Proof-Theoretic Analysis of Termination Proofs

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Introduction

In [Cichon 1990] the question has been discussed (and investigated) whether the order type of a termination ordering $\prec$ places a bound on the lengths of reduction sequences in rewrite systems reducing under $\prec$. It was claimed that at least in the cases of the recursive path ordering $\prec_{rpo}$ and the lexicographic path ordering $\prec_{lpo}$ the following theorem holds.

(0) If Λ is the order type of a termination ordering $\prec$ for a finite rewrite system $\mathcal{R}$ then the function G_Λ from the Slow-Growing Hierarchy bounds the lengths of reduction sequences in $\mathcal{R}$.

From (0) together with Girard's Hierarchy Comparison Theorem one derives

(I) If the rules of a finite rewrite system $\mathcal{R}$ are reducing under $\prec_{rpo}$ then the lengths of reduction sequences in $\mathcal{R}$ are bounded by some primitive recursive function.

(II) If the rules of a finite rewrite system $\mathcal{R}$ are reducing under $\prec_{lpo}$ then the lengths of reduction sequences in $\mathcal{R}$ are bounded by some function F_α from the fast-growing hierarchy below ω^ω.

Unfortunately the proof of (0) given in [Cichon 1990] (or [Cichon 1993]) contains a major error (namely Lemma 4.16 in [Cichon 1990], Lemma 6.9 in [Cichon 1993]). But in [Hofbauer 1990] and [Weiermann 199?], resp., by means of rather cumbersome calculations the correctness of (I) and (II) has been shown directly, i.e. without referring to the slow-growing hierarchy. In the present paper we give alternative proofs of (I) and (II) which avoid those cumbersome calculations and in addition provide very good insight into the relationship between the "strength" of a termination ordering $\prec$ and the derivation lengths in rewrite systems reducing under $\prec$. We show that if a finite rewrite system $\mathcal{R}$ is reducing under $\prec_{rpo}$ ($\prec_{lpo}$, resp.) then termination of $\mathcal{R}$ can be proved within the fragment Σ^0_1-IA (Π^0_2-IA, resp.) of Peano-Arithmetic PA. Combining this with the well-known proof-theoretical result on bounds for provable Π^0_2-sentences in fragments of PA (cf. [Parsons 1966]) yields (I) and (II).

Since the treatment of $\prec_{lpo}$ is particularly simple, we start with the proof of (II) which runs as follows. In §1 we carry out a termination (or wellfoundedness) proof for $\prec_{lpo}$ which as its main tool uses the Π^1_1-set $W := \bigcap\{X \subseteq T : \forall t(\forall s{\prec_{lpo}}t(s \in X) \to t \in X)\}$ (i.e. the so-called *accessible part* of $\prec_{lpo}$). Then in §2 we take advantage of the fact that for proving termination of a single *finite* rewrite system reducing under $\prec_{lpo}$ one does not need the full relation $\prec_{lpo}$, since every such system $\mathcal{R}$ is already reducing under a suitable "approximation" $\prec_k$ of $\prec_{lpo}$ (with $k \in \mathbb{N}$ depending on $\mathcal{R}$). The essential property of $\prec_k$ is that for every term t there are only finitely many predecessors $s \prec_k t$ and therefore the accessible part W_k of $\prec_k$ can already be defined by a Σ^0_1-formula. Moreover by replacing in the termination proof for $\prec_{lpo}$ all occurrences of $\prec_{lpo}$ and W by $\prec_k$, W_k, resp., one obtains a termination proof for $\prec_k$ which is formalizable in the fragment Π^0_2-IA of Peano-Arithmetic. It follows that if $\mathcal{R}$ is reducing under $\prec_k$ then Π^0_2-IA proves the Π^0_2-sentence saying that for every term t there exists an $l \in \mathbb{N}$ such that every $\mathcal{R}$-reduction sequence starting with t has length less than l. Now one applies the above mentioned result from classical proof-theory and obtains (II). The main idea for the just sketched proof (namely the transition from the Π^1_1-set W to the Σ^0_1-set(s) W_k)

comes from [Arai 1991] where a similar method has been used to establish one direction of the Hierarchy Comparison Theorem (cf. also [Schmerl 1981]).

In §3 we define suitable approximations $\prec_k$ for the recursive path ordering $\prec_{rpo}$ and prove wellfoundedness of $\prec_k$ within Σ^0_1-IA. Then (I) is established in the same way as (II).

§1 A termination proof for the lexicographic path ordering

Let $p \in \mathbb{N}$, and let $f_0, ..., f_p$ be function symbols where each f_ν has a fixed arity $\#(f_\nu)$.
Let T be the set of all terms built up from variables $v_0, v_1, ...$ by means of $f_0, ..., f_p$.
In the following s, t, s_i, t_i denote elements of T, and i, j, k, l, m, n denote natural numbers.

Abbreviation
By $\mathcal{A}(\prec, s, t)$ we abbreviate the following proposition:
t is of the form $f_\nu t_1...t_n$ and one of the following three cases holds
($\prec$ 1) $s \preceq t_j$ *for some* $j \in \{1, ..., n\}$
($\prec$ 2) $s = f_\mu s_1...s_m$ *with* $\mu < \nu$ *and* $s_1, ..., s_m \prec t$
($\prec$ 3) $s = f_\nu s_1...s_n$ *and there is a* $j \in \{1, ..., n\}$ *such that* $\forall i{<}j(s_i = t_i) \wedge s_j \prec t_j \wedge s_{j+1}, ..., s_n \prec t$.
As usual $s \preceq t$ abbreviates $s \prec t \vee s = t$.

Definition
The *lexicographic path ordering* $\prec_{lpo}$ on T is the least binary relation $\prec$ such that $\forall s, t(\mathcal{A}(\prec, s, t) \to s \prec t)$.

Remark : As an immediate consequence from this definition we get: $\forall s, t(s \prec_{lpo} t \to \mathcal{A}(\prec_{lpo}, s, t))$.

We now prove that $(T, \prec_{lpo})$ is wellfounded.

To simplify notation we write $\prec$ for $\prec_{lpo}$.

Definition
Let W be the *accessible part* of $(T, \prec)$, i.e. $W := \bigcap\{X \subseteq T : \forall t(\forall s{\prec}t(s \in X) \to t \in X)\}$.

Corollary
(W1) $\forall t(\forall s{\prec}t(s \in W) \leftrightarrow t \in W)$,
(W2) $\forall t{\in}W(\forall s{\prec}t\, F(s) \to F(t)) \to \forall t{\in}W\, F(t)$, for each predicate (formula) F.

Definition
$(s_1, ..., s_n) \prec^{lex} (t_1, ..., t_n) \;:\Longleftrightarrow\; \exists j{\in}\{1, ..., n\}[\, s_j \prec t_j \wedge \forall i{<}j(s_i = t_i)\,]$.

Lemma 1 (Transfinite induction over $(W^n, \prec^{lex})$)
$\forall t_1, ..., t_n \in W[\,\forall s_1, ..., s_n \in W(\,(s_1, ..., s_n) \prec^{lex} (t_1, ..., t_n) \to G(s_1, ..., s_n)\,) \;\to\; G(t_1, ..., t_n)\,] \to$
$\to \forall t_1, ..., t_n \in W\, G(t_1, ..., t_n)$.

Proof by induction on n:
1. $n = 1$: Trivial consequence of (W1),(W2).
2. $n > 1$: Abbreviations:
$\overline{G}(t_1) :\equiv \forall s_2, ..., s_n \in W\, G(t_1, s_2, ..., s_n)$,
$A :\equiv \forall t_1, ..., t_n \in W[\,\forall s_1, ..., s_n \in W(\,(s_1, ..., s_n) \prec^{lex} (t_1, ..., t_n) \to G(s_1, ..., s_n)\,) \to G(t_1, ..., t_n)\,]$,
$B :\equiv t_1 \in W \wedge \forall s_1 \prec t_1 \overline{G}(s_1)$,
$C :\equiv t_2, ..., t_n \in W \wedge \forall s_2, ..., s_n \in W(\,(s_2, ..., s_n) \prec^{lex} (t_2, ..., t_n) \to G(t_1, s_2, ..., s_n)\,)$.

Then we get
$B \wedge C \;\to\; t_1, ..., t_n \in W \wedge \forall s_1, ..., s_n \in W(\,(s_1, ..., s_n) \prec^{lex} (t_1, ..., t_n) \to G(s_1, ..., s_n)\,)$,
$A \wedge B \wedge C \;\to\; G(t_1, t_2, ..., t_n)$,
$A \wedge B \;\to\; \forall t_2, ..., t_n \in W[\,\forall s_2, ..., s_n \in W(\,(s_2, ..., s_n) \prec^{lex} (t_2, ..., t_n) \to G(t_1, s_2, ..., s_n)\,) \;\to\; G(t_1, t_2, ..., t_n)\,]$,
$A \wedge B \;\to\; \forall t_2, ..., t_n \in W\, G(t_1, t_2, ..., t_n)$, [by IH]
$A \;\to\; \forall t_1 \in W(\,\forall s_1 \prec t_1 \overline{G}(s_1) \;\to \overline{G}(t_1)\,)$,
$A \;\to\; \forall t_1 \in W\, \overline{G}(t_1)$ [by (W2)]. □

Lemma 2
$\forall t_1, ..., t_n \in W(\, f_\nu t_1 ... t_n \in W\,)$, where $n := \#(\nu)$.

Proof by induction on ν:
By Lemma 1 it suffices to prove:
$\forall t_1, ..., t_n \in W[\,\forall s_1, ..., s_n \in W((s_1, ..., s_n) \prec^{lex} (t_1, ..., t_n) \;\to\; f_\nu s_1 ... s_n \in W) \;\to\; f_\nu t_1 ... t_n \in W\,]$.
So let us assume that $t_1, ..., t_n \in W$ and $\forall s_1, ..., s_n \in W((s_1, ..., s_n) \prec^{lex} (t_1, ..., t_n) \;\to\; f_\nu s_1 ... s_n \in W)$ $(*)$.
By side induction on the build-up of s we prove: $s \prec f_\nu t_1 ... t_n \to s \in W$. Then (W1) yields $f_\nu t_1 ... t_n \in W$.
So let $s \prec t := f_\nu t_1 ... t_n$. Then one of the following three cases holds.
1. $s \preceq t_j$: In this case $s \in W$ follows from $t_j \in W$ by (W1).
2. $s = f_\mu s_1 ... s_m$ with $\mu < \nu$ and $s_1, ..., s_m \prec t$:
Then by SIH we have $s_1, ..., s_m \in W$ which by MIH yields $s \in W$.
3. $s = f_\nu s_1 ... s_n$ with $s_1 = t_1, ..., s_{j-1} = t_{j-1}, s_j \prec t_j$ and $s_{j+1}, ..., s_n \prec t$:
Then $(s_1, ..., s_n) \prec^{lex} (t_1, ..., t_n)$, and by SIH we have $s_1, ..., s_n \in W$.
Therefore the assumption $(*)$ yields $s \in W$. □

Lemma 3. $\forall t(\, t \in W\,)$.

Proof by induction on the build-up of t using Lemma 2.

Corollary. There is no infinite $\prec$-descending sequence $(t_i)_{i \in \mathbb{N}}$.

Proof: By (W2) one obtains for each $t \in W$: *There exists no infinite $\prec$-descending sequence $(t_i)_{i \in \mathbb{N}}$ with* $t_0 = t$. From this the claim follows by Lemma 3. □

§2 Proof-theoretic analysis

Now we analyze the just given wellfoundedness proof for $(T, \prec)$. The first observation is that we did not use the implication $\mathcal{A}(\prec, s, t) \to s \prec t$ but only its reverse direction (namely in the proof of Lemma 2). Secondly we observe that complete induction has only been used w.r.t. the following formulas $\Phi(x)$:
$\Phi(x) :\equiv (x \in W)$ [in the proof of Lemma 3],
$\Phi(x) :\equiv (x \prec f_\nu t_1 ... t_n \to x \in W)$ [in the proof of Lemma 2].
Further in the proof of Lemma 2 we used Lemma 1 for $G(t_1, ..., t_n) :\equiv (f_\nu t_1 ... t_n \in W)$, and in the proof of Lemma 1 we used (W2) for $F(t) :\equiv \overline{G}(t)$. Hence in the whole wellfoundedness proof the scheme (W2) is only needed for the formulas $F(t) :\equiv \forall t_2, ..., t_n \in W(f_\nu t\, t_2 ... t_n \in W)$.
Putting things together we obtain the following meta-theorem:
If $\prec$ is a primitive recursive relation on T such that Π^0_2-IA proves $\forall s, t(s \prec t \to \mathcal{A}(\prec, s, t))$ and if W is a Σ^0_1-set such that Π^0_2-IA proves (W1) and (W2) for all Π^0_2-formulas $F(t)$ then the wellfoundedness proof from §1 can be formalized in Π^0_2-IA, and thus Π^0_2-IA proves $\forall t(t \in W)$.

Below we define (for each $k \in \mathbb{N}$) a relation $\prec_k$ on T and a subset W_k of T such that $\prec_k$ and W_k satisfy the assumptions of the just stated meta-theorem (cf. Lemma 4). Hence this theorem yields Π^0_2-IA $\vdash \forall t(t \in W_k)$.

Definition of $|t|$ for each $t \in T$

1. $|v_i| := i$
2. $|f_\nu t_1...t_n| := \max\{n, |t_1|, ..., |t_n|\} + 1$.

Inductive Definition of $\prec_k$

$\mathcal{A}(\prec_k, s, t)$ & $|s| \leq k + |t| \implies s \prec_k t$.

Corollary: $s \prec_k t \implies \mathcal{A}(\prec_k, s, t)$ & $|s| \leq k + |t|$.

Remark

In the following we assume some canonical arithmetization of terms and identify each term with its numerical code (Gödel number). According to this T and $\prec_k$ are primitive recursive relations. Without loss of generality we may assume that there is an increasing primitive recursive function h such that $|t| \leq t < h(|t|)$ for all t. Hence $\forall s \prec_k t\, F(s) \Leftrightarrow \forall s < h(k+t)[\, s \prec_k t \to F(s)\,]$ and therefore $\forall s \prec_k t$ can be treated as a bounded quantifier.

Definition

$t \in (t_0, ..., t_{n-1}) \;:\Leftrightarrow\; \exists i < n(t = t_i)$

$\mathcal{D}_k := \{(t_0, ..., t_l) : \forall j \leq l \forall s \prec_k t_j(s \in (t_0, ..., t_{j-1}))\}$

$W_k := \{t : \exists d(d \in \mathcal{D}_k \wedge t \in d)\}$

The elements of $\mathcal{D}_k$ are called k-*derivations*.

Lemma 4

In Π^0_2-IA the following is provable

(W_k1) $\forall t(\forall s \prec_k t(s \in W_k) \leftrightarrow t \in W_k)$.

(W_k2) $\forall t{\in}W_k(\forall s \prec_k t\, F(s) \to F(t)) \;\to\; \forall t{\in}W_k\, F(t)$, for each Π^0_2-formula F.

Proof:

(W_k1) "$\Leftarrow$": obvious.

"$\Rightarrow$": (1) $\forall s \prec_k t \exists d(d \in \mathcal{D}_k \wedge s \in d) \to \exists d(d \in \mathcal{D}_k \wedge \forall s \prec_k t(s \in d))$.

Proof: Under the assumption $\forall s \prec_k t \exists d(d \in \mathcal{D}_k \wedge s \in d)$ one proves $\exists d(d \in \mathcal{D}_k \wedge \forall s < n(s \prec_k t \to s \in d))$ by induction on n. Since $s \prec_k t$ implies $s < h(k+t)$ this yields $\exists d(d \in \mathcal{D}_k \wedge \forall s \prec_k t(s \in d))$.

Assume that $d \in \mathcal{D}_k \wedge \forall s < n(s \prec_k t \to s \in d)$. If $n \prec_k t$ does not hold then $\forall s < n{+}1(s \prec_k t \to s \in d)$. If $n \prec_k t$ holds then by assumption there exists some $\tilde{d} \in \mathcal{D}_k$ with $n \in \tilde{d}$, and it follows that $d * \tilde{d} \in \mathcal{D}_k$ and $\forall s < n{+}1(s \prec_k t \to s \in d * \tilde{d})$.

By definition of $\mathcal{D}_k$ we have (2) $d \in \mathcal{D}_k \wedge \forall s \prec_k t(s \in d) \to d * (t) \in \mathcal{D}_k$.

From (1) and (2) we get $\forall s \prec_k t(s \in W_k) \to t \in W_k$.

(W_k2): Assume $\forall t{\in}W_k(\forall s \prec_k t\, F(s) \to F(t))$ and $t \in W_k$. Then $t \in (t_0, ..., t_l)$ for some k-derivation $(t_0, ..., t_l)$. By induction on i we prove $\forall i \leq l\, F(t_i)$. So let $j \leq l$. Then $\forall s \prec_k t_j(s \in (t_0, ..., t_{j-1}))$ and by IH $\forall i < j\, F(t_i)$. Hence $\forall s \prec_k t_j\, F(s)$ and therefore $F(t_j)$, since $t_j \in W_k$. □

As explained above the contents of §1 together with Lemma 4 yield Π^0_2-IA $\vdash \forall t(t \in W_k)$, i.e. Π^0_2-IA $\vdash \forall t \exists d(d \in \mathcal{D}_k \wedge t \in d)$. Therefore according to [Parsons 1966] there exists an $\alpha < \omega^\omega$ such that $\forall t \exists d \leq F_\alpha(t)(d \in \mathcal{D}_k \wedge t \in d)$ and consequently $\forall(t, t_1, ..., t_n)[\, t_n \prec_k \ldots \prec_k t_1 \prec_k t \to n < F_\alpha(t)\,]$. (Note that if $t_n \prec_k \ldots \prec_k t_1 \prec_k t$, and d is a k-derivation of t then $(t_n, ..., t_1, t)$ is a subsequence of d and thus $n < d$.)

Now let $\mathcal{R}$ be some finite rewrite system over T which is reducing under $\prec_{lpo}$, i.e. $\mathcal{R}$ is a finite subset of $\{(\ell,r) \in T \times T : r \prec_{lpo} \ell\}$. As usual $\to_{\mathcal{R}}$ denotes the rewrite relation generated by $\mathcal{R}$, i.e. $t \to_{\mathcal{R}} s$ iff there exists $(\ell,r) \in \mathcal{R}$ and a substitution θ such that s results from t by replacing one occurrence of $\ell\theta$ in t by $r\theta$. Below (in Lemma 7) we will prove that $\to_{\mathcal{R}}$ is contained in $\prec_k$ with $k := \max\{|r| : (\ell,r) \in \mathcal{R}\}$. Therefore the just established bound on the lengths of $\prec_k$-descending sequences is also a bound for the lengths of $\mathcal{R}$-reduction sequences. This finishes the proof of (II).

Lemma 5
$s \prec_{lpo} t \implies s\theta \prec_{|s|} t\theta$, for each substitution θ.

Proof: One first proves $s \prec_{lpo} t \Rightarrow |s\theta| \leq |s| + |t\theta|$ by induction on the definition of $\prec_{lpo}$. Using this one then obtains $s \prec_{lpo} t \Rightarrow s\theta \prec_{|s|} t\theta$ by another induction of this kind. □

Lemma 6
If $t = f_\nu t_1 ... t_n$ and $s = f_\nu t_1 ... t_{j-1} t'_j t_{j+1} ... t_n$ with $t'_j \prec_k t_j$ then $s \prec_k t$.

Proof: By ($\prec_k$ 1) we have $t_{j+1}, ..., t_n \prec_k t$, and from $t'_j \prec_k t_j$ we get $|t'_j| \leq k + |t_j| < k + |t|$. Hence $|s| \leq k + |t|$ and therefore $s \prec_k t$ by ($\prec_k$ 3). □

Lemma 7
$t \to_{\mathcal{R}} s \implies s \prec_k t$, with $k := \max\{|r| : (\ell,r) \in \mathcal{R}\}$.

Proof:
By Lemma 5 we have $r\theta \prec_{|r|} \ell\theta$ and thus $r\theta \prec_k \ell\theta$ for each $(\ell,r) \in \mathcal{R}$ and each substitution θ. From this together with Lemma 6 we obtain the assertion by induction on $|t|$. □

§3 Treatment of the recursive path ordering $\prec_{rpo}$

In this section we indicate briefly how a proof of (I) can be obtained by some minor modifications from the proof of (II) given in §1 and §2. We only present a list of definitions and lemmata and leave it to the reader to compose from that a proof of (I) by observing that all what follows can be formalized in Σ^0_1-IA.

$f_0, ..., f_p$ are now assumed to be varyadic function symbols.
T^* denotes the set of all finite sequences of terms $t \in T$.
Every term $t \in T$ is identified with the one element sequence $(t) \in T^*$. Hence $T \subseteq T^*$.
We use s, t as syntactic variables for elements of T, and a, b, c as syntactic variables for elements of T^*.
For $a = (t_1, ..., t_n)$ we set $|a| := \max\{n, |t_1|, ..., |t_n|\}$ and $f_\nu a := f_\nu t_1 ... t_n$. Hence $|f_\nu a| = |a| + 1$.
Further we define: $(s_0, ..., s_{m-1}) \approx (t_0, ..., t_{n-1}) \;:\Leftrightarrow\; m = n \wedge \exists\,\text{permutation } \pi \text{ of } n \; \forall i{<}n (t_i = s_{\pi(i)})$

We forget the definition of $\prec_k$ given in §2.

Inductive Definition of $b \prec_k a$ for $a, b \in T^*$
1. $s \preceq_k t_j \;\&\; j \in \{1, ..., n\} \implies s \prec_k f_\nu t_1 ... t_n$
2. $t = f_\nu t_1 ... t_n \;\&\; [\, b = f_\mu s_1 ... s_m \text{ with } \mu < \nu \text{ or } b = (s_1, ..., s_m)\,] \;\&\; s_1, ..., s_m \prec_k t \;\&\; |b| \leq k + |t| \implies b \prec_k t$
3. $t = f_\nu t_1 ... t_n \;\&\; s = f_\nu s_1 ... s_m \;\&\; (s_1, ..., s_m) \prec_k (t_1, ..., t_n) \;\&\; |s| \leq k + |t| \implies s \prec_k t$
4. $a \approx (t_0, ..., t_n) \;\&\; b \approx b_0 * \ldots * b_n \;\&\; n \geq 1 \;\&\; \forall i \leq n (b_i \preceq_k t_i) \;\&\; \exists i \leq n (b_i \prec_k t_i) \implies b \prec_k a$

Note that in rule 2 (and also in rule 3) $m = 0$ is allowed. Hence $(\,) \prec_k t$ for each $t \in T$.

Definition of $\prec_{rpo}$

The recursive path ordering $\prec_{rpo}$ on T^* is inductively defined by the same rules as $\prec_k$ only that in rule 2 and rule 3 the condition $|\cdot| \leq k + |t|$ is omitted.

Lemma 8

a) $b \prec_k t \implies |b| \leq k + |t|$.

b) $b \prec_k a \implies |b| \leq |a| \cdot (k + |a|)$.

Proof:

a) trivial.

b) Let $a \approx (t_0, ..., t_n)$ and $b \approx b_0 * \ldots * b_n$ with $\forall i \leq n(b_i \preceq_k t_i)$. Then $\forall i \leq n(|b_i| \leq k + |t_i| \leq k + |a|)$ and thus $|b| \leq |b_0| + \ldots + |b_n| \leq (n{+}1) \cdot (k + |a|) \leq |a| \cdot (k + |a|)$. □

Definition

$\mathcal{D}_k := \{(a_0, ..., a_l) : \forall j \leq l \forall c \prec_k a_j (c \in (a_0, ..., a_{j-1}))\}$

$W_k := \{a \in T^* : \exists d(d \in \mathcal{D}_k \wedge a \in d)\}$

Lemma 9

(W$_k$1) $\forall a(\forall b{\prec_k}a(b \in W_k) \leftrightarrow a \in W_k)$

(W$_k$2) $\forall a{\in}W_k(\forall b{\prec_k}a\, F(b) \to F(a)) \to \forall a{\in}W_k\, F(a)$, for all $F \in \Sigma^0_1$.

Lemma 10

If $c \prec_k a * b$ then there are a_1, b_1 such that $c \approx a_1 * b_1$ and $[a_1 = a \wedge b_1 \prec_k b]$ or $[a_1 \prec_k a \wedge b_1 \preceq_k b]$.

Proof:

Let $c \prec_k a * b$ with $a = (t_0, ..., t_{l-1})$ and $b = (t_l, ..., t_{n-1})$. Then there are $c_0, ...c_{n-1}$ with $c \approx c_0 * \ldots * c_{n-1}$ and $\forall i < n(c_i \preceq_k t_i)$. Let $a_1 := c_0 * \ldots * c_{l-1}$ and $b_1 := c_l * \ldots * c_{n-1}$. □

Lemma 11

$a \in W_k \wedge b \in W_k \to a * b \in W_k$.

Proof:

(1) $(c_0, ..., c_{n-1}) \in \mathcal{D}_k \wedge \forall x \prec_k a \forall i < n(x * c_i \in W_k) \to \forall i < n(a * c_i \in W_k)$.

Proof: We prove $a * c_i \in W_k$ by induction on i.

So let $i < n$ and $b := c_i$. We show $\forall c \prec_k a * b(c \in W_k)$.

Let $c \prec_k a * b$. By Lemma 10 we have $c \approx a_1 * b_1$ with $[a_1 = a \wedge b_1 \prec_k b]$ or $[a_1 \prec_k a \wedge b_1 \preceq_k b]$.

Case 1: $a_1 = a \wedge b_1 \prec_k b$. Then $b_1 = c_j$ with $j < i$. Hence, by I.H., $c \approx a * b_1 \in W_k$.

Case 2: $a_1 \prec_k a \wedge b_1 \preceq_k b$. Then $b_1 = c_j$ with $j \leq i$. Hence $a_1 * b_1 \in W_k$ by assumption.

From (1) and (W_k2) we get

(2) $(c_0, ..., c_{n-1}) \in \mathcal{D}_k \wedge a \in W_k \to \forall i < n(a * c_i \in W_k)$,

(3) $d \in \mathcal{D}_k \wedge a \in W_k \wedge b \in d \to a * b \in W_k$,

(4) $a \in W_k \wedge b \in W_k \to a * b \in W_k$. □

Lemma 12

$\forall a \in W_k(\, f_\nu a \in W_k\,)$.

Proof by induction on ν:

We prove $\forall a \in W_k(\,\forall b \prec_k a(f_\nu b \in W_k) \to f_\nu a \in W_k\,)$. The claim then follows by (W$_k$2).

So assume $a = (t_1, ..., t_n) \in W_k$ and $\forall b \prec_k a(\, f_\nu b \in W_k\,)$.

By side induction on the build-up of c we prove $c \in W_k$ for all $c \prec_k f_\nu a$. Then (W_k1) yields $f_\nu a \in W_k$.
Case 1: $c = c_0 * c_1$ with $c_0, c_1 \neq ()$.
Then $c_0, c_1 \prec_k f_\nu a$ and therefore by SIH $c_0, c_1 \in W_k$. From this we get $c \in W_k$ by Lemma 11.
Case 2: $c \in T$ and $c \preceq_k t_j$ with $1 \leq j \leq n$. Then $c \in W_k$ follows from $c \preceq_k t_j \preceq_k a \in W_k$ by (W_k1).
Case 3: $c = f_\mu s_1 ... s_m$ with $\mu < \nu$ and $s_1, ..., s_m \prec_k f_\nu a$ and $|c| \leq k + |f_\nu a|$.
Then $\tilde{c} := (s_1, ..., s_m) \prec_k f_\nu a$, and the SIH yields $\tilde{c} \in W_k$. From this we obtain $c = f_\mu \tilde{c} \in W_k$ by MIH.
Case 4: $c = f_\nu b$ with $b \prec_k a$. Then by assumption $c \in W_k$. □

Lemma 13
$\forall a (\, a \in W_k \,)$.
Proof by induction on the build-up of a using Lemma 11 and Lemma 12.

Lemma 14
$b \prec_{rpo} a \implies b\theta \prec_{|b|} a\theta$ for each substitution θ.

Lemma 15
If $\mathcal{R}$ is a finite rewrite system reducing under $\prec_{rpo}$ then the rewrite relation $\rightarrow_{\mathcal{R}}$ is contained in $\prec_k$ with $k := \max\{|r| : (\ell, r) \in \mathcal{R}\}$.

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