The space of located subsets

Tatsuji Kawai

Università di Padova

Second CORE meeting, 27 January 2017, LMU

The space of located subsets

We are interested in a point-free topology on the located subsets of some given structure.

Examples

- Extended Dedekind reals (L, U), i.e. extended with $+\infty, -\infty$.
  - $q \in U \iff (\exists q' < q)\, q' \in U$,
  - $p \in L \iff (\exists p' > p)\, p' \in L$,
  - $L \cap U = \emptyset$,
  - $p < q \implies p \in L \vee q \in U$.

  An extended Dedekind reals (L, U) is equivalent to a located (possibly unbounded) upper real U.
  - $q \in U \iff (\exists q' < q)\, q' \in U$,
  - (locatedness) $p < q \implies p \notin U \vee q \in U$.

- Compact (including $\emptyset$) subsets of a compact metric space (X, d) with the Hausdorff metric whose values are in the extended reals.

$$d(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}.$$

Outline

1. Located subsets of continuous lattices
2. Examples of located subsets
3. The (point-free) space of located subsets
4. Universal property

Continuous covers (Continuous lattices)

An **continuous cover** is a structure $\mathcal{S} = (S, \lhd, \mathsf{wb})$ where $\lhd \subseteq S \times \mathsf{Pow}(S)$ is a **cover** satisfying

$$\frac{a \in U}{a \lhd U}, \qquad \frac{a \lhd U \quad U \lhd V}{a \lhd V},$$

$$U \lhd V \overset{\text{def}}{\iff} (\forall a \in U)\, a \lhd V,$$

and wb is function $\mathsf{wb}\colon S \to \mathsf{Pow}(S)$ such that

1. $a \lhd \mathsf{wb}(a)$,
2. $(\forall b \in \mathsf{wb}(a))\, b \ll a$.

Here, $\ll$ is the **way-below** relation

$$b \ll a \overset{\text{def}}{\iff} \forall U \in \mathsf{Pow}(S)\, [a \lhd U \to (\exists U_0 \in \mathsf{Fin}(U))\, b \lhd U_0] .$$

Note that $a \ll b \iff (\exists A \in \mathsf{Fin}(S))\, a \lhd A \;\&\; A \subseteq \mathsf{wb}(b)$, so we can safely use the notation $\ll$ without compromising predicativity.

The $Sat(\mathcal{S}) \overset{\text{def}}{=} \{\mathcal{A}\, U \mid U \subseteq S\}$ where $\mathcal{A}\, U = \{a \in S \mid a \lhd U\}$ forms a **continuous lattice** with a base $\{\mathcal{A}\, B \mid B \in \mathsf{Fin}(S)\}$.

Fix a continuous cover $\mathcal{S} = (S, \lhd, \mathrm{wb})$. A subset $V \subseteq S$ is **splitting** if

$$a \lhd U \;\&\; a \in V \implies (\exists b \in U)\, b \in V.$$

Lemma. A subset $V \subseteq S$ is splitting iff

1. $a \lhd \{a_0, \ldots, a_{n-1}\} \;\&\; a \in V \implies (\exists i < n)\, a_i \in V$,
2. $a \in V \implies (\exists b \ll a)\, b \in V$.

Proof. ($\Rightarrow$) 1 is trivial. 2 is by $a \lhd \mathrm{wb}(a)$ and $b \in \mathrm{wb}(a) \implies b \ll a$.
($\Leftarrow$) Assume 1 & 2. Then $a \lhd U \;\&\; a \in V \overset{\text{by 2}}{\Longrightarrow} (\exists b \ll a)\, b \in V \overset{\text{def} \ll}{\Longrightarrow}$
$(\exists A \in \mathrm{Fin}(U))\, b \lhd A \overset{\text{by 1}}{\Longrightarrow} (\exists a \in A)\, a \in V$. □

A splitting subset $V \subseteq S$ is a **located** if $a \ll b \implies a \notin V \vee b \in V$.

Lemma. A subset $V \subseteq S$ is located iff $a \in \mathrm{wb}(b) \implies a \notin V \vee b \in V$.

Proof. ($\Leftarrow$) Suppose $a \ll b$. Since
$a \ll b \iff (\exists A \in \mathrm{Fin}(S))\, a \lhd A \subseteq \mathrm{wb}(b)$, either $A \subseteq S \setminus V$ or $b \in V$.
Hence, $a \notin V \vee b \in V$. □

Examples

Examples of located subsets

Example (Scott topology on Pow$(\mathbb{N})$)

$\mathcal{P}\omega = (\mathsf{Fin}(\mathbb{N}), \lhd_\omega, \mathsf{wb})$ where

$$A \lhd_\omega U \overset{\text{def}}{\iff} (\exists B \in U)\, B \subseteq A,$$

$$\mathsf{wb}(A) \overset{\text{def}}{=} \{B \in \mathsf{Fin}(S) \mid A \subseteq B\}\,.$$

- $V \subseteq \mathsf{Fin}(\mathbb{N})$ is splitting iff it is closed downwards w.r.t. $\subseteq$.
- A splitting subset V is located iff it is detachable (NB. $A \ll A$).

A splitting subset V corresponds to a subset $\bigcup V \in \mathsf{Pow}(\mathbb{N})$.
A located subset of $\mathcal{P}\omega$ corresponds to a detachable subset of $\mathbb{N}$.

Examples of located subsets

Example (Scott topology on the bounded upper reals)

$\mathcal{R}^u = (\mathbb{Q}, \lhd_u, \mathsf{wb})$ where

$$q \lhd_u U \overset{\text{def}}{\iff} (\forall p < q)\left(\exists q' \in U\right) p < q',$$
$$\mathsf{wb}(q) \overset{\text{def}}{=} \{p \in \mathbb{Q} \mid p < q\}.$$

- $V \subseteq \mathbb{Q}$ is splitting iff it is an upper real, i.e.

$$q \in V \iff (\exists p < q)\, p \in V.$$

- A splitting subset V is located iff it is a located upper real (extended real), i.e. $p < q \implies p \notin V \vee q \in V$.

Examples of located subsets

Example (Binary tree $\mathcal{C}$ (Formal Cantor space))

$\mathcal{C} = (\{0,1\}^*, \lhd_{\mathcal{C}}, \mathsf{wb})$ where

$$a \lhd_{\mathcal{C}} U \overset{\text{def}}{\iff} (\exists k \in \mathbb{N})\,(\forall c \in a[k])\,(\exists b \in U)\, b \preccurlyeq c$$
$$\iff U \text{ is a uniform bar of } a.$$

$$a[k] \overset{\text{def}}{=} \{a * b \mid |b| = k\},$$
$$\mathsf{wb}(a) \overset{\text{def}}{=} \{b \in \{0,1\}^* \mid a \preccurlyeq b\}.$$

- $V \subseteq \{0,1\}^*$ is splitting iff $a \in V \iff (\exists i \in \{0,1\})\, a * \langle i \rangle \in V$.
- A splitting subset V is located iff it is detachable (NB. $a \ll a$), i.e. it is a (possibly empty) "spread".

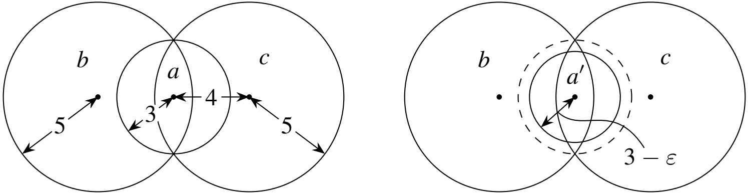
Examples of located subsets

Example (Locally compact metric spaces, Palmgren (2007))

Given a (Bishop) locally compact metric space (X, d), its **localic completion** is the continuous cover $\mathcal{M}(X) = (M_X, \lhd_X, \mathsf{wb})$ where

- $M_X \stackrel{\text{def}}{=} X \times \mathbb{Q}^{>0} = \left\{\mathsf{b}(x, \varepsilon) \mid x \in X \;\&\; \varepsilon \in \mathbb{Q}^{>0}\right\}$ with an order $\mathsf{b}(x, \varepsilon) <_X \mathsf{b}(y, \delta) \stackrel{\text{def}}{\iff} d(x, y) + \varepsilon < \delta.$
- $a \lhd_X U \stackrel{\text{def}}{\iff} (\forall b <_X a)\,(\exists A \in \mathsf{Fin}(U))\,\left(\exists \theta \in \mathbb{Q}^{>0}\right) b \sqsubset_\theta A,$ $b \sqsubset_\theta A \stackrel{\text{def}}{\iff} (\forall\, \mathsf{b}(x, \varepsilon) <_X b)\, \varepsilon < \theta \to (\exists a \in A)\, \mathsf{b}(x, \varepsilon) <_X a.$
- $\mathsf{wb}(a) \stackrel{\text{def}}{=} \{b \in M_X \mid b <_X a\}.$

Consider $\mathbb{R}^2$. We have $a \lhd_{\mathbb{R}^2} \{b, c\}$.



Examples of located subsets

Proposition

- *A splitting subset $V \subseteq M_X$ corresponds to a closed subset*

$$X_V \stackrel{\text{def}}{=} \{x \in X \mid (\forall\, \mathsf{b}(y,\delta) \in M_X)\, d(x,y) < \delta \to \mathsf{b}(y,\delta) \in V\}\,.$$

  A closed subset $Y \subseteq X$ corresponds to a splitting subset

$$V_Y \stackrel{\text{def}}{=} \{\mathsf{b}(x,\varepsilon) \in M_X \mid (\exists y \in Y)\, d(x,y) < \varepsilon\}\,.$$

  The correspondence is bijective.

- *(Coquand et al. (2011)) A splitting subset $V \subseteq M_X$ is located iff $X_V \subseteq X$ is semi-located, i.e. for each $x \in X$, the distance*

$$d(x, X_V) \stackrel{\text{def}}{=} \left\{q \in \mathbb{Q}^{>0} \mid (\exists y \in X_V)\, d(x,y) < q\right\}$$

  is a located upper real (we allow empty set to be semi-located).

The space of located subsets

A **geometric theory** $T = (P, R)$ over a set P of propositional symbols is a set R of axioms of the form

$$p_0 \wedge \cdots \wedge p_{n-1} \vdash \bigvee_{i \in I} q_0^i \wedge \cdots \wedge q_{n_i - 1}^i.$$

A **model** (ideal) of T is subset $\alpha \subseteq P$ such that

$$\{p_0, \ldots, p_{n-1}\} \subseteq \alpha \implies (\exists i \in I) \left\{q_0^i, \ldots, q_{n_i-1}^i\right\} \subseteq \alpha$$

for all axioms $p_0 \wedge \cdots \wedge p_{n-1} \vdash \bigvee_{i \in I} q_0^i \wedge \cdots \wedge q_{n_i-1}^i$ in T.

Problem. Given a continuous cover $\mathcal{S}$, find a geometric theory $T_{\mathcal{L}}$ whose models are the located subsets of $\mathcal{S}$.

Example (Theory of splitting subsets)

Let $\mathcal{S} = (S, \lhd, \mathrm{wb})$ be a continuous cover.
Recall that $V \subseteq S$ is splitting iff

1. $a \lhd \{a_0, \ldots, a_{n-1}\} \ \&\ a \in V \implies (\exists i < n)\, a_i \in V$,
2. $a \in V \implies (\exists b \ll a)\, b \in V$.

Thus, splitting subsets of $\mathcal{S}$ are the models of a geometric over S with the following axioms:

$$a \vdash \bigvee_{b \ll a} b, \qquad a \vdash \bigvee_{k<n} a_k \qquad (a \lhd \{a_0, \ldots, a_{n-1}\})$$

Non-example (Located subsets)

A locatedness $a \ll b \implies a \notin V \vee b \in V$ is not geometric.
A naive approach requires non-geometric axiom:

$$\top \vdash (a \to \bot) \vee b \qquad (a \ll b)$$

where $\top \overset{\text{def}}{=} \wedge\emptyset$.

But there is a way out ..., inspired by the following example

Example (Theory of extended Dedekind reals)

- Consider $\mathcal{R}^u = (\mathbb{Q}, \lhd_u, \text{wb})$ whose located subsets are the located (unbounded) upper reals.
- A located upper real is equivalent to an **extended Dedekind real** (L, U), a pair of disjoint lower and upper reals that is located: $p < q \implies p \in L \vee q \in U$.

Extended Dedekind reals are the models of a theory T_D over the propositional symbols $\{(p, +\infty) \mid p \in \mathbb{Q}\} \cup \{(-\infty, q) \mid q \in \mathbb{Q}\}$ with the following axioms:

$$(-\infty, q) \vdash \bigvee_{q' < q} (-\infty, q')$$

$$(-\infty, q) \vdash (-\infty, q') \qquad (q < q')$$

Dual axioms for $(p, +\infty)$ ....

$$(q, +\infty) \wedge (-\infty, q) \vdash \bot$$

$$\top \vdash (p, +\infty) \wedge (-\infty, q) \qquad (p < q),$$

$$\top \stackrel{\text{def}}{=} \wedge\emptyset, \quad \bot \stackrel{\text{def}}{=} \vee\emptyset.$$

Let $\mathcal{S} = (S, \lhd, \mathrm{wb})$ be a continuous cover. A **cut** of $\mathcal{S}$ is a pair (L, U) of subsets of S such that

1. $a \lhd \{a_0, \ldots, a_{n-1}\} \;\&\; a \in U \implies (\exists k < n)\, a_k \in U$,
2. $a \in U \implies (\exists b \ll a)\, b \in U$,
3. $a \lhd \{a_0, \ldots, a_{n-1}\} \;\&\; \{a_0, \ldots, a_{n-1}\} \subseteq L \implies a \in L$,
4. $a \in L \implies (\exists \{a_0, \ldots, a_{n-1}\} \gg a)\, \{a_0, \ldots, a_{n-1}\} \subseteq L$,
5. $L \cap U = \emptyset$,
6. $a \ll b \implies a \in L \vee b \in U$.

Note that U is a located subset of $\mathcal{S}$.

Proposition

There exists a bijective correspondence between the located subsets of $\mathcal{S}$ and the cuts of $\mathcal{S}$ given by

$$V \mapsto (L_V, V),$$

$$L_V \overset{\text{def}}{=} \{a \in S \mid (\exists \{a_0, \ldots, a_{n-1}\} \gg a)\, (\forall k < n)\, a_k \notin V\}.$$

Theory of located subsets

Given a continuous cover $\mathcal{S}$, define a geometric theory $T_{\mathcal{L}}$ over a propositional symbols $P = \{\mathbf{l}(a) \mid a \in S\} \cup \{\mathbf{u}(a) \mid a \in S\}$ consisting of axioms:

$$\mathbf{u}(a) \vdash \bigvee_{k<n} \mathbf{u}(a_k) \qquad (a \lhd \{a_0, \ldots, a_{n-1}\})$$

$$\mathbf{u}(a) \vdash \bigvee_{b \ll a} \mathbf{u}(b)$$

$$\mathbf{l}(a_0) \wedge \cdots \wedge \mathbf{l}(a_{n-1}) \vdash \mathbf{l}(a) \qquad (a \lhd \{a_0, \ldots, a_{n-1}\})$$

$$\mathbf{l}(a) \vdash \bigvee_{\{a_0,\ldots,a_{n-1}\} \gg a} \mathbf{l}(a_0) \wedge \cdots \wedge \mathbf{l}(a_{n-1})$$

$$\mathbf{l}(a) \wedge \mathbf{u}(a) \vdash \bot$$

$$\top \vdash \mathbf{l}(a) \vee \mathbf{u}(b) \qquad (a \ll b)$$

A model $\alpha \subseteq P$ corresponds to a cut of $\mathcal{S}$ via

$$\alpha \mapsto (\{a \mid \mathbf{l}(a) \in \alpha\}, \{a \mid \mathbf{u}(a) \in \alpha\}).$$

Universal property

a bit of topology . . . for specialists

Formal topology

A **formal topology** $\mathcal{S}$ is a triple $\mathcal{S} = (S, \lhd, \leq)$ where $(S, \leq)$ is a preorder and $\lhd \subseteq S \times \mathrm{Pow}(S)$ is called a **cover** on S such that

$$\frac{a \in U}{a \lhd U}, \quad \frac{a \leq b}{a \lhd b}, \quad \frac{a \lhd U \quad U \lhd V}{a \lhd V}, \quad \frac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow V},$$

for all $a, b \in S$ and $U, V \subseteq S$ where

$$U \downarrow V \stackrel{\text{def}}{=} {\downarrow}U \cap {\downarrow}V = \{c \in S \mid (\exists a \in U)\,(\exists b \in V)\, c \leq a \;\&\; c \leq b\}.$$

A geometric theory T over propositional symbols P determines a formal topology $\mathcal{S}_T = (\mathrm{Fin}(P), \lhd_T, \supseteq)$, where $\lhd_T$ is the smallest cover on $\mathrm{Fin}(P)$ such that

$$\{p_0, \ldots, p_{n-1}\} \lhd_T \left\{\{q_0^i, \ldots, q_{n_i-1}^i\} \mid i \in I\right\}$$

for each axiom $p_0 \wedge \cdots \wedge p_{n-1} \vdash \bigvee_{i \in I} q_0^i \wedge \cdots \wedge q_{n_i-1}^i$ in T.

Compact regular formal topologies

A formal topology $\mathcal{S} = (S, \lhd, \leq)$ is **regular** if

$$a \lhd \{b \in S \mid b \lll a\},$$

where $a \lll b \overset{\text{def}}{\iff} S \lhd a^* \cup \{b\}$ and $b^* \overset{\text{def}}{=} \{c \in S \mid b \downarrow c \lhd \emptyset\}$.
Intuitively, $b \lll a \iff$ " the closure of b is contained in a ".

A formal topology $\mathcal{S}$ is **compact** if

$$S \lhd U \implies (\exists A \in \mathsf{Fin}(U))\, S \lhd A.$$

Lemma (Johnstone (1982))

Every compact regular formal topology $\mathcal{S} = (S, \lhd, \leq)$ is a continuous cover $(S, \lhd, \mathsf{wb})$ with

$$\mathsf{wb}(a) \overset{\text{def}}{=} \{b \in S \mid b \lll a\}.$$

Morphisms

A **perfect map** between continuous covers $\mathcal{S} = (S, \lhd, \text{wb})$ and $\mathcal{S}' = (S', \lhd', \text{wb}')$ is a relation $r \subseteq S \times S'$ such that

1. $a \lhd' U \implies r^- \{a\} \lhd r^- U$,
2. $a \ll' b \implies r^- \{a\} \ll r^- \{b\}$.

Let **CCov** be the category of continuous covers and perfect maps.

A **continuous map** between formal topologies $\mathcal{S} = (S, \lhd, \leq)$ and $\mathcal{S}' = (S', \lhd', \leq')$ is a relation $r \subseteq S \times S'$ such that

1. $S \lhd r^- S'$,
2. $r^- \{a\} \downarrow r^- \{b\} \lhd r^-(a \downarrow' b)$,
3. $a \lhd' U \implies r^- \{a\} \lhd r^- U$.

Lemma

*Continuous maps between regular formal topologies are perfect. Hence, the category **KReg** of compact regular formal topologies and continuous maps is a full subcategory of **CCov**.*

The space of located subsets

Let $\mathcal{S}$ be a continuous cover, and let $\mathcal{L}(\mathcal{S})$ be the formal topology associated with the geometric theory $T_{\mathcal{L}}$; call $\mathcal{L}(\mathcal{S})$ **the space of located subsets** of $\mathcal{S}$.

Theory $T_{\mathcal{L}}$

Propositional symbols $\{\mathbf{l}(a) \mid a \in S\} \cup \{\mathbf{u}(a) \mid a \in S\}$ with axioms:

$$\mathbf{u}(a) \vdash \bigvee_{k<n} \mathbf{u}(a_k) \qquad (a \lhd \{a_0, \ldots, a_{n-1}\})$$

$$\mathbf{u}(a) \vdash \bigvee_{b \ll a} \mathbf{u}(b)$$

$$\mathbf{l}(a_0) \wedge \cdots \wedge \mathbf{l}(a_{n-1}) \vdash \mathbf{l}(a) \qquad (a \lhd \{a_0, \ldots, a_{n-1}\})$$

$$\mathbf{l}(a) \vdash \bigvee_{\{a_0, \ldots, a_{n-1}\} \gg a} \mathbf{l}(a_0) \wedge \cdots \wedge \mathbf{l}(a_{n-1})$$

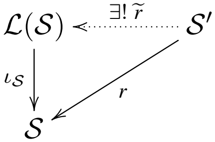
$$\mathbf{l}(a) \wedge \mathbf{u}(a) \vdash \bot$$

$$\top \vdash \mathbf{l}(a) \vee \mathbf{u}(b) \qquad (a \ll b)$$

Universal property

Proposition

1. *$\mathcal{L}(\mathcal{S})$ is a compact regular formal topology.*
2. *There exists a perfect map $\iota_{\mathcal{S}} \colon \mathcal{L}(\mathcal{S}) \to \mathcal{S}$ such that for any compact regular formal topology $\mathcal{S}'$ and a perfect map $r \colon \mathcal{S}' \to \mathcal{S}$, there exists a unique continuous map $\widetilde{r} \colon \mathcal{S}' \to \mathcal{L}(\mathcal{S})$ such that*



Theorem

The construction $\mathcal{L}(\mathcal{S})$ is the right adjoint to the forgetful functor ***KReg*** $\to$ ***CCov****.*

Lawson topology

Classically, the left adjoint to the forgetful functor **KReg** $\to$ **CCov** is defined by the Lawson topologies on continuous lattices.

Theorem

The space $\mathcal{L}(\mathcal{S})$ of located subsets of $\mathcal{S}$ represents the Lawson topology on $\mathcal{S}$.

The above adjunction induces a monad $K_{\mathcal{L}} = (\mathcal{L}, \eta_{\mathcal{L}}, \mu_{\mathcal{L}})$ on **KReg**. By an easy analogy to the classical domain theory, we have

Theorem

*The monad $K_{\mathcal{L}}$ induced by the adjunction is naturally isomorphic to the Vietoris monad on **KReg**.*

Note: Vietoris monad is a point-free extension of Hausdorff metric on compact subsets on a compact metric space.

Summary

- The notion located subset for continuous cover captures well-known examples of located subsets.
- Located subsets can be characterised geometrically via an equivalent notion of cuts.
- The space $\mathcal{L}(\mathcal{S})$ of located subsets of a continuous cover $\mathcal{S}$ is the Lawson topology on $\mathcal{S}$.
- The monad on **KReg** induced by the construction $\mathcal{L}(-)$ is the Vietoris monad on **KReg**.

References

Thierry Coquand, Erik Palmgren, and Bas Spitters. Metric complements of overt closed sets. *MLQ Math. Log. Q.*, 57(4): 373–378, 2011.

G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott . *Continuous Lattices and Domains*. Cambridge University Press, 2003.

Tatsuji Kawai. Lawson topology as the space of located subsets. preprint.

Erik Palmgren. A constructive and functorial embedding of locally compact metric spaces into locales. *Topology Appl.*, 154: 1854–1880, 2007.

Bas Spitters. Locatedness and overt sublocales. *Ann. Pure Appl. Logic*, 162(1):36–54, 2010.

Peter. T. Johnstone. *Stone Spaces*. Cambridge University Press, 1982.