Baire category theorem and nowhere differentiable continuous functions

Takako Nemoto

JAIST
third CORE meeting

26 January 2018

Abstract

In constructive mathematics, Baire Category Theorem has at least the following two forms:

A. For a sequence $\{U_n\}_{n=0}^{\infty}$ of dense open sets in a complete metric space X,

$$U = \bigcap_{n \in \mathbf{N}} U_n$$

is as well as dense in X.

B. For a sequence $\{V_n\}_{n=0}^{\infty}$ of nowhere dense closed sets in a complete metric space X,

$$V = \bigcup_{n \in \mathbf{N}} V_n$$

is as well as nowhere dense in X.

A is constructively provable. We will show that there exist nowhere differentiable continuous functions densely in $C[0,1]$, using A.

$\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{Q}$

- For natural number $\mathbf{N}$, we allow to use induction.
- By induction, we can prove that, for each $n, m \in \mathbf{N}$

  $n = m \vee \neg n = m;$

  $n < m \vee \neg n < m$ (equivalently, $n < m \vee m \leq n$).

- Integers $\mathbf{Z}$ and rationals $\mathbf{Q}$ can be coded by natural numbers. Therefore we also have, for each $p, q \in \mathbf{Q}$

  $p = q \vee \neg p = q;$

  $p < q \vee \neg(p < q)$ (equivalently, $p < q \vee q \leq p$).

$\mathbf{R}$ and functions on $\mathbf{R}$

- A sequence $x = (p_n)_n$ of rationals are *regular* if

$$\forall mn(|p_m - p_n| < 2^{-m} + 2^{-n})$$

- A regular sequence x of rationals is *real* ($x \in \mathbf{R}$).
  For $x = (p_n)_n$, $x_n = p_n$.
- The equality $=_\mathbf{R}$ is the following equivalence relation:

$$(p_n)_n =_\mathbf{R} (q_n)_n \overset{\text{def}}{\Longleftrightarrow} \forall n(|p_n - q_n| \leq 2^{-n+2})$$

The following are well-defined.

$$(x \pm_\mathbf{R} y)_n = x_{2n+1} \pm y_{2n+1} \quad |x|_n = |x_n|$$
$$\max\{x, y\}_n = \max\{x_n, y_n\} \quad \min\{x, y\}_n = \min\{x_n, y_n\}$$
$$(x \cdot_\mathbf{R} y)_n = x_{2kn+1} \cdot y_{2kn+1}, \quad \text{where } k = \max\{|x|_0 + 2, |y|_0 + 2\}$$

Order $<_{\mathbf{R}}$

Let x and y be reals.

Order $<_{\mathbf{R}}$

- x is positive if $\exists n(x_n > 2^{-n+2})$.
- x is negative if $\exists n(x_n < -2^{-n+2})$.
- $x <_{\mathbf{R}} y$ if $y -_{\mathbf{R}} x$ is positive.

Some properties of $<_{\mathbf{R}}$

- $x =_{\mathbf{R}} x' \wedge y =_{\mathbf{R}} y' \wedge x <_{\mathbf{R}} y \to x' <_{\mathbf{R}} y'$
- $\forall x, y \in \mathbf{R} \forall n(x_n < y_n \vee x_n = y_n \vee y_n < x_n)$.
- But we cannot prove $\forall x, y \in \mathbf{R}(x <_{\mathbf{R}} y \vee x =_{\mathbf{R}} y \vee y <_{\mathbf{R}} x)$ constructively (LPO).

Order $\leq_{\mathbf{R}}$

Let x and y be reals

Order $\leq_{\mathbf{R}}$

- $x \leq_{\mathbf{R}} y$ if $x -_{\mathbf{R}} y$ is *not* positive.

Some properties of $\leq_{\mathbf{R}}$

- $x =_{\mathbf{R}} x' \wedge y =_{\mathbf{R}} y' \wedge x \leq_{\mathbf{R}} y \to x' \leq_{\mathbf{R}} y'$
- $\forall x, y \in \mathbf{R}(x \leq_{\mathbf{R}} y \vee_{\mathbf{R}} y \leq_{\mathbf{R}} x)$ cannot be proved constructively (LLPO).
- $\forall x, y \in \mathbf{R}(x \leq_{\mathbf{R}} y \vee_{\mathbf{R}} \neg x \leq_{\mathbf{R}} y)$ cannot be proved constructively (WLPO).
- But $\forall x, y \in \mathbf{R}(\neg x <_{\mathbf{R}} y \to y \leq_{\mathbf{R}} x)$ *can* be proved constructively.

We omit $\mathbf{R}$ in $=_{\mathbf{R}}$, $+_{\mathbf{R}}$, $-_{\mathbf{R}}$, $<_{\mathbf{R}}$, $\leq_{\mathbf{R}}$.

How to make case divisions?

We can not use the following case division.

$$x <_{\mathbf{R}} y \vee x =_{\mathbf{R}} y \vee y <_{\mathbf{R}} x, \qquad x \leq_{\mathbf{R}} y \vee y \leq_{\mathbf{R}} x$$

What kind of case division is available?

How to make case divisions?

We can not use the following case division.

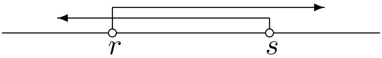
$$x <_{\mathbf{R}} y \vee x =_{\mathbf{R}} y \vee y <_{\mathbf{R}} x, \qquad x \leq_{\mathbf{R}} y \vee y \leq_{\mathbf{R}} x$$

What kind of case division is available?

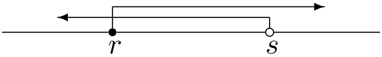
Lemma

For any $r <_{\mathbf{R}} s$, we have the following:

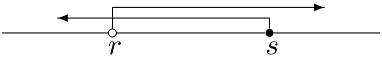
- $x <_{\mathbf{R}} s \vee r <_{\mathbf{R}} x$



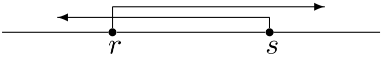
- $x <_{\mathbf{R}} s \vee r \leq_{\mathbf{R}} x$



- $x \leq_{\mathbf{R}} s \vee r <_{\mathbf{R}} x$



- $x \leq_{\mathbf{R}} s \vee r \leq_{\mathbf{R}} x$



For $a \leq_{\mathbf{R}} b$, we use the following notations:

$$(a, b) = \{x \in \mathbf{R} : a <_{\mathbf{R}} x <_{\mathbf{R}} b\} \quad [a, b] = \{x \in \mathbf{R} : a \leq_{\mathbf{R}} x \leq_{\mathbf{R}} b\}$$

Uniformly continuous function

- A uniformly continuous function $f : [0,1] \to \mathbf{R}$ consists of $\varphi : \mathbf{Q} \times \mathbf{N} \to \mathbf{Q}$ and $\nu : \mathbf{N} \to \mathbf{N}$ with the following properties:

  $$(f(p))_n = \varphi(p, n) \in \mathbf{R}$$
  $$\forall n \in \mathbf{N} \forall p, q \in \mathbf{Q}(|p - q| < 2^{-\nu(n)} \to |f(p) - f(q)| < 2^{-n}).$$

  For each $x \in [0,1]$, $f(x) \in \mathbf{R}$ is given by

  $$(f(x))_n = \varphi(\min\{\max\{x_{\mu(n)}, 0\}, 1\}, n+1),$$

  where $\mu(n) = \nu(n+1) + 1$.

Derivative and differentiability

- $f : \mathbf{R} \to \mathbf{R}$ is differentiable at x_0 if, for some $a \in \mathbf{R}$,

$$\forall k \exists l \forall x \left(|x - x_0| < \frac{1}{2^l} \to \left| \frac{f(x) - f(x_0)}{x - x_0} - a \right| \leq \frac{1}{2^k} \right).$$

Complete metric space

- A set X is *metric space* if there is $\rho : X \times X \to \mathbf{R}_{\geq 0}$ s.t.
  - $\rho(x, y) = 0$ iff $x = y$;
  - $\rho(x, y) = \rho(y, x)$;
  - $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.
- For a metric space X, a sequence $(x_n)_n$ from X is *regular* if

$$\forall mn(\rho(x_m, x_n) < 2^{-m} + 2^{-n}).$$

  The metric completion $\hat{X}$ of X consists of all regular sequences of X.
- The equality $=_{\hat{X}}$ is the following equivalence relation:

$$(x_n)_n =_{\hat{X}} (y_n)_n \overset{\text{def}}{\Longleftrightarrow} \forall n(|x_n - y_n| \leq 2^{-n+2})$$

- A metric space Y is a complete metric space if $\hat{Y} = Y$.

Some examples

- $\mathbf{R}$ is a complete metric space with $\rho(x, y) = |x - y|$.
- Let $C[0,1]$ be the set of all uniformly continuous $f : [0,1] \to \mathbf{R}$. Then $C[0,1]$ is a complete metric space with $\rho(f, g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$.
  - We need uniformity to show the existence of $\sup\{|f(x) - g(x)| : x \in [0,1]\}$.
  - To prove that continuous $f : [0,1] \to \mathbf{R}$ is uniformly continuous, we need some non-constructive principle (FAN)

Topological notions

Open & closed sets

For a complete metric space X,

- $U \subseteq X$ is *open* if, for each $x \in U$, there is $\varepsilon > 0$ s.t. $\mathcal{B}(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\} \subseteq U$.
- $V \subseteq X$ is *closed* if $x \in X$ satisfying that, for each $\varepsilon > 0$, there is $y \in \mathcal{B}(x, \varepsilon) \cap V$ is itself in V.

Dense & nowhere dense

- For $Y \subseteq X$, the set $\overline{Y} = \{x : \forall\varepsilon > 0 \exists y \in Y (y \in \mathcal{B}(x, \varepsilon))\}$ is the *closure* of Y.
- $Y \subseteq X$ is *dense* is if $\overline{Y} = X$.
- For $Y \subseteq X$, the set $Y^\circ = \{x : \exists\varepsilon > 0 \in Y (\mathcal{B}(x, \varepsilon) \subseteq Y)\}$ is the *interior* of Y.
- $Y \subseteq X$ is *nowhere dense* if $(\overline{Y})^\circ = \emptyset$.

Baire category theorem

There are several versions of Baire category theorem, which are equivalent over classical logic:

A. For a sequence $\{U_n\}_{n=0}^{\infty}$ of dense open sets in a complete metric space X,

$$U = \bigcap_{n \in \mathbf{N}} U_n$$

is as well as dense in X.

B. For a sequence $\{V_n\}_{n=0}^{\infty}$ of nowhere dense closed sets in a complete metric space X,

$$V = \bigcup_{n \in \mathbf{N}} V_n$$

is as well as nowhere dense in X.

A is constructively provable (cf. [1]).

Theorem in classical mathematics

- Let $C[0,1]$ be the set of all continuous $f : [0,1] \to \mathbf{R}$.
- Then $||f|| = \sup\{|f(x)| : x \in [0,1]\}$ is a norm on $C[0,1]$ and $d(f,g) = ||f-g||$ is a distance on $C[0,1]$.

Classical theorem (Banach)

There are densely many functions in $C[0,1]$ which are nowhere differentiable on $[0,1]$.

Sketch of the classical proof

1. Let $U_{m,n} = \{f \in C[0,1] : \varphi_{m,n}(f)\}$, where $\varphi_{m,n}(f)$ is

   $$\forall x \exists y \in [0,1] \left(0 < |y - x| < \frac{1}{m} \wedge \left| \frac{f(y) - f(x)}{y - x} \right| > n \right).$$

   If $f \in C[0,1]$ is differentiable in some $x \in [0,1]$, $f \notin U_{m,n}$ for some m, n.

Sketch of the classical proof

1. Let $U_{m,n} = \{f \in C[0,1] : \varphi_{m,n}(f)\}$, where $\varphi_{m,n}(f)$ is
$$\forall x \exists y \in [0,1] \left(0 < |y - x| < \frac{1}{m} \land \left| \frac{f(y) - f(x)}{y - x} \right| > n \right).$$
If $f \in C[0,1]$ is differentiable in some $x \in [0,1]$, $f \notin U_{m,n}$ for some m, n.
2. $U_{m,n}$ is open in $C[0,1]$.
   - If $U_{m,n}$ is not open, then there is $f \in U_{m,n}$ s.t. for any $k \in \mathbf{N}$ there is $g_k \notin U_{m,n}$ with $||f - g_k|| < 2^{-k}$.
   - $\lim_{k\to\infty} g_k = f$.
   - By Bolzano-Weierstrass, there is $x \in [0,1]$ s.t.
$$\forall y \in [0,1] \left(0 < |y - x| < \frac{1}{m} \to \left| \frac{f(y) - f(x)}{y - x} \right| \leq n \right).$$

Sketch of the classical proof

1. Let $U_{m,n} = \{f \in C[0,1] : \varphi_{m,n}(f)\}$, where $\varphi_{m,n}(f)$ is
$$\forall x \exists y \in [0,1] \left(0 < |y - x| < \frac{1}{m} \wedge \left| \frac{f(y) - f(x)}{y - x} \right| > n \right).$$
If $f \in C[0,1]$ is differentiable in some $x \in [0,1]$, $f \notin U_{m,n}$ for some m, n.
2. $U_{m,n}$ is open in $C[0,1]$.
   - If $U_{m,n}$ is not open, then there is $f \in U_{m,n}$ s.t. for any $k \in \mathbf{N}$ there is $g_k \notin U_{m,n}$ with $||f - g_k|| < 2^{-k}$.
   - $\lim_{k\to\infty} g_k = f$.
   - By Bolzano-Weierstrass, there is $x \in [0,1]$ s.t.
$$\forall y \in [0,1] \left(0 < |y - x| < \frac{1}{m} \to \left| \frac{f(y) - f(x)}{y - x} \right| \leq n \right).$$
3. $U_{m,n}$ is dense.
   - For any $f \in C[0,1]$ and $\varepsilon > 0$, there is piecewise-linear $p \in C[0,1]$ s.t. $||f - p|| < \varepsilon$.

Sketch of the classical proof

1. Let $U_{m,n} = \{f \in C[0,1] : \varphi_{m,n}(f)\}$, where $\varphi_{m,n}(f)$ is
$$\forall x \exists y \in [0,1] \left(0 < |y-x| < \frac{1}{m} \wedge \left| \frac{f(y)-f(x)}{y-x} \right| > n \right).$$
If $f \in C[0,1]$ is differentiable in some $x \in [0,1]$, $f \notin U_{m,n}$ for some m, n.
2. $U_{m,n}$ is open in $C[0,1]$.
   - If $U_{m,n}$ is not open, then there is $f \in U_{m,n}$ s.t. for any $k \in \mathbf{N}$ there is $g_k \notin U_{m,n}$ with $||f - g_k|| < 2^{-k}$.
   - $\lim_{k\to\infty} g_k = f$.
   - By Bolzano-Weierstrass, there is $x \in [0,1]$ s.t.
$$\forall y \in [0,1] \left(0 < |y-x| < \frac{1}{m} \rightarrow \left| \frac{f(y)-f(x)}{y-x} \right| \leq n \right).$$
3. $U_{m,n}$ is dense.
   - For any $f \in C[0,1]$ and $\varepsilon > 0$, there is piecewise-linear $p \in C[0,1]$ s.t. $||f - p|| < \varepsilon$.
4. By Baire category theorem A, $\bigcap_{m,n\in\mathbf{N}} U_{m,n}$ is dense.

Constructivising the proof

- Let $C[0,1]$ be the set of all uniformly continuous $f : [0,1] \to \mathbf{R}$.
- $||f|| = \sup\{|f(x)| : x \in [0,1]\}$ is a norm on $C[0,1]$ and $d(f,g) = ||f - g||$ is a distance on $C[0,1]$.
- We cannot prove $U_{m,n} = \{f \in C[0,1] : \varphi_{m,n}(f)\}$ is open for the following $\varphi_{m,n}(f)$:

$$\forall x \exists y \in [0,1] \left(0 < |y - x| < \frac{1}{m} \wedge \left| \frac{f(y) - f(x)}{y - x} \right| > n \right).$$

  - We cannot prove by contradiction.
  - We cannot prove Bolzano-Weierstrass constructively.

Constructivising the proof

- Let $\tilde{U}_{m,n} = \{f \in C[0,1] : \tilde{\varphi}_{m,n}(f)\}$, where $\tilde{\varphi}_{m,n}(f)$ is

$$\exists \varepsilon{>}0 \forall g{\in}C[0,1] \left(\begin{array}{l} ||f-g||{<}\varepsilon \to \\ \qquad \forall x{\in}[0,1] \neg\neg\exists t \in [0,1] \left(\begin{array}{r} 0 < |t-x| < \frac{1}{m} \\ \wedge \left| \frac{g(t)-g(x)}{t-x} \right| {>} n \end{array} \right) \end{array} \right)$$

  - If $f \in C[0,1]$ is differentiable at some $x \in [0,1]$, $f \notin \tilde{U}_{m,n}$ for some m, n.
- $\tilde{U}_{m,n}$ is open.
  - For $f \in \tilde{U}_{m,n}$ and $\varepsilon > 0$ witnessing $f \in \tilde{U}_{m,n}$, $\varepsilon' = \varepsilon - ||f-h|| > 0$ witnesses $h \in \tilde{U}_{m,n}$ for h with $||f-h|| < \varepsilon$.
- $\tilde{U}_{m,n}$ is dense.
  - For any $f \in C[0,1]$ and $\varepsilon > 0$, we have to find $g \in \tilde{U}_{m,n}$ s.t. $||f-g|| < \varepsilon$.

Constructivising the proof

- $p : [0,1] \to \mathbf{R}$ is *piecewise-linear* if there is a division $0 = a_0 < a_1 < \ldots < a_{n+1} = 1$ of $[0,1]$ s.t. p is linear on each $[a_i, a_{i+1}]$.
- Let $PL[0,1]$ be the set of all piecewise-linear $f \in C[0,1]$.

Lemma 1

If $p \in PL[0,1]$ and $|p'(x)| > n$ on all differentiable x, $p \in \tilde{U}_{m,n}$.

Proof.

Assume $0 = a_0 < a_1 < \ldots < a_{k+1} = 1$ and p is linear on each $[a_i, a_{i+1}]$.
For $g \in C[0,1]$ set

$$s = \min\left\{ \left| \frac{(p(a_{i+1}) - p(a_i))}{(a_{i+1} - a_i)} \right| - n : 0 \leq i \leq k \right\}$$

$$s' = \min(\{a_{i+1} - a_i : 0 \leq i \leq k\} \cup \{\frac{1}{m}\}), \qquad \varepsilon = s/16s'.$$

Then we have, for each g s.t. $||f - g|| < \varepsilon$,

$$\forall x \in [0,1] \neg\neg\exists t \in [0,1] \left(0 < |t - x| < \frac{1}{m} \wedge \left(\left| \frac{g(t) - g(x)}{t - x} \right| > n \right) \right).$$

Let $||g - p|| < \varepsilon$ and $\delta_x = \min\{|x - a_i| : 0 \leq i \leq k + 1\}$.
For each $x \in [0, 1]$, we have $\delta_x < s'/2 \vee s'/4 \leq \delta_x$.

Let $||g - p|| < \varepsilon$ and $\delta_x = \min\{|x - a_i| : 0 \le i \le k+1\}$.
For each $x \in [0,1]$, we have $\delta_x < s'/2 \vee s'/4 \le \delta_x$.
Case 1. $\delta_x < s'/2$: Take a_i s.t. $|x - a_i| < s'/2$.
If $a_i \le x$, then p has the slope b on $[x, x + \frac{s'}{2}]$.
By $|b| > n$, we have $b > n \vee b < -n$. If $b > n$, then

$$\begin{aligned}\frac{g(x+\frac{s'}{2}) - g(x)}{x + \frac{s'}{2} - x} \ge& \frac{2}{s'}\left(p(x+\frac{s'}{2}) - \varepsilon - (p(x) + \varepsilon)\right)\\ =& b - \frac{4}{s'}\varepsilon \ge s + n - \frac{s}{4} > n.\end{aligned}$$

Let $||g-p|| < \varepsilon$ and $\delta_x = \min\{|x-a_i| : 0 \le i \le k+1\}$.
For each $x \in [0,1]$, we have $\delta_x < s'/2 \vee s'/4 \le \delta_x$.
Case 1. $\delta_x < s'/2$: Take a_i s.t. $|x-a_i| < s'/2$.
If $a_i \le x$, then p has the slope b on $[x, x+\frac{s'}{2}]$.
By $|b| > n$, we have $b > n \vee b < -n$. If $b > n$, then

$$\begin{aligned}\frac{g(x+\frac{s'}{2})-g(x)}{x+\frac{s'}{2}-x} \ge& \frac{2}{s'}\left(p(x+\frac{s'}{2})-\varepsilon-(p(x)+\varepsilon)\right)\\ =& b-\frac{4}{s'}\varepsilon \ge s+n-\frac{s}{4} > n.\end{aligned}$$

Similarly, we have $\left|\frac{g(x-\frac{s'}{2})-g(x)}{x-\frac{s'}{2}-x}\right| > n$ when $b < -n$ or $x < a$. Hence

Let $||g-p||<\varepsilon$ and $\delta_x=\min\{|x-a_i|:0\leq i\leq k+1\}$.
For each $x\in[0,1]$, we have $\delta_x<s'/2\vee s'/4\leq\delta_x$.
Case 1. $\delta_x<s'/2$: Take a_i s.t. $|x-a_i|<s'/2$.
If $a_i\leq x$, then p has the slope b on $[x,x+\frac{s'}{2}]$.
By $|b|>n$, we have $b>n\vee b<-n$. If $b>n$, then

$$\begin{aligned}\frac{g(x+\frac{s'}{2})-g(x)}{x+\frac{s'}{2}-x}\geq&\frac{2}{s'}\left(p(x+\frac{s'}{2})-\varepsilon-(p(x)+\varepsilon)\right)\\=&b-\frac{4}{s'}\varepsilon\geq s+n-\frac{s}{4}>n.\end{aligned}$$

Similarly, we have $\left|\frac{g(x-\frac{s'}{2})-g(x)}{x-\frac{s'}{2}-x}\right|>n$ when $b<-n$ or $x<a$. Hence

$$a_i\leq x\vee x<a_i\to\exists t\in[0,1]\left(0<|t-x|<\frac{1}{m}\wedge\left|\frac{g(t)-g(x)}{t-x}\right|>n\right),\text{ and}$$

$$\neg\neg(a_i\leq x\vee x<a_i)\to\neg\neg\exists t\in[0,1]\left(0<|t-x|<\frac{1}{m}\wedge\left|\frac{g(t)-g(x)}{t-x}\right|>n\right).$$

Let $||g-p|| < \varepsilon$ and $\delta_x = \min\{|x-a_i| : 0 \le i \le k+1\}$.
For each $x \in [0,1]$, we have $\delta_x < s'/2 \vee s'/4 \le \delta_x$.
Case 1. $\delta_x < s'/2$: Take a_i s.t. $|x-a_i| < s'/2$.
If $a_i \le x$, then p has the slope b on $[x, x+\frac{s'}{2}]$.
By $|b| > n$, we have $b > n \vee b < -n$. If $b > n$, then

$$\begin{aligned}\frac{g(x+\frac{s'}{2})-g(x)}{x+\frac{s'}{2}-x} \ge& \frac{2}{s'}\left(p(x+\frac{s'}{2})-\varepsilon-(p(x)+\varepsilon)\right)\\ =& b-\frac{4}{s'}\varepsilon \ge s+n-\frac{s}{4} > n.\end{aligned}$$

Similarly, we have $\left|\frac{g(x-\frac{s'}{2})-g(x)}{x-\frac{s'}{2}-x}\right| > n$ when $b < -n$ or $x < a$. Hence

$$a_i \le x \vee x < a_i \to \exists t \in [0,1]\left(0<|t-x|<\frac{1}{m} \wedge \left|\frac{g(t)-g(x)}{t-x}\right|>n\right), \text{and}$$

$$\neg\neg(a_i \le x \vee x < a_i) \to \neg\neg\exists t \in [0,1]\left(0<|t-x|<\frac{1}{m} \wedge \left|\frac{g(t)-g(x)}{t-x}\right|>n\right).$$

By $\neg\neg(a_i \le x \vee x < a_i)$, we have the right-hand side of $\to$.

Let $||g - p|| < \varepsilon$ and $\delta_x = \min\{|x - a_i| : 0 \leq i \leq k+1\}$.
For each $x \in [0,1]$, we have $\delta_x < s'/2 \vee s'/4 \leq \delta_x$.

Let $||g-p|| < \varepsilon$ and $\delta_x = \min\{|x-a_i| : 0 \le i \le k+1\}$.
For each $x \in [0,1]$, we have $\delta_x < s'/2 \vee s'/4 \le \delta_x$.
Case 2. $s'/4 \le \delta_x$: p has the slope b on $[x-\frac{s'}{4}, x+\frac{s'}{4}]$. By $|b| > n$, we have $b > n \vee b < -n$. If $b > n$, then

$$\begin{aligned}
\frac{g(x+\frac{s'}{4})-g(x)}{x+\frac{s'}{4}-x} \ge & \frac{4}{s'}\left(p(x+\frac{s'}{4})-\varepsilon-(p(x)+\varepsilon)\right) \\
= & \frac{4}{s'}\left(\left(p(x+\frac{s'}{4})-p(x)\right)-2\varepsilon\right) \\
= & b-\frac{8}{s'}\varepsilon \\
\ge & s+n-\frac{s}{2} > n.
\end{aligned}$$

Let $||g-p|| < \varepsilon$ and $\delta_x = \min\{|x-a_i| : 0 \le i \le k+1\}$.
For each $x \in [0,1]$, we have $\delta_x < s'/2 \vee s'/4 \le \delta_x$.
Case 2. $s'/4 \le \delta_x$: p has the slope b on $[x-\frac{s'}{4}, x+\frac{s'}{4}]$. By $|b| > n$, we have $b > n \vee b < -n$. If $b > n$, then

$$\begin{aligned}\frac{g(x+\frac{s'}{4})-g(x)}{x+\frac{s'}{4}-x} \geq& \frac{4}{s'}\left(p(x+\frac{s'}{4})-\varepsilon-(p(x)+\varepsilon)\right)\\ =&\frac{4}{s'}\left(\left(p(x+\frac{s'}{4})-p(x)\right)-2\varepsilon\right)\\ =&b-\frac{8}{s'}\varepsilon\\ \geq& s+n-\frac{s}{2} > n.\end{aligned}$$

Similarly, we have $\frac{g(x+\frac{a}{4})-g(x)}{x+\frac{a}{4}-x} < -n$ when $b < -n$.

Constructivising the proof

Lemma 2

$PL[0,1]$ is dense in $C[0,1]$.

Proof.

Let $f \in C[0,1]$ and $\varepsilon > 0$. Take $k \in \mathbf{N}$ s.t.

$$\forall x, y \in [0,1](|x-y| < \frac{1}{k} \to |f(x) - f(y)| < \varepsilon/3).$$

Let $a_i = i/k$ for $0 \leq i \leq k$ and define $p_0 : [0,1] \cap \mathbf{Q} \to \mathbf{R}$ by

$$p_0(x) = k(f(a_{i+1}) - f(a_i))(x - a_i) + f(a_i).$$

We can extend this p_0 to $p \in PL[0,1]$ by defining $p(x) = \{p(x_i)\}_{i=0}^{\infty}$ for $x = \{x_i\}_{i=0}^{\infty}$. Then there is a_i s.t. $|x - a_i| < \frac{1}{k}$ and

$$|p(x) - f(a_i)| < \frac{2}{3}\varepsilon.$$

Since $|f(x) - f(a_i)| < \varepsilon/3$, we have $|p(x) - f(x)| < \varepsilon$. □

Constructivising the proof

Lemma 3

For each $f \in C[0,1]$, $\varepsilon > 0$ and n, there is $p \in PL[0,1]$ s.t. $||f - p|| < \varepsilon$ and $|p'(x)| > n$ for all differentiable x.

Proof.

Let $f \in C[0,1]$. By Lemma 2, there are k and $p \in PL[0,1]$ $||f - p|| < \varepsilon/2$ and p is linear on each $[\frac{i}{k}, \frac{i+1}{k}]$.
Take $M \in \mathbf{N}$ s.t. $|p'(x)| < M$ for all differentiable $x \in [0,1]$ and $l > 2(M+n)/\varepsilon$. There is $q(x) \in PL[0,1]$ s.t. $|q(x)| \leq 1$ for all $[0,1]$ and $q'(x) = \pm k$ for all differentiable $x \in [0,1]$. Let

$$g(x) = p(x) + \frac{\varepsilon}{2} q(x).$$

Since $||f - p|| < \varepsilon/2$ and $||g - p|| < \varepsilon/2$, we have $||f - g|| < \varepsilon$.
For each differentiable $x \in [0,1]$, we have

$$|g'(x)| = \left| p'(x) + \frac{\varepsilon}{2} q'(x) \right| \geq \left| |p'(x)| - \frac{\varepsilon}{2} k \right| = \left| |p'(x)| - (M+n) \right| > n.$$

□

Constructivising the proof

By Lemma 1, 2 and 3, $\tilde{U}_{m,n}$ is dense in $C[0,1]$.

Theorem

There are densely many functions in $C[0,1]$ which are nowhere differentiable on $[0,1]$.

Proof.

Since $\tilde{U}_{m,n}$ is dense open in $C[0,1]$,
$\bigcap_{m,n\in\mathbf{N}} U_{m,n}$ is also dense in $C[0,1]$ by Baire category theorem A.
If $f \in C[m,n]$ is differentiable at some x, then $f \notin \tilde{U}_{m,n}$ for some m,n. Therefore $f \in \bigcap_{m,n\in\mathbf{N}} U_{m,n}$ is nowhere differentiable. □

Some observations

- In classical proof, we used

$$\forall x \exists y \in [0,1] \left(0 < |y - x| < \frac{1}{m} \wedge \left| \frac{f(y) - f(x)}{y - x} \right| > n \right)$$

$$U_{m,n} = \{ f \in C[0,1] : \varphi_{m,n}(f) \} .$$

For each $f \in U_{m,n}$, how to calculate $\varepsilon > 0$ s.t. $\mathcal{B}(f, \varepsilon) \subseteq U_{m,n}$? What information of f is needed?

- It is easy for $f \in U_{m,n} \cap PL[0,1]$.
- For general $f \in U_{m,n}$, how to calculate the following values?

$$\inf \left\{ \sup \left\{ \left| \frac{f(y) - f(x)}{y - x} \right| - n : 0 < |y - x| < \frac{1}{m} \right\} : x \in [0,1] \right\}$$

$$\inf \left\{ \sup \left\{ |x - y| : 0 < |y - x| < \frac{1}{m} \wedge \left| \frac{f(y) - f(x)}{y - x} \right| > n \right\} : x \in [0,1] \right\}$$

References

1. E. Bishop, *Foundations of Constructive Analysis*, Academic Press (1967).
2. D. Marker, *Most Continuous Functions are Nowhere Differentiable*, Lecture notes, `http://homepages.math.uic.edu/~marker/math414/fs.pdf` (2004).

Acknowledgment

The authors thank the Japan Society for the Promotion of Science (JSPS), Core-to-Core Program (A. Advanced Research Networks) for supporting the research.