Boundaries and Separation

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Suppose we start at a point ξ in the interior of a located subset C of a normed space X and move linearly towards a point z in the metric complement of C. Are we able to tell when we are crossing the boundary of C?
In general, the constructive answer is *no*.
However, our geometric intuition suggests that when C is convex, we might succeed in pinpointing boundary crossing points.

Our context is a normed space X. Note that if $x, y \in X$, then $x \neq y$ (x and y are **distinct**) means $\|x - y\| > 0$.

A subset C of X has three types of complement:

- the **logical complement**

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- the **metric/apartness complement**

$$-C = \{x \in X : \exists_{r>0} \forall_{y \in C} (\|x - y\| \geq r)\}$$

Our boundary-crossing theorem uses three geometric lemmas about convexity.

Lemma 1 *Let C be a convex subset of X, ξ an interior point of C, and r a positive number such that $B(\xi, r) \subset C$. Let $z \neq \xi$, and let $z' = t\xi + (1-t)z$ where $0 < t < 1$. If $B(z, tr)$ intersects C, then $B(z', t^2 r) \subset C$.*

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Lemma 2 *Let C be an open convex subset of X such that $C \cup -C$ is dense in X. Let $\xi \in C$ and $z \in -C$. Then $(C \cup -C) \cap [\xi, z]$ is dense in $[\xi, z]$.*

The third lemma is almost trivial, yet remarkably useful.

Lemma 3 *Let* x_1, x_2 *be distinct points of* X*; let* $x_3 = \lambda x_1 + (1-\lambda)\,x_2$ *with* $\lambda \neq 0, 1$*. For all* $\alpha, \beta > 0$*, if* $\|x - x_1\| < \alpha / |\lambda|$ *and* $\|y - x_2\| < \beta / |1-\lambda|$*, then*

$$\|\lambda x + (1-\lambda) y - x_3\| < \alpha + \beta.$$

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One application:

Proposition 1 *If* C *is an inhabited open convex subset of* X*, then* $-C$ *is dense in* $\sim C$*.*

A more significant application of Lemma 3 is in the proof of our boundary crossing theorem:

Theorem 1 *Let C be an open convex subset of a Banach space X, such that $C \cup -C$ is dense in X, and let $\xi \in C$. For each $z \in -C$ and each $t \in [0,1]$ write*

$$z_t = t\xi + (1-t)z.$$

Then the following hold:

(a) $\gamma(\xi, z) = \inf\{t \in [0,1] : z_t \in C\}$ *exists, and* $0 < \gamma(\xi, z) < 1$.

(b) $z_{\gamma(\xi,z)}$ *is the unique intersection of* $[\xi, z]$ *with the boundary* ∂C *of* C.

(c) *If* $\gamma(\xi, z) < t \leq 1$, *then* $z_t \in C$.

(d) *If* $0 \leq t < \gamma(\xi, z)$, *then* $z_t \in -C$.

Moreover, the boundary crossing map $(\xi, z) \rightsquigarrow z_{\gamma(\xi,z)}$ *of* $C \times -C$ *into* ∂C *is continuous.*

A subset C of a vector space X over $\mathbf{K}$ is called a **cone** if for all $x, y \in C$ and all $t > 0$, both $x + y$ and tx belong to C. In that case, C is convex.

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If K is a convex subset of X, then the set

$$c(K) = \{tx : x \in K, t > 0\}$$

is a cone—the **cone generated by the convex set** K.

If X is a normed space and K is open, then so is $c(K)$.

Lemma 4 *Let K be a bounded, located, convex subset of a normed space X such that $\rho(0, K) > 0$. Then $c(K)$ is located.*

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Proof. Given $x_0 \in X$, for all $x \in X$ and $t > 0$ we have

$$\|x_0 - tx\| \geq t\|x\| - \|x_0\|,$$

so

$$\rho(x_0, tK) \geq t\rho(0, K) - \|x_0\| \to \infty \text{ as } t \to \infty.$$

We can therefore find $\tau > 0$ such that $\rho(x_0, c(K)) = \rho(x_0, \tau K)$. □

Lemma 5 *Let K and L be open cones in a normed space X such that $K \cup L$ is dense in X and $K \subset \sim L$. Then*

(i) $K \subset -L$ *and* $L \subset -K$,

(ii) $K \cup -K$ *and* $L \cup -L$ *are dense in* X, *and*

(iii) K *and* L *have a common boundary—namely,* $\overline{K} \cap \overline{L}$.

If also $L = \{-x : x \in K\}$, *then* ∂K *is a subspace of* X.

By a **half-space** of a normed space X we mean a convex subset K such that ∂K is a hyperplane and the set

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The **Basic Separation Theorem**:

Theorem 2 *Let X be a separable normed space, K_0 a bounded, located, open, convex subset of X such that $\rho(0, K_0) > 0$, and x_0 a point of X such that $-x_0 \in K_0$. Then there exists an open half-space K of X such that $K_0 \subset K$, $\rho(x_0, K) > 0$, and the boundary of K is a located subspace of X that is a hyperplane with associated vector x_0.*

The (full) **Separation Theorem**:

Theorem 3 *Let A and B be bounded convex subsets of a separable normed space X such that the* **algebraic difference**

$$\{y - x : x \in A,\ y \in B\}$$

is located and the **mutual distance**

$$d = \inf\{\|y - x\| : x \in A,\ y \in B\}$$

is positive. Then for each $\varepsilon > 0$ there exists a normed linear functional u on X, with norm 1, such that

$$\operatorname{Re} u(y) > \operatorname{Re} u(x) + d - \varepsilon \quad (x \in A, y \in B)\,.$$

The **Berger-Svindland Separation Theorem:**

Theorem 4 *Let C, Y be convex subsets of $\mathbf{R}^n$ such that*

(i) *C is convex and compact;*

(ii) *Y is convex, closed, and located;*

(iii) *$x \neq y$ for all $x \in C$ and $y \in Y$.*

Then there exist $p \in \mathbf{R}^n$ and real α, β such that

$$\langle p, x \rangle < \alpha < \beta < \langle p, y \rangle$$

for all $x \in C$ and $y \in Y$.

A crucial step in the proof is showing that

$$\inf\{\|x-y\| : x \in C, y \in Y\} > 0. \tag{1}$$

Under what conditions can we show that if C, Y are located convex subsets of a normed space satisfying (iii), then (1) holds?

Recall the following:

- **Bishop's Lemma:** if Y is an inhabited, complete, located subset of a metric space X, then for each $x \in X$ such that $x \neq y$ implies that $\rho(x, Y) > 0$.

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- **Bishop's Lemma:** if Y is an inhabited, complete, located subset of a metric space X, then for each $x \in X$ such that $x \neq y$ implies that $\rho(x, Y) > 0$.
- A convex subset C of a normed space X is **uniformly rotund** if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, x' \in C$ and $\|x - x'\| > \varepsilon$, then $\frac{1}{2}(x + x') + z \in C$ for all $z \in X$ with $\|z\| \leqslant \delta$.

We now have a weak generalisation of Bishop's Lemma.

Theorem 5 *Let K, L be inhabited, complete convex subsets of a normed space X such that*

(a) *K is uniformly rotund,*

(b) *L contains at least two distinct points, and*

(c) *$d \equiv \inf_{x \in K} \rho(x, L)$ exists.*

Then there exist $x_\infty \in K$ and $y_\infty \in L$ such that if $x_\infty \neq y_\infty$, then d is positive.

Theorem 5 is at least interesting, and perhaps useful.

But we should note that if K is compact and contains at least two distinct points, then, by uniform rotundity, K includes a ball centred at their midpoint; that ball, being closed and located in K, is compact, so the space X is finite-dimensional.

- D.S. Bridges: ‘The construction of a continuous demand function for uniformly rotund preferences’, J. Math. Economics **21**, 217–227, 1992. (*for uniform rotundity)*

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- Josef Berger and Gregor Svindland, ‘Convexity and constructive infima’, Arch. Math. Logic **55**, 873–881, 2016. DOI 10.1007/s00153-016-0502-y *(for a general result about infima of positive convex functions)*