

A polygonal presentation of $Pic(\overline{\mathfrak{M}}_{0,n})$

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Abstract

This article gives a combinatorial presentation of the Picard group of $\overline{\mathfrak{M}}_{0,n}$. In the first section, we recall Keel's well-known presentation of $Pic(\overline{\mathfrak{M}}_{0,n})$ using boundary divisors of $\overline{\mathfrak{M}}_{0,n}$ as generators, and describe a basis for $Pic(\overline{\mathfrak{M}}_{0,n})$ recently discovered by Keel and Gibney. In the second section, we present a theorem which gives an explicit and very simple expression for every boundary divisor in terms of this basis, thereby yielding a new presentation with a minimal set of relations.

1 The Picard group of $\overline{\mathfrak{M}}_{0,n}$

We begin by presenting definitions and theorems about divisors on $\overline{\mathfrak{M}}_{0,n}$, the compactification of the moduli space of genus 0 n -pointed curves over $\mathbb{C}$.

Definition 1.1. *Let X be a smooth manifold, and let $Div(X)$ be the group formally generated by Weil divisors on X . The Picard group, $Pic(X)$, is the quotient of $Div(X)$ by the principal divisors.*

We have the following characterization/definition of the Picard group of Weil divisors on $\overline{\mathfrak{M}}_{0,n}$.

Theorem 1.2. *[Ke] The Picard group, $Pic(\overline{\mathfrak{M}}_{0,n})$, is isomorphic to the quotient $Div(\overline{\mathfrak{M}}_{0,n})/\sim$, where $\sim$ denotes numerical equivalence of divisors.*

$\overline{\mathfrak{M}}_{0,n}$ denotes the Deligne-Mumford compactification of $\mathfrak{M}_{0,n}$ [DM]. The points of $\mathfrak{M}_{0,n}$ are Riemann spheres with n marked points modulo isomorphism. Let $S_n = \{1, \dots, n\}$ denote the alphabet used to index the set of marked points, $Z_n = \{z_1, \dots, z_n\}$, on $\mathfrak{M}_{0,n}$. The boundary divisor, $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$, parametrizes stable curves of type $(0, n)$. The stable curves are curves which are nodal Riemann surfaces of genus zero such that each component has at least three marked or singular points. In this article we will use the term boundary divisor to refer to the irreducible components of $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$.

The (irreducible) boundary divisors can be combinatorially enumerated by specifying a partition of S_n into two subsets, A and $S_n \setminus A$, with $2 \leq |A| \leq n-2$ in the following way. Any simple closed loop on a sphere with n marked points partitions the points of Z_n (indexed by S_n) into two subsets as in the left hand side of figure 1. Pinching this loop to a point yields a nodal topological surface

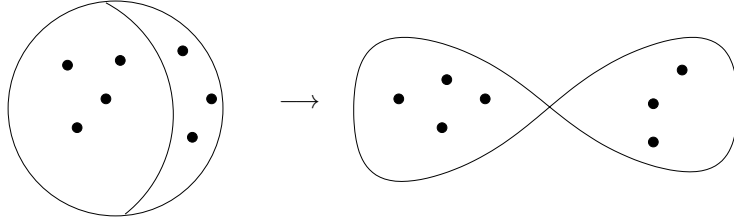


Figure 1: A point on a boundary divisor in $\overline{\mathcal{M}}_{0,n}$

as in the right hand side of figure 1. The stable curves of this type are obtained by putting all possible complex structures on this topological surface. A single boundary component parametrizes these stable curves for a given pinched loop.

We denote by d_A the boundary divisor in which the loop pinches the subset indexed by $A \subset S_n$, hence $d_A = d_{S_n \setminus A}$. We denote the set of (irreducible) boundary divisors on $\overline{\mathcal{M}}_{0,n}$ by D^n . This set has cardinality $2^{n-1} - 1 - n$.

Example 1.3. *The set of boundary divisors, D^4 , on $\overline{\mathcal{M}}_{0,4}$ contains the three divisors, $d_{1,2}, d_{1,3}$ and $d_{1,4}$.*

The set of boundary divisors, D^5 , on $\overline{\mathcal{M}}_{0,5}$ contains the 10 divisors, d_A , where $A \subset S_5$ has cardinality 2.

We now introduce the geometric notation for a boundary divisor which will be useful in the proofs in this paper. We will frequently refer to cyclic orderings, denoted $(i_1, \dots, i_n)$, on the set S_n . One may think of a cyclic ordering as an n -sided, oriented polygon, whose vertices are labeled by the elements of S_n such that i_{p+1} follows i_p in the clockwise direction, and hence $(i_1, \dots, i_n) = (i_2, \dots, i_n, i_1) = \dots = (i_n, i_1, \dots, i_{n-1})$. We denote by P_n the n -sided, oriented polygon whose cyclic ordering is $(1, \dots, n)$ and call this the standard ordering.

Notation 1.4. *To obtain a geometric representation of the boundary divisor d_A associated to $A = \{i_1, \dots, i_p\}$, we partition A into subsets, $B_1, \dots, B_k$, of indices which are adjacent on the polygon P_n , with the condition that there is a unique minimal k . In other words, there is no union of two sets, B_a and B_b that form an adjacent block on P_n . Then, the set $\{B_1, \dots, B_k\}$ inherits a cyclic ordering from P_n . Without loss of generality, we may assume that this cyclic ordering corresponds to the subscripts on the B_i and that the number 1 is an element both of B_1 and of A . Given these conditions, we have that for each B_i there exists a gap, G_i , of adjacent indices on P_n which are not in A and which are exactly the indices between B_i and B_{i+1} on P_n . Then we denote the divisor d_A by the $2k$ -sided polygon whose vertices are cyclically labeled by $(B_1, G_1, B_2, G_2, \dots, G_{k-1}, B_k, G_k)$. In the case where $k = 1$, we draw the divisor polygon as circle with two points, B_1 and G_1 .*

Example 1.5. *For $n = 10$ two divisors are represented in figure 2. The divisor $d_{\{1,2,4,7,10\}}$ is represented by a hexagon and the divisor $d_{\{1,2,3,9,10\}}$ is represented by a two-pointed circle.*

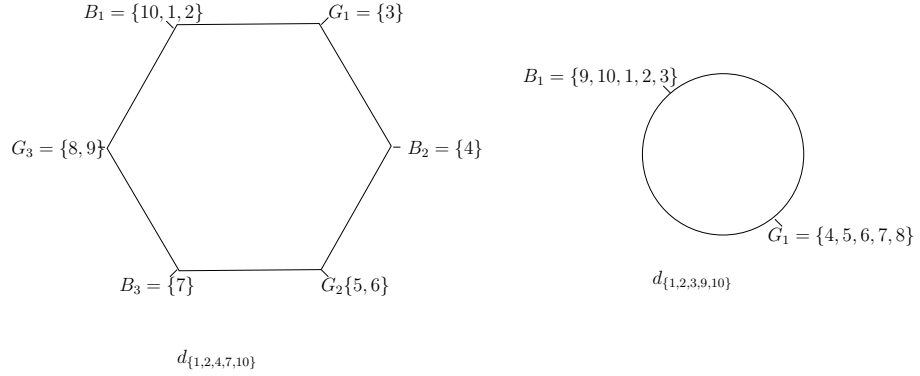


Figure 2: Boundary divisors in $\overline{\mathfrak{M}}_{0,10}$

Theorem 1.6. [Ke] A presentation of the Picard group, $\text{Pic}(\overline{\mathfrak{M}}_{0,n})$, is given by taking the classes, δ_A of the boundary divisors, $d_A \in D^n$ as generators, subject to the following relations: for any four distinct elements, i, j, k, l in S_n ,

$$\sum_{\substack{i,j \in A \\ k,l \notin A}} \delta_A = \sum_{\substack{i,k \in A \\ j,l \notin A}} \delta_A = \sum_{\substack{i,l \in A \\ j,k \notin A}} \delta_A.$$

The following proposition due to A. Gibney [GM] specifies a basis for $\text{Pic}(\overline{\mathfrak{M}}_{0,n})$ and also yields an expression for its dimension.

Proposition 1.7. Let $(i_1, \dots, i_n)$ denote any cyclic ordering. Then a basis for $\text{Pic}(\overline{\mathfrak{M}}_{0,n})$ is given by the divisors defined by nonempty subsets of marked points on the n -gon which do not form an adjacent set of vertices on the n -gon. We call this set of divisors the **non-adjacent basis** for the cyclic ordering $(i_1, \dots, i_n)$.

In the notation 1.4, this proposition means that the set of divisors

$$\{(B_1, G_1, \dots, B_k, G_k) : k \geq 2\}$$

is a basis for $\text{Pic}(\overline{\mathfrak{M}}_{0,n})$. Therefore the main purpose of this paper is to find an expression for the divisors denoted (B_1, G_1) .

Remarks. The following combinatorial formula for the dimension follows immediately from counting the elements of a non-adjacent basis:

$$\dim(\text{Pic}(\overline{\mathfrak{M}}_{0,n})) = 2^{n-1} - 1 - n - \binom{n}{2} + n = 2^{n-1} - 1 - \binom{n}{2}. \quad (1)$$

This dimension was calculated by S. Keel in [Ke] as the dimension of the first Chow group of $\overline{\mathfrak{M}}_{0,n}$.

Example 1.8. Consider the standard ordering, $(1, 2, 3, 4, 5)$. Then the non-adjacent basis for $\text{Pic}(\overline{\mathfrak{M}}_{0,5})$ for this ordering is given by the five divisor classes

$$\delta_{\{1,3\}}, \delta_{\{1,4\}}, \delta_{\{2,4\}}, \delta_{\{2,5\}}, \delta_{\{3,5\}}.$$

Proof. (of Proposition 1.7). Without loss of generality, we prove this proposition for the standard cyclic ordering, $(1, 2, \dots, n)$. We restate this theorem by saying that the set of divisor classes, $\{\delta_A$ such that by notation 1.4, $A = B_1 \sqcup \dots \sqcup B_k$ where $k \geq 2\}$, forms a basis for $Pic(\overline{\mathfrak{M}}_{0,n})$. We will prove this proposition by showing that all divisors corresponding to $k = 1$ may be written as divisors corresponding to $k \geq 2$ and then by justifying that this set of “non-adjacent” divisor classes indeed forms a basis.

Consider any divisor class, δ_I where I is a set of adjacent indices taken modulo n , $\{m, m+1, \dots, m+p\}$. By taking $[i, j, k, l] = [m-1, m, m+p, m+p+1]$, we have the following relation from theorem 1.6:

$$\sum_{\substack{m-1, m+p \in A \\ m, m+p+1 \notin A}} \delta_A = \sum_{\substack{m-1, m+p+1 \in A \\ m, m+p \notin A}} \delta_A. \quad (2)$$

The left-hand side of equation (2) is the sum of divisor classes, δ_A where A is a set of non-adjacent indices, since the partition of any such A is at least two sets of consecutive indices, one containing $m-1$ and the other containing $m+p$. The sets containing $m-1$ and $m+p$ cannot be the same, because any consecutive ordering containing both $m-1$ and $m+p$ must contain either m or $m+p+1$, but there are no such terms on the left-hand side.

The right-hand side is the sum of divisor classes, δ_A , such that A contains $m-1$ and $m+p+1$, and such that both m and $m+p$ are contained in gaps. The only divisor class term indexed by a consecutive set is δ_A , $A = S_n \setminus I$. Therefore for any set of adjacent indices I , we may write equation (2) as

$$\sum_p \delta_{A_p} = \delta_I + \sum_q \delta_{B_q}$$

where the A_p, B_q are all elements of the non-adjacent basis, and hence these divisor classes span the Picard group.

Furthermore, there are exactly $2^{n-1} - 1 - \binom{n}{2}$ non-adjacent divisors, which is the dimension of the Picard group [Ke], so these elements indeed form a basis. $\square$

2 A polygonal presentation of $Pic(\overline{\mathfrak{M}}_{0,n})$

In this section, we give a simple expression of each boundary divisor in $Pic(\overline{\mathfrak{M}}_{0,n})$ in terms of any non-adjacent basis. This yields a new and very simple presentation for $Pic(\overline{\mathfrak{M}}_{0,n})$ with a minimal set of relations.

Theorem 2.1. *Let γ denote a cyclic ordering $(i_1, \dots, i_n)$ on S_n . Then $Pic(\overline{\mathfrak{M}}_{0,n})$ is generated by the set of classes, δ_I , of boundary divisors of $\overline{\mathfrak{M}}_{0,n}$, subject to the relations*

$$\delta_I = \sum_{J \in \mathcal{J}} \delta_J - \sum_{K \in \mathcal{K}} \delta_K, \quad (3)$$

where I denotes a consecutive subset of points for the ordering γ , $\mathcal{J}$ denotes the set of non-adjacent subsets

$$J = B_1 \cup \dots \cup B_j$$

of $\{1, \dots, n\}$ such that I is equal to a “segment” of even length $B_i, G_i, \dots, B_k, G_k$ or $G_i, B_{i+1}, G_{i+1}, \dots, G_k, B_{k+1}$ of $(B_1, G_1, \dots, B_N, G_N)$, and $\mathcal{K}$ denotes the set of non-adjacent subsets $K = B_1 \cup \dots \cup B_j$ such that I is equal to a “segment” of odd length $B_i, G_i, \dots, B_k, G_k, B_{k+1}$ or $G_i, B_{i+1}, G_{i+1}, \dots, B_k, G_k$ of $(B_1, G_1, \dots, B_N, G_N)$.

The beauty of the theorem is more easily seen by rephrasing it as: the coefficients of any divisor in the basis of the Picard group given by a cyclic ordering can be calculated by the parity of the defining blocks of the divisor. The precise statement of the theorem does not do justice to its simplicity, as illustrated in the following example.

Example 2.2. We have the following expression for the divisor, $\delta_{\{1,2,3\}}$, in the basis of $\text{Pic}(\overline{\mathcal{M}}_{0,6})$ given by the cyclic ordering $(1, 2, 3, 4, 5, 6)$:

$$\delta_{\{1,2,3\}} = -\delta_{\{1,3\}} + \delta_{\{1,4\}} + \delta_{\{3,6\}} - \delta_{\{4,6\}} + \delta_{\{1,2,4\}} - \delta_{\{1,3,5\}} + \delta_{\{1,4,5\}}.$$

Proof. Without loss of generality, we may prove this theorem on the standard cyclic ordering $\gamma = (1, 2, \dots, n)$. Let Δ_γ be set of divisors which is a basis for $\text{Pic}(\overline{\mathcal{M}}_{0,n})$ with respect to the cyclic order γ by proposition 1.7. We denote by $\delta_{B_1 \dots B_N} = [B_1, G_1, \dots, B_N, G_N]$ an element of Δ_γ . Let $I = (m, \dots, m+p)$ (where indices are taken modulo n) be a consecutive subset for the cyclic order γ .

Then we may restate the theorem as follows. One can express δ_I as the linear combination of elements of Δ_γ :

$$\delta_I = \sum C_{B_1, \dots, B_{J_k}}^I [B_1, G_1, \dots, B_{J_k}, G_{J_k}], \quad (4)$$

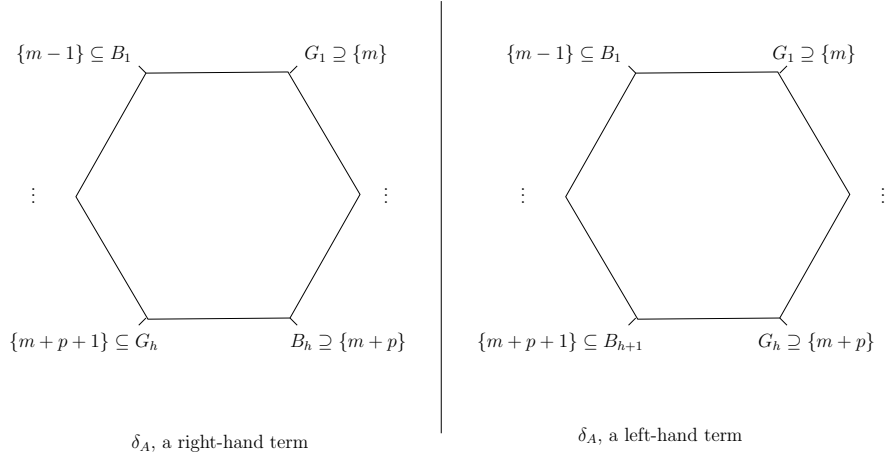
and the coefficients are given by

$$C_{B_1, \dots, B_N}^I = \begin{cases} 1 & I = \bigcup_{p=1}^j B_{i+p} \cup G_{i+p} \\ & \left(\text{or } I = G_i \cup \left(\bigcup_{p=1}^j B_{i+p} \cup G_{i+p} \right) \cup B_{i+j+1} \right) \\ -1 & I = \left(\bigcup_{p=1}^j B_{i+p} \cup G_{i+p} \right) \cup B_{i+j+1} \\ & \left(\text{or } I = G_i \cup \left(\bigcup_{p=1}^j B_{i+p} \cup G_{i+p} \right) \right) \\ 0 & \text{otherwise,} \end{cases} \quad (5)$$

where for $1 < p < j$, $i+p$ is taken modulo n .

To prove this theorem, we take $[i, j, k, l] = [m-1, m, m+p, m+p+1]$ as in equation (2),

$$\sum_{\substack{m-1, m+p \in A \\ m, m+p+1 \notin A}} \delta_A = \sum_{\substack{m-1, m+p+1 \in A \\ m, m+p \notin A}} \delta_A. \quad (6)$$



I claim that the left-hand side of equation (6) is a sum of all of the divisor classes, δ_A , such that A the I is the disjoint union of an even number (≥ 2), of B_p, G_q (as in notation 1.4) and the right-hand side of this equation is the sum of δ_A such that I can be expressed as the disjoint union of an odd number (≥ 1) of B_p, G_q . The justification of this claim proves the theorem.

Any divisor class, δ_A , which is a term on the left-hand side of equation (6) has a polygonal representation as in figure 3. Therefore I is the disjoint union of the even number of sets, $G_1 \sqcup B_2 \sqcup G_2 \sqcup \cdots \sqcup G_{h-1} \sqcup B_h$. While any divisor class, δ_A , which is a term on the right-hand side of (6) has a polygonal representation as in figure 3, so that I is the disjoint union of an odd number of sets, $G_1 \sqcup B_2 \sqcup \cdots \sqcup B_h \sqcup G_h$.

By bringing all terms except δ_I over to the left-hand side of equation (6), we have the desired expression in the statement of the theorem. □

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