in \mathbb{R}^4$, it follows that $(\Lambda_2, a_2)(\Lambda_1, a_1) \in \mathcal{P}$.

- Associativity:
  Let $(\Lambda_1, a_1), (\Lambda_2, a_2), (\Lambda_3, a_3) \in \mathcal{P}$.

$$(\Lambda_1, a_1)((\Lambda_2, a_2)(\Lambda_3, a_3)) = (\Lambda_1, a_1)(\Lambda_2\Lambda_3, \Lambda_2 a_3 + a_2) = (\Lambda_1\Lambda_2\Lambda_3, \Lambda_1\Lambda_2 a_3 + \Lambda_1 a_2 + a_1) \tag{291}$$

$$((\Lambda_1, a_1)(\Lambda_2, a_2))(\Lambda_3, a_3) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)(\Lambda_3, a_3) = (\Lambda_1\Lambda_2\Lambda_3, \Lambda_1\Lambda_2 a_3 + \Lambda_1 a_2 + a_1) \tag{292}$$

  Thus:

$$(\Lambda_1, a_1)((\Lambda_2, a_2)(\Lambda_3, a_3)) = ((\Lambda_1, a_1)(\Lambda_2, a_2))(\Lambda_3, a_3) \tag{293}$$

- Identity Element:
  $(\mathbb{1}, 0) \in \mathcal{P}$ since $\mathbb{1} \in \mathcal{P}$ and $0 \in \mathcal{L}$

$$(\mathbb{1}, 0)(\Lambda, a) = (\Lambda, a) = (\Lambda, a)(\mathbb{1}, 0) \tag{294}$$

- Inverse Element:
  $(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a) \in \mathcal{P}$ since $\Lambda^{-1} \in \mathcal{P}$ and $-\Lambda^{-1}a \in \mathcal{L}$

$$(\Lambda, a)^{-1}(\Lambda, a) = (\Lambda^{-1}, -\Lambda^{-1}a)(\Lambda, a) = (\Lambda^{-1}\Lambda, \Lambda^{-1}a + (-\Lambda^{-1}a)) = (\mathbb{1}, 0) \tag{295}$$

$$(\Lambda, a)(\Lambda^{-1}, -\Lambda^{-1}a) = (\Lambda\Lambda^{-1}, \Lambda(-\Lambda^{-1}a) + a) = (\mathbb{1}, 0) \tag{296}$$

It can be shown that the Poincaré group is also a Lie group.
In the previous chapter, we established the connection between rotations and boosts and their corresponding generators, starting from the Lorentz algebra. Now, by considering the Poincaré group, one can construct its associated Lie algebra, known as the Poincaré algebra. This allows to find the generators also for time and space translations, which turn out to be the Dirac Hamiltonian

$$H = -i\vec{\alpha} \cdot \vec{\nabla} + \beta m \qquad \text{where } \beta = \gamma^0 \text{ and } \alpha^i = \gamma^0\gamma^i = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \tag{297}$$

and the momentum operator

$$\vec{P} = -i\vec{\nabla} \tag{298}$$

In this way, one would obtain a direct connection between the space-time symmetries and the time evolution of corresponding spinor fields, among the Dirac evolution, on the Dirac evolution, on the basis of a mathematical represantation theoretic discussion only.
Because of the appearance of derivatives in the generators, e.g. (283), (297) or (298), a mathematical rigorous discussion requires functional analysis and is beyond the scope of this thesis.

A Appendix

For the axioms and principles of quantum mechanics, we follow (Woit, 2024, Chapter 1.2):

Axiom A.1 (States)**.** *The state of a quantum mechanical system is given by a non-zero vector in a complex vector space $\mathcal{H}$ with Hermitian inner product $\langle \cdot, \cdot \rangle$.*

Axiom A.2 (Quantum Observable)**.** *The observables of a quantum mechanical system are given by self-adjoint linear operators on $\mathcal{H}$.*

Axiom A.3 (Dynamics)**.** *There is a distinguished quantum observable, the Hamiltonian H. Time evolution of states $|\psi(t)\rangle \in H$ is given by the Schrödinger equation*

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \tag{299}$$

The operator H has eigenvalues that are bounded below.

Principle A.4. *States for which the value of an observable can be characterized by a well-defined number are the states that are eigenvectors for the corresponding self-adjoint operator. The value of the observable in such a state will be a real number, the eigenvalue of the operator.*

Principle A.5. *Given an observable O and two unit-norm states $|\psi_1\rangle$ and $|\psi_2\rangle$ that are eigenvectors of O with distinct eigenvalues λ_1 and λ_2*

$$O|\psi_1\rangle = \lambda_1|\psi_1\rangle, \quad O|\psi_2\rangle = \lambda_2|\psi_2\rangle \tag{300}$$

the complex linear combination state

$$c_1|\psi_1\rangle + c_2|\psi_2\rangle \tag{301}$$

will not have a well-defined value for the observable O. If one attempts to measure this observable, one will get either λ_1 or λ_2, with probabilities

$$\frac{|c_1^2|}{|c_1^2| + |c_2^2|} \tag{302}$$

and

$$\frac{|c_2^2|}{|c_1^2| + |c_2^2|} \tag{303}$$

respectively.

One sees that all vectors

$$|\hat{\psi}\rangle = \{\lambda|\psi\rangle \mid \lambda \in \mathbb{C}, \psi \in \mathcal{H}\} \tag{304}$$

correspond to the same state of the physical system because they lead to the same observable predictions. $\hat{\psi}$ is called a ray. The set of all rays forms a projective space, denoted by $\hat{\mathcal{H}}$.

Definition A.1 (Projective Group Representation). *A projective representation of a group G is a pair (V, ρ) of a vector space V and a homomorphism*

$$\rho : G \to PGL(V),\ \ g \mapsto \rho(g) \tag{305}$$

where PGL is the projective general linear group.

A projective representation can be understood as a set of maps $\pi(g) \in GL(V)$ such that $\forall g, h \in G$:

$$\rho(g)\rho(h) = c(g, h)\rho(g * h) \tag{306}$$

with some phase factor $c(g, h) \in \{e^{i\theta} \mid \theta \in \mathbb{R}\}$.

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