The Computational Content of the Brouwer Fixed Point Theorem Revisited

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joint work with

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Outline

The Brouwer Fixed Point Theorem

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Theorem (Brouwer 1911)

Every continuous function $f : [0,1]^n \to [0,1]^n$ *has a fixed point* x*, i.e., a point* $x \in [0,1]^n$ *with* $f(x) = x$*.*

Luitzen E.J. Brouwer (1881-1966)

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- By $\mathrm{BFT}_n : \mathcal{C}_n \rightrightarrows [0,1]^n$ we denote the operation defined by $\mathrm{BFT}_n(f) := \{x \in [0,1]^n : f(x) = x\}$ for $n \in \mathbb{N}$.

Theorem (Orevkov 1963, Baigger 1985)

There exists a computable function $f : [0,1]^2 \to [0,1]^2$ that has no computable fixed point $x \in [0,1]^2$.

- The proof is essentially based on a reduction to a Kleene tree (equivalently, to the existence of two computably inseparable c.e. sets).

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Reverse Mathematics

Theorem (Simpson 1999)

Over RCA_0 the following are equivalent in second order arithmetic:

- *Weak Kőnig's Lemma WKL_0.*
- *The Brouwer Fixed Point Theorem.*

- Neither uniform nor resource sensitive!

Theorem (Ishihara 2006)

Using intuitionistic logic the following are equivalent:

- *Weak Kőnig's Lemma WKL.*
- *The Lesser Limited Principle of Omniscience LLPO.*
- *The Intermediate Value Theorem.*
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- Very uniform, but even less resource sensitive!

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The Weihrauch Lattice

Mathematical Problems

- We consider partial multi-valued functions $f :\subseteq X \rightrightarrows Y$ as mathematical problems.
- We assume that the underlying spaces X and X are represented spaces, hence notions of computability and continuity are well-defined.
- Every theorem of the form

$$(\forall x \in X)(\exists y \in Y)(x \in D \Longrightarrow P(x, y))$$

can be identified with $F :\subseteq X \rightrightarrows Y$ with $\mathrm{dom}(F) := D$ and $F(x) := \{y \in Y : P(x, y)\}$.
- Weak Kőnig's Lemma is the mathematical problem

$$\mathsf{WKL} :\subseteq \mathrm{Tr} \rightrightarrows 2^{\mathbb{N}}, T \mapsto [T]$$

with $\mathrm{dom}(\mathsf{WWKL}) := \{T \in \mathrm{Tr} : T \text{ infinite}\}$.

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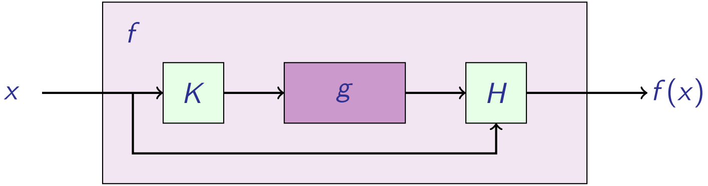
Weihrauch Reducibility

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be two mathematical problems.



- f is called Weihrauch reducible to g, in symbols $f \leq_{\mathrm{W}} g$, if there are computable $H :\subseteq X \times W \rightrightarrows Y$ and $K :\subseteq X \rightrightarrows Z$ such that $H(\mathrm{id}, gK) \subseteq f$ and $\mathrm{dom}(f) \subseteq \mathrm{dom}(H(\mathrm{id}, gK))$.
- f is called strongly Weihrauch reducible to g, in symbols $f \leq_{\mathrm{sW}} g$, if there are computable $H :\subseteq W \rightrightarrows Y$ and $K :\subseteq X \rightrightarrows Z$ such that $HgK \subseteq f$ and $\mathrm{dom}(f) \subseteq \mathrm{dom}(HgK)$.

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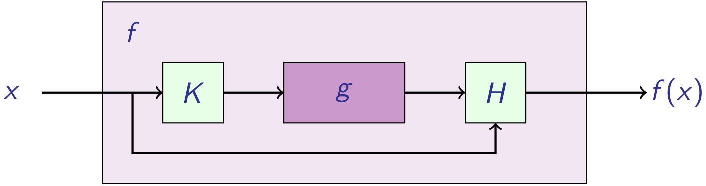

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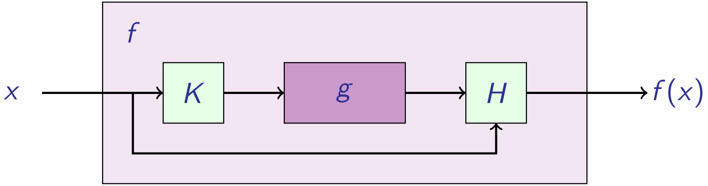

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Algebraic Operations in the Weihrauch Lattice

Definition

Let f, g be two mathematical problems. We consider:

- $f \times g$: both problems are available in parallel (Product)
- $f \sqcup g$: both problems are available, but for each instance one has to choose which one is used (Coproduct)
- $f \sqcap g$: given an instance of f and g, only one of the solutions will be provided (Sum)
- $f * g$: f and g can be used consecutively (Comp. Product)
- $g \to f$: this is the simplest problem h such that f can be reduced to $g * h$ (Implication)
- f^*: f can be used any given finite number of times in parallel (Star)
- $\widehat{f}$: f can be used countably many times in parallel (Parallelization)
- f': f can be used on the limit of the input (Jump)

Some Formal Definitions

Definition

For $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq W \rightrightarrows Z$ we define:

- $f \times g :\subseteq X \times W \rightrightarrows Y \times Z$, $(x, w) \mapsto f(x) \times g(w)$ (Product)
- $f \sqcup g :\subseteq X \sqcup W \rightrightarrows Y \sqcup Z$, $z \mapsto \begin{cases} f(z) \text{ if } z \in X \\ g(z) \text{ if } z \in W \end{cases}$ (Coproduct)
- $f \sqcap g :\subseteq X \times W \rightrightarrows Y \sqcup Z$, $(x, w) \mapsto f(x) \sqcup g(w)$ (Sum)
- $f^* :\subseteq X^* \rightrightarrows Y^*$, $f^* = \bigsqcup_{i=0}^{\infty} f^i$ (Star)
- $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$, $\widehat{f} = \mathsf{X}_{i=0}^{\infty} f$ (Parallelization)

- Weihrauch reducibility induces a lattice with the coproduct $\sqcup$ as supremum and the sum $\sqcap$ as infimum.
- Parallelization and star operation are closure operators in the Weihrauch lattice.
- With $\sqcup, \times, ^*$ one obtains a Kleene algebra.
- The Weihrauch lattice is neither a Brouwer nor a Heyting algebra (Higuchi und Pauly 2012).

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Choice

Definition

By C_X we denote the choice problem of a space X, i.e., the problem given a closed subset $A \subseteq X$ to find a point in A.

- CC_X is the C_X restricted to connected sets.
- PWCC_X is C_X restricted to pathwise connected sets.
- XC_X is C_X restricted to convex sets.
- PC_X is C_X restricted to sets of positive measure.

Example

- $\mathsf{C}_2 \equiv_{\mathrm{sW}} \mathsf{LLPO}$,
- $\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \widehat{\mathsf{LLPO}}$,
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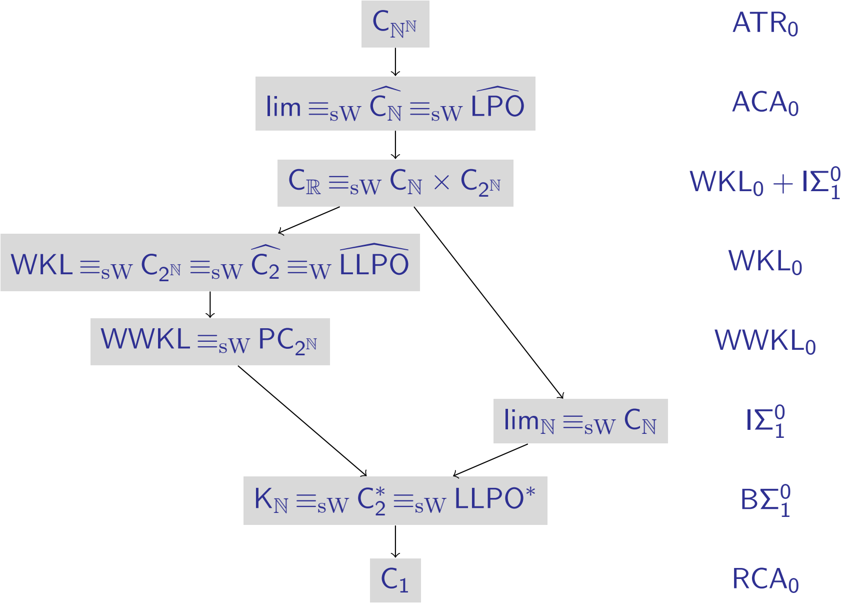
Basic Complexity Classes and Reverse Mathematics



The Classification

The Brouwer Fixed Point Theorem

Theorem

$\mathrm{BFT}_n \equiv_{\mathrm{sW}} \mathrm{CC}_{[0,1]^n}$ *for all* $n \in \mathbb{N}$.

Proof. (Sketch) "$\geq_{\mathrm{sW}}$".

- Given a connected closed set $\emptyset \neq A \subseteq [0,1]^n$ we determine a continuous function $f : [0,1]^n \to [0,1]^n$ that has exactly A as its set of fixed points.
- We use a compactly decreasing sequence (A_i) of bi-computable, effectively path-connected closed sets A_i such that $A = \bigcap_{i=0}^{\infty} A_i$.
- We use the sequence (A_i) to construct functions $g_i : [0,1]^n \to [0,1]^n$ and $f := \mathrm{id} + 2^{-4} \sum_{i=0}^{\infty} g_i$ with the property that A is the set of fixed points of f.
- Note: with a lot of careful extra calculations one can even construct f such that it has Lipschitz constant $L = 6$.

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Proof. (Sketch) "$\leq_{\mathrm{sW}}$".

- Given f we can compute $A = (f - \mathrm{id}_{[0,1]^n})^{-1}\{0\}$.
- It is sufficient to find a connectedness component of A.
- Using a tree of rational complexes we can find such a component since $\mathrm{ind}(f, R)$ is computable for rational complexes R (Joe S. Miller 2002).

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The Brouwer Fixed Point Theorem

Theorem

$\mathsf{BFT}_n \equiv_{\mathrm{sW}} \mathsf{CC}_{[0,1]^n}$ *for all* $n \in \mathbb{N}$.

- How does this equivalence class depend on $n \in \mathbb{N}$?

Proposition (Intermediate Value Theorem)

$\mathsf{BFT}_1 \equiv_{\mathrm{sW}} \mathsf{CC}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{IVT}$.

- It is clear that

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  holds for all $n \in \mathbb{N}$.
- Is this reduction chain strictly increasing?

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A Geometric Construction

Theorem

$\mathsf{CC}_{[0,1]^n} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]}$ *for all* $n \geq 3$.

Proof. The map

$$A \mapsto (A \times [0,1] \times \{0\}) \cup (A \times A \times [0,1]) \cup ([0,1] \times A \times \{1\})$$

is computable and maps *any* non-empty closed $A \subseteq [0,1]$ to a *connected* non-empty closed $B \subseteq [0,1]^3$. Given a point $(x, y, z) \in B$, one can find a point in A, in fact, $x \in A$ or $y \in A$ and which one is true can be determined with z. □

Corollary

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Corollary (Orevkov 1963, Baigger 1985)

There exists a computable function $f : [0,1]^2 \to [0,1]^2$ *that has no computable fixed point* $x \in [0,1]^2$.

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An Inverse Limit Construction for Dimension 2

Theorem

$\mathsf{CC}_{[0,1]^2} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]}$.

Proof. (Sketch) In fact we use $\mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \widehat{\mathsf{LLPO}}$ and given an instance p of this problem, we construct a connected set

$$\begin{aligned} A &= \{x \in B_0 : (\forall n \in \mathbb{N})\ f_{n-1}^{-1} \circ \ldots \circ f_0^{-1}(x) \in E_n(p)\} \subseteq [0,1]^2 \\ &= \bigcap_{n=0}^{\infty} (f_0 \circ \ldots \circ f_{n-1})(E_n(p)) \end{aligned}$$

so that a point $x \in A$ allows us to compute a solution to $\widehat{\mathsf{LLPO}}(p)$.

Here the $f_n : B_{n+1} \hookrightarrow S_n$ are computable embeddings of certain blocks B_n into certain "snakes" $S_n \subseteq B_n$. The set $E_n(p) \subseteq B_n$ reflects the information given in a certain portion of p.

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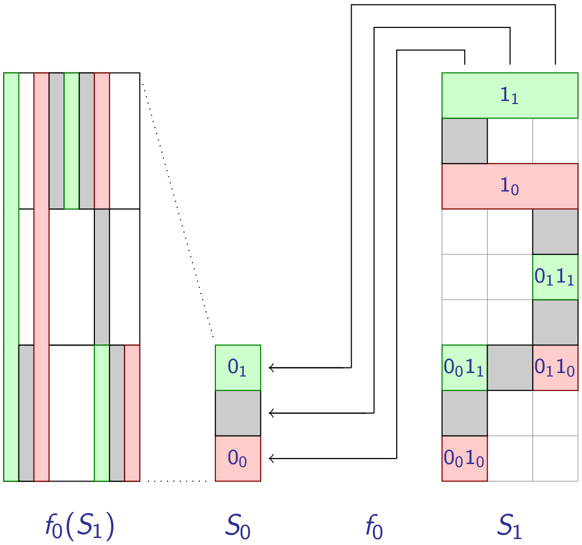

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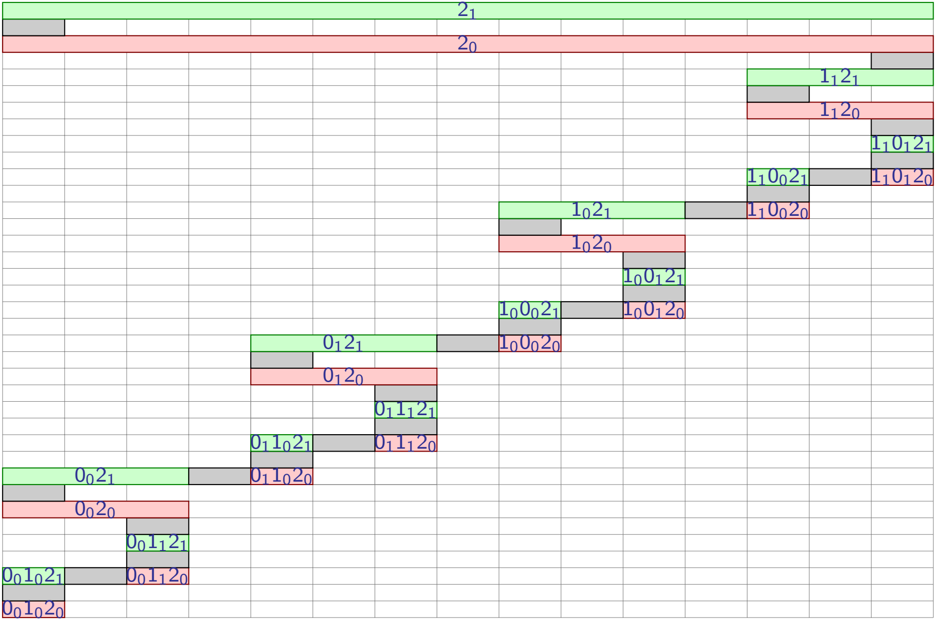

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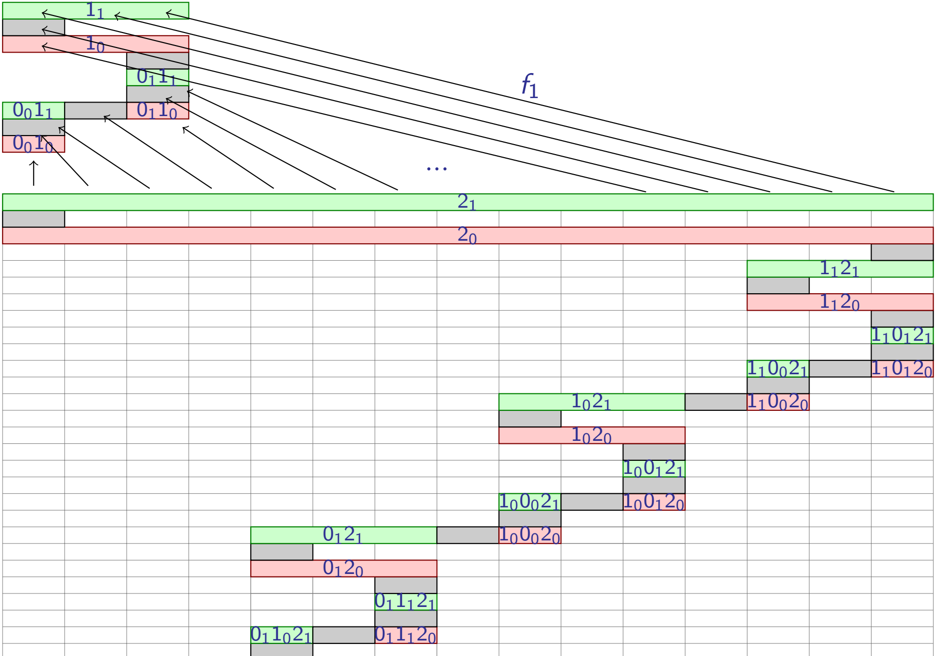

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- The set A constructed by the inverse limit construction is not pathwise connected in general.

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$\mathsf{PWCC}_{[0,1]^2} <_{\mathrm{sW}} \mathsf{PWCC}_{[0,1]^3}$?

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By $\mathrm{BFT}_{n,L}$ we denote the Brouwer Fixed Point Theorem restricted to maps that are Lipschitz continuous with constant L.

- $\mathrm{BFT}_{n,L}$ with $L < 1$ is the Banach Fixed Point Theorem (for contractions $f : [0,1]^n \to [0,1]^n$) and hence computable.
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$\mathrm{BFT}_{n,L} \equiv_{\mathrm{sW}} \mathrm{BFT}_n$ *for all* $n \in \mathbb{N}$ *and* $L > 1$.

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Theorem (Eike Neumann 2015)

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Hence we have a trichotomy for the Brouwer Fixed Point Theorem depending on the Lipschitz constant for $n \geq 2$:

- For $L < 1$ it is computable.
- For $L = 1$ it gets increasingly more difficult with increasing dimension n.
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Fixed Point Theorems in the Weihrauch Lattice

$\mathsf{BFT}_{n+2} \equiv_{\mathrm{sW}} \mathsf{BFT}_{n+2,L>1} \equiv_{\mathrm{sW}} \mathsf{CC}_{[0,1]^{n+2}} \equiv_{\mathrm{sW}} \mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \widehat{\mathsf{LLPO}}$ — Brouwer Fixed Point Theorem

↓

⋮

↓

$\mathsf{BFT}_{3,L=1} \equiv_{\mathrm{W}} \mathsf{XC}_{[0,1]^3}$

↓ — Brouwder-Göhde-Kirk Fixed Point Theorem

$\mathsf{BFT}_{2,L=1} \equiv_{\mathrm{W}} \mathsf{XC}_{[0,1]^2}$

↓

$\mathsf{BFT}_1 \equiv_{\mathrm{sW}} \mathsf{BFT}_{1,L\geq 1} \equiv_{\mathrm{sW}} \mathsf{IVT} \equiv_{\mathrm{sW}} \mathsf{CC}_{[0,1]} = \mathsf{XC}_{[0,1]}$ — Intermediate Value Theorem

↓

$\mathsf{K}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_2^* \equiv_{\mathrm{sW}} \mathsf{LLPO}^*$

↓

$\mathsf{BFT}_{n,L<1} \equiv_{\mathrm{W}} \mathsf{C}_1$ — Banach Fixed Point Theorem

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