THE DOUBLE SHUFFLE RELATIONS BETWEEN MULTIPLE ZETA VALUES

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1. Multiple Zeta Values

First I will introduce the real numbers known as multizeta values. The multizeta values can be considered as multivariate functions on positive integers and also multivariate functions on the P^n where P is the set $\{0,1\}$. Considered as functions on positive integers, the multizeta values satisfy the product relation of stuffle, and considered as the other type, the multizeta values satisfy the shuffle product relation.

Definition 1. Given a sequence of positive integers, $k_1, \dots, k_d$, we associate a real number,

$$\zeta(k_1, \dots, k_d) = \sum_{n_1 > n_2 > \cdots > n_d > 0} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}.$$

This sum converges iff $k_1 > 1$, so we put on the extra condition that $k_1 \geq 2$.

Definition 2. Alternative notation for a zeta value. To a sequence of integers $k_1, \dots, k_d$ we associate a sequence of 0's and 1's as follows. To every integer k_i associate the sequence 0,0,0,...0,1 $= 0^{k_i - 1}1$. Then concatenate all of the respective sequences to obtain a sequence of 0's and 1's of length $k_1 + k_2 + \cdots + k_d$,

$$k_1, \dots, k_d = 0^{k_1-1}10^{k_2-1}1 \dots 0^{k_d-1}1$$

and we say that

$$\zeta(k_1, \dots, k_d) = \zeta(0^{k_1-1}10^{k_2-1}1 \dots 0^{k_d-1}1).$$

In order for the zeta value to converge, we require $k_1 > 1$ which is equivalent to requiring that the sequence starts with a 0. By definition, the sequence ends in a 1. We call such sequence a convergent sequence and suppose for the rest of the exposé that every given sequence is a convergent one.

Definition 3. The weight of a zeta value, $\zeta(\underline{\epsilon})$, is the length of $\underline{\epsilon}$ which is equal to $\sum k_i$.

The goal of this talk is to show you firstly that if you multiply multizeta values, you obtain a $\mathbb{Q}$ sum of multizeta values which is not unique. This non-uniqueness gives relations between these otherwise mysterious numbers. It is conjectured that these two relations give all relations on multizeta values.

2. The Stuffle Relation

First, I will give you an example of the stuffle relation. Consider $\zeta(2)\zeta(3)$,

$$\zeta(2)\zeta(3) = \sum_i \frac{1}{i^2} \sum_j \frac{1}{j^3} \tag{1}$$
$$= \sum_{i>j>0} \frac{1}{i^2 j^3} + \sum_{j>i>0} \frac{1}{j^3 i^2} + \sum_{i=j>0} \frac{1}{i^2 j^3} \tag{2}$$
$$= \sum_{i>j>0} \frac{1}{i^2 j^3} + \sum_{j>i>0} \frac{1}{j^3 i^2} + \sum_i \frac{1}{i^5} \tag{3}$$
$$= \zeta(2,3) + \zeta(3,2) + \zeta(5). \tag{4}$$

We call this the stuffle product of the two sequences

$$(2) * (3) = (2,3) + (3,2) + (5).$$

In general the stuffle product of two sequences of positive integers is the sum over all of the sequences obtained by shuffling the original ones together, in other words making a new sequence which is a permutation of the concatenation of the two original sequences that preserves the order of the original sequences, and then eventually adding adjacent terms one of which comes from the first and the other of which comes from the second, i.e. "stuffing" integers into the same spot.

Examples 4.

$$(2,1) * (3) = (2,1,3) + (2,3,1) + (3,2,1) \tag{5}$$
$$+ (2,4) + (5,1). \tag{6}$$

The stuffle product is the commutative product defined recursively on two sequences,

$$(a_1, ..., a_i) * \emptyset = (a_1, ..., a_i)$$
$$(a_1, ..., a_i) * (b_1, ..., b_j) = a_1 \cdot ((a_2, ..., a_i) * (b_1, ..., b_j)) + b_1 \cdot ((a_1, ..., a_i) * (b_2, ..., b_j))$$
$$+ (a_1 + b_1) \cdot ((a_2, ..., a_1) * (b_2, ...b_j)).$$

The number of terms in the sum is

$$\sum_{l=0}^{\min\{i,j\}} \binom{i+j-l}{\max\{i,j\}} \binom{\max\{i,j\}}{l}.$$

If $\underline{k}$ is a term in the stuffle product of two sequences, $\underline{k}_1 * \underline{k}_2$, we use the notation, $\underline{k} \in \underline{k}_1 * \underline{k}_2$.

Theorem 5 (Euler)**.** *Let $\underline{k}_1$ and $\underline{k}_2$ be two sequences of positive integers. Then the product of the multizeta values,*

$$\zeta(\underline{k}_1)\zeta(\underline{k}_2) = \sum_{\underline{k} \in \underline{k}_1 * \underline{k}_2} \zeta(\underline{k}).$$

The proof is a direct calculation.

Corollary 6. *The $\mathbb{Q}$ vector space of multiple zeta values forms a $\mathbb{Q}$ algebra, which we call Z.*

3. The Shuffle Relation

To show this relation on multiple zeta values, we use a theorem of Kontsevich to show how multiple zeta values can be expressed as integrals.

Theorem 7. *Let $\underline{\epsilon}$ be a sequence of 0's and 1's of length (weight) n and let Δ be the real simplex $0 < t_1 < t_2 < \cdots < t_n < 1$. Then,*

$$\zeta(\underline{\epsilon}) = (-1)^d \int_\Delta \frac{d\underline{t}}{\Pi(t_{n-i+1} - \epsilon_i)}.$$

Proof. We will do the proof for $\zeta(2)$, and you can easily extend the method to sequences of greater length by induction. Write $\zeta(2) = \zeta(0,1)$. Now calculate the integral,

$$\begin{aligned}
\int_{0<t_1<t_2<1} \frac{dt_1 dt_2}{t_2(1-t_1)} &= \sum_{i=0}^{\infty} \int_{0<t_1<t_2<1} \frac{t_1^i dt_1 dt_2}{t_2} \\
&= \sum_{i=0}^{\infty} \frac{1}{i+1} \int_{0<t_2<1} \frac{t_2^{i+1} dt_2}{t_2} \\
&= \sum_i \frac{1}{i+1} \int t_2^i dt2 \\
&= \sum_i \frac{1}{(i+1)^2} \\
&= \zeta(2).
\end{aligned}$$

□

Definition 8. Given any two sequences, $\underline{\epsilon}_1, \underline{\epsilon}_2$, of combined length n, we define the shuffle product,

$$\underline{\epsilon}_1 \sqcup\!\sqcup \underline{\epsilon}_2 = \sum_{\sigma \in \mathfrak{S}(n) \text{ s.t. } \sigma \text{ preserves the order of the two original sequences}} \sigma(\underline{\epsilon}_1 \cdot \underline{\epsilon}_2).$$

Examples 9.

$$\begin{aligned}
(0,1) \sqcup\!\sqcup (0,1) &= (0,1) \sqcup\!\sqcup (0',1') \\
&= (0,1,0',1') + (0,0',1,1') + (0,0',1',1) + (0',0,1,1') + (0',0,1',1) + (0',1',0,1) \\
&= 2(0,1,0,1) + 4(0,0,1,1).
\end{aligned}$$

This product is easier than the stuffle product. It derives it's name from shuffling cards.

Theorem 10 (The shuffle product on multiple zeta values)**.** *[Kontsevich] Given two convergent sequences, $\underline{\epsilon}_1, \underline{\epsilon}_2$, we have the following identity on the product of zeta values,*

$$\begin{aligned}\zeta(\underline{\epsilon}_1)\zeta(\underline{\epsilon}_2) &= \zeta(\underline{\epsilon}_1 ⧢ \underline{\epsilon}_2)\\ &= \sum_{\underline{\epsilon}\in\underline{\epsilon}_1 ⧢ \underline{\epsilon}_2} \zeta(\underline{\epsilon}).\end{aligned}$$

Example 11. By the calculation above, we have $\zeta(2)\zeta(2) = 2\zeta(2,2) + 4\zeta(3,1)$.

Proof. The product

$$(-1)^{d_1+d_2}\int_{\Delta_1}\frac{d\underline{t}}{\Pi(t_{n-i+1}-\epsilon_i)} \times \int_{\Delta_2}\frac{d\underline{t}'}{\Pi(t'_{m-i+1}-\epsilon_i)}$$

is just the differential form $\frac{d\underline{t}d\underline{t}'}{\Pi(t_{n-i+1}-\epsilon_i)\Pi(t'_{m-i+1}-\epsilon_i)}$ integrated over the product of the simplices, $\Delta_1 \times \Delta_2$. The product of these simplices is the shuffle sum of simplices, since the order of Δ_1 and Δ_2 must be preserved. By doing a variable change to bring each simplex in the sum back to the standard simplex, $0 < t_1 < \cdots < t_n < t'_1 < \cdots < t'_m < 1$, we obtain the shuffle sum of zeta values. Let's do this with a small weight example.

$$\begin{aligned}
&\int_{0<t_1<t_2<1}\frac{dt_1dt_2}{t_2(1-t_1)} \times \int_{0<t_3<t_4<1}\frac{dt_3dt_4}{t_4(1-t_3)}\\
&= \int_{0<t_1<t_2<1,\ 0<t_3<t_4<1}\frac{dt_1dt_2dt_3dt_4}{t_4(1-t_3)t_2(1-t_1)}\\
&= \int_{0<t_1<t_2<t_3<t_4<1} + \int_{0<t_1<t_3<t_2<t_4<1} + \int_{0<t_1<t_3<t_4<t_2<1}\\
&\quad + \int_{0<t_3<t_1<t_2<t_4<1} + \int_{0<t_3<t_1<t_4<t_2<1}\\
&\quad + \int_{0<t_3<t_4<t_1<t_2<1}\frac{dt_1dt_2dt_3dt_4}{t_4(1-t_3)t_2(1-t_1)}\\
&= \int_{0<t_1<t_2<t_3<t_4<1}\frac{dt_1dt_2dt_3dt_4}{t_4(1-t_3)t_2(1-t_1)} + \frac{dt_1dt_2dt_3dt_4}{t_4(1-t_2)t_3(1-t_1)}\\
&\quad + \frac{dt_1dt_2dt_3dt_4}{t_3(1-t_2)t_4(1-t_1)} + \frac{dt_1dt_2dt_3dt_4}{t_4(1-t_1)t_3(1-t_2)}\\
&\quad + \frac{dt_1dt_2dt_3dt_4}{t_3(1-t_1)t_4(1-t_2)} + \frac{dt_1dt_2dt_3dt_4}{t_2(1-t_1)t_4(1-t_3)}.\\
&= 2\zeta(2,2) + 4\zeta(3,1).
\end{aligned}$$

□

Now we have another relation on Z, and we can use these "quadratic" relations to find linear relations among multizeta values.

Examples 12. By the last theorem, we know that $\zeta(2)^2 = 2\zeta(2,2)+4\zeta(3,1)$. From the stuffle theorem, we have that $\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$. Therefore, we have the linear relation,

$$\zeta(3,1) = \frac{1}{4}\zeta(4).$$

The structure of this $\mathbb{Q}$ algebra however is somewhat of a mystery.
There are many conjectures about this.

Conjecture 13. There are no algebraic relations among multiple zeta values of different weight. Therefore, Z forms a graded algebra.

The proof of this conjecture implies that multiple zeta values are transcendental.

Conjecture 14. All relations on multizeta values come from the stuffle and shuffle relations.

Conjecture 15 (Zagier)**.** If we assume conjecture 13, the dimension of the graded n part of Z, d_n, is given by the recurrence formula,

$$d_n = d_{n-2} + d_{n-3}.$$

References

[1] H. Furusho, *The multiple zeta value algebra and the stable derivation algebra*, Publications of the Research Institute for Mathematical Sciences (Kyoto), Vol. 39 (2003), p. 695

[2] M. Hoffman, *References on multizeta values and Euler sums*, http://www.usna.edu/Users/math/meh/biblio.html

[3] M. Kontsevich and D. Zagier, *Periods*, IHES/M/01/22

[4] V. Zudilin, *Algebraic relations for multiple zeta values*, Russian Math. Surveys, Vol. 58:1 (2003), p. 1