Ordinal notations and fundamental sequences

Wilfried Buchholz

Universität München

buchholz@mathematik.uni-muenchen.de

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§1 Fundamental sequences for ordinals $< \varepsilon_{\Omega+1}$

Abbreviations.

$\mathsf{Succ} := \{\alpha+1 : \alpha \in On\}$, $P := \{\omega^\alpha : \alpha \in On\}$;

$\alpha =_{NF} \gamma + \Omega^\eta\beta \;:\Leftrightarrow\; \alpha = \gamma + \Omega^\eta\beta$ with $\gamma = \Omega^{\eta+1}\cdot\tilde{\gamma}$ and $0 < \beta < \Omega$.

Definition (Fundamental sequences for limit ordinals $< \varepsilon_{\Omega+1}$)

Let $\alpha =_{NF} \gamma + \Omega^\eta\beta \in Lim$.

$$\alpha[x] := \begin{cases} \gamma + \Omega^\eta x \\ \gamma + \Omega^\eta\beta_0 + \Omega^{\eta[x]} \\ \gamma + \Omega^\eta\beta_0 + \Omega^{\eta_0}x \end{cases} \quad \tau(\alpha) := \begin{cases} \beta & \text{if } \beta \in Lim \cap \Omega \\ \tau(\eta) & \text{if } \beta = \beta_0{+}1 \ \&\ \eta \in Lim \\ \Omega & \text{if } \beta = \beta_0{+}1 \ \&\ \eta = \eta_0{+}1 \end{cases}$$

We also set $\tau(\alpha+1) := 1$ and $\tau(0) := 0$.

Proposition.

If $\alpha \in Lim$ then $(\alpha[x])_{x<\tau(\alpha)}$ is a fundamental sequence for α, i.e.,

$\forall x, y(x < y < \alpha \Rightarrow \alpha[x] < \alpha[y] < \alpha)$ and $\alpha = \sup_{x<\tau(a)} \alpha[x]$.

Definition. $\alpha^* := \begin{cases} 0 & \text{if } \alpha = 0 \\ \max\{\gamma^*, \eta^*, \beta\} & \text{if } \alpha =_{NF} \gamma + \Omega^\eta\beta \end{cases}$.

Lemma 1.1. For $\alpha \in Lim$ and $x < \tau(\alpha)$ the following holds

(a) $x \le \alpha[x]^*$;

(b) $\tau(\alpha) < \Omega \;\Rightarrow\; \tau(\alpha) \le \alpha^*$;

(c) $\alpha[x]^* \le \max\{\alpha^*, x\}$ and $(\tau(\alpha) < \Omega \;\Rightarrow\; \alpha[x]^* \le \alpha^*)$;

(d) $\tau(\alpha) \ne \alpha^* \ \&\ 1 \le x \;\Rightarrow\; \alpha^* \le \alpha[x]^* + 1$;

(e) $\alpha[x] \le \delta < \alpha \;\Rightarrow\; \alpha[x]^* \le \delta^*$.

§2 The function $\vartheta : \varepsilon_{\Omega+1} \to \Omega$

In the following, $\alpha, \gamma, \delta, \xi, \eta, \theta$ denote ordinals $< \varepsilon_{\Omega+1}$, and β denotes ordinals $< \Omega$.

Definition. $\vartheta\alpha := \min\{\theta \in P : \alpha^* < \theta \ \&\ \forall \xi < \alpha(\xi^* < \theta \Rightarrow \vartheta\xi < \theta)\}$

Lemma 2.1.
(a) $\alpha^* < \vartheta\alpha$.
(b) $\alpha_0 < \alpha \ \&\ \alpha_0^* < \vartheta\alpha \ \Rightarrow\ \vartheta\alpha_0 < \vartheta\alpha$.
(c) $\vartheta\alpha_0 = \vartheta\alpha_1 \ \Rightarrow\ \alpha_0 = \alpha_1$.
(d) $\beta_0 < \beta < \Omega \ \Rightarrow\ \vartheta(\Omega\alpha + \beta_0) < \vartheta(\Omega\alpha + \beta)$.

Proof:
(c) Assume $\alpha_0 < \alpha_1$. Then $\alpha_0 < \vartheta\alpha_1$ (which yields $\vartheta\alpha_0 < \vartheta\alpha_1$) or $\vartheta\alpha_1 \le \alpha_0^* < \vartheta\alpha_0$.
(d) $(\Omega\alpha + \beta_0)^* = \max\{(\Omega\alpha)^*, \beta_0\} \le \max\{(\Omega\alpha)^*, \beta\} = (\Omega\alpha + \beta)^* < \vartheta(\Omega\alpha + \beta)$.

Lemma 2.2. $\forall \alpha < \varepsilon_{\Omega+1}(\vartheta\alpha < \Omega)$.

Proof by transfinite induction on α:
We show that $X_\alpha := \{\theta < \Omega : \forall \xi < \alpha(\xi^* < \theta \Rightarrow \vartheta(\xi) < \theta)\}$ is closed unbounded in Ω.
closed: Let $\emptyset \ne U \subseteq X_\alpha$. $\xi < \alpha \ \&\ \xi^* < \sup(U) \ \Rightarrow\ \xi < \alpha \ \&\ \xi^* < \theta$ for some $\theta \in U \ \Rightarrow\ \vartheta\xi < \theta \le \sup(U)$.
unbounded: Let $\beta_0 < \Omega$ be given. We set $\beta_{n+1} := \sup^+\{\vartheta\xi : \xi < \alpha \ \&\ \xi^* < \beta_n\}$.
$\beta_n < \Omega \ \Rightarrow\ card(\{\xi : \xi^* < \beta_n\}) < \Omega \ \Rightarrow\ \beta_{n+1} < \Omega$. Hence $\beta := \sup_{n\in\omega} \beta_n < \Omega$.
Moreover it can easily be seen that $\beta \in X_\alpha$.

Definition.
$C(\alpha, \beta) :=$ the least set X such that $\beta \cup \{\xi + \eta : \xi, \eta \in X\} \cup \{\vartheta(\xi) : \xi < \alpha \ \&\ \xi^* \in X\} \subseteq X$.

Lemma 2.3.
(a) $C(\alpha, \alpha^*{+}1) = C(\alpha, \vartheta(\alpha)) = \vartheta(\alpha)$.
(b) $\beta \in \vartheta(\alpha) \cap P \implies \beta \le \alpha^*$ or $\beta = \vartheta(\xi)$ with $\xi < \alpha$.

Proof:
(1) $C(\alpha, \vartheta(\alpha)) = \vartheta(\alpha)$. [This follows immediately from the definitions.]

(2) $C(\alpha, \alpha^*{+}1) = \gamma \in On \implies \gamma = \vartheta(\alpha)$.

Proof: "$\subseteq$": $\alpha^* < \vartheta(\alpha) \ \Rightarrow\ C(\alpha, \alpha^*{+}1) \subseteq C(\alpha, \vartheta(\alpha)) = \vartheta(\alpha)$.
"$\supseteq$": By definition of $C(\alpha, \alpha^*{+}1)$, $\gamma \in P \ \&\ \alpha^* < \gamma \ \&\ \{\vartheta(\xi) : \xi < \alpha \ \&\ \xi^* \in \gamma\} \subseteq \gamma$. Hence $\vartheta(\alpha) \le \gamma$.

HS 2: $C(\alpha, \beta) \in On$.

Proof by induction on α. By side induction on γ we prove: $\gamma \in C(\alpha, \beta) \ \Rightarrow\ \gamma \subseteq C(\alpha, \beta)$.
1. $\gamma \in \beta$: trivial. 2. $\gamma = \gamma_0 + \gamma_1$ mit $\gamma_0, \gamma_1 \in C(\alpha, \beta)$: By SIH $\gamma_0 \cup \gamma_1 \subseteq C(\alpha, \beta)$ which together with $\gamma_0 \in C(\alpha, \beta)$ yields $\gamma_0 + \gamma_1 \subseteq C(\alpha, \beta)$.
3. $\gamma = \vartheta(\xi)$ with $\xi < \alpha \ \&\ \xi^* \in C(\alpha, \beta)$: Using $\xi^* < \gamma$ by SIH we get $\xi^*{+}1 \subseteq C(\alpha, \beta)$, hence $C(\alpha, \xi^*{+}1) \subseteq C(\alpha, \beta)$. By MIH and HS1 we have $\gamma = \vartheta(\xi) = C(\xi, \xi^*{+}1)$.

(b) follows immediately from (a) and the definition of $C(\alpha, \alpha^*{+}1)$.

From now we only consider ordinals $\alpha < \varepsilon_{\Omega+1}$ with $\alpha^* < \vartheta(\varepsilon_{\Omega+1})$. Especially $\beta < \vartheta(\varepsilon_{\Omega+1})$.

Then for each $\beta \in P$ there exists a $\xi < \varepsilon_{\Omega+1}$ with $\beta = \vartheta\xi$.

Definition. For $\alpha = \Omega \cdot \tilde{\alpha}$ let

$\mathrm{FIX}(\alpha) := \{\vartheta(\gamma) : \alpha + \Omega \leq \gamma \;\&\; \alpha^* < \vartheta(\gamma)\}$.

$\vartheta^*_\alpha(0) := \begin{cases} \tau(\alpha) & \text{if } \alpha[1]^* < \alpha^* = \tau(\alpha) = \vartheta(\alpha_1) \text{ with } \alpha < \alpha_1 \\ 0 & \text{otherwise} \end{cases}$, $\quad \vartheta^*_\alpha(\beta) := \begin{cases} \vartheta(\alpha + \beta_0) & \text{if } \beta = \beta_0{+}1 \\ \beta & \text{if } \beta \in \mathrm{Lim} \end{cases}$.

Lemma 2.4. Let $\alpha = \Omega \cdot \tilde{\alpha}$ and $\beta \in \mathsf{Succ} \cup \mathrm{FIX}(\alpha)$.

(a) $(\alpha + \beta)^* \leq \vartheta^*_\alpha(\beta)$.

(b) $\forall \delta < \beta(\vartheta(\alpha + \delta) \leq \vartheta^*_\alpha(\beta))$.

(c) $\beta \in \mathrm{FIX}(\alpha) \implies \forall \delta < \beta(\vartheta(\alpha + \delta) < \beta)$.

Proof:

(a) 1. $\beta = \beta_0{+}1$: $(\alpha + \beta)^* = \max\{\alpha^*, \beta_0{+}1\} \leq \vartheta(\alpha + \beta_0)$.

2. $\alpha^* < \beta = \vartheta(\gamma)$: Then $(\alpha + \beta)^* = \beta = \vartheta^*_\alpha(\beta)$.

(c) Let $\delta = \vartheta(\gamma_0) < \beta = \vartheta(\gamma)$ where $\alpha + \Omega \leq \gamma$ and $\alpha^* < \vartheta\gamma$.

Then $\alpha + \delta < \alpha + \Omega \leq \gamma$ and $(\alpha + \delta)^* = \max\{\alpha^*, \delta\} < \vartheta\gamma$. Hence $\vartheta(\alpha + \delta) < \vartheta\gamma = \beta$.

(b) Due to (c), it suffices to prove the claim for $\beta = \beta_0 + 1$.

$\delta < \beta \;\Rightarrow\; \alpha + \delta \leq \alpha + \beta_0 \overset{2.1\mathrm{d}}{\Rightarrow} \vartheta(\alpha + \delta) \leq \vartheta(\alpha + \beta_0) = \vartheta^*_\alpha(\beta_0)$.

Lemma 2.5. Let $\alpha = \Omega \cdot \tilde{\alpha} \;\&\; \beta \in \mathrm{Lim} \setminus \mathrm{FIX}(\alpha)$.

(a) $\exists \delta < \beta(\beta \leq \vartheta(\alpha + \delta))$.

(b) $\vartheta(\alpha + \beta) = \sup_{x<\beta} \vartheta(\alpha + x) =: \theta$.

Proof:

(a) 1. If $\beta = \beta_0 + \beta_1$ with $0 < \beta_1 \leq \beta_0$, then $\beta_0 < \beta < \vartheta(\alpha + \beta_0)$.

2. If $\beta = \vartheta\gamma \leq \alpha^*$, then $0 < \beta \leq \alpha^* < \vartheta(\alpha + 0)$.

3. If $\alpha^* < \beta = \vartheta\gamma$ with $\gamma < \alpha + \Omega$, then $\gamma^* + 1 < \beta = \vartheta\gamma < \vartheta(\alpha + \gamma^* + 1)$.

(b) 1. $x < \beta \;\Rightarrow\; \alpha + x < \alpha + \beta \;\&\; (\alpha + x)^* = \max\{\alpha^*, x\} \leq (\alpha + \beta)^* < \vartheta(\alpha + \beta)$. Hence $\theta \leq \vartheta(\alpha + \beta)$.

2. By (a) we have an $x_0 < \beta$ such that $\beta < \vartheta(\alpha + x_0)$, and so $(\alpha + \beta)^* = \max\{\alpha^*, \beta\} < \vartheta(\alpha + x_0)$.

$\xi < \alpha + \beta \;\&\; \xi^* < \theta \;\Rightarrow\; \exists x < \beta(\xi < \alpha + x \;\&\; \xi^* < \vartheta(\alpha + x)) \;\Rightarrow\; \exists x < \beta(\vartheta\xi < \vartheta(\alpha + x) \leq \theta) \;\Rightarrow\; \vartheta\xi < \theta$.

Hence $\vartheta(\alpha + \beta) \leq \theta$.

Lemma 2.6. $\beta \in \mathsf{Succ} \cup \mathrm{FIX}(0) \implies \vartheta(\beta) = \sup_{n<\omega} \vartheta^*_0(\beta) \cdot n =: \theta$.

Proof:

Obviously $\vartheta^*_0(\beta) \cdot n < \vartheta(\beta)$. By induction on δ we prove $\forall \delta < \vartheta(\beta)(\delta < \theta)$:

1. $\delta = \delta_0 + \delta_1$: Immediate by IH. 2. $\delta \leq \beta$: Then (by 2.4a) $\delta \leq \vartheta^*_0(\beta)$.

3. $\delta = \vartheta(\gamma)$ with $\gamma < \beta$: Then $\vartheta(\gamma) \leq \vartheta^*_0(\beta)$ by 2.4b.

Lemma 2.7.

If $\alpha = \Omega \cdot \tilde{\alpha}$, $\tau(\alpha) < \Omega$, and $\beta \in \{0\} \cup \mathsf{Succ} \cup \mathrm{FIX}(\alpha)$, then $\vartheta(\alpha+\beta) = \sup_{x<\tau(\alpha)} \vartheta(\alpha[x] + \vartheta^*_\alpha(\beta)) =: \theta$.

Proof:

1. $x < \tau(\alpha) \Rightarrow \alpha[x]+\vartheta^*_\alpha(\beta)<\alpha \,\&\, (\alpha[x]+\vartheta^*_\alpha(\beta))^* \overset{1.1c}{\le} \max\{\alpha^*, \vartheta^*_\alpha(\beta)\}<\vartheta(\alpha+\beta) \Rightarrow \vartheta(\alpha[x]+\vartheta^*_\alpha(\beta)) < \vartheta(\alpha+\beta)$.

2. By induction on δ we prove $\forall\delta < \vartheta(\alpha+\beta)(\delta < \theta)$. Assume $\delta = \vartheta(\gamma) < \vartheta(\alpha+\beta)$.

2.1. $\gamma = \alpha + \eta$ with $\eta < \beta$: Then $\vartheta(\gamma) \le \vartheta^*_\alpha(\beta)$ by 2.4b.

2.2. $\gamma < \alpha$: Then there exists $x < \tau(\alpha)$ such that $\gamma < \alpha[x]$ and (by IH) $\gamma^* < \vartheta(\alpha[x] + \vartheta^*_\alpha(\beta))$.
Hence $\vartheta(\gamma) < \vartheta(\alpha[x] + \vartheta^*_\alpha(\beta)) \le \theta$.

2.3. $\delta \le (\alpha+\beta)^*$:

2.3.1. $0 < \beta$: Then $\delta \le \vartheta^*_\alpha(\beta)$ by 2.4a.

2.3.2. $0 = \beta$ and $\alpha^* \le \vartheta^*_\alpha(0)$: Then $\delta \le \vartheta^*_\alpha(\beta)$ follows from $\delta \le \alpha^*$.

2.3.3. $0 = \beta$ and $\vartheta^*_\alpha(0) < \alpha^*$: Then $\vartheta^*_\alpha(0) = 0$ and $\delta \le \alpha^*$ and $\neg(\alpha[1]^* < \alpha^* = \tau(\alpha) = \vartheta(\alpha_1)$ with $\alpha_1 > \alpha)$.
To prove: $\alpha^* < \theta$. One easily sees that $\tau(\alpha) = \sup_{x<\tau(\alpha)} x \le \sup_{x<\tau(\alpha)} \vartheta(\alpha[x]) \le \theta$.
(i) $\alpha^* \le \alpha[1]^* + 1$: Then $\alpha^* < \theta$, since $\alpha[1]^* < \vartheta\alpha[1] \le \theta$ and $1 = \vartheta 0 < \vartheta\alpha[1] \in P$.
(ii) $\alpha^* = \tau(\alpha) \notin P$: Then $\alpha^* < \theta$ follows from $\tau(\alpha) \le \theta \,\&\, \theta \in P$.
(iii) otherwise: From $\alpha[1]^* + 1 < \alpha^*$ by L.1.1d we get $\tau(\alpha) = \alpha^* \in P$. Hence $\tau(\alpha) = \vartheta(\alpha_1)$ with $\alpha_1 < \alpha$.
Now there exists an $x < \tau(\alpha) = \vartheta(\alpha_1)$ with $\alpha_1 < \alpha[x] \,\&\, \alpha_1^* \le x < \vartheta(\alpha[x])$, and so $\alpha^* = \vartheta(\alpha_1) < \theta$.

Lemma 2.8. If $\tau(\alpha) = \Omega$, $\beta \in \{0\} \cup \mathsf{Succ} \cup \mathrm{FIX}(\alpha)$, and $x_0 := \vartheta^*_\alpha(\beta)$, $x_{n+1} := \vartheta\alpha[x_n]$, then
(a) $x_n < x_{n+1} < \vartheta(\alpha+\beta)$ for all n; (b) $\vartheta(\alpha+\beta) = \sup_{n<\omega} x_n =: \theta$.

Proof:

(a) $x_n \le \alpha[x_n]^* < \vartheta\alpha[x_n] = x_{n+1}$. — Proof of $x_n < \vartheta(\alpha+\beta)$ by induction on n:

1. $x_0 = \vartheta^*_\alpha(\beta) < \vartheta(\alpha+\beta)$.

2. $x_n < \vartheta(\alpha+\beta) \overset{1.1c}{\Rightarrow} \alpha[x_n] < \alpha \,\&\, \alpha[x_n]^* < \vartheta(\alpha+\beta) \Rightarrow x_{n+1} = \vartheta\alpha[x_n] < \vartheta(\alpha+\beta)$.

(b) We have $\theta \overset{(a)}{\le} \vartheta(\alpha+\beta)$, $\theta \in P$ and
either $(\alpha+\beta)^* \le \vartheta^*_\alpha(\beta) = x_0 < \theta$ or $(\alpha+\beta)^* = \alpha^* \le \alpha[x_1]^* + 1 \le \vartheta\alpha[x_1] = x_2 < \theta$.
It remains to prove $\forall\gamma < \alpha+\beta(\gamma^* < \theta \Rightarrow \vartheta\gamma < \theta)$: Let $\gamma < \alpha+\beta \,\&\, \gamma^* < x_{n+1}$.

1. $\gamma < \alpha$. Assume $\alpha[x_{n+1}] \le \gamma$. Then $x_{n+1} \overset{L.1.1a}{\le} \alpha[x_{n+1}]^* \overset{L.1.1e}{\le} \gamma^* < x_{n+1}$. Contradiction.
Hence $\gamma < \alpha[x_{n+1}]$ and so $\vartheta\gamma < x_{n+2} < \theta$.

2. $\gamma = \alpha + \beta_0$ with $\beta_0 < \beta$: Then by 2.4b $\vartheta\gamma \le \vartheta^*_\alpha(\beta) = x_0 < \theta$.

DEFINITION of $\delta\{n\}$ for $\delta \in \mathit{Lim} \cap \vartheta(\varepsilon_{\Omega+1})$ and $n < \omega$.

I. If $\delta =_{NF} \delta_0 + ... + \delta_k$ then $\delta\{n\} := \delta_0 + ... + \delta_{k-1} + \delta_k\{n\}$.

II. If $\delta = \vartheta(\alpha+\beta)$ with $\alpha = \Omega\cdot\tilde{\alpha}$ and $\beta \in \mathit{Lim} \setminus \mathrm{FIX}(\alpha)$, then $\delta\{n\} := \vartheta(\alpha + \beta\{n\})$

III. If $\delta = \vartheta(\alpha+\beta)$ with $\alpha = \Omega\cdot\tilde{\alpha}$ and $\beta \in \{0\} \cup \mathsf{Succ} \cup \mathrm{FIX}(\alpha)$ and $\tau := \tau(\alpha)$ then

$$\delta\{n\} := \begin{cases} \vartheta^*_\alpha(\beta)\cdot(n+1) & \text{if } \tau = 0 \\ \vartheta(\alpha[\cdot])^{n+1}(\vartheta^*_\alpha(\beta)) & \text{if } \tau = \Omega \\ \vartheta(\alpha[\tau\{n\}] + \vartheta^*_\alpha(\beta)) & \text{if } 0 < \tau < \Omega \end{cases}$$

THEOREM. From Lemmata 2.5–2.8 it follows that $\delta = \sup_{n<\omega} \delta\{n\}$, for each $\delta \in \mathit{Lim} \cap \vartheta(\varepsilon_{\Omega+1})$.