Beweistheorie

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§1 The Tait calculus PL1 for classical 1^{st}-order predicate logic without equality

Formal language of PL1

Basic symbols:

1. Variables $v_0, v_1, v_2...$ (denoted by $x, y, z, x_1, ...$)
2. $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$

Let $\mathcal{L}$ be some fixed (countable) language, i.e. set of function and predicate symbols where each symbol $p \in \mathcal{L}$ has a certain *arity* $\#(p) \in \mathbb{N}$.

From now on all syntactic notions such as terms, formulas, sequents,.... are defined with respect to $\mathcal{L}$.

Inductive definition of *terms*

1. Every variable is a term.
2. If f is an n-ary function symbol ($n \geq 0$) and $t_1, ..., t_n$ are terms then the string $ft_1...t_n$ is a term.

Abbreviation: $\mathsf{Vars} :=$ set of all variables; $\mathrm{T} :=$ set of all terms t.

An *atomic formula* is an expression $pt_1...t_n$ where p is an n-ary predicate symbol and $t_1, ..., t_n$ are terms.

An expression of the form A or $\neg A$, where A is an atomic formula, is called a (positive or negative) *literal.*

Inductive definition of *formulas*

1. Every literal is a formula.
2. If A, B are formulas then also $\wedge AB$ and $\vee AB$ are formulas.
3. If A is a formula then $\forall x A$ and $\exists x A$ are formulas.

As usual we write $A \wedge B$, $A \vee B$ for $\wedge AB$, $\vee AB$.

Syntactic variables:

r, s, t for terms; A, B, C, D, F, G for formulas; $\diamond$ for $\wedge$, $\vee$; Q for $\forall$, $\exists$.

Definition of the *negation* $\mathsf{neg}(A)$ of a formula A

1. If A is atomic then $\mathsf{neg}(A) := \neg A$ and $\mathsf{neg}(\neg A) := A$.
2. $\mathsf{neg}(A \wedge B) := \mathsf{neg}(A) \vee \mathsf{neg}(B)$, $\mathsf{neg}(A \vee B) := \mathsf{neg}(A) \wedge \mathsf{neg}(B)$.
3. $\mathsf{neg}(\forall x A) := \exists x\, \mathsf{neg}(A)$, $\mathsf{neg}(\exists x A) := \forall x\, \mathsf{neg}(A)$.

Corollary. $\mathsf{neg}(A)$ is a formula, and $\mathsf{neg}(\mathsf{neg}(A)) = A$.

Notation.

(i) From now on we write $\neg A$ for $\mathsf{neg}(A)$.

(ii) $A_1 \to ... \to A_n \to B := \neg A_1 \vee (\neg A_2 \vee (... \vee (\neg A_n \vee B)...))$.

For each *expression* (i.e. term or formula) E we define the set $\mathrm{FV}(E)$ of its free variables in the usual way. If $\mathcal{X}$ is a set of terms and/or formulas then $\mathrm{FV}(\mathcal{X}) := \bigcup\{\mathrm{FV}(E) : E \in \mathcal{X}\}$.

Formulas A, A' which differ only in the names of their bound variables will be identified. This is sometimes expressed by saying that A and A' are α-equivalent.

A *substitution* is a mapping $\theta : \mathsf{Vars} \to \mathrm{T}$ with $\mathrm{dom}(\theta) := \{x \in \mathsf{Vars} : \theta(x) \neq x\}$ finite.

The *updates* θ_y^t of θ are defined by $\theta_y^t(x) := \begin{cases} t & \text{if } x = y \\ \theta(x) & \text{otherwise} \end{cases}$.

$(x_1/t_1, ..., x_n/t_n)$ denotes the substituion θ with $\theta(x) = \begin{cases} t_i & \text{if } x = x_i \text{ with } 1 \le i \le n \\ x & \text{otherwise} \end{cases}$.

If $\theta = (x_1/t_1, ..., x_n/t_n)$ and E is an expression then $E\theta$ denotes the result of simultaneously substituting the terms $t_1, \ldots, t_n$ for the variables $x_1, \ldots, x_n$ respectively. In using the substitution notation we shall tacitly assume that a suitable renaming of bound variables is carried out, so that whenever x_i occurs free in the range of a quantifier $\mathsf{Q}y$ then $y \notin \mathrm{FV}(t_i)$. In §XXX we will give a thorough treatment of these things.

We also write $E_{x_1,...,x_n}(t_1, ..., t_n)$ instead of $E(x_1/t_1, ..., x_n/t_n)$.

Locally we shall adopt the following convention. In an argument, once a formula has been introduced as $A(x)$, i.e. A with a designated free variable x, we write $A(t)$ for $A_x(t)$, and similarly with more variables.

Definition of $\mathrm{rk}(A)$

1. $\mathrm{rk}(A) := 0$, if A is a literal.
2. $\mathrm{rk}(A \diamond B) := \max\{\mathrm{rk}(A), \mathrm{rk}(B)\} + 1$.
3. $\mathrm{rk}(\mathsf{Q}xA) := \mathrm{rk}(A) + 1$.

Corollary. $\mathrm{rk}(\neg A) = \mathrm{rk}(A) = \mathrm{rk}(A\theta)$.

We shall derive finite sets of formulas (so-called *sequents*), denoted by $\Gamma, \Delta, ...$. The intended meaning of Γ is the disjunction of all formulas in Γ. We use the notation A, Γ, Δ for $\{A\} \cup \Gamma \cup \Delta$, etc.

An *inference* is a a finite tupel of sequents $(\Gamma_0, ..., \Gamma_{n-1}, \Gamma)$ written as $\dfrac{\Gamma_0 \ \ldots \ \Gamma_{n-1}}{\Gamma}$.

If $n = 0$ the inference (as well as the sequent Γ) is called an *axiom.*

A *rule* $\mathfrak{R}$ is an inference scheme like, e.g., $\dfrac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B}$. A single instance of a rule $\mathfrak{R}$ is called an $\mathfrak{R}$-inference. We also think of a rule $\mathfrak{R}$ as the set of all its instances, i.e., the set of all $\mathfrak{R}$-inferences.

The rules of the system $\mathrm{PL1}$ **are**

$$(\mathrm{LogAx}) \quad \Gamma, A, \neg A \quad \text{if } A \text{ atomic} \qquad \textit{(logical axioms)}$$

$$(\wedge) \quad \frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma, A_0 \wedge A_1} \qquad\qquad (\vee) \quad \frac{\Gamma, A_k}{\Gamma, A_0 \vee A_1} \quad (k \in \{0, 1\})$$

$$(\forall) \quad \frac{\Gamma, A}{\Gamma, \forall x A} \quad \text{if } x \notin \mathrm{FV}(\Gamma) \qquad\qquad (\exists) \quad \frac{\Gamma, A_x(t)}{\Gamma, \exists x A}$$

$$(\mathrm{Cut}) \quad \frac{\Gamma, C \quad \Gamma, \neg C}{\Gamma}$$

The *principal formulas* in (LogAx) are A and $\neg A$. In $(\wedge)$, $(\vee)$, $(\forall)$, $(\exists)$ the principal formula is $A \wedge B$, $A \vee B$, $\forall x A$ and $\exists x A$, respectively. (Cut) has no principal formula. The C in (Cut) is called a *cut-formula.* The displayed formulas in the premiss of an inference are called its *minor formulas.* For example, $(\wedge)$ with principle formula $A \wedge B$ has the minor formulas A, B. The variable x in $(\forall)$ is called the *eigenvariable* of the respective inference. Due to our convention on identifying α-equivalent formulas we have $\forall x A = \forall y A_x(y)$ if $y \notin \mathrm{FV}(\forall x A)$. Hence, if $y \notin \mathrm{FV}(\Gamma, \forall x A)$ then also $\dfrac{\Gamma, A_x(y)}{\Gamma, \forall x A}$ is a correct $\forall$-inference.

Note that if $\dfrac{\Delta}{\Gamma, A}$ is an $(\vee)$-, $(\forall)$-, or $(\exists)$-inference with principal formula A and minor formula A_0, then we do not necessarily have $\Delta = \Gamma, A_0$; but we only know $\Delta = \Gamma', A_0$ with $\Gamma', A = \Gamma, A$. Similarly for $(\wedge)$.

Definition.

A *derivation* is a tree of sequents generated from the above axioms and rules.

The sequent at the root of a derivation d is called its *endsequent*.

d is called a *derivation of* Γ if Γ is its endsequent.

Examples:

$$\dfrac{\dfrac{\dfrac{\dfrac{Rft,\neg Rft}{\neg Rfft\vee Rft,\neg Rft}\,(\vee)}{\neg Rfft\vee Rft,\neg Rft\vee Rt}\,(\vee)}{\exists x(\neg Rfx\vee Rx),\neg Rft\vee Rt}\,(\exists)}{\exists x(\neg Rfx\vee Rx)}\,(\exists)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\neg Rxy, Rxy}{\exists y\neg Rxy, Rxy}\,(\exists)}{\exists y\neg Rxy,\exists xRxy}\,(\exists)}{\forall x\exists y\neg Rxy,\exists xRxy}\,(\forall)}{\forall x\exists y\neg Rxy,\forall y\exists xRxy}\,(\forall)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{G,\neg F_0,F_0 \quad F_1,\neg F_1}{G,\neg F_0,F_1,F_0\wedge\neg F_1}\,(\wedge)}{G,\neg F_0,F_1}\,(\exists) \qquad F_2,\neg F_2}{G,\neg F_0,F_2,F_1\wedge\neg F_2}\,(\wedge)}{G,\neg F_0,F_2}\,(\exists)$$

In the third example we have used the following abbreviations: $G := \exists x(F(x)\wedge\neg F(\mathsf{S}\,x))$, i.e. $G = \neg\forall x(F(x)\to F(\mathsf{S}\,x))$. $F_0 := F(0)$, $F_1 := F(\mathsf{S}\,0)$, etc.

Actually this not a completely correct PL1-derivation, since it contains $\wedge$-inferences $\dfrac{\Gamma_0,A \quad \Gamma_1,B}{\Gamma_0,\Gamma_1,A\wedge B}$ with $\Gamma_0\neq\Gamma_1$. But in an obvious way it can be taken as a shorthand for the following PL1-derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{G,\neg F_0,F_2,F_1,F_0 \quad G,\neg F_0,F_2,F_1\neg F_1}{G,\neg F_0,F_2,F_1,F_0\wedge\neg F_1}\,(\wedge)}{G,\neg F_0,F_2,F_1}\,(\exists) \qquad G,\neg F_0,F_2,F_1,\neg F_2}{G,\neg F_0,F_2,F_1\wedge\neg F_2}\,(\wedge)}{G,\neg F_0,F_2}\,(\exists)$$

Definition.

The *cut-rank* of a derivation d is $\mathsf{crk}(d) := \sup\{\mathrm{rk}(C)+1 : C \text{ cut-formula of } d\}$.

d is called *cutfree* if $\mathsf{crk}(d)=0$.

The *height* $\mathsf{hgt}(d)$ of a derivation d is defined recursively by

$\mathsf{hgt}(d) := \sup_{i<n}(\mathsf{hgt}(d_i)+1)$ where $d_0,...,d_{n-1}$ are the *immediate subderivations* of d $(0\le n\le 2)$.

The last (bottommost) inference of d is denoted by $\mathsf{last}(d)$.

Abbreviations.

$d\vdash^k_m\Gamma \;:\Longleftrightarrow\;$ d is a derivation of Γ with $\mathsf{hgt}(d)\le k$ and $\mathsf{crk}(d)\le m$;

$\mathrm{PL1}\vdash^k_m\Gamma \;:\Longleftrightarrow\;$ $d\vdash^k_m\Gamma$ for some PL1-derivation d.

For any inference $\mathcal{I}$ we set $\mathrm{Eig}(\mathcal{I}) = \begin{cases}\{x\} & \text{if } \mathcal{I} \text{ has the eigenvariable } x\\ \emptyset & \text{if } \mathcal{I} \text{ has no eigenvariable}\end{cases}$.

Definition

A rule $\mathfrak{R}$ is *closed under substitution* iff the following holds for every $\mathfrak{R}$-inference $\mathcal{I} = \dfrac{\Gamma_0\dots\Gamma_{n-1}}{\Gamma}$:

If θ is a substitution such that $(\mathrm{Eig}(\mathcal{I})=\{x\} \Rightarrow x\theta\in\mathsf{Var}\setminus\mathrm{FV}(\Gamma\theta))$, then
$\mathcal{I}\theta := \dfrac{\Gamma_0\theta\dots\Gamma_{n-1}\theta}{\Gamma\theta}$ is an $\mathfrak{R}$-inference too.

Lemma 1.0. The rules of PL1 are closed under substitution. Proof: cf. .....

Lemma 1.1. (Substitution)

$\mathrm{PL1} \vdash^k_m \Gamma \implies \mathrm{PL1} \vdash^k_m \Gamma\theta$.

Proof by induction on k:

Let $d \vdash^k_m \Gamma$.

1. Assume that $\mathsf{last}(d)$ is a $\forall$-inference. Then $k > 0$ and $\Gamma = \Gamma_0, \forall x A$ with $\vdash^{k-1}_m \Gamma_0, A$ and $x \notin \mathrm{FV}(\Gamma)$:
Choose $y \notin \mathrm{FV}(\Gamma\theta)$ and let $\tilde{\theta} := \theta^y_x$.
I.H. $\Rightarrow \vdash^{k-1}_m \Gamma_0\tilde{\theta}, A\tilde{\theta} \stackrel{\text{L.1.0}}{\Rightarrow} \vdash^k_m \Gamma\tilde{\theta}$. From $x \notin \mathrm{FV}(\Gamma)$ it follows that $\Gamma\tilde{\theta} = \Gamma\theta$.

2. In all other cases the claim follows immediately from the I.H. and L.1.0.

Lemma 1.2. (Weakening)

$\mathrm{PL1} \vdash^k_m \Gamma \ \& \ \Gamma \subseteq \Gamma' \implies \mathrm{PL1} \vdash^k_m \Gamma'$.

Proof by induction on k:

1. Assume $k > 0$ and $\Gamma = \Gamma_0, \forall x A$ with $\vdash^{k-1}_m \Gamma_0, A$ and $x \notin \mathrm{FV}(\Gamma_0)$:
Choose $y \notin \mathrm{FV}(\Gamma')$. $\vdash^{k-1}_m \Gamma_0, A \stackrel{\text{L.1.1}}{\Rightarrow} \vdash^{k-1}_m \Gamma_0, A_x(y) \stackrel{\text{I.H.}}{\Rightarrow} \vdash^{k-1}_m \Gamma', A_x(y) \stackrel{(\forall)}{\Rightarrow} \vdash^k_m \Gamma'$.
For the last step note that $\forall x A \in \Gamma'$ and $y \notin \mathrm{FV}(\Gamma')$.

2. In all other cases the claim follows immediately from the I.H. (Note that if $\Gamma \subseteq \Gamma'$ and $\frac{\ldots\Gamma_i\ldots}{\Gamma}$ is an inference (LogAx), ($\wedge$), ($\vee$), ($\exists$) or (Cut) then $\frac{\ldots\Gamma', \Gamma_i\ldots}{\Gamma'}$ is an inference of the same kind.)

Corollary.

$\mathrm{PL1} \vdash^k_m \Gamma$ iff one of the following cases holds

(LogAx) $\{A, \neg A\} \subseteq \Gamma$ for some atomic A,

($\wedge$) $A_0 \wedge A_1 \in \Gamma$ and $\mathrm{PL1} \vdash^{k-1}_m \Gamma, A_i$ for each $i \in \{0, 1\}$,

($\vee$) $A_0 \vee A_1 \in \Gamma$ and $\mathrm{PL1} \vdash^{k-1}_m \Gamma, A_i$ for some $i \in \{0, 1\}$,

($\forall$) $\forall x A \in \Gamma$ and $\mathrm{PL1} \vdash^{k-1}_m \Gamma, A$ with $x \notin \mathrm{FV}(\Gamma)$,

($\exists$) $\exists x A \in \Gamma$ and $\mathrm{PL1} \vdash^{k-1}_m \Gamma, A_x(t)$,

(Cut) $\mathrm{PL1} \vdash^{k-1}_m \Gamma, C \ \& \ \mathrm{PL1} \vdash^{k-1}_m \Gamma, \neg C$ with $\mathrm{rk}(C) < m$.

In all cases except (LogAx) it is tacitly assumed that $k > 0$.

Lemma 1.3. (Inversion)

(a) $\mathrm{PL1} \vdash^k_m \Gamma, \forall x A \implies \mathrm{PL1} \vdash^k_m \Gamma, A_x(t)$;

(b) $\mathrm{PL1} \vdash^k_m \Gamma, A_0 \wedge A_1 \implies \mathrm{PL1} \vdash^k_m \Gamma, A_i$ for $i = 0, 1$;

(c) $\mathrm{PL1} \vdash^k_m \Gamma, A \vee B \implies \mathrm{PL1} \vdash^k_m \Gamma, A, B$.

Proof of (a) by induction on k:

1. Assume that $\forall x A$ is principal part of the last inference of the given derivation.
Then this has to be a $\forall$-inference, and we have $\vdash^{k-1}_m \Gamma, \forall x A, A$ with $x \notin \mathrm{FV}(\Gamma)$.
By L.1.1 we obtain $\vdash^{k-1}_m \Gamma, \forall x A, A_x(t)$, and then, by I.H., $\vdash^{k-1}_m \Gamma, A_x(t)$.
2. $\vdash^{k-1}_m \Gamma, \forall x A, B$ with $\forall y B \in \Gamma$ and $y \notin \mathrm{FV}(\Gamma, \forall x A)$: Let $z \notin \mathrm{FV}(\Gamma, A_x(t))$.
$\vdash^{k-1}_m \Gamma, \forall x A, B \stackrel{\text{L.1.1}}{\Longrightarrow} \vdash^{k-1}_m \Gamma, \forall x A, B_y(z) \stackrel{\text{I.H.}}{\Longrightarrow} \vdash^{k-1}_m \Gamma, A_x(t), B_y(z) \stackrel{(\forall)}{\Longrightarrow} \vdash^k_m \Gamma, A_x(t)$.
3. In all other cases the claim is trivial or follows immediately from the I.H.

Cutelimination

Lemma 1.4.

$\mathrm{PL1} \vdash^k_m \Gamma, C \;\;\&\;\; \mathrm{PL1} \vdash^l_m \Gamma, \neg C \;\;\&\;\; \mathrm{rk}(C) \le m \implies \mathrm{PL1} \vdash^{k+l}_m \Gamma.$

Proof by induction on $k + l$:

Assume $d \vdash^k_m \Gamma, C$ and $e \vdash^l_m \Gamma, \neg C$.

1. C is not a principal formula of $\mathsf{last}(d)$:

Let $A_1, ..., A_n$ $(n \le 2)$ be the minor formulas of $\mathsf{last}(d)$, so that we have $\vdash^{k-1}_m \Gamma, C, A_i$ for $i = 1, ..., n$.

Then the claim is obtained as shown in the following diagram

$$\dfrac{\dfrac{\vdash^{k-1}_m \Gamma, C, A_i \qquad \dfrac{\vdash^l_m \Gamma, \neg C}{\vdash^l_m \Gamma, \neg C, A_i}\,\text{L.1.2}}{\ldots\ \vdash^{k-1+l}_m \Gamma, A_i\ \ldots}\,\text{IH}}{\vdash^{k+l}_m \Gamma}\,\mathsf{last}(d)$$

For $n = 0$ we have $\mathsf{last}(d) = \mathrm{LogAx}$, and the diagram reduces to $\dfrac{}{\vdash^{k+l}_m \Gamma}\,(\mathrm{LogAx})$.

1'. $\neg C$ is not a principal formula of $\mathsf{last}(e)$: symmetric to 1.

2. C is principal formula of $\mathsf{last}(d)$, and $\neg C$ is principal formula of $\mathsf{last}(e)$:

2.1. C is a literal:

Then $\{C, \neg C\} \subseteq \Gamma \cup \{C\}$ and $\{C, \neg C\} \subseteq \Gamma \cup \{\neg C\}$. Hence $\{C, \neg C\} \subseteq \Gamma$, and therefore $\vdash^{k+l}_m \Gamma$.

2.2. $C = \exists x A$: Then $\neg C = \forall x \neg A$ and we have $\vdash^{k-1}_m \Gamma, C, A_x(t)$ and $\vdash^{l-1}_m \Gamma, \neg C, \neg A$ with $x \notin \mathrm{FV}(\Gamma, \neg C)$.

Now the claim is obtained as shown in the following diagram:

$$\dfrac{\dfrac{\vdash^{k-1}_m \Gamma, C, A_x(t) \qquad \dfrac{\vdash^l_m \Gamma, \neg C}{\vdash^l_m \Gamma, \neg C, A_x(t)}\,\text{L.1.2}}{\vdash^{k-1+l}_m \Gamma, A_x(t)}\,\text{IH} \qquad \dfrac{\dfrac{\vdash^k_m \Gamma, C}{\vdash^k_m \Gamma, C, \neg A_x(t)}\,\text{L.1.2} \qquad \dfrac{\vdash^{l-1}_m \Gamma, \neg C, \neg A}{\vdash^{l-1}_m \Gamma, \neg C, \neg A_x(t)}\,\text{L.1.1}}{\vdash^{k+l-1}_m \Gamma, \neg A_x(t)}\,\text{IH}}{\vdash^{k+l}_m \Gamma}\,(\mathrm{Cut})$$

For the last step note that $\mathrm{rk}(A_x(t)) < \mathrm{rk}(C) \le m$.

2.2'. $C = \forall x A$ or $A_0 \wedge A_1$ or $A_0 \vee A_1$: analogous to 2.2.

Theorem 1.5 (Cut-Elimination).

$\mathrm{PL1} \vdash^k_{m+1} \Gamma \implies \mathrm{PL1} \vdash^{2^k}_m \Gamma.$

Proof by induction on k:

Let $d \vdash^k_{m+1} \Gamma$ and assume that $\mathsf{last}(d) = \dfrac{\Gamma, C \quad \Gamma, \neg C}{\Gamma}$ with $\mathrm{rk}(C) = m$.

$\vdash^{k-1}_{m+1} \Gamma, C$ and $\vdash^{k-1}_{m+1} \Gamma, \neg C \overset{\text{I.H.}}{\Rightarrow} \vdash^{2^{k-1}}_m \Gamma, C$ and $\vdash^{2^{k-1}}_m \Gamma, \neg C \overset{\text{L.1.4}}{\Longrightarrow} \vdash^{2^{k-1}+2^{k-1}}_m \Gamma$.

In all other cases the claim follows immediately from the I.H.

Definition. B is a *subformula* of A if B can be obtained from A by finitely many steps of the kind $\mathsf{Q}xA \mapsto A_x(t)$ or $A_0 \diamond A_1 \mapsto A_i$. Especially A is a subformula of itself.

Remark. Cutfree derivations are distinguished by the following

Subformula property

If d is a cutfree derivation of Γ then every formula occurring in d is a subformula of some $A \in \Gamma$.

Partial Cut Elimination

Let $\mathfrak{S}^+$ be a family of additional inference rules of the form $\dfrac{\Gamma,\Delta_0 \quad \dots \quad \Gamma,\Delta_{n-1}}{\Gamma,\Delta}$, and being closed under substitution. As in PL1, formulas $A \in \Delta$ [$A \in \Delta_i$, resp.] are called the principal [minor, resp.] formulas of the respective inference. Some of these rules may be restricted by a "variable condition" of the kind that a certain variable (called the *eigenvariable* of the resp. inference) must not occur free in the conclusion Γ,Δ.

Let $\mathfrak{S} := \mathrm{PL1} + \mathfrak{S}^+$.

$\Phi := \Phi(\mathfrak{S}) := \{A : A \text{ is a principal formula of an } \mathfrak{S}^+\text{-inference}\}$

Remark. Since the rules of $\mathfrak{S}^+$ are closed under substitution, we have $\forall\theta\forall A \in \Phi(A\theta \in \Phi)$.

Let $\overline{\Phi} := \Phi \cup \{\neg A : A \in \Phi\}$.

Definition of $\mathrm{rk}_\Phi(A)$

1. $\mathrm{rk}_\Phi(A) := -1$, if $A \in \overline{\Phi}$;
2. $\mathrm{rk}_\Phi(A) := 0$, if A is a literal and $A \notin \overline{\Phi}$;
3. $\mathrm{rk}_\Phi(A \diamond B) := \max\{\mathrm{rk}_\Phi(A), \mathrm{rk}_\Phi(B)\} + 1$, if $A \diamond B \notin \overline{\Phi}$;
4. $\mathrm{rk}_\Phi(\mathsf{Q}xA) := \mathrm{rk}_\Phi(A) + 1$, if $\mathsf{Q}xA \notin \overline{\Phi}$.

Remark. $\mathrm{rk}_\Phi(A\theta) \le \mathrm{rk}_\Phi(A)$ and $\mathrm{rk}_\Phi(\neg A) = \mathrm{rk}_\Phi(A)$.

Definition. The relation $\mathfrak{S} \vdash^k_{\Phi,m} \Gamma$ is defined in the same way as $\mathrm{PL1} \vdash^k_m \Gamma$, but with the difference that now m refers to rk_Φ instead of rk.

Lemma 1.6.

(a) $\mathfrak{S} \vdash_{\Phi,m} \Gamma \implies \mathfrak{S} \vdash_{\Phi,m} \Gamma\theta$;

(b) $\mathfrak{S} \vdash^k_{\Phi,m} \Gamma \;\&\; \Gamma \subseteq \Gamma' \implies \mathfrak{S} \vdash^k_{\Phi,m} \Gamma'$.

Lemma 1.7

$\mathfrak{S} \vdash^k_{\Phi,m} \Gamma, C \;\&\; \mathfrak{S} \vdash^l_{\Phi,m} \Gamma, \neg C \;\&\; 0 \le \mathrm{rk}_\Phi(C) \le m \implies \mathfrak{S} \vdash^{k+l}_{\Phi,m} \Gamma$.

Proof by induction on $k + l$:

The proof proceeds almost literally as the proof of Lemma 1.4. In case 2.2 one concludes from $0 \le \mathrm{rk}_\Phi(C)$ that $C, \neg C \notin \Phi$ and therefore (by the above assumption on $\mathfrak{S}$) $\mathsf{last}(d)$ and $\mathsf{last}(e)$ are inferences of PL1.

Theorem 1.8 (Partial Cut-Elimination). $\mathfrak{S} \vdash^k_{\Phi,m+1} \Gamma \;\&\; m \ge 0 \implies \mathfrak{S} \vdash^{2^k}_{\Phi,m} \Gamma$.

Proof: As for Theorem 1.4.

From Theorem 1.8 it follows that every $\mathfrak{S}$-derivation can be transformed into a $\mathfrak{S}$-derivation of the same sequent where all cut-formulas have Φ-rank -1, i.e. belong to $\overline{\Phi}$.

Lemma 1.9 (Inversion).

(a) $\forall xA \notin \Phi \;\&\; \mathfrak{S} \vdash^k_{\Phi,m} \Gamma, \forall xA \implies \mathfrak{S} \vdash^k_{\Phi,m} \Gamma, A_x(t)$;

(b) $A_0 \wedge A_1 \notin \Phi \;\&\; \mathfrak{S} \vdash^k_{\Phi,m} \Gamma, A_0 \wedge A_1 \implies \mathfrak{S} \vdash^k_{\Phi,m} \Gamma, A_i$ for $i = 0, 1$;

(c) $A \vee B \notin \Phi \;\&\; \mathfrak{S} \vdash^k_{\Phi,m} \Gamma, A \vee B \implies \mathfrak{S} \vdash^k_{\Phi,m} \Gamma, A, B$.

Proof as for L.1.3: In (a) the condition $\forall xA \notin \Phi$ guarantees that $\forall xA$ cannot be the principal formula of an inference other than $(\forall)$. Similarly for (b),(c).

Completeness of PL1

Assuming that $\mathcal{L}$ is countable we will prove the completeness of PL1 without cut rule. This (together with the correctness of PL1) yields a so-called *semantical cut elimination proof* for PL1.

Definition

$\Sigma \models C \iff C$ is a logical consequence from Σ (Σ a set of formulas)

$\Sigma \models \{A_1, ..., A_n\} \iff \Sigma \models A_1 \vee ... \vee A_n$

$\models \Gamma \iff \emptyset \models \Gamma$.

Theorem 1.10

$\models \Gamma \implies \mathrm{PL1} \vdash_0 \Gamma$.

Corollary

$\Sigma \models C \implies$ There are $A_1, ..., A_n \in \Sigma$ such that $\mathrm{PL1} \vdash_0 \neg A_1, ..., \neg A_n, C$.

Proof: $\Sigma \models C \Rightarrow$ There are $A_1, ..., A_n \in \Sigma$ with $\{A_1, ..., A_n\} \models C \Rightarrow \models \neg A_1, ..., \neg A_n, C$.

Proof of Theorem 1.10:

AX := set of finite sequences $(A_0, ..., A_l)$ such that there is a prime formula A with $\{A, \neg A\} \subseteq \{A_0, ..., A_l\}$.

Let Π be a fixed finite sequence of formulas .

$t_0, t_1, ...$: enumeration of Ter.

μ, ν are ranging over finite 0-1-sequences (elements of $\{0,1\}^{<\omega}$).

$\mu \sqsubseteq \nu :\Leftrightarrow \mu$ is an initial segment of ν (i.e. $\nu = \mu * \tau$ for some $\tau \in \{0,1\}^{<\omega}$)

For each $\nu \in \{0,1\}^{<\omega}$ we define a finite sequence of formulas Π_ν.

The definition proceeds by recursion on $lh(\nu)$.

1. $\Pi_{\langle\,\rangle} := \Pi$,

Let $n = lh(\nu)$, and assume that Π_μ is already defined for each $\mu \sqsubseteq \nu$.

2. $\Pi_\nu \in \mathrm{AX}$ or all formulas in Π_ν are literals: $\Pi_{\nu*\langle i\rangle} := \Pi_\nu$,

3. $\Pi_\nu = \Pi', A, \Pi'' \notin \mathrm{AX}$, and $\mathrm{rk}(A) > 0$ while all formulas in Π' are literals:

3.1. $A = A_0 \wedge A_1$: $\Pi_{\nu*\langle i\rangle} := \Pi', A_i, \Pi''$,

3.2. $A = A_0 \vee A_1$: $\Pi_{\nu*\langle i\rangle} := \Pi', A_0, A_1, \Pi''$,

3.3. $A = \forall x B$: $\Pi_{\nu*\langle i\rangle} := \Pi', B_x(y), \Pi''$, where y is the first variable not in $\mathrm{FV}(\Pi_\nu)$,

3.4. $A = \exists x B$: $\Pi_{\nu*\langle i\rangle} := \Pi', B_x(t_k), \Pi'', A$, where k is minimal s.t. ($x \in \mathrm{FV}(B) \Rightarrow \forall \mu \sqsubseteq \nu\, B_x(t_k) \notin \Pi_\mu$).

Remark. Each $\Pi \in \mathrm{AX}$ is an axiom of PL1. $\dfrac{\Pi_{\nu*\langle 0\rangle} \quad \Pi_{\nu*\langle 1\rangle}}{\Pi_\nu}$ (in case 3.1), $\dfrac{\Pi_{\nu*\langle 0\rangle}}{\Pi_\nu}$ (in cases 3.3, 3.4) is an inference of PL1. In case 3.2, Π_ν is obtained from $\Pi_{\nu*\langle 0\rangle}$ by two ($\vee$)-inferences.

Assumption:

(A) $(i_n)_{n\in\mathbb{N}}$ is a 0-1-sequence such that $\forall n \in \mathbb{N} (\, \Pi_{\langle i_0, ..., i_{n-1}\rangle} \notin \mathrm{AX}\,)$.

Abbreviation: $\nu(n) := \langle i_0, ..., i_{n-1}\rangle$, $\mathcal{F} := \bigcup_{n\in\mathbb{N}} \Pi_{\nu(n)}$

Definition: If not all formulas in Π_ν are literals let $\mathrm{df}(\Pi_\nu)$ be the first formula in Π_ν which is not a literal.

Proposition 1. $\Pi_{\nu(n)} = \Pi', A, \Pi'' \ \&\ \mathrm{rk}(A) > 0 \implies \exists k \geq n(\,\mathrm{df}(\Pi_{\nu(k)}) = A\,)$.

Proof by induction on the number of logical symbols $\wedge, \vee, \forall, \exists$ occurring in Π'.

Proposition 2.

(a) $\mathrm{rk}(A) = 0 \implies A \notin \mathcal{F}$ or $\neg A \notin \mathcal{F}$,

(b) $A_0 \wedge A_1 \in \mathcal{F} \implies A_0 \in \mathcal{F}$ or $A_1 \in \mathcal{F}$,

(c) $A_0 \vee A_1 \in \mathcal{F} \implies A_0 \in \mathcal{F}$ and $A_1 \in \mathcal{F}$,

(d) $\forall x B \in \mathcal{F} \implies \exists u \in \mathsf{Vars}(\, B_x(y) \in \mathcal{F}\,)$,

(e) $\exists x B \in \mathcal{F} \implies \forall t \in \mathrm{Ter}(\, B_x(t) \in \mathcal{F}\,)$.

Proof:

(a) $\mathrm{rk}(A) = 0 \ \&\ A \in \Pi_{\nu(n)} \ \&\ \neg A \in \Pi_{\nu(m)} \ \Rightarrow\ \forall k \geq \max\{m, n\}(A, \neg A \in \Pi_{\nu(k)})$.

Assume that $A \in \mathcal{F}$ with $\mathrm{rk}(A) > 0$. Then $A = \mathrm{df}(\Pi_{\nu(k)})$ for some k.

(b) $A = A_0 \wedge A_1$: Then $A_i \in \Pi_{\nu(k)*\langle i\rangle}$ for $i = 0, 1$, and therefore $A_0 \in \Pi_{\nu(k+1)}$ or $A_1 \in \Pi_{\nu(k+1)}$.

(c) $A = A_0 \vee A_1$: Then $A_0, A_1 \in \Pi_{\nu(k)*\langle i\rangle}$ for $i = 0, 1$, and therefore $A_0 \in \Pi_{\nu(k+1)}$ and $A_1 \in \Pi_{\nu(k+1)}$.

(d) $A = \forall x B$: Then $B_x(y) \in \Pi_{\nu(k)*\langle i\rangle}$ for some $y \in \mathsf{Vars}$.

(e) $A = \exists x B$: Then $\forall n \geq k(\, \exists x B \in \Pi_{\nu(n)}\,)\ (*)$.

By induction on m we prove $B_x(t_m) \in \mathcal{F}$:

Assume that $B(t_i) \in \mathcal{F}$ for all $i < m$. By Proposition 1 and $(*)$ there is an $n \geq k$ such that $\mathrm{df}(\Pi_{\nu(n)}) = \exists x B$ and $\forall i < m \exists j \leq n(\, B(t_i) \in \Pi_{\nu(j)}\,)\ (**)$.

By definition of $\Pi_{\nu(n+1)}$ we get $\big(\forall j \leq n(\, B(t_m) \notin \Pi_{\nu(j)}\,) \Rightarrow B(t_m) \in \Pi_{\nu(n+1)}\big)$ and thus $B(t_m) \in \mathcal{F}$.

$|\mathcal{M}| := \mathrm{Ter},\ \ f^{\mathcal{M}}(s_1, ..., s_n) := f s_1 ... s_n,\ \ p^{\mathcal{M}}(s_1, ..., s_n) :\Leftrightarrow p s_1, ..., s_n \notin \mathcal{F}$

Let $\xi : \mathsf{Var} \to \mathrm{Ter}$ such that $\xi(x) = x$ for $x \in \mathsf{Vars}$.

Proposition 3.

(a) $t^{\mathcal{M}}[\xi] = t$ for each $t \in \mathrm{Ter}$,

(b) $A \in \mathcal{F} \ \Rightarrow\ \mathcal{M} \not\models A[\xi]$.

Proof of (b) by induction on $\mathrm{rk}(A)$:

$\forall x B \in \mathcal{F} \ \Rightarrow\ B_x(y) \notin \mathcal{F}$ for some $y \in \mathsf{Vars} \overset{\mathrm{IH}}{\Rightarrow} \mathcal{M} \not\models B_x(y)[\xi] \ \Rightarrow\ \mathcal{M} \not\models (\forall x B)[\xi]$.

$\exists x B \in \mathcal{F} \ \Rightarrow\ B_x(t) \in \mathcal{F}$ for all $t \in \mathrm{Ter} \overset{\mathrm{IH}}{\Rightarrow} \mathcal{M} \not\models B_x(t)[\xi]$ for all $t \in \mathrm{Ter} \overset{(a)}{\Rightarrow}$

$\Rightarrow\ \mathcal{M} \not\models B[\xi^t_x]$ for all $t \in \mathrm{Ter} \ \Rightarrow\ \mathcal{M} \not\models (\exists x B)[\xi]$.

Now, since $\Pi \subseteq \mathcal{F}$, Proposition 3b yields $\not\models \Pi$.

Now assume $\models \Pi$. Then the above assumption **(A)** is false, and it follows that for every 0-1-sequence $(i_n)_{n\in\mathbb{N}}$ there exists an n with $\Pi_{\langle i_0,\dots,i_{n-1}\rangle} \in \mathrm{AX}$.

By Königs Lemma (and since $\Pi_\nu \in \mathrm{AX}$ implies $\Pi_{\nu*\langle i\rangle} \in \mathrm{AX}$) the set $\{\nu : \Pi_\nu \notin \mathrm{AX}\}$ is finite.

Let $m := \max\{lh(\nu) : \Pi_\nu \notin \mathrm{AX}\} + 1$.

By induction on $m \dot{-} lh(\nu)$ one easily proves $\mathrm{PL1} \vdash_0 \Pi_\nu$ (cf. the *Remark* following the definition of Π_ν).

Hence $\mathrm{PL1} \vdash_0 \Pi$.

§2 An application of partial cut elimination; provably recursive functions of PRA and $\mathrm{I}\Sigma_1$

The axiom system PRA of primitive recursive arithmetic

Inductive Definition of sets PR^n of n-ary function symbols

(PR 1) $\mathbf{0}^n \in \mathrm{PR}^n$ $(n \geq 0)$, $\mathsf{S} \in \mathrm{PR}^1$, $\mathrm{I}_i^n \in \mathrm{PR}^n$ $(1 \leq i \leq n)$.

(PR 2) $h \in \mathrm{PR}^m$ & $g_1, ..., g_m \in \mathrm{PR}^n$ & $m, n \geq 1 \implies (\circ h g_1 ... g_m) \in \mathrm{PR}^n$.

(PR 3) $g \in \mathrm{PR}^n$ & $h \in \mathrm{PR}^{n+2} \implies (\mathrm{R} gh) \in \mathrm{PR}^{n+1}$.

Abbreviation: $\mathrm{PR} := \bigcup_{n \in \mathbb{N}} \mathrm{PR}^n$, $\mathbf{0} := \mathbf{0}^0$.

$\mathcal{L}_0 := \mathrm{PR} \cup \{=\}$, where $=$ is a binary relation symbol (equality).

The $\mathcal{L}_0$-terms $\mathbf{0}$, $\mathsf{S}\,\mathbf{0}$, $\mathsf{S}\,\mathsf{S}\,\mathbf{0}$,.... are called numerals. For $n \in \mathbb{N}$ let $\underline{n} := \overbrace{\mathsf{S} \ldots \mathsf{S}}^{n}\mathbf{0}$.

$T_0 :=$ set of all closed $\mathcal{L}_0$-terms.
If $t \in T_0$ then $t^{\mathcal{N}}$ denotes its canonical *value*. Hence $\underline{n}^{\mathcal{N}} = n$.

$\mathrm{TRUE}_0 :=$ set of all true closed literals of $\mathcal{L}_0$ $[= \{s{=}t : s, t \in T_0 \,\&\, s^{\mathcal{N}} = t^{\mathcal{N}}\} \cup \{\neg(s{=}t) : s, t \in T_0 \,\&\, s^{\mathcal{N}} \neq t^{\mathcal{N}}\}]$

By QF we denote the set of all *quantifierfree* $\mathcal{L}_0$-formulas.

The **axioms** of PRA are the universal closures of the following $\mathcal{L}_0$-formulas:

$x{=}x$
$x{=}y \to A \to A_x(y)$, for each atomic $\mathcal{L}_0$-formula A
$\neg(\mathsf{S}\,x{=}\mathbf{0})$
$\mathsf{S}\,x{=}\,\mathsf{S}\,y \to x{=}y$
$\mathbf{0}^n x_1 ... x_n{=}\mathbf{0}$
$\mathrm{I}_i^n x_1 ... x_n{=}x_i$
$(\circ h g_1 ... g_m) x_1 ... x_n{=}h g_1 x_1 ... x_n \ldots g_m x_1 ... x_n$
$(\mathrm{R} gh) x_1 ... x_n 0{=}g x_1 ... x_n$
$(\mathrm{R} gh) x_1 ... x_n\, \mathsf{S}\, y{=}h x_1 ... x_n y (\mathrm{R} gh) x_1 ... x_n y$
$F_x(0) \to \forall x(F \to F_x(\mathsf{S}\,x)) \to F$, for each $F \in \mathsf{QF}$.

The corresponding Tait-style system PRA is an extension of PL1 given by the axioms

(G1) $\Gamma, t{=}t$

(G2) $\Gamma, \neg(s{=}t), \neg A_x(s), A_x(t)$, for each atomic $\mathcal{L}_0$-formula A

(S0) $\Gamma, \neg(\mathsf{S}\,t{=}\mathbf{0})$

(S1) $\Gamma, \neg(\mathsf{S}\,s{=}\,\mathsf{S}\,t), s{=}t$

(PR0) $\Gamma, \mathbf{0}^n t_1 ... t_n{=}\mathbf{0}$

(PR1) $\Gamma, \mathrm{I}_i^n t_1 ... t_n{=}t_i$

(PR2) $\Gamma, (\circ h g_1 ... g_m) t_1 ... t_n{=}h g_1 t_1 ... t_n \ldots g_m t_1 ... t_n$

(PR3.0) $\Gamma, (\mathrm{R} gh) t_1 ... t_n 0{=}g t_1 ... t_n$

(PR3.1) $\Gamma, (\mathrm{R} gh) t_1 ... t_n\, \mathsf{S}\, s{=}h t_1 ... t_n s (\mathrm{R} gh) t_1 ... t_n s$

and the QF-*induction rule* $\dfrac{\Gamma, \neg F, F_x(\mathsf{S}\,x)}{\Gamma, \neg F(0), F_x(t)}$ $(F \in \mathsf{QF}$ and $x \notin \mathrm{FV}(\Gamma, F_x(t)))$.

One easily sees that the rules of PRA are closed under substitution (cf. §3).

Therefore 1.6-1.9 apply to PRA with $\Phi = \mathsf{QF}$.

The following Lemma shows the Tait system PRA proves exactly the logical consequences of PRA.

Lemma 2.1.

$\mathsf{PRA} \vdash \Gamma \iff \mathrm{PL1} \vdash \neg A_1, ..., \neg A_n, \Gamma$ for some $A_1, ..., A_n \in \mathrm{PRA}$.

Proof: Exercise.

Some special function symbols

There are function symbols $+, \cdot, \dot{-} \in \mathrm{PR}^2$ and $\mathsf{prd} \in \mathrm{PR}^1$ such that the following equations are axioms of PRA: $s+0 = s$, $s+\mathsf{S}\,t = \mathsf{S}(s+t)$, $s\cdot 0 = 0$, $s\cdot\mathsf{S}\,t = s{\cdot}t+s$, $\mathsf{prd}\,0 = 0$, $\mathsf{prd}\,\mathsf{S}\,t = t$, $s \dot{-} 0 = s$, $s \dot{-} \mathsf{S}\,t = \mathsf{prd}(s \dot{-} t)$.

Lemma 2.2. The following formulas are provable in PRA.

(a) $x \neq 0 \to x = \mathsf{S}\,\mathsf{prd}\,x$

(b) $x + y = y + x \land (x + y) + z = x + (y + z)$

(c) $x + y = 0 \leftrightarrow x = 0 \land y = 0$

(d) $x \cdot y = y \cdot x \land (x \cdot y) \cdot z = x \cdot (y \cdot z) \land x \cdot (y + z) = x \cdot y + x \cdot z$

(e) $x \cdot y = 0 \leftrightarrow x = 0 \lor y = 0$

(f) $\mathsf{S}\,x \dot{-} \mathsf{S}\,y = x \dot{-} y$

(g) $x + 1 = \mathsf{S}\,x \land x \cdot 1 = x \land \mathsf{prd}\,1 = 0 \land x \dot{-} 1 = \mathsf{prd}\,x$ (where $1 := \mathsf{S}\,0$)

Proof (sketch):

(a) QF-Ind: $0 = 0 \Rightarrow 0 \neq 0 \to 0 = \mathsf{S}\,\mathsf{prd}\,0$. $x = \mathsf{prd}\,\mathsf{S}\,x \Rightarrow \mathsf{S}\,x = \mathsf{S}\,\mathsf{prd}\,\mathsf{S}\,x$.

(b),(d) Proof by At-Ind, i.e. induction with atomic induction formula F.

(c) "←": $0 + 0 = 0$. "→": $y \neq 0 \to x + y \overset{(a)}{=} x + \mathsf{S}\,\mathsf{prd}\,y = \mathsf{S}(x + \mathsf{prd}\,y) \neq 0$.

(e) "←": $y = 0 \to x \cdot y = 0$, $x = 0 \to x \cdot y = y \cdot x = 0$;

"→": $x \neq 0 \land y \neq 0 \to x \cdot y = x \cdot \mathsf{S}\,\mathsf{prd}\,y = x \cdot \mathsf{prd}\,y + x = x \cdot \mathsf{prd}\,y + \mathsf{S}\,\mathsf{prd}\,x = \mathsf{S}(...) \neq 0$.

(f) At-Ind: $\mathsf{S}\,x \dot{-} \mathsf{S}\,0 = \mathsf{prd}(\mathsf{S}\,x \dot{-} 0) = \mathsf{prd}\,\mathsf{S}\,x = x = x \dot{-} 0$. $\mathsf{S}\,x \dot{-} \mathsf{S}\,\mathsf{S}\,y = \mathsf{prd}(\mathsf{S}\,x \dot{-} \mathsf{S}\,y) \overset{\mathrm{IH}}{=} \mathsf{prd}(x \dot{-} y) = x \dot{-} \mathsf{S}\,y$.

(g) $x + \mathsf{S}\,0 = \mathsf{S}(x + 0) = \mathsf{S}\,x$, $x \cdot \mathsf{S}\,0 = x \cdot 0 + x = 0 + x = x + 0 = x$, $\mathsf{prd}\,\mathsf{S}\,0 = 0$, $x \dot{-} \mathsf{S}\,0 = \mathsf{prd}(x \dot{-} 0) = \mathsf{prd}\,x$.

Lemma 2.3. The following formulas are provable in PRA.

(a) $x \neq 0 \leftrightarrow 1 \dot{-} x = 0$

(b) $y \dot{-} x \neq 0 \to y = x + (y \dot{-} x)$

(c) $y \dot{-} (x + y) = 0$

(d) $(x + y) \dot{-} y = x$

(e) $x \dot{-} y = 0 \to x = y \lor y \dot{-} x \neq 0$

(f) $x \dot{-} y = 0 \to y = x + (y \dot{-} x)$

Proof:

(a) By At-Ind we obtain $0 \dot{-} x = 0$ $[0 \dot{-} 0 = 0,\ 0 \dot{-} \mathsf{S}\,x = \mathsf{prd}(0 \dot{-} x) = \mathsf{prd}\,0 = 0]$.

$x \neq 0 \to 1 \dot{-} x = \mathsf{S}\,0 \dot{-} \mathsf{S}\,\mathsf{prd}\,x = 0 \dot{-} \mathsf{prd}\,x = 0$; $x = 0 \to 1 \dot{-} x = 1 \neq 0$.

(b) QF-Ind: 1. $y = 0 + (y \dot- 0)$.

2. $y \dot- \mathsf{S}\,x \neq 0 \Rightarrow \mathsf{prd}(y \dot- x) \neq 0 \Rightarrow y \dot- x \neq 0 \Rightarrow y \dot- x = \mathsf{S}\,\mathsf{prd}(y \dot- x) = \mathsf{S}(y \dot- \mathsf{S}\,x)$,
$y \dot- \mathsf{S}\,x \neq 0 \Rightarrow y \overset{\text{IH}}{=} x + (y \dot- x) = x + \mathsf{S}(y \dot- \mathsf{S}\,x) = \mathsf{S}\,x + (y \dot- \mathsf{S}\,x)$.

(c) At-Ind: 1. $0 \dot- (x + 0) = 0 \dot- x = 0$.

2. $\mathsf{S}\,y \dot- (x + \mathsf{S}\,y) = \mathsf{S}\,y \dot- \mathsf{S}(x + y) = y \dot- (x + y) \overset{\text{IH}}{=} 0$.

(d) At-Ind: 1. $(x + 0) \dot- 0 = x$. 2. $(x + \mathsf{S}\,y) \dot- \mathsf{S}\,y = \mathsf{S}(x + y) \dot- \mathsf{S}\,y = (x + y) \dot- y \overset{\text{IH}}{=} x$.

(e) QF-Ind: 1. $x \dot- 0 = 0 \to x = 0$.

2. $x \dot- \mathsf{S}\,y = 0 \Rightarrow \mathsf{prd}(x \dot- y) = 0 \Rightarrow x \dot- y = 0 \vee x \dot- y = \mathsf{S}\,0$ [since $x \dot- y \neq 0 \to x \dot- y = \mathsf{S}\,\mathsf{prd}(x \dot- y)$]
$\overset{\text{I.H.}}{\Rightarrow}$ $x = y \vee y \dot- x \neq 0 \vee x \dot- y = \mathsf{S}\,0$.
$x = y \Rightarrow \mathsf{S}\,y \dot- x = (1 + y) \dot- y \overset{\text{(d)}}{=} 1 \neq 0$.
$y \dot- x \neq 0 \overset{\text{(b)}}{\Rightarrow} y = x + (y \dot- x) \Rightarrow \mathsf{S}\,y \dot- x = (x + \mathsf{S}(y \dot- x)) \dot- x \overset{\text{(d)}}{=} \mathsf{S}(y \dot- x) \neq 0$.
$x \dot- y = \mathsf{S}\,0 \Rightarrow x \dot- y \neq 0 \overset{\text{(b)}}{\Rightarrow} x = y + (x \dot- y) = y + \mathsf{S}\,0 = \mathsf{S}\,y$.

(f) $x = y \Rightarrow y = x = x + (x \dot- x) = x + (y \dot- x)$; $y \dot- x \neq 0 \overset{\text{(b)}}{\Rightarrow} y = x + (y \dot- x)$.

Lemma 2.4.

(a) For each $A \in \mathsf{QF}$ there exists a term t_A such that $\mathsf{PRA} \vdash A \leftrightarrow t_A = 0$.

(b) For each $A \in \mathsf{QF}$ and terms t_0, t_1 there exist a term $d_A(t_0, t_1)$ such that
$\mathsf{PRA} \vdash A \to d_A(t_0, t_1) = t_0$ and $\mathsf{PRA} \vdash \neg A \to d_A(t_0, t_1) = t_1$.

(c) For each PR-term t with $\mathrm{FV}(t) \subseteq \{x_1, ..., x_n\}$ there is a function symbol $f \in \mathrm{PR}^n$ such that
$\mathsf{PRA} \vdash t = f x_1 ... x_n$.

Proof:

(a) $t_{r=s} := (s \dot- r) + (r \dot- s)$ $[s \neq r \overset{2.3\text{e}}{\to} s \dot- r \neq 0 \vee r \dot- s \neq 0]$;

$t_{\neg(r=s)} := 1 \dot- t_{r=s}$; $t_{A \wedge B} := t_A + t_B$; $t_{A \vee B} := t_A \cdot t_B$.

(b) Let $d_A(t_0, t_1) := t_0 \cdot (1 \dot- t_A) + t_1 \cdot (1 \dot- (1 \dot- t_A))$.

(c) cf. Logic I.

Abbreviations.

$s \leq t := (s \dot- t = 0)$; $s < t := (s \leq t \wedge s \neq t)$.
$\exists x \leq t A := \exists x (x \leq t \wedge A)$ and $\forall x \leq t A := \forall x (x \leq t \to A)$ if $x \notin \mathrm{FV}(t)$.

Lemma 2.5. The following formulas are provable in PRA.

(a) $x \leq y \leftrightarrow y = x + (y \dot- x) \leftrightarrow \exists z (y = x + z)$;

(b) $x \leq x \wedge (x \leq y \wedge y \leq x \to x = y)$;

(c) $x \leq y \wedge y \leq z \to x \leq z$;

(d) $x \leq y \vee y \leq x$;

(e) $0 \leq x$;

(f) $x < y \leftrightarrow \neg(y \leq x)$;

(g) $x < \mathsf{S}\,y \leftrightarrow x \leq y$;

(h) $(x < y \wedge y \leq z) \vee (x \leq y \wedge y < z) \to x < z$.

Proof: (a) $x \dot- y = 0 \to y = x + (y \dot- x) \to \exists z(y = x + z)$. $y = x + z \to x \dot- y = x \dot- (z + x) = 0$.

(b) $x \dot- x = x \dot- (0 + x) = 0$. $x \dot- y = 0 \to x = y \vee y \dot- x \neq 0$, hence $x \dot- y = 0 = y \dot- x \to x = y$.

(c) $x \leq y \wedge y \leq z \Rightarrow \exists u, v(x + u = y \wedge y + v = z) \Rightarrow \exists u, v(z = x + u + v) \Rightarrow x \leq z$.

(d) $\neg(x \leq y) \Rightarrow x \dot- y \neq 0 \Rightarrow x = y + (x \dot- y) \Rightarrow y \leq x$.

(e) $x = 0 + x$.

(f) $x < y \leftrightarrow x \leq y \wedge x \neq y \leftrightarrow x \dot- y = 0 \wedge x \neq y \overset{(e)}{\to} y \dot- x \neq 0 \leftrightarrow \neg(y \leq x)$.

$y \dot- x \neq 0 \to x \neq y \wedge y = x + (y \dot- x)$; $y = x + z \to x \dot- y = x \dot- (x + z) = 0$.

(g) "$\to$": $x < \mathsf{S}\, y \Rightarrow \neg(\mathsf{S}\, y \leq x) \Rightarrow u := \mathsf{S}\, y \dot- x \neq 0 \Rightarrow \mathsf{S}\, y = x + u = x + \mathsf{S}\,\mathsf{prd}\, u = \mathsf{S}(x + \mathsf{prd}\, u) \Rightarrow y = x + \mathsf{prd}\, u$.

"$\leftarrow$": $x \dot- y = 0 \to x \dot- \mathsf{S}\, y = \mathsf{prd}(x \dot- y) = 0$. $x = \mathsf{S}\, y \wedge x \leq y \to 1 = (1 + y) \dot- y = \mathsf{S}\, y \dot- y = 0$.

(h) $x < y \wedge y \leq z \Rightarrow x \leq y \wedge y \leq z \Rightarrow x \leq z$. $x = z \wedge y \leq z \Rightarrow y \leq x \Rightarrow \neg(x < y)$.

$x \leq y \wedge y < z \Rightarrow x \leq y \wedge y \leq z \Rightarrow x \leq z$. $x = z \wedge x \leq y \Rightarrow z \leq y \Rightarrow \neg(y < z)$.

Lemma 2.6 (Pairing)

There are function symbols $\pi \in \mathrm{PR}^2$, $\pi_1, \pi_2 \in \mathrm{PR}^1$ such that $\mathsf{PRA} \vdash \pi_i \pi x_1 x_2 = x_i \wedge \pi\pi_1 x \pi_2 x = x$.

Proof:

We argue informally, but so that all steps are easily formalizable in PRA.

$\pi(a, b) := f(a + b) + b$ with $f(n) := \sum_{i \leq n} i$ (i.e., $f(0) = 0$, $f(n{+}1) = f(n) + n + 1$).

$$h(0) := 0,\ h(k + 1) := \begin{cases} h(k) + 1 & \text{if } f(h(k) + 1) \leq k{+}1 \\ h(k) & \text{otherwise} \end{cases}$$

(1) $fh(k) \leq k < f(h(k) + 1)$

Induction step:

Case 1. $fh(k + 1) = f(h(k) + 1) \leq k + 1 \overset{\text{IH}}{\leq} f(h(k) + 1) = fh(k + 1) < f(h(k + 1) + 1)$.

Case 2. $fh(k + 1) = fh(k) \overset{\text{IH}}{\leq} k < k + 1 < f(h(k) + 1) = f(h(k + 1) + 1)$.

Definition. $\pi_2(k) := k \dot- fh(k)$, $\pi_1(k) := h(k) \dot- \pi_2(k)$.

$k < f(h(k) + 1) = f(h(k)) + h(k) + 1 \ \&\ fh(k) \leq k \ \Rightarrow\ \pi_2(k) \leq h(k) \ \Rightarrow\ \pi_1(k) + \pi_2(k) = h(k)$.

$\pi(\pi_1(k), \pi_2(k)) = f(\pi_1(k) + \pi_2(k)) + \pi_2(k) = fh(k) + (k \dot- fh(k)) = k$.

Let $k := \pi(a, b)$. Then $f(a + b) \leq k < f(a + b + 1)$ and therefore $h(k) = a + b$.

$\pi_2(k) = (f(a + b) + b) \dot- f(a + b) = b$ and $\pi_1(k) = (a + b) \dot- b = a$.

Lemma 2.7. (Bounded μ-operator)

Let $A \in \mathsf{QF}$ and $\mathrm{FV}(A) \subseteq \{x_1, ..., x_n, y\}$. Then there exists an $f \in \mathrm{PR}^{n+1}$ such that the following formulas are provable in PRA (where $A(t) := A_y(t)$):

(a) $\exists z \leq y A(z) \to A(f\vec{x}y) \wedge \forall z{<}f\vec{x}y\, \neg A(z)$ (i.e. $\exists z \leq y A(z) \to f\vec{x}y = \min\{z : A(z)\}$).

(b) $\exists z \leq y A(z) \leftrightarrow A(f\vec{x}y)$

(c) $\neg A(0) \wedge A(y) \to \neg A(p) \wedge A(p{+}1)$, where $p := f\vec{x}y \dot- 1$.

Notation. $\overline{\mu}_{z \leq y} A(z) := f\vec{x}y$.

Proof:

By Lemma 2.4 there is a function symbol $h \in \mathrm{PR}^{n+2}$ such that

$\mathsf{PRA} \vdash A(z) \vee \neg A(\mathsf{S}\, y) \to hx_1...x_n yz = z$ and $\mathsf{PRA} \vdash \neg A(z) \wedge A(\mathsf{S}\, y) \to hx_1...x_n yz = \mathsf{S}\, y$.

Let $f := (\mathrm{R}0^n h)$. Then the following formulas are provable in PRA

(1) $f\vec{x}0 = 0$

(2) $A(f\vec{x}y) \vee \neg A(\mathsf{S}\, y) \to f\vec{x}\,\mathsf{S}\, y = f\vec{x}y$

(3) $\neg A(f\vec{x}y) \wedge A(\mathsf{S}\, y) \to f\vec{x}\,\mathsf{S}\, y = \mathsf{S}\, y$

(4) $f\vec{x}y \leq y$

(5) $A(y) \to A(f\vec{x}y)$

Proof by induction on y: 1. $y = 0$: trivial.
2. $A(\mathsf{S}\, y) \overset{(2),(3)}{\Rightarrow} (A(f\vec{x}y) \wedge f\vec{x}\,\mathsf{S}\, y = f\vec{x}y) \vee f\vec{x}\,\mathsf{S}\, y = \mathsf{S}\, y \Rightarrow A(f\vec{x}\,\mathsf{S}\, y)$.

(6) $A(y_0) \wedge y_0 \leq y \to f\vec{x}y_0 = f\vec{x}y$

Proof by induction on y: 1. $y = y_0$: trivial.
2. $y = \mathsf{S}\, z \wedge y_0 \leq z$: $A(y_0) \overset{(5)+\mathrm{IH}}{\Rightarrow} A(f\vec{x}y_0) \wedge f\vec{x}y_0 = f\vec{x}z \Rightarrow f\vec{x}y = f\vec{x}\,\mathsf{S}\, z \overset{(2)}{=} f\vec{x}z = f\vec{x}y_0$.

(7) $z < f\vec{x}y \to \neg A(z)$

Proof: $A(z) \wedge z < f\vec{x}y \overset{(4)}{\Rightarrow} A(z) \wedge z < y \overset{(6),(4)}{\Rightarrow} f\vec{x}y = f\vec{x}z \leq z$. Contradiction.

Now (a),(b) follow from (4)-(7).

Proof of (c): $\neg A(0) \wedge A(y) \overset{(5)}{\Rightarrow} \neg A(0) \wedge A(f\vec{x}y) \Rightarrow f\vec{x}y \neq 0 \Rightarrow f\vec{x}y = p{+}1 \overset{(7)}{\Rightarrow} \neg A(p)$.

Definition.

The Δ_0-*formulas* are generated from $\mathcal{L}_0$-literals by means of $\wedge$, $\vee$, $\forall x \leq t$, $\exists x \leq t$ $(x \notin \mathrm{FV}(t))$.

The Σ-*formulas* are generated from $\mathcal{L}_0$-literals by means of $\wedge$, $\vee$, $\forall x \leq t$ $(x \notin \mathrm{FV}(t))$, $\exists x$.

A formula is called Σ_1-*formula* if it is in QF or has the form $\exists x A$ with $A \in \mathsf{QF}$.

Lemma 2.8.

For each Δ_0-formula A there exists a PR-term t_A such that $\mathsf{PRA} \vdash A \leftrightarrow t_A = 0$.

Proof:

By Lemma 2.4a it suffices to prove that every Δ_0-formula is equivalent to a quantifierfree formula.

1. Let $A = \exists y \leq sB(y)$ with $\mathrm{FV}(A) = \{\vec{x}\}$. By I.H. there is a $C(y) \in \mathsf{QF}$ with $\mathrm{FV}(C) \subseteq \{\vec{x}, y\}$ and $\vdash B(y) \leftrightarrow C(y)$. By Lemma 2.7 there is a function symbol f such that $\vdash \exists z{\leq}yC(z) \leftrightarrow C(f\vec{x}y)$.
Hence $\vdash \exists y{\leq}sB(y) \leftrightarrow \exists y{\leq}sC(y) \leftrightarrow C(f\vec{x}s)$.

2. If $A = \forall x \leq sB$ then (as we have just shown) $\vdash \exists x \leq s\neg B \leftrightarrow C$ (for some $C \in \mathsf{QF}$) and hence $\vdash A \leftrightarrow \neg C$.

Definition. A recursive function $f : \mathbb{N}^n \to \mathbb{N}$ called *provably recursive* (or *provably total*) *in* $\mathfrak{S}$ if there is a Σ_1-formula $A(\vec{x}, y)$ such that (i) $\mathfrak{S} \vdash \forall\vec{x}\exists y\, A(\vec{x}, y)$ and (ii) $\forall \vec{a}, b \in \mathbb{N}(\, f(\vec{a}) = b \Leftrightarrow \mathcal{N} \models A[\vec{a}, b]\,)$.

Theorem 2.9

If $\mathsf{PRA} \vdash \Gamma, \exists y A$ where $A \in \mathsf{QF}$ and $\forall$ does not occur in Γ, then $\mathsf{PRA} \vdash \Gamma, A_y(t)$ for some PR-term t.

Proof:

By Theorem 1.8 (partial cut elimination) we have (for some k) $\mathsf{PRA} \vdash^k_{\mathsf{QF},0} \Gamma, \exists y A$, i.e. there exists a PRA-derivation of $\Gamma, \exists y A$ where all cut formulas are quantifierfree. We proceed by induction on such a derivation.

1. $\Gamma, \exists y A$ is an axiom: Then Γ is an axiom too, and we may set $t := 0$.

2. $\vdash^{k-1}_{\mathsf{QF},0} \Gamma, \exists y A, B_0$ and $\vdash^{k-1}_{\mathsf{QF},0} \Gamma, \exists y A, B_1$ with $B_0 \wedge B_1 \in \Gamma$:
Let $t := d_{A(t_0)}(t_0, t_1)$ where by I.H. $\vdash \Gamma, A(t_i), B_i$ $(i = 0, 1)$ (1).

$\vdash A(t_0) \to t = t_0 \;\Rightarrow \vdash A(t_0) \to A(t)$ (2).
$\vdash \neg A(t_0) \to t = t_1 \;\Rightarrow \vdash \neg A(t_0) \to (A(t_1) \to A(t)) \overset{(2)}{\Rightarrow} \vdash A(t_1) \to A(t)$ (3).

From (1), (2), (3) we obtain $\vdash \Gamma, A(t), B_i$ (for $i = 0, 1$) and then $\vdash \Gamma, A(t)$.

3. $\vdash^{k-1}_{\mathsf{QF},0} \Gamma, \exists y A, B_k$ with $B_0 \vee B_1 \in \Gamma$: Let $t := t_0$, where by I.H. $\vdash \Gamma, A(t_0), B_k$.
4. $\vdash^{k-1}_{\mathsf{QF},0} \Gamma, \exists y A, B_x(s)$ with $\exists x B \in \Gamma$: As 3.
5. $\vdash^{k-1}_{\mathsf{QF},0} \Gamma, \exists y A, A(s)$: Let $t := d_{A(t_0)}(t_0, s)$ where by I.H. $\vdash \Gamma, A(t_0), A(s)$.
As in 2. we obtain $\vdash A(t_0) \to A(t)$ and $\vdash A(s) \to A(t)$. Hence $\vdash \Gamma, A(t)$.

6. $\vdash^{k-1}_{\mathsf{QF},0} \Gamma, \exists y A, B$ and $\vdash^{k-1}_{\mathsf{QF},0} \Gamma, \exists y A, \neg B$ with $B \in \mathsf{QF}$:
Let $t := d_{A(t_0)}(t_0, t_1)$, where by I.H. $\vdash \Gamma, A(t_0), B$ and $\vdash \Gamma, A(t_1), \neg B$.
As in 2. we obtain $\vdash \Gamma, A(t), B$ und $\vdash \Gamma, A(t), \neg B$. Hence $\vdash \Gamma, A(t)$.

7. $\Gamma = \Gamma', \neg F(0), F(s)$ and $\vdash^{k-1}_{\mathsf{QF},0} \Gamma', \neg F(x), F(\mathsf{S}\,x), \exists y A$:

We have to prove $\vdash \Gamma', A(t), \neg F(0), F(s)$ for some term t.

We set $r_0 := \overline{\mu}_{z \le s} \neg F(z) \dot{-} 1$.

Then $\vdash \neg F(0), F(s), F(r_0)$ and $\vdash \neg F(0), F(s), \neg F(\mathsf{S}\,r_0)$.

[[By Lemma 2.7c, $\vdash F(0) \wedge \neg F(y) \to F(p(y)) \wedge \neg F(p(y)+1)$ with $p(y) := \overline{\mu}_{z \le y} \neg F(z) \dot{-} 1$.]]

Now we conclude as follows

$$\dfrac{\dfrac{\text{IV}\;\dfrac{\text{Subst}\;\dfrac{\Gamma', \exists y A(y), \neg F(x), F(\mathsf{S}\,x)}{\Gamma', \exists y A(y), \neg F(r_0), F(\mathsf{S}\,r_0)}}{\Gamma', A(t), \neg F(r_0), F(\mathsf{S}\,r_0)} \qquad \neg F(0), F(s), F(r_0)}{\Gamma', A(t), \neg F(0), F(s), F(\mathsf{S}\,r_0)}\,\text{Cut} \qquad \neg F(0), F(s), \neg F(\mathsf{S}\,r_0)}{\Gamma', A(t), \neg F(0), F(s)}\,\text{Cut}$$

Corollary 2.9. The provably recursive functions of PRA are exactly the primitive recursive functions.

Proof: Obviously every primitive recursive function f is provably recursive in PRA: let $A(\vec{x}, y) := (f\vec{x} = y)$. Now let A be a Σ_1-formula with (i) $\mathsf{PRA} \vdash \forall \vec{x} \exists y A(\vec{x}, y)$ and (ii) $\forall \vec{a}, b (f(\vec{a}) = b \Leftrightarrow \mathcal{N} \models A[\vec{a}, b])$. First notice that the statement of Theorem 2.9 also holds for $A \in \Sigma_1$: $\vdash \Gamma, \exists y \exists x B(x, y) \implies \vdash \Gamma, \exists z B(\pi_1 z, \pi_2 z) \overset{\text{Th.2.9}}{\implies} \vdash \Gamma, B(\pi_1 t, \pi_2 t) \implies \vdash \Gamma, \exists x B(x, \pi_2 t)$. Then we conclude as follows: $\mathsf{PRA} \vdash \forall \vec{x} \exists y A(\vec{x}, y) \Rightarrow \mathsf{PRA} \vdash \exists y A(\vec{x}, y) \Rightarrow \mathsf{PRA} \vdash A(\vec{x}, t(\vec{x}))$ for some PR-term t. Then $\forall \vec{a} (\mathcal{N} \models A(\vec{x}, t(\vec{x}))[\vec{a}])$ and thus, by (ii), $\forall \vec{a} (f(\vec{a}) = t^{\mathcal{N}}[\vec{a}])$.

Definition of $\mathrm{I}\Sigma_1$

The (Tait style) system $\mathrm{I}\Sigma_1$ is the same as PRA only that the QF-induction rule is replaced by the

Σ_1-*induction rule* $\dfrac{\Gamma, F_x(0) \quad \Gamma, \neg F, F_x(\mathsf{S}\,x)}{\Gamma, F_x(t)}$ $(F \in \Sigma_1$ and $x \notin \mathrm{FV}(\Gamma, F_x(t)))$

Remark. As for PRA we see that the results 1.6-1.9 apply to $\mathrm{I}\Sigma_1$, now with $\Phi = \Sigma_1$.

Theorem 2.10

$\mathrm{I}\Sigma_1 \vdash \Gamma$ with $\Gamma \subseteq \Sigma_1 \implies \mathsf{PRA} \vdash \Gamma$.

Beweis:

By Theorem 1.8 we have (for some k) $\mathrm{I}\Sigma_1 \vdash^k_{\Sigma_1,0} \Gamma$, i.e. there exists an $\mathrm{I}\Sigma_1$-derivation of Γ where all cut formulas are in $\overline{\Sigma_1}$. We proceed by induction on such a derivation.

1. $\mathrm{I}\Sigma_1 \vdash^{k-1}_{\Sigma_1,0} \exists x A, \Gamma$ and $\mathrm{I}\Sigma_1 \vdash^{k-1}_{\Sigma_1,0} \forall x \neg A, \Gamma$ with $A \in \mathsf{QF}$:

$\mathrm{I}\Sigma_1 \vdash^{k-1}_{\Sigma_1,0} \exists x A, \Gamma \overset{\text{IH}}{\Longrightarrow} \mathsf{PRA} \vdash \Gamma, \exists x A \quad (*)$.

$\mathrm{I}\Sigma_1 \vdash^{k-1}_{\Sigma_1,0} \forall x \neg A, \Gamma \overset{\text{Inversion}}{\Longrightarrow} \mathrm{I}\Sigma_1 \vdash^{k-1}_{\Sigma_1,0} \neg A_x(y), \Gamma \overset{\text{IH}}{\Rightarrow} \mathsf{PRA} \vdash \neg A_x(y), \Gamma \Rightarrow \mathsf{PRA} \vdash \forall x \neg A, \Gamma \overset{(*)}{\Rightarrow} \mathsf{PRA} \vdash \Gamma$.

2. $\Gamma = \Gamma', \exists x A(x,t)$ and $\mathrm{I}\Sigma_1 \vdash^{k-1}_{\Sigma_1,0} \Gamma', \exists x A(x,0)$, $\mathrm{I}\Sigma_1 \vdash^{k-1}_{\Sigma_1,0} \Gamma', \neg\exists x A(x,y), \exists x A(x, \mathsf{S}\,y)$ with $y \notin \mathrm{FV}(\Gamma)$:

(Due to our general conventions we also have $x \notin \mathrm{FV}(t)$ and $x, y \notin \mathrm{FV}(A(0,0))$.)

By Inversion and I.H. we obtain $\mathsf{PRA} \vdash \Gamma', \exists x A(x,0)$ and $\mathsf{PRA} \vdash \Gamma', \neg A(x,y), \exists x A(x, \mathsf{S}\,y)$ where w.l.o.g. $x \notin \mathrm{FV}(\Gamma')$.

From this it follows by Theorem 2.9 that $\mathsf{PRA} \vdash \Gamma', A(q,0)$ and $\mathsf{PRA} \vdash \Gamma', \neg A(x,y), A(p(x,y), \mathsf{S}\,y)$ for certain PR-terms p, q.

There exists a PR-term $r(y)$ such that $\mathsf{PRA} \vdash r(0) = q \wedge r(\mathsf{S}\,y) = p(r(y), y)$.

Now we obtain $\mathsf{PRA} \vdash \Gamma', \neg A(r(y), y), A(r(\mathsf{S}\,y), \mathsf{S}\,y)$, and then $\mathsf{PRA} \vdash \Gamma', \neg A(r(0), 0), A(r(t), t)$ by (Ind).

Together with $\mathsf{PRA} \vdash r(0) = q$ and $\mathsf{PRA} \vdash \Gamma', A(q,0)$ this yields $\mathsf{PRA} \vdash \Gamma', A(r(t), t)$.

From this we conclude $\mathsf{PRA} \vdash \Gamma', \exists x A(x,t)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{r(\mathsf{S}\,y) = p(r(y),y) \qquad \dfrac{\dfrac{\Gamma', \neg A(x,y), \exists x A(x, \mathsf{S}\,y)}{\Gamma', \neg A(x,y), A(p(x,y), \mathsf{S}\,y)}\,(Th.2.9)}{\Gamma', \neg A(r(y),y), A(p(r(y),y), \mathsf{S}\,y)}\,(Subst)}{\Gamma', \neg A(r(y),y), A(r(\mathsf{S}\,y), \mathsf{S}\,y)}}{\Gamma', \neg A(r(0),0), A(r(t),t)}\,(Ind) \qquad r(0) = q}{\Gamma', \neg A(q,0), A(r(t),t)} \qquad \Gamma', A(q,0)}{\dfrac{\Gamma', A(r(t),t)}{\Gamma', \exists x A(x,t)}\,(\exists)}\,\mathrm{Cut}$$

3. In all other cases the claim follows immediately from the I.H.

Definition. A formula of the form $\forall \vec{x} \exists \vec{y} A$ with $A \in \mathsf{QF}$ is called a Π^0_2-formula.

Corollary 2.10.

$\mathrm{I}\Sigma_1$ is Π^0_2-conservative over PRA, i.e. $\mathrm{I}\Sigma_1$ is an extension of PRA which proves the same Π^0_2-sentences as PRA. Especially, $\mathrm{I}\Sigma_1$ has the same provably recursive functions as PRA.

Lemma 2.11.

For each Σ-formula C there is a Σ_1-formula C' such that $\mathsf{PRA} \vdash C' \to C$ and $\mathrm{I}\Sigma_1 \vdash C \to C'$.

Proof:

Let C^z be the Δ_0-formula resulting from C when every unbounded quantifier is bounded by z (z being a new variable). Let $C' := \exists z C_0(z)$ where $C_0(z) \in \mathsf{QF}$ such that $\mathsf{PRA} \vdash C_0(z) \leftrightarrow C^z$.

Obviously $\mathsf{PRA} \vdash \exists z C^z \to C$, for each Σ-formula C.

By induction on Σ-formulas one proves $\mathrm{I}\Sigma_1 \vdash C \to \exists z C^z$. We only treat the crucial case $C = \forall x \le t B$. Then $C^z = \forall x \le t B^z$. By I.H. $\mathrm{I}\Sigma_1 \vdash B \to \exists y B^y$ and thus $\mathrm{I}\Sigma_1 \vdash C \to \forall x \le t \exists y B^y$. Now by the Proposition below we obtain $\mathrm{I}\Sigma_1 \vdash C \to \exists z \forall x \le t B^z$

Proposition. $\mathrm{I}\Sigma_1 \vdash \forall x \le x_1 \exists y B^y \to \exists z \forall x \le x_1 B^z$ for each Σ-formula B

Proof (in $\mathrm{I}\Sigma_1$): Assume $\forall x \le x_1 \exists y B(x)^y \quad (*)$.
Now by induction on x_0 we prove $\exists z \forall x \le x_0 (x \le x_1 \to B(x)^z)$.
Due to Lemma 2.8 this induction is admissible in $\mathrm{I}\Sigma_1$.

Start: $\vdash B(0)^y \to \forall x \le 0 (x \le x_1 \to B(x)^y) \;\Rightarrow \vdash \exists y B(0)^y \to \exists z \forall x \le 0 (x \le x_1 \to B(x)^z))$.

Step: By IH we have a z with $\forall x \le x_0 (x \le x_1 \to B(x)^z)$.
1. Assume $\mathsf{S}\,x_0 \le x_1$. Then by $(*)$ there is a y with $B(\mathsf{S}\,x_0)^y$. Let $z_1 := \max\{z, y\}$.
Then $\forall x \le \mathsf{S}\,x_0 (x \le x_1 \to B(x)^{z_1})$.
(Here we have used that for every Σ-formula B one has $\vdash z \le z_1 \wedge B^z \to B^{z_1}$.)

2. Otherwise $\forall x (\mathsf{S}\,x_0 \le x_1 \to B(x)^0)$.

§3 A general framework for variable binding and substitution

In this section we will give a thorough treatment of *substitution* which has been somewhat unprecise and up in the air up to now. At the beginning of §1 we have said that α-equivalent formulas (i.e., one which coincide after a suitable renaming of bound variables) will be identified, so that formally spoken formulas would be equivalence classes. This approach will not be pursued further. Instead we will modify the mechanism of variable binding by making use of so called de Bruijn indices instead of bound variables. So we come to a notion of formula where α-equivalence is just identity, and substitution can be carried out without renaming. We first present a general "theory" of variable binding and substitution, and after that consider the language of 1st order predicate logic as a special case.

Let us assume the following pairwise disjoint sets of *basic symbols*.

Vars : infinite set of *variables*, denoted by $x, y, z, ...$;

$\{\circ_k : k \in \mathbb{N}\}$: set of *de Bruijn indices*;

$\mathcal{F}$: set of *function symbols*, denoted by f;

$\mathcal{B}$: set of *binding symbols (binders)*, denoted by $\flat$.

For every $f \in \mathcal{F}$ an *arity* $\#(f) \in \mathbb{N}$ is fixed; further we set $\#(\circ_k) := 0$ and $\#(\flat) := 1$.

$\mathcal{F}' := \{\circ_k : k \in \mathbb{N}\} \cup \mathcal{F} \cup \mathcal{B}$, $\mathcal{F}'_m := \{h \in \mathcal{F}' : \#(h) = m\}$, $\mathcal{F}_m := \mathcal{F}'_m \cap \mathcal{F}$.

Inductive Definition of the set $\mathcal{T}' = \mathcal{T}'(\mathsf{Vars}; \mathcal{F}; \mathcal{B})$ of quasiterms

1. $\mathsf{Vars} \subseteq \mathcal{T}'$;
2. $h \in \mathcal{F}'_m$ & $t_1, ..., t_m \in \mathcal{T}' \implies ht_1...t_m \in \mathcal{T}'$.

Notation: We use r, s, t to denote quasiterms.

Definition.

$\mathrm{FV}(t) :=$ set of all variables occurring in t,

$\mathrm{lh}(t) :=$ length of t as string of basic symbols.

Definition of $t_x[n] \in \mathcal{T}'$ for $t \in \mathcal{T}'$

1. For $t \in \mathsf{Vars} \cup \{\circ_k : k \in \mathbb{N}\}$: $t_x[n] := \begin{cases} \circ_n & \text{if } t = x \\ t & \text{otherwise} \end{cases}$;
2. $(ft_1...t_m)_x[n] := f(t_1)_x[n]...(t_m)_x[n]$;
3. $(\flat r)_x[n] := \flat r_x[n{+}1]$.

Definition. $\flat x.r := \flat r_x[0]$.

Remark. (B0) $\mathrm{FV}(\flat x.r) = \mathrm{FV}(r) \setminus \{x\}$.

Proof: $\mathrm{FV}(\flat x.r) = \mathrm{FV}(\flat r_x[0]) = \mathrm{FV}(r_x[0]) = \mathrm{FV}(r) \setminus \{x\}$.

Inductive Definition of the set $\mathcal{T} = \mathcal{T}(\mathsf{Vars}; \mathcal{F}; \mathcal{B})$ of terms

1. $\mathsf{Vars} \subseteq \mathcal{T}$;
2. $f \in \mathcal{F}_m$ & $t_1, ..., t_m \in \mathcal{T} \implies ft_1...t_m \in \mathcal{T}$;
3. $\flat \in \mathcal{B}$ & $r \in \mathcal{T} \implies \flat x.r \in \mathcal{T}$.

Definition

A *substitution* is a mapping $\theta : \mathcal{T}' \to \mathcal{T}'$, $t \mapsto t\theta$ such that

(i) $x\theta \in \mathcal{T}$ for all $x \in \mathsf{Vars}$,

(ii) $(ht_1...t_m)\theta = h(t_1\theta)...(t_m\theta)$ for all $ht_1...t_m \in \mathcal{T}' \setminus \mathsf{Vars}$.

SUB := set of all substitutions. $\epsilon := \mathrm{id}_{\mathcal{T}'}$.

We use θ, θ' to denote substitutions.

Lemma 3.1.

(S0) $\forall t \in \mathcal{T}(t\theta \in \mathcal{T})$;

(S1) $\forall x \in \mathrm{FV}(t)(x\theta = x\theta') \iff t\theta = t\theta'$;

(S2) $\epsilon \in \mathrm{SUB}$;

(S3) $\theta, \theta' \in \mathrm{SUB} \implies \theta \circ \theta' \in \mathrm{SUB}$.

(S4) For every $\theta \in \mathrm{SUB}$, $x \in \mathsf{Vars}$, $s \in \mathcal{T}$ there is a unique $\theta_x^s \in \mathrm{SUB}$ with $y\theta_x^s := \begin{cases} s & \text{if } y = x \\ y\theta & \text{otherwise} \end{cases}$.

The proof of (S0) will be given below. The other statements are easily seen, where for (S3) one uses (S0).

Remark. $\forall x \in \mathsf{Vars}(x\theta = x\theta') \implies \theta = \theta'$. [cf. (S1)]

Notation: $t_x(s) := t(x/s) := t\,\epsilon_x^s$

Remark. $x \notin \mathrm{FV}(t) \Rightarrow t_x(s) = t$. [cf. (S1),(S2)]

Lemma 3.2.

(a) $x \neq y \Rightarrow (x \in \mathrm{FV}(t) \Leftrightarrow t\epsilon_x^y \neq t)$.

(b) $\mathrm{FV}(t\theta) = \bigcup_{z \in \mathrm{FV}(t)} \mathrm{FV}(z\theta)$.

(c) $y \notin \mathrm{FV}((\flat x.r)\theta) \implies r\theta_x^y \epsilon_y^s = r\theta_x^s$.

(d) $x \in \mathrm{FV}(t) \implies \mathrm{FV}(t_x(s)) = (\mathrm{FV}(t) \setminus \{x\}) \cup \mathrm{FV}(s)$.

(e) $y \notin \mathrm{FV}(\flat x.r) \implies r_x(y)_y(s) = r_x(s)$.

(f) $r_x(s)\theta = r\theta_x^{s\theta}$.

(g) $y \notin \mathrm{FV}((\flat x.r)\theta) \implies r_x(s)\theta = (r\theta_x^y)_y(s\theta)$.

Proof:

(a) $x \notin \mathrm{FV}(t) \Leftrightarrow \forall z \in \mathrm{FV}(t)(z \neq x) \overset{\mathrm{Def.}\,\epsilon_x^y}{\Leftrightarrow} \forall z \in \mathrm{FV}(t)(z\epsilon_x^y = z) \overset{(S1),(S2)}{\Leftrightarrow} t\epsilon_x^y = t$.

(b) Let $x \neq y$. Then: $x \notin \mathrm{FV}(t\theta) \overset{(a)}{\Leftrightarrow} t\theta\epsilon_x^y = t\theta \overset{(S1),(S3)}{\Leftrightarrow} \forall z \in \mathrm{FV}(t)(z\theta\epsilon_x^y = z\theta) \overset{(a)}{\Leftrightarrow} \forall z \in \mathrm{FV}(t)(x \notin \mathrm{FV}(z\theta))$.

(c) 1. $x\theta_x^y\epsilon_y^s = y\epsilon_y^s = s = x\theta_x^s$.

2. $x \neq z \in \mathrm{FV}(r) \Rightarrow z \in \mathrm{FV}(\flat x.r) \overset{(b)}{\Rightarrow} y \notin \mathrm{FV}(z\theta) \Rightarrow z\theta_x^y\epsilon_y^s = z\theta\epsilon_y^s \overset{(S1),(S2)}{=} z\theta = z\theta_x^s$.

Now the claim follows by (S1),(S3).

(d) $\mathrm{FV}(t_x(s)) \overset{(b)}{=} \bigcup_{z \in \mathrm{FV}(t)} \mathrm{FV}(z_x(s)) = \bigcup_{z \in \mathrm{FV}(t) \setminus \{x\}} \{z\} \cup \mathrm{FV}(s) = (\mathrm{FV}(t) \setminus \{x\}) \cup \mathrm{FV}(s)$.

(e) follows from (c) with $\theta = \epsilon$.

(f) 1. $x_x(s)\theta = s\theta = x\theta_x^{s\theta}$. 2. $y \neq x \Rightarrow y_x(s)\theta = y\theta = y\theta_x^{s\theta}$. Now the claim follows by (S1),(S3).

(g) $r_x(s)\theta \overset{(f)}{=} r\theta_x^{s\theta} \overset{(c)}{=} r\theta_x^y\epsilon_y^{s\theta}$.

Lemma 3.3. For $r, r' \in \mathcal{T}$ the following holds

(B1) $\flat x.r = \flat x.r' \implies r = r'$.

(B2) $y \notin \mathrm{FV}((\flat x.r)\theta) \implies (\flat x.r)\theta = \flat y.r\theta^y_x$.

The proof will be given below.

Lemma 3.4. For $r, r' \in \mathcal{T}$ we have

(a) $y \notin \mathrm{FV}(\flat x.r) \implies \flat x.r = \flat y.r_x(y)$.

(b) $\flat x.r = \flat y.r' \implies r' = r_x(y)$.

(c) $\flat x.r = \flat y.r' \iff \forall s \in \mathcal{T}(r_x(s) = r'_y(s))$.

Proof:

(a) follows from (B2) with $\theta := \epsilon$.

(b) $\flat x.r = \flat y.r' \overset{\text{(B0)}}{\Rightarrow} y \notin \mathrm{FV}(\flat x.r) \overset{\text{(B2)}}{\Rightarrow} \flat y.r' = \flat x.r = \flat y.r_x(y) \overset{\text{(B1)}}{\Rightarrow} r' = r_x(y)$. (c) "$\Rightarrow$": $\flat x.r = \flat y.r' \overset{\text{(b)}}{\Rightarrow}$ $y \notin \mathrm{FV}(\flat x.r)$ & $r_x(y) = r' \overset{\text{L.3.2e}}{\Rightarrow} r_x(s) = r_x(y)_y(s) = r'_y(s)$.

"$\Leftarrow$": $r = r_x(x) = r'_y(x) \overset{\text{L.3.2d}}{\Rightarrow} \mathrm{FV}(r) = \mathrm{FV}(r'_y(x)) \subseteq (\mathrm{FV}(r') \setminus \{y\}) \cup \{x\} \Rightarrow$

$\Rightarrow y \notin \mathrm{FV}(r) \setminus \{x\} = \mathrm{FV}(\flat x.r) \overset{\text{(a)}}{\Rightarrow} \flat x.r = \flat y.r_x(y) = \flat y.r'_y(y) = \flat y.r'$.

Definition.

$\flat\mathcal{T} := \{\flat x.r : x \in \mathsf{Vars}\ \&\ r \in \mathcal{T}\}$

$\beta : \flat\mathcal{T} \times \mathcal{T} \to \mathcal{T}$, $\beta(\flat x.r, s) := r_x(s)$ (due to Lemma 3.4c, β is well defined)

Lemma 3.5.

(a) $r, s \in \mathcal{T} \Rightarrow (\flat x.r)\theta \in \flat\mathcal{T}\ \&\ \beta((\flat x.r)\theta, s) = r\theta^s_x$.

(b) $t \in \flat\mathcal{T}\ \&\ y \notin \mathrm{FV}(t) \Rightarrow t = \flat y.\beta(t, y)$.

(c) $t \in \flat\mathcal{T}\ \&\ s \in \mathcal{T} \Rightarrow \beta(t, s)\theta = \beta(t\theta, s\theta)$.

Proof:

(a) Let $y \notin \mathrm{FV}((\flat x.r)\theta)$. Then $\beta((\flat x.r)\theta, s) = \beta(\flat y.r\theta^y_x, s) = (r\theta^y_x)_y(s) \overset{\text{L3.2c}}{=} r\theta^s_x$.

(b) Let $t = \flat x.r$. Then $t = \flat y.r_x(y) = \flat y.\beta(t, y)$.

(c) Let $t = \flat x.r$. $\beta((\flat x.r)\theta, s\theta) \overset{\text{(a)}}{=} r\theta^{s\theta}_x \overset{\text{3.2f}}{=} r_x(s)\theta = \beta(\flat x.r, s)\theta$.

Remark. Given a term $\flat x.r$ and a substitution θ with $\mathrm{dom}(\theta) = \{y : y \neq y\theta\}$ finite, one may assume w.l.o.g. that $x \notin \mathrm{FV}((\flat x.r)\theta) \cup \mathrm{dom}(\theta)$, and so $(\flat x.r)\theta = \flat x.r\theta^x_x = \flat x.r\theta$.

Proof of (S0), (B1), (B2)

Inductive definition of sets $\mathcal{T}_n = \mathcal{T}_n(\mathsf{Vars}; \mathcal{F}; \mathcal{B})$ of quasiterms

1. $\mathsf{Vars} \cup \{\circ_k : k < n\} \subseteq \mathcal{T}_n$;
2. $f \in \mathcal{F}_m\ \&\ t_1, \ldots, t_m \in \mathcal{T}_n \implies f t_1 \ldots t_m \in \mathcal{T}_n$;
3. $r \in \mathcal{T}_{n+1}\ \&\ \flat \in \mathcal{B} \Rightarrow \flat r \in \mathcal{T}_n$.

Remark. $n < m \Rightarrow \mathcal{T}_n \subseteq \mathcal{T}_m$. $\mathcal{T}' = \bigcup_{n \in \mathbb{N}} \mathcal{T}_n$.

Lemma 3.6.

(a) $t \in \mathcal{T}_n \;\Rightarrow\; t_x[n] \in \mathcal{T}_{n+1}$.

(b) $\mathcal{T} = \mathcal{T}_0$.

(c) $t \in \mathcal{T}_n \;\Rightarrow\; t\theta \in \mathcal{T}_n$.

(d) $t, t' \in \mathcal{T}_n \;\&\; t_x[n] = t'_x[n] \;\Rightarrow\; t = t'$.

(e) $t \in \mathcal{T}_n \;\&\; y = x\theta \notin \bigcup_{z \in \mathrm{FV}(t)\setminus\{x\}} \mathrm{FV}(z\theta) \;\Rightarrow\; t_x[n]\theta = (t\theta)_y[n]$.

Proof:

(a) obvious.

(b) "$\mathcal{T} \subseteq \mathcal{T}_0$": $r \in \mathcal{T}_0 \overset{(a)}{\Rightarrow} r_x[0] \in \mathcal{T}_1 \;\Rightarrow\; \flat x.r = \flat r_x[0] \in \mathcal{T}_0$.

"$\mathcal{T}_0 \subseteq \mathcal{T}$": One easily proves: (*) $t' \in \mathcal{T}_{n+1} \;\&\; x \notin \mathrm{FV}(t') \;\Rightarrow\; \exists t \in \mathcal{T}_n (t' = t_x[n])$.

Now by induction on $\mathrm{lh}(t)$ on proves $(t \in \mathcal{T}_0 \Rightarrow t \in \mathcal{T})$:

Let $t = \flat r'$ with $r' \in \mathcal{T}_1$. Take $x \notin \mathrm{FV}(r')$. Then by (*) there is an $r \in \mathcal{T}_0$ with $r' = r_x[0]$.

Now $\mathrm{lh}(r) = \mathrm{lh}(r') < \mathrm{lh}(t)$ and therefore by I.H. $r \in \mathcal{T}$ and thus $t = \flat r_x[0] = \flat x.r \in \mathcal{T}$.

(c) follows from (b) by induction on $\mathcal{T}_n$.

(d) 1. $t \in \mathsf{Vars} \cup \{\circ_k : k < n\}$: Then also $t' \in \mathsf{Vars} \cup \{\circ_k : k < n\}$.

1.1. $t = x$: $t'_x[n] = t_x[n] = \circ_n \;\Rightarrow\; t' = x$.

1.2. $t \neq x$: $t'_x[n] = t_x[n] = t \neq \circ_n \;\Rightarrow\; t' = t'_x[n] = t$.

2. $t = \flat r$ with $r \in \mathcal{T}_{n+1}$: Then $t' = \flat r'$ with $r' \in \mathcal{T}_{n+1}$ and $\flat r_x[n{+}1] = t_x[n] = t'_x[n] = \flat r'_x[n{+}1]$.

Hence $r_x[n{+}1] = r'_x[n{+}1]$ and by I.H. $r = r'$ which yields $t = t'$.

(e) 1. $t = x$: $t_x[n]\theta = \circ_n = y_y[n] = (x\theta)_y[n]$.

2. $x \neq t \in \mathsf{Vars} \cup \{\circ_k : k < n\}$: $t_x[n]\theta = t\theta = (t\theta)_y[n]$, since $y \notin \mathrm{FV}(t\theta)$.

3. $t = \flat r$ with $r \in \mathcal{T}_{n+1}$: $t_x[n]\theta = \flat r_x[n{+}1]\theta \overset{\mathrm{IH}}{=} \flat (r\theta)_y[n{+}1] = (\flat(r\theta))_y[n] = (t\theta)_y[n]$.

(S0) $t \in \mathcal{T} \overset{3.6\mathrm{b}}{\Rightarrow} t \in \mathcal{T}_0 \overset{3.6\mathrm{c}}{\Rightarrow} t\theta \in \mathcal{T}_0 \overset{3.6\mathrm{b}}{\Rightarrow} t\theta \in \mathcal{T}$.

(B1) $r, r' \in \mathcal{T} \;\&\; \flat x.r = \flat x.r' \overset{3.6\mathrm{b}}{\Rightarrow} r, r' \in \mathcal{T}_0 \;\&\; r_x[0] = r'_x[0] \overset{3.6\mathrm{d}}{\Rightarrow} r = r'$.

(B2) $y \notin \mathrm{FV}((\flat x.r)\theta) \overset{\mathrm{L.3.2b}}{=} \bigcup_{z \in \mathrm{FV}(\flat x.r)} \mathrm{FV}(z\theta) \overset{(\mathrm{B0})}{=} \bigcup_{z \in \mathrm{FV}(r)\setminus\{x\}} \mathrm{FV}(z\theta) \;\Rightarrow$

$\Rightarrow\; (\flat x.r)\theta \overset{(\mathrm{B0}),(\mathrm{S1})}{=} (\flat x.r)\theta^y_x = \flat\, r_x[0]\theta^y_x \overset{3.6\mathrm{b,e}}{=} \flat(r\theta^y_x)_y[0] = \flat y.r\theta^y_x$.

Now we come back to the language of 1st order predicate logic.

Let $\mathsf{Vars} := \{v_0, v_1, ...\}$, and $\mathcal{L}$ a 1st order language as introduced in §1.

$\mathcal{L}$-terms and $\mathcal{L}$-formulas are introduced literally as in §1, but with the difference that now $\mathsf{Q}xA$ is considered as a shorthand for $\mathsf{Q}x.A$, i.e., $\mathcal{L}$-terms and $\mathcal{L}$-formulas are considered as elements of $\mathcal{T}(\mathsf{Vars}; \mathcal{F}; \mathcal{B})$ with $\mathcal{F} := \mathcal{L} \cup \{\neg, \vee, \wedge\}$ and $\mathcal{B} := \{\forall, \exists\}$.

In the following, θ ranges over substitutions having the property that $x\theta$ is an $\mathcal{L}$-term for each $x \in \mathsf{Vars}$. Moreover we assume that $\mathrm{dom}(\theta) := \{x \in \mathsf{Vars} : x\theta \neq x\}$ *is finite.*

Lemma 3.7

(a) If t is an $\mathcal{L}$-term then $t\theta$ is an $\mathcal{L}$-term.

(b) If C is an $\mathcal{L}$-formula then $C\theta$ is an $\mathcal{L}$-formula.

Proof by induction on the definition of $\mathcal{L}$-terms and $\mathcal{L}$-formulas:
The only nontrivial case is $C = \forall x A$. Choose $y \notin \mathrm{FV}(C\theta)$. Then $C\theta = \forall y A\theta^y_x$, and by I.H. $A\theta^y_x$ is an $\mathcal{L}$-formula. This yields the claim.

Definition of the truth value $[\![C]\!]^{\mathcal{M}}_{\xi}$ of a formula C in an interpretation $(\mathcal{M}, \xi)$

Let $\mathcal{M}$ be an $\mathcal{L}$-structure with universe M, and let ξ, η range over $\mathcal{M}$-assignments,
i.e., functions $\xi : \mathsf{Vars} \to M$.

For each $\mathcal{M}$-assignment ξ, the *value* $[\![t]\!]^{\mathcal{M}}_{\xi} \in M$ of an $\mathcal{L}$-term t and the *truth value* $[\![A]\!]^{\mathcal{M}}_{\xi} \in \{0,1\}$ of an $\mathcal{L}$-formula are define as usual. Only the quantifier case requires some additional care; here we make use of some previously fixed function v which assigns to each formula C a variable $\mathsf{v}(C) \notin \mathrm{FV}(C)$:

If $C = {}^{\forall}_{\exists} x A$ *with* $x = \mathsf{v}(C)$ *then* $[\![C]\!]^{\mathcal{M}}_{\xi} := {}^{\min}_{\max}\{[\![A]\!]^{\mathcal{M}}_{\xi^a_x} : a \in M\}$.

Of course, this definition is only reasonable if $[\![C]\!]^{\mathcal{M}}_{\xi}$ does not depend on the choice of x, i.e., if $[\![\forall x A]\!]^{\mathcal{M}}_{\xi} = \min\{[\![A]\!]^{\mathcal{M}}_{\xi^a_x} : a \in M\}$ also in case that $x \neq \mathsf{v}(\forall x A)$. This will be shown now.

Lemma 3.8. $\forall z \in \mathrm{FV}(C)(\xi(z) = \eta(z)) \implies [\![C]\!]^{\mathcal{M}}_{\xi} = [\![C]\!]^{\mathcal{M}}_{\eta}$.

Proof: If $C = \forall x A$ with $x = \mathsf{v}(C)$ then $[\![C]\!]_{\xi} = \min\{[\![A]\!]_{\xi^a_x} : a \in M\} \overset{\mathrm{IH}}{=} \min\{[\![A]\!]_{\eta^a_x} : a \in M\} = [\![C]\!]_{\eta}$.

Lemma 3.9. $\forall z \in \mathrm{FV}(C)([\![z\theta]\!]^{\mathcal{M}}_{\xi} = [\![z]\!]_{\eta}) \implies [\![C\theta]\!]^{\mathcal{M}}_{\xi} = [\![C]\!]_{\eta}$.

Proof: Let $C = \forall x A$ with $x = \mathsf{v}(C)$; then $C\theta = \forall y A\theta^y_x$ with $y = \mathsf{v}(C\theta)$.

$[\![C\theta]\!]_{\xi} = [\![\forall y A\theta^y_x]\!]_{\xi} = \min\{[\![A\theta^y_x]\!]_{\xi^a_y} : a \in M\} \overset{\mathrm{IH}+(*)}{=} \min\{[\![A]\!]_{\eta^a_x} : a \in M\} = [\![C]\!]_{\eta}$.

$(*)$ 1. $[\![x\theta^y_x]\!]_{\xi^a_y} = a = [\![x]\!]_{\eta^a_x}$.

2. If $z \in \mathrm{FV}(A) \setminus \{x\} = \mathrm{FV}(C)$ then $y \notin \mathrm{FV}(z\theta)$ (since $y \notin \mathrm{FV}(C\theta)$) and thus
$[\![z\theta^y_x]\!]_{\xi^a_y} = [\![z\theta]\!]_{\xi^a_y} \overset{\mathrm{L.3.8}}{=} [\![z\theta]\!]_{\xi} = [\![z]\!]_{\eta} = [\![z]\!]_{\eta^a_x}$.

Lemma 3.10. $[\![\forall x A]\!]_{\xi} = \min\{[\![A]\!]_{\xi^a_x} : a \in M\}$

Proof: Let $y := \mathsf{v}(\forall x A)$. Then $\forall x A = \forall y A_x(y)$, and thus
$[\![\forall x A]\!]_{\xi} = \min\{[\![A_x(y)]\!]_{\xi^a_y} : a \in M\} \overset{\mathrm{L.3.9}+(*)}{=} \min\{[\![A]\!]_{\xi^a_x} : a \in M\}$.

$(*)$: cf. $(*)$ with $\theta := \epsilon$ in the proof of 3.9.

Lemma 3.11.

All rules of PRA and $\mathrm{I}\Sigma_1$ are closed under substitution.

Proof:

1. $\dfrac{\Gamma, A}{\Gamma, \forall x A}$ with $x \notin \mathrm{FV}(\Gamma)$:

In this case we only have to consider substitutions θ with $y := x\theta \in \mathsf{Vars} \setminus \mathrm{FV}(\Gamma\theta, (\forall x A)\theta)$.

$y = x\theta \notin \mathrm{FV}((\forall x A)\theta) \;\Rightarrow\; (\forall x A)\theta = \forall y\, A\theta^y_x = \forall y\, A\theta$.

2. $\dfrac{\Gamma, A_x(t)}{\Gamma, \exists x A}$: Let $y \notin \mathrm{FV}((\exists x A)\theta)$. Then $(\exists x A)\theta = \exists y\, A\theta^y_x$ and $A_x(t)\theta = (A\theta^y_x)_y(t\theta)$.

3. $(\neg(s=t), \neg A_x(s), A_x(t))\theta = \neg(s\theta = t\theta), \neg(A\theta^y_x)_y(s\theta), (A\theta^y_x)_y(t\theta)$, if $y \notin \mathrm{FV}((\forall x A)\theta)$.

4. $\dfrac{\Gamma, \neg F, F_x(\mathsf{S}\,x)}{\Gamma, \neg F_x(0), F_x(t)}$ or $\dfrac{\Gamma, F_x(0) \quad \Gamma, \neg F, F_x(\mathsf{S}\,x)}{\Gamma, F_x(t)}$ with $x \notin \mathrm{FV}(\Gamma, F_x(t))$:

Then (as in 1.) $y = x\theta \notin \mathrm{FV}(\Gamma\theta, \neg F_x(t)\theta)$ which implies $y \notin \mathrm{FV}((\forall x F)\theta)$.

Hence $F_x(t)\theta = (F\theta)_y(t\theta)$, $F_x(0)\theta = (F\theta)_y(0)$, $(F_x(\mathsf{S}\,x))\theta = (F\theta)_y((\mathsf{S}\,x)\theta) = (F\theta)_y(\mathsf{S}\,y)$.

Moreover if $F \in \mathsf{QF}$ [Σ_1, resp.] then $F\theta \in \mathsf{QF}$ [Σ_1, resp.]

§4 An alternative presentation of the Tait style sequent calculus

We introduce a new notion of derivation for the Tait style sequent calculus which differs from the usual one (introduced in §1) in so far as the new derivations have so-called inference symbols (denoting inferences) and not sequents assigned to their nodes. The sequent "belonging" to a certain node τ of a derivation d is not explicitely displayed, but can be computed by tree recursion from d (similarly as the free assumptions in a natural deduction style derivation). This approach is particularly useful for our further purposes.

Proof systems

A proof system $\mathfrak{S}$ is given by

- a set of formal expressions called *inference symbols* (syntactic variable $\mathcal{I}$)
- for each inference symbol $\mathcal{I}$ a set $|\mathcal{I}|$ (the *arity* of $\mathcal{I}$), a sequent $\Delta(\mathcal{I})$ and a family of sequents $(\Delta_\iota(\mathcal{I}))_{\iota\in|\mathcal{I}|}$. The elements of $\Delta(\mathcal{I})$ [$\bigcup_{\iota\in|\mathcal{I}|}\Delta_\iota(\mathcal{I})$] are called the *principal formulas* [*minor formulas*] of $\mathcal{I}$.
- for each inference symbol $\mathcal{I}$ a set $\mathrm{Eig}(\mathcal{I})$ which is either empty or a singleton $\{x\}$ with $x \in \mathsf{Vars}\backslash\mathrm{FV}(\Delta(\mathcal{I}))$; in the latter case x is called the *eigenvariable* of $\mathcal{I}$.

NOTATION

By writing

$(\mathcal{I}) \quad \dfrac{\ldots\Delta_\iota\ldots(\iota\in I)}{\Delta}$ [$!x!$]

we express that $\mathcal{I}$ is an inference symbol with $|\mathcal{I}| = I$, $\Delta(\mathcal{I}) = \Delta$, $\Delta_\iota(\mathcal{I}) = \Delta_\iota$, $\mathrm{Eig}(\mathcal{I}) = \emptyset$ [$\mathrm{Eig}(\mathcal{I}) = \{x\}$].

If $|\mathcal{I}| = \{0,...,n\}$ we write $\dfrac{\Delta_0 \;\; \Delta_1 \;\ldots\; \Delta_n}{\Delta}$, instead of $\dfrac{\ldots\Delta_\iota\ldots(\iota\in I)}{\Delta}$.

Inference symbols $\mathcal{I}$ with $|\mathcal{I}| = \emptyset$ will be called *axioms*.

By writing " $(\mathcal{I})\;\; \Delta$ " we declare $\mathcal{I}$ as an axiom with $\Delta(\mathcal{I}) := \Delta$.

For almost all inference symbols (except axioms) the sequents $\Delta(\mathcal{I}), \Delta_\iota(\mathcal{I})$ are singletons or empty.

Example:

By (Cut_C) $\dfrac{C \qquad \neg C}{\emptyset}$ we express that for each formula C, the expression $\mathcal{I} := \mathsf{Cut}_C$ is an inference symbol with $|\mathcal{I}| = \{0,1\}$, $\Delta(\mathcal{I}) = \emptyset$, $\Delta_0(\mathcal{I}) = \{C\}$, $\Delta_1(\mathcal{I}) = \{\neg C\}$.

NOTATION

$\dfrac{...\Gamma_\iota...(\iota\in I)}{\Gamma}\mathcal{I} \;:\Longleftrightarrow\; |\mathcal{I}| = I \;\&\; \Delta(\mathcal{I}) \subseteq \Gamma \;\&\; \forall\iota \in I(\Gamma_\iota \subseteq \Gamma \cup \Delta_\iota(\mathcal{I}))$ (Γ *is derived from* $(\Gamma_\iota)_{\iota\in I}$ *by* $\mathcal{I}$)

Especially $\dfrac{...\Gamma, \Delta_\iota(\mathcal{I})...(\iota\in I)}{\Gamma, \Delta(\mathcal{I})}\mathcal{I}$.

Inductive definition of $\mathfrak{S}$-derivations

If $\mathcal{I}$ is an inference symbol of $\mathfrak{S}$, and $(d_\iota)_{\iota\in|\mathcal{I}|}$ is a family of $\mathfrak{S}$-*derivations* such that $\mathrm{Eig}(\mathcal{I})\cap\mathrm{FV}(\Gamma)=\emptyset$ where

$$\Gamma := \Delta(\mathcal{I}) \cup \bigcup_{\iota\in|\mathcal{I}|}(\Gamma(d_\iota)\setminus\Delta_\iota(\mathcal{I})),$$

then $d := \mathcal{I}(d_\iota)_{\iota\in|\mathcal{I}|}$ (or $\mathcal{I}d_0...d_{n-1}$ if $|\mathcal{I}| = \{0,...,n-1\}$) is an $\mathfrak{S}$-*derivation* with

$\Gamma(d) := \Gamma$ *(endsequent of d)*,

$\mathsf{last}(d) := \mathcal{I}$ *(last inference (symbol) of d)*,

$\mathsf{crk}(d) := \sup(\{\mathrm{rk}(\mathcal{I})\}\cup\{\mathsf{crk}(d_\iota) : \iota\in|\mathcal{I}|\})$ where $\mathrm{rk}(\mathcal{I}) := \begin{cases}\mathrm{rk}(C){+}1 & \text{if } \mathcal{I}=\mathsf{Cut}_C\\ 0 & \text{otherwise}\end{cases}$ *(cut-rank of d)*,

$\mathsf{hgt}(d) := \sup\{\mathsf{hgt}(d_\iota)+1 : \iota\in|\mathcal{I}|\}$ *(height of d)*.

Until further notice we will only consider derivations with $\mathsf{crk}(d) < \omega$.

Abbreviations

$\mathfrak{S}\ni d\vdash^\alpha_m \Gamma \;:\Longleftrightarrow\;$ d is an $\mathfrak{S}$-derivation with $\Gamma(d)\subseteq\Gamma$, $\mathsf{crk}(d)\le m$, $\mathsf{hgt}(d)\le\alpha$;

$\mathfrak{S}\vdash^\alpha_m \Gamma \;:\Longleftrightarrow\; \mathfrak{S}\ni d\vdash^\alpha_m\Gamma$ for some $\mathfrak{S}$-derivation d.

The meaning of $\mathfrak{S}\ni d\vdash\Gamma$ and $\mathfrak{S}\vdash\Gamma$ should now be clear.

Remark

If $(\forall\iota\in|\mathcal{I}|)\,\mathfrak{S}\ni d_\iota\vdash\Gamma,\Delta_\iota(\mathcal{I})$ where $\mathcal{I}\in\mathfrak{S}$ and $\mathrm{Eig}(\mathcal{I})\cap\mathrm{FV}(\Gamma)=\emptyset$,

then $\mathfrak{S}\ni\mathcal{I}(d_\iota)_{\iota\in|\mathcal{I}|}\vdash\Gamma,\Delta(\mathcal{I})$.

Definition. A proof system $\mathfrak{S}$ is called *finitary* if all its inference symbols have finite arity; otherwise $\mathfrak{S}$ is called *infinitary*.

The finitary proof system PL1

$(\mathsf{Ax}_{A,\neg A})$ $A,\neg A$ if A is a literal

$(\bigwedge_{A_0\wedge A_1})$ $\dfrac{A_0 \qquad A_1}{A_0\wedge A_1}$ $\qquad$ $(\bigvee^k_{A_0\vee A_1})$ $\dfrac{A_k}{A_0\vee A_1}$ $(k\in\{0,1\})$

$(\bigwedge^x_{\forall xA})$ $\dfrac{A}{\forall xA}$ $!x!$ $\qquad$ $(\bigvee^t_{\exists xA})$ $\dfrac{A_x(t)}{\exists xA}$ $(t\in\mathrm{Ter})$

(Cut_C) $\dfrac{C \qquad \neg C}{\emptyset}$

Displaying derivations:

To increase readability we often write derivations in tree form, i.e. we write $\dfrac{d_0\ldots d_n}{\mathcal{I}}$ instead of $\mathcal{I}d_0...d_n$.

Another way of representing derivations is to write them as trees of sequents (as before) and to display the respective inference symbols at the right or left end of each inference line. Mostly we will not show the full inference symbol $\mathcal{I}$ but only some kind of abbreviation (e.g. the outermost logical symbol of the principal formula of $\mathcal{I}$) or nothing.

Example:

$d = \bigvee^{S0}_G \bigwedge_{F(S0)\wedge\neg F(SS0)} \bigvee^0_G \bigwedge_{F(0)\wedge\neg F(S0)} \mathsf{Ax}_{F(0)} \mathsf{Ax}_{F(S0)}\ \mathsf{Ax}_{F(SS0)} =$

$$= \quad \dfrac{\dfrac{\dfrac{\dfrac{\mathsf{Ax}_{F(0)} \quad \mathsf{Ax}_{F(S0)}}{\bigwedge_{F(0)\wedge\neg F(S0)}}}{\bigvee^{0}_{G}} \qquad \mathsf{Ax}_{F(SS0)}}{\bigwedge_{F(S0)\wedge\neg F(SS0)}}}{\bigvee^{S0}_{G}} \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{\neg F(0), G, F(0) \quad \neg F(S0), F(S0)}{\neg F(0), G, F(0)\wedge\neg F(S0), F(S0)}\,(\wedge)}{\neg F(0), G, F(S0)}\,(\exists) \qquad \neg F(SS0), F(SS0)}{\neg F(0), G, F(S0)\wedge\neg F(SS0), F(SS0)}\,(\wedge)}{\neg F(0), G, F(SS0)}\,(\exists)$$

where $G := \exists x(F(x) \wedge \neg F(Sx))$,

The proof system Z of 1st order arithmetic

The *language* of Z is $\mathcal{L}_0(\mathcal{X}) := \mathcal{L}_0 \cup \{X_0, X_1, ...\}$, where $X_0, X_1, ...$ are unary predicate symbols; we call them *set variables*. But note that they are not considered as variables in the proper sense (e.g. $\mathrm{FV}(X_i\mathbf{0}) = \emptyset$). We use X as syntactic variable for $X_0, X_1, ...$

Recall that the logical axioms (LogAx) and the PRA-axioms (G1)-(PR3.1) had all been presented in the form Γ, Δ with arbitrary Γ. We call Δ the *principal part* of the respective axiom.

$\mathcal{AX}(\mathsf{Z})$:= set of all principal parts Δ of axioms (LogAx), (G1)-(PR3.1) in the extended language.

The *inference symbols of* Z are those of PL1 plus

$(\mathsf{Ax}_\Delta) \qquad \Delta \quad \text{for } \Delta \in \mathcal{AX}(\mathsf{Z})$,

$(\mathsf{Ind}^{x,t}_F) \;\; \dfrac{\neg F, F_x(\mathsf{S}\,x)}{\neg F_x(0), F_x(t)} \quad !x!$.

In this section $\mathcal{I}$ is always an inference symbol of Z, and $d, d', ...$ denote Z-derivations.

Definition

$\mathsf{Ax}_\Delta\theta := \mathsf{Ax}_{\Delta\theta}$, $\bigwedge_A\theta := \bigwedge_{A\theta}$, $\bigvee^k_A\theta := \bigvee^k_{A\theta}$, $\mathsf{Cut}_C\theta := \mathsf{Cut}_{C\theta}$, $\bigvee^s_{\exists xA}\theta := \bigvee^{s\theta}_{(\exists xA)\theta}$,
$\bigwedge^x_{\forall xA}\theta := \bigwedge^{x\theta}_{(\forall xA)\theta}$, $\mathsf{Ind}^{x,t}_F\theta := \mathsf{Ind}^{x\theta,t\theta}_{F\theta}$.

In the last two cases it is required that $x\theta \in \mathsf{Vars} \setminus \mathrm{FV}(\Delta(\mathcal{I})\theta)$.

Then the following holds for every Z-inference $\mathcal{I}$:

– $\mathcal{I}\theta$ is a Z-inferences belonging to the same rule as $\mathcal{I}$, (cf. proof of L....)

– $|\mathcal{I}\theta| = |\mathcal{I}|$ and $\mathrm{Eig}(\mathcal{I}\theta) = \mathrm{Eig}(\mathcal{I})\theta$,

– $\Delta(\mathcal{I}\theta) = \Delta(\mathcal{I})\theta$,

– $(\forall i \in |\mathcal{I}|)\; \Delta_i(\mathcal{I}\theta) = \Delta_i(\mathcal{I})\theta$.

Definition of $d\theta$

For $d = \mathcal{I}d_0...d_{n-1}$ we set $d\theta := (\mathcal{I}\tilde{\theta})d_0\tilde{\theta}\ldots d_{n-1}\tilde{\theta}$ with

$$\tilde{\theta} := \begin{cases} \theta^y_x & \text{if } \mathrm{Eig}(\mathcal{I}) = \{x\} \quad \text{where } y \in \mathsf{Var} \setminus \mathrm{FV}(\Gamma(d)\theta) \\ \theta & \text{if } \mathrm{Eig}(\mathcal{I}) = \emptyset \end{cases}$$

In the first case, if $x \notin \mathrm{FV}(\Gamma(d)\theta)$ we take $y := x$.

Remark. Let $d = \mathcal{I}d_0...d_{n-1}$.

(a) $d\epsilon = d$.

(b) If $\theta = \epsilon^t_x$ with $\mathrm{FV}(t) = \emptyset$ then $d\theta = \begin{cases} d & \text{if } \mathrm{Eig}(\mathcal{I}) = \{x\} \\ \mathcal{I}\theta d_0\theta...d_{n-1}\theta & \text{otherwise} \end{cases}$

Proof of (b):

We have $x \notin \mathrm{FV}(\Gamma(d)\theta)$ and therefore $\tilde{\theta} = \begin{cases} \theta^x_x = \epsilon & \text{if } \mathrm{Eig}(\mathcal{I}) = \{x\} \\ \theta^z_z = \theta & \text{if } \mathrm{Eig}(\mathcal{I}) = \{z\} \neq \{x\} \\ \theta & \text{if } \mathrm{Eig}(\mathcal{I}) = \emptyset \end{cases}$

Lemma 4.1. $\mathsf{Z} \ni d \vdash \Gamma \ \Rightarrow\ \mathsf{Z} \ni d\theta \vdash \Gamma\theta$.

Proof: Let $d = \mathcal{I}d_0...d_{n-1}$. W.l.o.g. $\Gamma(d) = \Gamma$.

(1) $\Gamma\tilde{\theta} = \Gamma\theta$, since $\mathrm{Eig}(\mathcal{I}) \cap \mathrm{FV}(\Gamma) = \emptyset$.

(2) $\mathrm{Eig}(\mathcal{I}\tilde{\theta}) \cap \mathrm{FV}(\Gamma\theta) = \emptyset$, since $\mathrm{Eig}(\mathcal{I}\tilde{\theta}) = \mathrm{Eig}(\mathcal{I})\tilde{\theta}$ and $(\mathrm{Eig}(\mathcal{I}) = \{x\} \Rightarrow x\tilde{\theta} \notin \mathrm{FV}(\Gamma\theta))$.

$d \vdash \Gamma \ \Rightarrow\ \Delta(\mathcal{I}) \subseteq \Gamma \ \&\ \forall i < n(d_i \vdash \Gamma, \Delta_i(\mathcal{I})) \overset{\mathrm{IH}+(1)}{\Rightarrow} \Delta(\mathcal{I})\tilde{\theta} \subseteq \Gamma\theta \ \&\ \forall i < n(d_i\tilde{\theta} \vdash \Gamma\theta, \Delta_i(\mathcal{I})\tilde{\theta}) \ \Rightarrow\ \Delta(\mathcal{I}\tilde{\theta}) \subseteq \Gamma\theta \ \&\ \forall i < n(d_i\tilde{\theta} \vdash \Gamma\theta, \Delta_i(\mathcal{I}\tilde{\theta})) \overset{(2)}{\Rightarrow} d\theta \vdash \Gamma\theta$.

Definition. $\mathrm{FV}(\mathcal{I}) := \begin{cases} \mathrm{FV}(\Delta(\mathcal{I})) \cup \mathrm{FV}(t) & \text{if } \mathcal{I} = \bigvee^t_{\exists x A} \text{ or } \mathsf{Ind}^{x,t}_F \\ \mathrm{FV}(\Delta(\mathcal{I})) & \text{otherwise} \end{cases}$

Remark. $\mathrm{Eig}(\mathcal{I}) \cap \mathrm{FV}(\mathcal{I}) = \emptyset$.

Definition. $\mathrm{FV}(\mathcal{I}d_0...d_{n-1}) := \mathrm{FV}(\mathcal{I}) \cup \bigcup_{i<n}(\mathrm{FV}(d_i) \setminus \mathrm{Eig}(\mathcal{I}))$

Lemma 4.2

(a) $\mathrm{FV}(\Gamma(d)) \subseteq \mathrm{FV}(d)$,

(b) $\mathrm{FV}(d(x/t)) = \mathrm{FV}(d) \setminus \{x\}$, if $t \in T_0$.

Proof: Let $d = \mathcal{I}d_0...d_{n-1}$.

a) $\Gamma(d) = \Delta(\mathcal{I}) \cup \bigcup_{i<n}(\Gamma(d_i) \setminus \Delta_i(\mathcal{I}))$ and $\mathrm{FV}(\Gamma(d)) \cap \mathrm{Eig}(\mathcal{I}) = \emptyset$ $(*)$.

$\mathrm{FV}(\Delta(\mathcal{I})) \subseteq \mathrm{FV}(\mathcal{I}) \subseteq \mathrm{FV}(d)$.

$\mathrm{FV}(\Gamma(d_i) \setminus \Delta_i(\mathcal{I})) \overset{(*)}{\subseteq} \mathrm{FV}(\Gamma(d_i)) \setminus \mathrm{Eig}(\mathcal{I}) \overset{\mathrm{IH}}{\subseteq} \mathrm{FV}(d_i) \setminus \mathrm{Eig}(\mathcal{I}) \subseteq \mathrm{FV}(d)$.

b) Abb.: $\theta := (x/t)$.

1. $\mathrm{Eig}(\mathcal{I}) = \{x\}$: Then $d\theta = d$ and $x \notin \mathrm{FV}(d)$. Hence $\mathrm{FV}(d\theta) = \mathrm{FV}(d) = \mathrm{FV}(d) \setminus \{x\}$.

2. Otherwise: Then $d\theta = \mathcal{I}\theta d_0\theta...d_{n-1}\theta$, and by IH $\mathrm{FV}(d_i\theta) = \mathrm{FV}(d\theta) \setminus \{x\}$.

Moreover one easily verifies that $\mathrm{FV}(\mathcal{I}\theta) = \mathrm{FV}(\mathcal{I}) \setminus \{x\}$.

Hence $\mathrm{FV}(d\theta) = \mathrm{FV}(\mathcal{I}\theta) \cup \bigcup_i(\mathrm{FV}(d_i\theta) \setminus \mathrm{Eig}(\mathcal{I}\theta)) \overset{\mathrm{IH}}{=}$

$(\mathrm{FV}(\mathcal{I}) \setminus \{x\}) \cup \bigcup_i \big((\mathrm{FV}(d_i) \setminus \{x\}) \setminus \mathrm{Eig}(\mathcal{I})\big) =$

$\big((\mathrm{FV}(\mathcal{I}) \cup \bigcup_i(\mathrm{FV}(d_i) \setminus \mathrm{Eig}(\mathcal{I}))\big) \setminus \{x\} = \mathrm{FV}(d) \setminus \{x\}$.

Definition

A Z-derivation d is called *closed* iff $\mathrm{FV}(d) = \emptyset$.

Lemma 4.3

(a) Each Z-derivation d can be transformed into a Z-derivation d' with $\Gamma(d') \subseteq \Gamma(d)$ and $\mathrm{FV}(d') \subseteq \mathrm{FV}(\Gamma(d))$; in particular, d' is closed if $\Gamma(d)$ is closed.

(b) If $d = \mathcal{I}d_0...d_{n-1}$ is closed and $\mathrm{Eig}(\mathcal{I}) = \emptyset$ then $d_0, ..., d_{n-1}$ are closed.

(c) If $d = \mathcal{I}d_0$ is closed and $\mathrm{Eig}(\mathcal{I}) = \{x\}$ then $d_0(x/t)$ is closed for each $t \in T_0$.

Proof:

(a) Induction on the cardinality of $\mathrm{FV}(d)$:

If $\mathrm{FV}(d) \subseteq \mathrm{FV}(\Gamma(d))$ then $d' := d$.

Now assume that $x \in \mathrm{FV}(d) \setminus \mathrm{FV}(\Gamma(d))$. Then $\Gamma(d(x/0)) \subseteq \Gamma(d)(x/0) = \Gamma(d)$,

and (by L.4.2b) $\mathrm{FV}(d(x/0)) = \mathrm{FV}(d) \setminus \{x\}$. Hence the claim follows by IH.

(b) $\mathrm{FV}(d_i) \subseteq \mathrm{FV}(d) \cup \mathrm{Eig}(\mathcal{I})$.

(c) $\mathrm{FV}(d_0) \subseteq \mathrm{FV}(d) \cup \{x\} = \{x\} \ \Rightarrow\ \mathrm{FV}(d_0(x/t)) = \mathrm{FV}(d_0) \setminus \{x\} = \emptyset$.

§5 Proof theoretic analysis of Z via the infinitary system Z^∞

Definition

Let R be an $\mathcal{L}_0$-formula with $\mathrm{FV}(R) = \{x, y\}$ such that the relation $\prec := \{(m,n) \in \mathbb{N}^2 : \mathbb{N} \models R_{y,x}(m,n)\}$ is wellfounded.

By recursion over $\prec$ one defines the $\prec$-*norm* $|n|_\prec$ of $n \in \mathbb{N}$:

$|n|_\prec := \sup\{|m|_\prec + 1 : m \prec n\}$.

$\|\prec\| := \sup\{|n|_\prec + 1 : n \in \mathbb{N}\}$.

Abbreviations.

$|t|_\prec := |t^{\mathcal{N}}|_\prec$ for $t \in T_0$,

$s \prec t := R_{y,x}(s,t)$, $\forall y{\prec}t\, F(y) := \forall y(y{\prec}t \to F(y))$.

We use $\mathcal{F}$ to denote expressions $\lambda x F$ (F a formula). For $\mathcal{F} = \lambda x F$ we set $\mathcal{F}(t) := F_x(t)$.

$\mathrm{Prog}_\prec(\mathcal{F}) := \forall x(\forall y{\prec}x\mathcal{F}(y) \to \mathcal{F}(x))$,

$\mathrm{TI}_\prec(\mathcal{F}, t) := \mathrm{Prog}_\prec(\mathcal{F}) \to \forall x{\prec}t\mathcal{F}(x)$,

$\mathrm{TI}_\prec(\mathcal{F}) := \mathrm{Prog}_\prec(\mathcal{F}) \to \forall x\mathcal{F}(x)$.

Finally $\mathrm{TI}_\prec(X) := \mathrm{TI}_\prec(\lambda x X x)$, etc.

In this section we will show that ε_0 is the least ordinal α such that *transfinite induction up to* α is not provable in Z; more precisely we will establish the following

Results.

(I) $\mathsf{Z} \vdash \mathrm{TI}_\prec(X) \implies \| \prec \| < \varepsilon_0$ (for any arithmetic $\prec$).

(II) For each $\alpha < \varepsilon_0$ there is a primitive recursive wellordering $\prec_\alpha$ of ordertype α such that $\mathsf{Z} \vdash \mathrm{TI}_{\prec_\alpha}(X)$.

Sketch of the proof of (I):

We define an infinitary proof system Z^∞, which (essentially) results from Z by

(i) replacing each inference symbol $\bigwedge^x_{\forall xA}$ by its infinitary version

$$(\bigwedge\nolimits_{\forall xA}) \quad \frac{\dots A_x(t) \dots (t{\in}T_0)}{\forall xA} \qquad (\omega\text{-rule})$$

(ii) adding the axioms A (for $A \in \mathrm{TRUE}_0$) and $Xs, \neg Xt$ (for $s, t \in T_0$ with $s^{\mathcal{N}} = t^{\mathcal{N}}$)

Then we prove the following Theorems which together yield the above result.

(Embedding) $\mathsf{Z} \vdash \Gamma \implies \mathsf{Z}^\infty \vdash^{\omega^2}_m \Gamma$ for some $m < \omega$.

(Cut Elimination) $\mathsf{Z}^\infty \vdash^\alpha_{m+1} \Gamma \implies \mathsf{Z}^\infty \vdash^{3^\alpha}_m \Gamma$.

(Boundedness) $\mathsf{Z}^\infty \vdash^\beta_0 \mathrm{TI}_\prec(X) \implies \|\prec\| \leq 2^\beta$.

The infinitary proof system Z^∞

The language of Z^∞ consists of all *closed* $\mathcal{L}_0(\mathcal{X})$-formulas (sentences).

We introduce the following relation $\simeq$ between $\mathcal{L}_0(\mathcal{X})$-sentences and (possibly infinitary) conjunctions or disjunctions of $\mathcal{L}_0(\mathcal{X})$-sentence:

$A_0 \wedge A_1 \simeq \bigwedge_{i\in\{0,1\}} A_i$, $\quad \forall x A \simeq \bigwedge_{t\in T_0} A_x(t)$, $\quad A_0 \vee A_1 \simeq \bigvee_{i\in\{0,1\}} A_i$, $\quad \exists x A \simeq \bigvee_{t\in T_0} A_x(t)$

Then we have

– $A \simeq *_{\iota\in J} A_\iota \;\&\; \iota \in J \implies \mathrm{rk}(A_\iota) < \mathrm{rk}(A)$,

– $A \simeq *_{\iota\in J} A_\iota \implies \neg(A) \simeq \overline{*}_{\iota\in J} \neg A_\iota$, where $\overline{\bigvee} := \bigwedge$, $\overline{\bigwedge} := \bigvee$,

Definition

$\mathcal{AX}(\mathsf{Z}^\infty) :=$ set of all sequents Δ such that

– all elements of Δ are closed literals,

– $\Delta \cap \mathrm{TRUE}_0 \neq \emptyset$ or Δ contains a subset $\{Xs, \neg Xt\}$ with $s^{\mathcal{N}} = t^{\mathcal{N}}$.

Note that $\{\Delta \in \mathcal{AX}(\mathsf{Z}) : \mathrm{FV}(\Delta) = \emptyset\} \subseteq \mathcal{AX}(\mathsf{Z}^\infty)$.

Remark. $\Delta', \Delta'' \in \mathcal{AX}(\mathsf{Z}^\infty) \implies (\Delta' \setminus \{C\}) \cup (\Delta'' \setminus \{\neg C\}) \in \mathcal{AX}(\mathsf{Z}^\infty)$

Proof: Assume $\Delta' \setminus \{C\} \notin \mathcal{AX}(\mathsf{Z}^\infty)$ and $\Delta'' \setminus \{\neg C\} \notin \mathcal{AX}(\mathsf{Z}^\infty)$.
Then (w.l.o.g.) $C = Xt$ and $\neg Xs \in \Delta' \setminus \{C\}$ and $Xr \in \Delta'' \setminus \{\neg C\}$ with $s^{\mathcal{N}} = t^{\mathcal{N}} = r^{\mathcal{N}}$.

Z^∞-inferences

(Ax_Δ) $\quad \Delta \quad$ if $\Delta \in \mathcal{AX}(\mathsf{Z}^\infty)$

$(\bigwedge_A)$ $\quad \dfrac{\ldots A_\iota \ldots (\iota\in J)}{A} \quad$ if $A \simeq \bigwedge_{\iota\in J} A_\iota$

$(\bigvee_A^\mu)$ $\quad \dfrac{A_\mu}{A} \quad$ if $A \simeq \bigvee_{\iota\in J} A_\iota$ and $\mu \in J$

(Cut_C) $\quad \dfrac{C \quad \neg C}{\emptyset}$

(Rep) $\quad \dfrac{\emptyset}{\emptyset}$

Remark. At moment we could do without Rep inferences. They will become important later.

NOTATION

Until further notice we use $d, d_0, d_1, e, \ldots$ as syntactic variables for Z^∞-derivations.

$$d \simeq \left\{ \begin{array}{c} d_\iota \\ | \\ \dfrac{\ldots \Gamma_\iota : \alpha_\iota \ldots}{\Gamma : \alpha}\mathcal{I} \end{array} \right. \;:\Leftrightarrow\; d = \mathcal{I}(d_\iota)_{\iota\in|\mathcal{I}|} \;\&\; \forall \iota \in |\mathcal{I}|(\Gamma(d_\iota) \subseteq \Gamma_\iota \;\&\; \mathsf{hgt}(d_\iota) \le \alpha_\iota < \alpha) \;\&\; \frac{\ldots\Gamma_\iota\ldots(\iota\in|\mathcal{I}|)}{\Gamma}\mathcal{I}\,,$$

where $\dfrac{\ldots\Gamma_\iota\ldots(\iota\in|\mathcal{I}|)}{\Gamma}\mathcal{I} \;:\Leftrightarrow\; \Delta(\mathcal{I}) \subseteq \Gamma \;\&\; \forall \iota \in |\mathcal{I}|(\Gamma_\iota \subseteq \Gamma, \Delta_\iota(\mathcal{I}))$.

Note that $d \simeq \left\{ \dfrac{\ldots\ldots}{\Gamma : \alpha}\mathcal{I} \right.$ implies $\Gamma(d) \subseteq \Gamma$ and $\mathsf{hgt}(d) \le \alpha$.

Theorem and Definition 5.1

For each formula C we define an operator $\mathcal{R}_C$ such that:

$d \vdash^{\alpha}_{m} \Gamma, C \;\;\&\;\; e \vdash^{\beta}_{m} \Gamma, \neg C \;\;\&\;\; \mathrm{rk}(C) \le m \implies \mathcal{R}_C(d,e) \vdash^{\alpha\#\beta}_{m} \Gamma.$

Proof:

$\mathcal{R}_C(d,e)$ is defined by recursion on $\alpha\#\beta$.

1. $C \notin \Delta(\mathsf{last}(d))$: Then $d \simeq \begin{cases} \begin{array}{c} d_\iota \\ | \\ \dots\Gamma, C, \Delta_\iota : \alpha_\iota \dots (\iota{\in}I) \\ \hline \Gamma, C : \alpha \end{array} \mathcal{I} \end{cases}$ where $\mathcal{I} := \mathsf{last}(d)$, $I := |\mathcal{I}|$.

By IH we get $\mathcal{R}_C(d_\iota, e) \vdash^{\alpha_\iota\#\beta}_{m} \Gamma, \Delta_\iota$ for all $\iota \in I$. Further we have $\alpha_\iota\#\beta < \alpha\#\beta$ for all $\iota \in I$.

Hence $\mathcal{R}_C(d,e) := \mathcal{I}(\mathcal{R}_C(d_\iota,e))_{\iota\in I} \simeq \begin{cases} \begin{array}{c} \mathcal{R}_C(d_\iota,e) \\ | \\ \dots\Gamma, \Delta_\iota : \alpha_\iota\#\beta \dots (\iota{\in}I) \\ \hline \Gamma : \alpha\#\beta \end{array} \mathcal{I} \end{cases}$ is a derivation as required.

1'. $\neg C \notin \Delta(\mathsf{last}(e))$: symmetric to 1.

2. $C \in \Delta(\mathsf{last}(d))$ and $\neg C \in \Delta(\mathsf{last}(e))$:

2.1. C is a literal: Then $\mathsf{last}(d) = \mathsf{Ax}_{\Delta'}$ and $\mathsf{last}(e) = \mathsf{Ax}_{\Delta''}$ with $\Delta' \in \mathcal{AX}(\mathsf{Z}^\infty)$ and $\Delta'' \in \mathcal{AX}(\mathsf{Z}^\infty)$. Hence $\Delta := (\Delta' \setminus \{C\}) \cup (\Delta'' \setminus \{\neg C\}) \subseteq \Gamma$, and $\Delta \in \mathcal{AX}(\mathsf{Z}^\infty)$ (cf. above). We set $\mathcal{R}_C(d,e) := \mathsf{Ax}_\Delta$.

2.2. $C \simeq \bigvee_{\iota\in J} C_\iota$:

Then $\neg C \simeq \bigwedge_{\iota\in J} \neg C_\iota$ and $d \simeq \begin{cases} \begin{array}{c} d_0 \\ | \\ \Gamma, C, C_\mu : \alpha_0 \\ \hline \Gamma, C : \alpha \end{array} \bigvee^{\mu}_{C} \end{cases}$, $e \simeq \begin{cases} \begin{array}{c} e_\iota \\ | \\ \dots\Gamma, \neg C, \neg C_\iota : \beta_\iota \dots (\iota{\in}J) \\ \hline \Gamma, \neg C : \beta \end{array} \bigwedge_{\neg C} \end{cases}$.

By IH we get $\mathcal{R}_C(d_0,e) \vdash^{\alpha_0\#\beta}_{m} \Gamma, C_\mu$ and $\mathcal{R}_C(d,e_\mu) \vdash^{\alpha\#\beta_\mu}_{m} \Gamma, \neg C_\mu$.

Further $\mathrm{rk}(C_\mu) < \mathrm{rk}(C) \le m$.

Hence $\mathcal{R}_C(d,e) := \mathsf{Cut}_{C_\mu} \mathcal{R}_C(d_0,e)\mathcal{R}_C(d,e_\mu) \simeq \begin{cases} \begin{array}{cc} \mathcal{R}_C(d_0,e) & \mathcal{R}_C(d,e_\mu) \\ | & | \\ \Gamma, C_\mu : \alpha_0\#\beta & \Gamma, \neg C_\mu : \alpha\#\beta_\mu \\ \hline \multicolumn{2}{c}{\Gamma : \alpha\#\beta} \end{array} \mathsf{Cut}_{C_\mu} \end{cases}$.

2.2'. $C \simeq \bigwedge_{\iota\in J} C_\iota$: symmetric to (Case 2.2).

Theorem and Definition 5.2

We define an operator $\mathcal{E}$ such that the following holds: $d \vdash^{\alpha}_{m+1} \Gamma \implies \mathcal{E}(d) \vdash^{3^\alpha}_{m} \Gamma$.

Proof:

1. $d \simeq \begin{cases} \begin{array}{cc} d_0 & d_1 \\ | & | \\ \Gamma, C : \alpha_0 & \Gamma, \neg C : \alpha_1 \\ \hline \multicolumn{2}{c}{\Gamma : \alpha} \end{array} \mathsf{Cut}_C \end{cases}$: Then $\mathrm{rk}(C) \le m$ and, by IH, $\mathcal{E}(d_0) \vdash^{3^{\alpha_0}}_{m} \Gamma, C$ and $\mathcal{E}(d_1) \vdash^{3^{\alpha_1}}_{m} \Gamma, \neg C$.

Hence $\mathcal{R}_C(\mathcal{E}(d_0), \mathcal{E}(d_1)) \vdash^{3^{\alpha_0}\#3^{\alpha_1}}_{m} \Gamma$ by Theorem 5.1. So we could define $\mathcal{E}(d)$ to be $\mathcal{R}_C(\mathcal{E}(d_0), \mathcal{E}(d_1))$. But for reasons which become clear later we set $\mathcal{E}(d) := \mathsf{Rep}\, \mathcal{R}_C(\mathcal{E}(d_0), \mathcal{E}(d_1))$.

2. otherwise: $\mathcal{E}(d) := \mathcal{I}(\mathcal{E}(d_\iota))_{\iota\in I} \simeq \begin{cases} \begin{array}{c} \mathcal{E}(d_\iota) \\ | \\ \dots\Gamma, \Delta_\iota : 3^{\alpha_\iota} \dots (\iota{\in}I) \\ \hline \Gamma : 3^\alpha \end{array} \mathcal{I} \end{cases}$ if $d = \begin{cases} \begin{array}{c} d_\iota \\ | \\ \dots\Gamma, \Delta_\iota : \alpha_\iota \dots (\iota{\in}I) \\ \hline \Gamma : \alpha \end{array} \mathcal{I} \end{cases}$

Theorem 5.3 (Inversion)
(a) $A \simeq \bigwedge_{\iota\in J} A_\iota$ & $\mathsf{Z}^\infty \vdash^\alpha_m \Gamma, A \implies \mathsf{Z}^\infty \vdash^\alpha_m \Gamma, A_\iota$ for each $\iota \in J$.
(b) $\mathsf{Z}^\infty \vdash^\alpha_m \Gamma, A_0 \vee A_1 \implies \mathsf{Z}^\infty \vdash^\alpha_m \Gamma, A_0, A_1$.

Definition. $\mathcal{L}_0(X) := \mathcal{L}_0 \cup \{X\}$,
A closed $\mathcal{L}_0(X)$-formula (sequent) is called X-positive if it contains no subformula $\neg Xt$.
$\models^\alpha A \;:\Leftrightarrow\; (\mathcal{N}, \{n : |n|_\prec < \alpha\}) \models A$,
$\models^\alpha \{A_1, ..., A_k\} \;:\Leftrightarrow\; \models^\alpha A_1 \vee ... \vee A_k$

Theorem 5.4 (Boundedness). Let Γ be X-positive.
$\mathsf{Z}^\infty \vdash^\beta_1 \neg\mathrm{Prog}_\prec(X), \neg X s_1, ..., \neg X s_k, \Gamma$ & $|s_1|_\prec, ..., |s_k|_\prec \le \alpha \implies \models^{\alpha+2^\beta} \Gamma$.

Corollary. $\mathsf{Z}^\infty \vdash^\beta_1 \mathrm{TI}_\prec(X) \implies \|{\prec}\| \le 2^\beta$.
Proof of the Corollary:
$\mathsf{Z}^\infty \vdash^\beta_1 \mathrm{TI}_\prec(X) \overset{5.3}{\Rightarrow} \mathsf{Z}^\infty \vdash^\beta_1 \neg\mathrm{Prog}_\prec(X), X\underline{n}$ for all $n \overset{5.4}{\Rightarrow} |n|_\prec < 2^\beta$ for all $n \Rightarrow \|{\prec}\| \le 2^\beta$.
Proof of Lemma 5.4 by induction on β:
Abbreviations: $\Lambda := \{\neg X s_1, ..., \neg X s_k\}$.
Let $d \vdash^\beta_1 \neg\mathrm{Prog}(X), \Lambda, \Gamma$.
1.1. $\mathsf{last}(d) = \mathsf{Ax}_\Delta$ and $\Delta \cap \mathrm{TRUE}_0 \neq \emptyset$: Then $\Gamma \cap \mathrm{TRUE}_0 \neq \emptyset$ and the claim is trivial.
1.2. $\mathsf{last}(d) = \mathsf{Ax}_\Delta$ and $Xt, \neg Xs \subseteq \Delta$ with $t^\mathcal{N} = s^\mathcal{N}$:
Then $Xt \in \Gamma$ and $\neg Xs \in \Lambda$. Hence $|t|_\prec = |s|_\prec \le \alpha < \alpha + 2^\beta$ and thus $\models^{\alpha+2^\beta} \Gamma$.
2. $\mathsf{last}(d) = \bigvee^{s_0}_{\neg\mathrm{Prog}(X)}$: Then $\vdash^{\beta_0}_1 \neg\mathrm{Prog}(X), \Lambda, \Gamma, \forall y{\prec}s_0\, Xy \wedge \neg X s_0$ with $\beta_0 < \beta$.
By 5.3a (Inversion) we get (1) $\vdash^{\beta_0}_1 \neg\mathrm{Prog}(X), \Lambda, \Gamma, \forall y{\prec}s_0\, Xy$, and (2) $\vdash^{\beta_0}_1 \neg\mathrm{Prog}(X), \neg X s_0, \Lambda, \Gamma$.
By IH from (1) we get $\models^{\alpha+2^{\beta_0}} \Gamma, \forall y{\prec}s_0\, Xy$.
(Case 1) $\models^{\alpha+2^{\beta_0}} \Gamma$: Then also $\models^{\alpha+2^\beta} \Gamma$, since $\beta_0 \le \beta$ and Γ is X-positive.
(Case 2) $\models^{\alpha+2^{\beta_0}} \forall y{\prec}s_0\, Xy$: Then $|m|_\prec < \alpha + 2^{\beta_0}$ for all $m \prec s_0^\mathcal{N}$, i.e. $|s_0|_\prec \le \alpha + 2^{\beta_0}$.
From this together with $|s_1|_\prec, ..., |s_k|_\prec \le \alpha$ and (2) by IH we obtain $\models^{\alpha+2^{\beta_0}+2^{\beta_0}} \Gamma$, and thus $\models^{\alpha+2^\beta} \Gamma$.
3. $\mathsf{last}(d) = \bigwedge_C$ with $C \simeq \bigwedge_{\iota\in I} C_\iota \in \Gamma$: Then, for all $\iota \in I$, $\vdash^{\beta_\iota}_1 \neg\mathrm{Prog}(X), \Lambda, \Gamma, C_\iota$ and $\beta_\iota < \beta$. Hence, by IH, $\models^{\alpha+2^{\beta_\iota}} \Gamma, C_\iota$ for all ι. Since Γ is X-positive, this implies $\models^{\alpha+2^\beta} \Gamma, C_\iota$ for all ι, and thus $\models^{\alpha+2^\beta} \Gamma, C$.
4. $\mathsf{last}(d) = \bigvee^\mu_C$ with $C \in \Gamma$: as 3.
5. $\mathsf{last}(d) = \mathsf{Rep}$: immediate by IH.
6. $\mathsf{last}(d) = \mathsf{Cut}_C$ with $C \in \mathrm{TRUE}_0$: Then by I.H. $\models^{\alpha+2^{\beta_0}} \Gamma, \neg C$ and thus $\models^{\alpha+2^\beta} \Gamma$.
7. $\mathsf{last}(d) = \mathsf{Cut}_{Ys}$ with $Y \neq X$: Then $\vdash^{\beta_0}_1 \neg\mathrm{Prog}(X), \Lambda, \Gamma, \neg Ys$ which implies $\vdash^{\beta_0}_1 \neg\mathrm{Prog}(X), \Lambda, \Gamma, \neg(0=0)$.
Now the claim follows as in 6.
8. $\mathsf{last}(d) = \mathsf{Cut}_{Xs_0}$: Then (1) $\vdash^{\beta_0}_1 \neg\mathrm{Prog}(X), \Lambda, \Gamma, X s_0$ and (2) $\vdash^{\beta_0}_1 \neg\mathrm{Prog}(X), \neg X s_0, \Lambda, \Gamma$.
By IH from (1) we get $\models^{\alpha+2^{\beta_0}} \Gamma, X s_0$.
(Case 1) $\models^{\alpha+2^{\beta_0}} \Gamma$: Then also $\models^{\alpha+2^\beta} \Gamma$.
(Case 2) $\models^{\alpha+2^{\beta_0}} X s_0$: Then $|s_0|_\prec < \alpha + 2^{\beta_0}$. From this together with $|s_1|_\prec, ..., |s_k|_\prec \le \alpha$ and (2) by IH we obtain $\models^{\alpha+2^{\beta_0}+2^{\beta_0}} \Gamma$, and thus $\models^{\alpha+2^\beta} \Gamma$.

Embedding of Z in Z^∞

From now on we denote Z^∞-derivations by $\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{c}$.

For each closed Z-derivation d we define its interpretation $d^\infty \in \mathsf{Z}^\infty$ as follows:

1. $(\bigwedge^x_{\forall x A} d_0)^\infty := \bigwedge_{\forall x A}\big(d_0(x/t)^\infty\big)_{t \in T_0}$,
2. $(\mathsf{Ind}^{x,t}_F d_0)^\infty := \begin{cases} \boldsymbol{c}^{x,t}_F & \text{if } n = 0 \\ \mathsf{Cut}_{F(\underline{n})} \boldsymbol{e}_{n-1} \boldsymbol{c}^{x,t}_F & \text{if } n > 0 \end{cases}$ where

   $n := t^{\mathcal{N}}$, $\boldsymbol{c}^{x,t}_F \vdash^{2\mathrm{rk}(F)}_0 \neg F_x(\underline{n}), F_x(t)$, $\boldsymbol{e}_0 := d_0(x/0)^\infty$, $\boldsymbol{e}_i := \mathsf{Cut}_{F(\underline{i})} \boldsymbol{e}_{i-1} d_0(x/\underline{i})^\infty$ for $i > 0$.

$$(\mathsf{Ind}^{x,t}_F d_0)^\infty = \cfrac{\cfrac{\cfrac{\cfrac{d_0(x/0)^\infty \quad d_0(x/1)^\infty}{\mathsf{Cut}_{F(1)}} \quad d_0(x/2)^\infty}{\vdots} \quad d_0(x/n{-}1)^\infty}{\mathsf{Cut}_{F(n-1)}} \quad \boldsymbol{c}^{x,t}_F}{\mathsf{Cut}_{F(n)}}$$

3. Otherwise: $(\mathcal{I} d_0 ... d_{n-1})^\infty := \mathcal{I} d_0^\infty \dots d_{n-1}^\infty$.

Definition of $\tilde{o}(d)$ and $\mathsf{dg}(d)$ for each Z-derivation d

Let $d = \mathcal{I} d_0 ... d_{n-1}$.

$\tilde{o}(d) := \begin{cases} \tilde{o}(d_0) + \omega & \text{if } \mathcal{I} = \mathsf{Ind}^{x,t}_F \\ \sup_{i<n}(\tilde{o}(d_i){+}1) & \text{otherwise} \end{cases}$

$\mathsf{dg}(d) := \max(\{\mathsf{dg}(\mathcal{I})\} \cup \{\mathsf{dg}(d_i) : i < n\})$ where $\mathsf{dg}(\mathcal{I}) := \begin{cases} \mathrm{rk}(C){+}1 & \text{if } \mathcal{I} = \mathsf{Cut}_C \text{ or } \mathsf{Ind}^{x,t}_C \\ 0 & \text{otherwise} \end{cases}$

Corollary. $\tilde{o}(d\theta) = \tilde{o}(d)$ and $\mathsf{dg}(d\theta) = \mathsf{dg}(d)$.

Theorem 5.5.

If d is a closed Z-derivation then $\mathsf{Z}^\infty \ni d^\infty \vdash^{\tilde{o}(d)}_{\mathsf{dg}(d)} \Gamma(d)$.

Corollary. $\mathsf{Z} \vdash \Gamma$ and Γ closed $\Longrightarrow$ $\mathsf{Z}^\infty \vdash^{\omega \cdot k}_m \Gamma$ for some $k, m < \omega$.

Proof: Let $\Gamma := \Gamma(d)$.

1. $d = \bigwedge^x_{\forall x A} d_0$: Then for each $t \in T_0$, $\mathsf{Z} \ni d_0(x/t) \vdash \Gamma, A_x(t)$ where $d_0(x/t)$ is closed.
Further we have $\alpha := \tilde{o}(d_0(x/t)) = \tilde{o}(d_0) < \tilde{o}(d)$ and $\mathsf{dg}(d_0(x/t)) = \mathsf{dg}(d_0) \le \mathsf{dg}(d)$.
I.H. $\Longrightarrow$ $\mathsf{Z}^\infty \ni d_0(x/t)^\infty \vdash^\alpha_{\mathsf{dg}(d)} \Gamma, A_x(t)$ $(\forall t \in T_0)$ $\Longrightarrow$ $\mathsf{Z}^\infty \ni d^\infty \vdash^{\tilde{o}(d)}_{\mathsf{dg}(d)} \Gamma$.

2. $d = \mathsf{Ind}^{x,t}_F d_0$: Since d is closed, $t \in T_0$. Let $n := t^{\mathcal{N}}$. As above we get (by I.H.) for all $i \in \mathbb{N}$, $\mathsf{Z}^\infty \ni d_0(x/\underline{i})^\infty \vdash^\alpha_m \Gamma, \neg F_x(\underline{i}), F_x(\underline{i{+}1})$ where $\alpha := \tilde{o}(d_0)$ and $m := \mathsf{dg}(d)$. Now by induction on i we get $\boldsymbol{e}_i \vdash^{\alpha+i}_m \Gamma, F_x(\underline{i{+}1})$: (i) $\boldsymbol{e}_0 = d_0(x/0)^\infty \vdash^\alpha_m \Gamma, F_x(1)$ (note that $\neg F_x(0) \in \Gamma$). (ii) $i > 0$: $d_0(x/\underline{i})^\infty \vdash^\alpha_m \Gamma, \neg F(\underline{i}), F(\underline{i{+}1})$ and $\boldsymbol{e}_{i-1} \vdash^{\alpha+i-1}_m \Gamma, F(\underline{i})$ $\overset{\mathrm{rk(F)}<\mathrm{m}}{\Longrightarrow}$ $\boldsymbol{e}_i = \mathsf{Cut}_{F(\underline{i})} \boldsymbol{e}_{i-1} d_0(x/\underline{i})^\infty \vdash^{\alpha+i}_m \Gamma, F(\underline{i{+}1})$.

$n = 0$: $d^\infty = \boldsymbol{c}^{x,t}_F \vdash^{\alpha+\omega}_0 \neg F(0), F(t)$.

$n > 0$: $\boldsymbol{e}_{n-1} \vdash^{\alpha+n-1}_m \Gamma, F(\underline{n})$ and $\boldsymbol{c}^{x,t}_F \vdash^k_0 \neg F(\underline{n}), F(t)$ $\overset{\mathrm{rk}(F)<m}{\Longrightarrow}$ $d^\infty = \mathsf{Cut}_{F(\underline{n})} \boldsymbol{e}_{n-1} \boldsymbol{c}^{x,t}_F \vdash^{\alpha+\omega}_m \Gamma, F(t)$ $(= \Gamma)$.

3. $d = \mathcal{I} d_0 ... d_{n-1}$ otherwise: Then, since d is closed, $d_0, ..., d_{n-1}$ are closed, and $\mathcal{I}$ is also an inference symbol of Z^∞. Hence the claim follows immediately from the I.H.

Theorem 5.6 $\mathsf{Z} \vdash \mathrm{TI}_\prec(X)$ $\Longrightarrow$ $\|{\prec}\| < \varepsilon_0$. [Proof by 5.5, 5.2, 5.4(Corollary).]

Fragments of Z

Let Z_m be the subsystem of Z where the induction rule is restricted to formulas F with $\mathrm{rk}_{\mathsf{QF}}(F) < m$.

We are now going to sharpen Theorem 5.6 by showing that if $\mathsf{Z}_m \vdash \mathrm{TI}_\prec(X)$ ($m \geq 1$, $\prec$ transitive) then $\|\prec\| < \omega_{m+1}$, where $\omega_0 := 1$, $\omega_{n+1} := \omega^{\omega_n}$. For this we need a sharper version of Theorem 5.4 (due to Beckmann).

Definitions.

For $U \subseteq \mathbb{N}$ let $|n|_U := \sup\{|i|_U{+}1 : i \prec n \,\&\, i \notin U\}$, and $U^\alpha := \{n \in \mathbb{N} : |n|_U < \alpha\} \cup U$.

$\models^\alpha_U A \;:\Leftrightarrow\; (\mathcal{N}, U^\alpha) \models A$,

$\models^\alpha_U \{A_1, ..., A_k\} \;:\Leftrightarrow \models^\alpha_U A_1 \vee ... \vee A_k$

Lemma 5.7.

$\prec$ transitive $\&$ $U' = U \cup \{m\}$ $\&$ $|m|_U \leq \alpha_0 < \alpha \implies \forall n(|n|_U \leq |n|_{U'} + 1)$ $\&$ $(U')^{\alpha_0} \subseteq U^\alpha$.

Theorem 5.8.

For transitive $\prec$ and X-positive Γ we have:

$\mathsf{Z}^\infty \vdash^\alpha_1 \neg\mathrm{Prog}_\prec(X), \neg X s_1, ..., \neg X s_k, \Gamma \implies \models^\alpha_U \Gamma$ with $U := \{s_1^{\mathcal{N}}, ..., s_k^{\mathcal{N}}\}$.

Proof by induction on α using Lemma 5.7.

Definition

$\alpha_0(\beta) := \beta$, $\alpha_{m+1}(\beta) := \alpha^{\alpha_m(\beta)}$. $\omega_m := \omega_m(1)$, i.e. $\omega_0 = 1$, $\omega_1 = \omega$, $\omega_2 = \omega^\omega$, $\omega_3 = \omega^{\omega^\omega}$ $(= \omega^{(\omega^\omega)})$, ...

Theorem 5.9. $\prec$ transitive and $\mathsf{Z}_m \vdash \mathrm{TI}_\prec(X) \implies \|\prec\| < \begin{cases} \omega^2 & \text{if } m = 0 \\ \omega_{m+1} & \text{if } m > 0 \end{cases}$

Proof:

Let Γ be closed. By partial cut-elimination and Lemma 4.3, if $\mathsf{Z}_m \vdash \Gamma$ then there exists a closed Z_m-derivation d of Γ with $\mathrm{rk}_{\mathsf{QF}}(C) < m$ for all cut or induction formulas C.

Then also $\mathrm{rk}_{\mathsf{QF}}(C) < m$ for all cut-formulas C of d^∞.

So we have $d^\infty \vdash^{\omega\cdot k}_{\mathsf{QF},m} \Gamma$ for some $k \geq 1$.

By inspecting the proofs of 5.1 and 5.2 one sees that the following holds:

$\boldsymbol{d} \vdash^\alpha_{\mathsf{QF},n+1} \Gamma \implies \mathcal{E}(\boldsymbol{d}) \vdash^{3^\alpha}_{\mathsf{QF},n} \Gamma$.

Hence $\mathcal{E}^m(d^\infty) \vdash^{3_m(\omega\cdot k)}_{\mathsf{QF},0} \Gamma$. Moreover there is an $\ell \in \mathbb{N}$ so that $\mathrm{rk}(C) \leq \ell$ for all cut formulas C of $\mathcal{E}^m(d^\infty)$.

Abbreviation. $\mathsf{Z}^\infty \vdash^\alpha_{n,0} \Gamma \;:\Leftrightarrow\; \boldsymbol{d} \vdash^\alpha_n \Gamma$ for some Z^∞-derivation $\boldsymbol{d}$ in which all cut formulas are quantifierfree.

The following propositions are easily proved (where (1) and (2) are needed for the proof of (3)):

(1) $\mathsf{Z}^\infty \vdash^\alpha_{n,0} \Gamma, A_0 \wedge A_1 \;\Rightarrow\; \mathsf{Z}^\infty \vdash^\alpha_{n,0} \Gamma, A_i$;

(2) $\mathsf{Z}^\infty \vdash^\alpha_{n,0} \Gamma, A_0 \vee A_1 \;\Rightarrow\; \mathsf{Z}^\infty \vdash^\alpha_{n,0} \Gamma, A_0, A_1$;

(3) $\mathsf{Z}^\infty \vdash^\alpha_{n+1,0} \Gamma \;\&\; n \geq 1 \;\Rightarrow\; \mathsf{Z}^\infty \vdash^{2\cdot\alpha}_{n,0} \Gamma$.

By the above we have $\mathsf{Z}^\infty \vdash^{3_m(\omega\cdot k)}_{\ell+1,0} \Gamma$. From this by (3) we get $\mathsf{Z}^\infty \vdash^{2^\ell \cdot 3_m(\omega\cdot k)}_1 \Gamma$.

So we have: $\mathsf{Z}_m \vdash \mathrm{TI}_\prec(X) \;\Rightarrow\; \mathsf{Z}^\infty \vdash^\alpha_1 \mathrm{TI}_\prec(X) \overset{5.8+5.3}{\Rightarrow} \|\prec\| \leq \alpha = 2^\ell \cdot 3_m(\omega \cdot k) < \begin{cases} \omega^2 & \text{if } m = 0 \\ \omega_{m+1} & \text{if } m \geq 1 \end{cases}$

Arithmetization of ordinals $< \varepsilon_0$

In the following a, b, c, x, y, z denote natural numbers.

We assume that $\mathbb{N}^{<\omega} \to \mathbb{N}$, $(a_0, ..., a_{n-1}) \mapsto \langle a_0, ..., a_{n-1}\rangle$
is a bijective coding of finite sequences of natural numbers such that
(1) $0 = \langle\,\rangle$ and $a_i < \langle a_0, ..., a_n\rangle < \langle a_0, ..., a_{n+1}\rangle$ for $i \le n$;
(2) For each n the function $\mathbb{N}^{n+1} \to \mathbb{N}$, $(a_0, ..., a_n) \mapsto \langle a_0, ..., a_n\rangle$ is primitive recursive;
(3) The function $* : \mathbb{N}^2 \to \mathbb{N}$, $\langle a_0, ..., a_{m-1}\rangle * \langle b_0, ..., b_{n-1}\rangle := \langle a_0, ..., a_{m-1}, b_0, ..., b_{n-1}\rangle$ is primitive recursive.

Definition of $(\mathrm{OT}, \prec)$

By simultaneous recursion we define a set $\mathrm{OT} \subseteq \mathbb{N}$ of *ordinal notations* and a binary relation $\prec$ on OT.

1. $a \in \mathrm{OT} \iff a = \langle a_0, ..., a_{n-1}\rangle$ with $a_0, ..., a_{n-1} \in \mathrm{OT}$ and $a_{n-1} \preceq \ldots \preceq a_0$.
2. $a \prec b$ iff $a = \langle a_0, ..., a_{m-1}\rangle \in \mathrm{OT}$, $b = \langle b_0, ..., b_{n-1}\rangle \in \mathrm{OT}$ and one of the following two cases holds
   (i) $m < n$ and $a_i = b_i$ for all $i < m$;
   (ii) $\exists k < \min\{m, n\}[\, a_k \prec b_k \ \&\ a_i = b_i \text{ for } i < k\,]$.

Remark. OT and $\prec$ are primitive recursive.

Definition of $o : \mathrm{OT} \to \varepsilon_0$
$o(\langle a_0, ..., a_{n-1}\rangle) := \omega^{o(a_0)} + \ldots + \omega^{o(a_{n-1})}$

Lemma 5.10. o maps $(\mathrm{OT}, \prec)$ isomorphic onto $(\varepsilon_0, <)$.

Corollary. $(\mathrm{OT}, \prec)$ is a wellordering and $|a|_\prec = o(a)$ for $a \in \mathrm{OT}$.

Proof of $|a|_\prec = o(a)$: If $a \in \mathrm{OT}$ then $|a|_\prec = \sup\{|b|_\prec{+}1 : b \prec a\} \overset{\mathrm{IH}}{=} \sup\{o(b){+}1 : b \prec a\} \overset{5.10}{=} o(a)$.

Definition.
For $a, b, k \in \mathbb{N}$ we set $a \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^b{\cdot}k := a * \langle\underbrace{b, \ldots, b}_{k}\rangle$ and $a \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^b := a \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^b{\cdot}1$, $\boldsymbol{\omega}^b := \langle b\rangle$.

Remark. If $a \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^b{\cdot}k \in \mathrm{OT}$ then $o(a \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^b{\cdot}k) = o(a) + \omega^{o(b)}{\cdot}k$.

Lemma 5.11. Provably in PRA we have
(a) $(\mathrm{OT}, \prec)$ is a linear ordering.
(b) If $a \prec c \prec a \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^b$ then there are $d \prec b$ and $k \in \mathbb{N}$ such that $c \prec a \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^d{\cdot}k$.

Proof of (b):
Let $a = \langle a_0, ..., a_{m-1}\rangle$ and $c = \langle c_0, ..., c_{n-1}\rangle$. From $\langle a_0, ..., a_{m-1}\rangle \prec \langle c_0, ..., c_{n-1}\rangle \prec \langle a_0, ..., a_{m-1}, b\rangle$ it follows that that $m < n$, $a_i = c_i$ for $i < m$, and $c_{n-1} \preceq \ldots \preceq c_m \prec b$. Hence $c \prec a \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^{c_m} \cdot (n{-}m{+}1)$.

Provability of $\mathrm{TI}_\prec$

$\Sigma_0 := \Pi_0 :=$ set of all quantifierfree $\mathcal{L}_0(\mathcal{X})$-formulas.

$\Sigma_{m+1} := \{\exists x A : A \in \Pi_n\}$, $\Pi_{m+1} := \{\forall x A : A \in \Sigma_m\}$,

Π_n-IA denotes the subsystem of Z where the induction rule is restricted to Π_n-formulas.

As shown in the exercises, Π_n-IA and Σ_n-IA prove the same sequents.

In the following we understand Π_n and Σ_n modulo provable equivalence in Π_0-IA.

In this sense Π_{n+1} (Σ_{n+1}, resp.) is closed under $\wedge$, $\vee$, $\forall$ ($\exists$, resp.). Moreover $\Sigma_n \cup \Pi_n \subseteq \Sigma_{n+1} \cap \Pi_{n+1}$.

Definition. $\overline{\mathcal{F}}(y) := \forall x(\forall z \prec x\mathcal{F}(z) \to \forall z \prec x \,\widehat{\oplus}\, \boldsymbol{\omega}^y\, \mathcal{F}(z))\}$.

Remark. If $\mathcal{F} \in \Pi_{n+1}$ then $\overline{\mathcal{F}} \in \Pi_{n+2}$.

Proof: $\overline{\mathcal{F}}(y) = \forall x(A \vee B)$ with $A \in \Sigma_{n+1} \subseteq \Pi_{n+2}$, $B \in \Pi_{n+1} \subseteq \Pi_{n+2}$.

Lemma 5.12. Π_{n+1}-IA $\vdash \mathrm{Prog}_{\prec}(\mathcal{F}) \to \mathrm{Prog}_{\prec}(\overline{\mathcal{F}})$ for $\mathcal{F} \in \Pi_{n+1}$.

Proof:

Assumptions: (1) $\mathrm{Prog}(\mathcal{F})$, (2) $\forall y \prec b\,\overline{\mathcal{F}}(y)$, (3) $\forall z \prec a\,\mathcal{F}(z)$.

To prove: $\forall z \prec a \,\widehat{\oplus}\, \boldsymbol{\omega}^b\, \mathcal{F}(z)$. Let $G(y,k) := \forall z \prec a \,\widehat{\oplus}\, \boldsymbol{\omega}^{y}{\cdot}k\, \mathcal{F}(z)$.

Then $\overline{\mathcal{F}}(y) \to (G(y,k) \to G(y,k{+}1))$, and from (3) we get $G(y,0)$.

Now by Π_{n+1}-IA we obtain $\overline{\mathcal{F}}(y) \to \forall k G(y,k)$, and then (4) $\forall y \prec b \forall k G(y,k)$ by (2).

Now let $z \prec a \,\widehat{\oplus}\, \boldsymbol{\omega}^b$. If $z \preceq a$ then $\mathcal{F}(z)$ by (1) and (3).

If $a \prec z$ then $\exists y \prec b \exists k(z \prec a \,\widehat{\oplus}\, \omega^y{\cdot}k)$ and thus $\mathcal{F}(z)$ by (4).

Definition.

For $\alpha < \varepsilon_0$ let Π_n-TI(α) denote the axiom scheme $\mathrm{TI}_{\prec}(\mathcal{F}, \underline{\ulcorner\alpha\urcorner})$ $(\mathcal{F} \in \Pi_n)$, where $\ulcorner\alpha\urcorner := o^{-1}(\alpha) \in \mathrm{OT}$.

Lemma 5.13. Π_{n+1}-IA $+ \Pi_{n+2}$-TI$(\alpha) \vdash \Pi_{n+1}$-TI$(\omega^\alpha)$.

Proof:

Let $\mathcal{F} \in \Pi_{n+1}$. Then $\overline{\mathcal{F}} \in \Pi_{n+2}$, and the claim follows from Π_{n+1}-IA $\vdash \mathrm{Prog}(\mathcal{F}) \to \mathrm{Prog}(\overline{\mathcal{F}})$,

Π_{n+2}-TI$(\alpha) \vdash \mathrm{Prog}(\overline{\mathcal{F}}) \to \overline{\mathcal{F}}(\underline{\ulcorner\alpha\urcorner})$, and Π_0-IA $\vdash \overline{\mathcal{F}}(\underline{\ulcorner\alpha\urcorner}) \to \forall z \prec \underline{\ulcorner\omega^\alpha\urcorner}\mathcal{F}(z)$.

Corollary. Π_{n+1}-IA $\vdash \Pi_2$-TI$(\omega_n(k))$ and Π_1-TI$(\omega_{n+1}(k))$.

Proof:

Π_{n+1}-IA $+ \Pi_{n+2}$-TI$(k) \vdash \Pi_{n+1}(\omega_1^k) \vdash \ldots \vdash \Pi_2$-TI$(\omega_n^k) \vdash \Pi_1$-TI$(\omega_{n+1}^k)$.

It remains to prove Π_{n+1}-IA $\vdash \Pi_{n+2}$-TI(k).

Actually we show (by meta-induction on k) Π_0-IA $\vdash \mathrm{TI}(\mathcal{F}, \underline{\ulcorner k\urcorner})$ for any $\mathcal{F}$.

Obviously Π_0-IA $\vdash \forall z \prec \underline{\ulcorner 0\urcorner}\mathcal{F}(z)$ and Π_0-IA $\vdash \mathrm{Prog}(\mathcal{F}) \to \forall z \prec \underline{\ulcorner k\urcorner}\mathcal{F}(z) \to \forall z \prec \underline{\ulcorner k{+}1\urcorner}\mathcal{F}(z)$ for each $k \in \mathbb{N}$.

Hence Π_0-IA $\vdash \mathrm{Prog}(\mathcal{F}) \to \forall z \prec \underline{\ulcorner k\urcorner}\mathcal{F}(z)$ for each $k \in \mathbb{N}$.

Theorem 5.14.

(a) Π_{m+1}-IA $\vdash \Pi_1$-TI(α) for each $\alpha < \omega_{m+2}$.

(b) Π_{m+1}-IA $\nvdash \Pi_1$-TI(ω_{m+2}).

Proof of (b):

Note that Π_{m+1}-IA is contained in Z_{m+1}. Let $\alpha < \varepsilon_0$ and $\prec_\alpha := \{(i,j) : i \prec j \prec \underline{\ulcorner\alpha\urcorner}\}$.

Π_{m+1}-IA $\vdash \Pi_1$-TI$(\alpha) \overset{(*)}{\Longrightarrow} \mathsf{Z}_{m+1} \vdash \mathrm{TI}_{\prec_\alpha}(X) \implies \alpha = \|{\prec_\alpha}\| < \omega_{m+2}$.

$(*)$ $\mathrm{Prog}_{\prec_\alpha}(X) = \forall x(\forall y \prec_\alpha x(y \in X) \to x \in X) \implies \forall x(\forall y \prec x(y \in X) \to x \in X) = \mathrm{Prog}_{\prec}(X) \overset{\Pi_n\text{-IA}(\alpha)}{\Longrightarrow}$

$\forall x \prec \underline{\ulcorner\alpha\urcorner}(x \in X) \implies \forall x(x \in X)$, since $[x \not\prec \underline{\ulcorner\alpha\urcorner} \Rightarrow \forall y \prec_\alpha x(y \in X) \Rightarrow x \in X]$.

§6 Notations for infinitary derivations; the proof system Z^*

Since for every true $\mathcal{L}_0$-sentence there exists a Z^∞-derivation of heigth $< \omega$, the method of §5 (by which we have shown the unprovability of $\mathrm{TI}_{\varepsilon_0}(X)$) is not suitable for showing the unprovability (in Z) of $\mathcal{L}_0$-sentences. Especially one cannot obtain bounds on the provably recursive functions of Z by this method. One way to achieve this would be introduce an effective version of Z^∞ where infinitary derivations are coded by indices for recursive functions. Here we choose a different way where the finite derivations of an extension Z^* of Z serve as codes (notations) for Z^∞-derivations.

The proof-system Z^*

The system Z^* results from Z by adding the inference symbol $(\mathsf{E})\ \dfrac{\emptyset}{\emptyset}$
and defining $\mathrm{o}(h)$ and $\mathsf{deg}(h)$ for Z^*-derivations $h = \mathcal{I}h_0...h_{n-1}$ as follows

$$\mathrm{o}(h) := \begin{cases} \mathrm{o}(h_0)\#\mathrm{o}(h_1) & \text{if } \mathcal{I} = \mathsf{Cut}_C \\ \mathrm{o}(h_0)\cdot\omega & \text{if } \mathcal{I} = \mathsf{Ind}_C^{x,t} \\ 3^{\mathrm{o}(h_0)} & \text{if } \mathcal{I} = \mathsf{E} \\ \sup_{i<n}(\mathrm{o}(h_i){+}1) & \text{otherwise} \end{cases}, \quad \mathsf{deg}(h) := \begin{cases} \mathsf{deg}(h_0) \dot- 1 & \text{if } \mathcal{I} = \mathsf{E} \\ \max(\{\mathsf{deg}(\mathcal{I})\} \cup \{\mathsf{deg}(h_i) : i < n\}) & \text{otherwise} \end{cases}$$

$$\text{where } \mathsf{deg}(\mathcal{I}) := \begin{cases} \mathrm{rk}(C) & \text{if } \mathcal{I} = \mathsf{Cut}_C \text{ or } \mathsf{Ind}_C^{x,t} \\ 0 & \text{otherwise} \end{cases}$$

We use $h, h_0, ...$ as syntactic variables for Z^*-derivations.

Remark: The definitions of $\mathrm{o}(h)$ and $\mathsf{deg}(h)$ are motivated by the interpretation $h \mapsto h^{\boldsymbol{\omega}}$ (introduced below) and Theorems 5.1,5.2. For example, since, according to Theorem 5.2, $\mathrm{o}(\mathcal{E}(h_0^{\boldsymbol{\omega}})) \le 3^{\mathrm{o}(h_0^{\omega})}$ and $\mathsf{deg}(\mathcal{E}(h_0^{\boldsymbol{\omega}})) \le \mathsf{deg}(h_0^{\boldsymbol{\omega}}) \dot- 1$ holds, we have defined $\mathrm{o}(\mathsf{E}h_0) := 3^{\mathrm{o}(h_0)}$ and $\mathsf{deg}(\mathsf{E}h_0) := \mathsf{deg}(h_0) \dot- 1$.

The definition of $\mathcal{I}\theta$ and $d\theta$ for inference symbols and derivations of Z is extended to Z^* by $\mathsf{E}\theta := \mathsf{E}$.
Closed derivations are defined as in Z.

Lemma 6.1.

(a) $\mathrm{o}(h\theta) = \mathrm{o}(h)$ and $\mathsf{deg}(h\theta) = \mathsf{deg}(h)$.

(b) $\mathsf{Z}^* \ni h \vdash \Gamma \;\Rightarrow\; \mathsf{Z}^* \ni h\theta \vdash \Gamma\theta$.

(c) If $h = \mathcal{I}h_0...h_{n-1}$ is closed and $\mathrm{Eig}(\mathcal{I}) = \emptyset$ then $h_0, ..., h_{n-1}$ are closed.

(d) If $h = \mathcal{I}h_0$ is closed and $\mathrm{Eig}(\mathcal{I}) = \{x\}$ then $h_0(x/t)$ is closed for each $t \in T_0$.

Interpretation of Z^* in Z^∞

For each closed Z^*-derivation h we define its interpretation $h^{\boldsymbol{\omega}} \in \mathsf{Z}^\infty$ as follows:

1. $(\bigwedge^x_{\forall xA} h_0)^{\boldsymbol{\omega}} := \bigwedge_{\forall xA}\big(h_0(x/t)^{\boldsymbol{\omega}}\big)_{t\in T_0}$,
2. $(\mathsf{Cut}_C h_0 h_1)^{\boldsymbol{\omega}} := \mathcal{R}_C(h_0^{\boldsymbol{\omega}}, h_1^{\boldsymbol{\omega}})$,
3. $(\mathsf{E}h_0)^{\boldsymbol{\omega}} := \mathcal{E}(h_0^{\boldsymbol{\omega}})$,
4. $(\mathsf{Ind}_F^{x,t} h_0)^{\boldsymbol{\omega}} := \begin{cases} \mathsf{Rep}\, \boldsymbol{c}_F^{x,t} & \text{if } n = 0 \\ \mathsf{Rep}\, \mathcal{R}_{F(\underline{n})}(\boldsymbol{e}_{n-1}, \boldsymbol{c}_F^{x,t}) & \text{if } n > 0 \end{cases}$ where

   $n := t^{\mathcal{N}}$, $\boldsymbol{c}_F^{x,t} \vdash_0^{<\omega} \neg F_x(\underline{n}), F_x(t)$, $\boldsymbol{e}_0 := h_0(x/0)^{\boldsymbol{\omega}}$, $\boldsymbol{e}_i := \mathcal{R}_{F(\underline{i})}(\boldsymbol{e}_{i-1}, h_0(x/\underline{i})^{\boldsymbol{\omega}})$ for $i > 0$.
5. Otherwise: $(\mathcal{I}h_0...h_{n-1})^{\boldsymbol{\omega}} := \mathcal{I}h_0^{\boldsymbol{\omega}} \ldots h_{n-1}^{\boldsymbol{\omega}}$.

Remark

With the help of Theorems 5.1,5.2 one easily verifies that $h^{\boldsymbol{\omega}}$ is a Z^∞-derivation with $h^{\boldsymbol{\omega}} \vdash^{\mathrm{o}(h)}_{\mathsf{deg}(h)} \Gamma(h)$.

Let us look at the Ind-case:

Let $\alpha_0 := \mathrm{o}(h_0)$, $\alpha_{i+1} := \alpha_i \# \alpha_0$ and $m := \mathsf{deg}(h) = \max\{\mathrm{rk}(F), \mathsf{deg}(h_0)\}$.

From $h_0(x/i)^{\boldsymbol{\omega}} \vdash^{\alpha_0}_m \Gamma, \neg F(i), F(i{+}1)$ and $\boldsymbol{e}_{i-1} \vdash^{\alpha_{i-1}}_m \Gamma, \neg F(0), F(i)$ we obtain $\boldsymbol{e}_i = \mathcal{R}_{F(\underline{i})}(\boldsymbol{e}_{i-1}, h_0(x/\underline{i})^{\boldsymbol{\omega}}) \vdash^{\alpha_i}_m \Gamma, \neg F(0), F(i+1)$.

$n > 0$: From $\boldsymbol{e}_{n-1} \vdash^{\alpha_{n-1}}_m \Gamma, \neg F(0), F(n)$ and $\boldsymbol{c}^{x,t}_F \vdash^k_0 \neg F(\underline{n}), F(t)$ we obtain $\mathsf{Rep}\mathcal{R}_{F(\underline{n})}(\boldsymbol{e}_{n-1}, \boldsymbol{c}^{x,t}_F) \vdash^{\alpha_0 \cdot \omega}_m \Gamma, \neg F(0), F(t)$.

$n = 0$: Then $\boldsymbol{c}^{x,t}_F \vdash^{\alpha_0 \cdot \omega}_0 \neg F(0), F(t)$. $(1 \le \alpha_0 \Rightarrow \omega \le \alpha_0 \cdot \omega)$

Definition of $\mathsf{tp}(h)$ and $h[\iota]$ for closed Z^*-derivations h and $\iota \in |\mathsf{tp}(h)|$

By recursion on the build-up of h we define a Z^∞-inference $\mathsf{tp}(h)$ and closed Z^*-derivation(s) $h[\iota]$ in such a way that

$$h^{\boldsymbol{\omega}} = \mathsf{tp}(h)\Big(h[\iota]^{\boldsymbol{\omega}}\Big)_{\iota \in |\mathsf{tp}(h)|} = \frac{\ldots h[\iota]^{\boldsymbol{\omega}} \ldots (\iota {\in} |\mathsf{tp}(h)|)}{\mathsf{tp}(h)}$$

The definition clauses for $h = \mathsf{Cut}_C h_0 h_1$ and $h = \mathsf{E} h_0$ can be read off from the corresponding clauses in the definitions of $\mathcal{R}_C$ and $\mathcal{E}$.

1.1. $h = \mathsf{Ax}_\Delta$: $\mathsf{tp}(h) := \mathsf{Ax}_\Delta$.

1.2. $h = \bigwedge_C h_0 h_1$: $\mathsf{tp}(h) := \bigwedge_C$, $h[i] := h_i$.

1.3. $h = \bigwedge^x_C h_0$: $\mathsf{tp}(h) := \bigwedge_C$, $h[t] := h_0(x/t)$.

1.4. $h = \bigvee^\iota_C h_0$: $\mathsf{tp}(h) := \bigvee^\iota_C$, $h[0] := h_0$.

2. $h = \mathsf{Ind}^{x,t}_F h_0$: $\mathsf{tp}(h) := \mathsf{Rep}$, $h[0] := \begin{cases} c^{x,t}_F & \text{if } n = 0 \\ \mathsf{Cut}_{F(\underline{n})} e_{n-1} c^{x,t}_F & \text{if } n > 0 \end{cases}$ where

$n := t^{\mathcal{N}}$, $\mathsf{Z} \ni c^{x,t}_F \vdash_1 \neg F(\underline{n}), F(t)$, $e_0 := h_0(x/0)$, $e_i := \mathsf{Cut}_{F(\underline{i})} e_{i-1} h_0(x/\underline{i})$ for $i > 0$.

3. $h = \mathsf{E} h_0$:

3.1. $\mathsf{tp}(h_0) = \mathsf{Cut}_C$: $\mathsf{tp}(h) := \mathsf{Rep}$, $h[0] := \mathsf{Cut}_C \mathsf{E} h_0[0] \mathsf{E} h_0[1]$,

3.2. otherwise: $\mathsf{tp}(h) := \mathsf{tp}(h_0)$, $h[\iota] := \mathsf{E} h_0[\iota]$.

4. $h = \mathsf{Cut}_C h_0 h_1$:

4.1. $C \notin \Delta(\mathsf{tp}(h_0))$: $\mathsf{tp}(h) := \mathsf{tp}(h_0)$, $h[\iota] := \mathsf{Cut}_C h_0[\iota] h_1$.

4.2. $\neg C \notin \Delta(\mathsf{tp}(h_1))$: $\mathsf{tp}(h) := \mathsf{tp}(h_1)$, $h[\iota] := \mathsf{Cut}_C h_0 h_1[\iota]$.

4.3. $C \in \Delta(\mathsf{tp}(h_0))$ and $\neg C \in \Delta(\mathsf{tp}(h_1))$:

4.3.0. $\mathrm{rk}(C) = 0$: $\mathsf{tp}(h) := \mathsf{Ax}_\Delta$ with $\Delta := (\Delta(\mathsf{tp}(h_0)) \setminus \{C\}) \cup (\Delta(\mathsf{tp}(h_1)) \setminus \{\neg C\})$.

4.3.1. $C = \exists x A$: Then $\mathsf{tp}(h_0) = \bigvee^t_C$ for some $t \in T_0$, and $\mathsf{tp}(h_1) = \bigwedge_{\neg C}$.
$\mathsf{tp}(h) := \mathsf{Cut}_{A_x(t)}$, $h[0] := \mathsf{Cut}_C h_0[0] h_1$, $h[1] := \mathsf{Cut}_C h_0 h_1[t]$.

4.3.2. $C = \forall x A$ or $A_0 \wedge A_1$ or $A_0 \vee A_1$: analogous to 4.3.1.

Definition. $h \vdash^\alpha_m \Gamma \;:\Longleftrightarrow\;$ h is a closed Z^*-derivation with $\Gamma(h) \subseteq \Gamma$, $\mathrm{o}(h) \le \alpha$, $\mathsf{deg}(h) \le m$.

Theorem 6.2

If $h \overset{\alpha}{\underset{m}{\vdash}} \Gamma$ and $\mathcal{I} = \mathsf{tp}(h)$ then the following holds:

(a) $\Delta(\mathcal{I}) \subseteq \Gamma$,

(b) $\mathcal{I} = \mathsf{Cut}_C \Rightarrow \mathrm{rk}(C) < m$,

(c) For each $\iota \in |\mathcal{I}|$: $h[\iota] \overset{\alpha_\iota}{\underset{m}{\vdash}} \Gamma, \Delta_\iota(\mathcal{I})$ with $\alpha_\iota < \alpha$.

Proof by straightforward induction on the build-up of h:

W.l.o.g. $\mathrm{FV}(\Gamma) = \emptyset$.

1. $h = \mathsf{Ax}_\Delta$: Then $\mathcal{I} = \mathsf{Ax}_\Delta$ and thus $\Delta(\mathcal{I}) = \Delta = \Gamma(h) \subseteq \Gamma$ and $|\mathcal{I}| = \emptyset$.

2.1. $h = \bigwedge^y_{\forall x A} h_0$: Then $\mathcal{I} = \bigwedge_{\forall x A}$, $\Delta(\mathcal{I}) = \{\forall x A\} \subseteq \Gamma$ and $h_0 \overset{\alpha_0}{\underset{m}{\vdash}} \Gamma, A_x(y)$ with $\alpha_0 < \alpha$.
By L.6.1a,b we get $h[t] = h_0(y/t) \overset{\gamma}{\underset{m}{\vdash}} \Gamma, A_x(t)$ for each $t \in T_0$.

2.2. $h = \bigvee^t_{\exists x A} h_0$: Then $t \in T_0$, $\mathcal{I} = \bigvee^t_{\exists x A}$, $\Delta(\mathcal{I}) = \{\exists x A\} \subseteq \Gamma$ and $h_0 \overset{\alpha_0}{\underset{m}{\vdash}} \Gamma, A_x(t)$ with $\alpha_0 < \alpha$.

2.3. $h = \bigwedge_{A_0 \wedge A_1} h_0 h_1$ or $h = \bigvee^k_{A_0 \vee A_1}$: analogous to 2.1 and 2.2.

4.1. $h = \mathsf{Cut}_C h_0 h_1$ with $C = \exists x A$, $\mathsf{tp}(h_0) = \bigvee^t_C$, $\mathsf{tp}(h_1) = \bigwedge_{\neg C}$: Then $t \in T_0$ and $\mathsf{tp}(h) = \mathsf{Cut}_{A(t)}$.

Let $\gamma := \mathrm{o}(h_0)$, $\beta := \mathrm{o}(h_1)$.

Then $h_0 \overset{\gamma}{\underset{m}{\vdash}} \Gamma, C$, $h_1 \overset{\beta}{\underset{m}{\vdash}} \Gamma, \neg C$ and $\mathrm{rk}(A(t)) < \mathrm{rk}(C) \le m$.
By IH we obtain $h_0[0] \overset{\gamma_0}{\underset{m}{\vdash}} \Gamma, C, A(t)$ with $\gamma_t < \gamma$, and $h_1[t] \overset{\beta_t}{\underset{m}{\vdash}} \Gamma, \neg C, \neg A(t)$ with $\beta_t < \beta$.
Hence $h[0] = \mathsf{Cut}_C h_0[0] h_1 \overset{\gamma_0 \# \beta}{\underset{m}{\vdash}} \Gamma, A(t)$ and $h[1] = \mathsf{Cut}_C h_0 h_1[t] \overset{\gamma \# \beta_t}{\underset{m}{\vdash}} \Gamma, \neg A(t)$
with $\gamma_0 \# \beta$, $\gamma \# \beta_t < \gamma \# \beta = \mathrm{o}(h) \le \alpha$.

4.2. $h = \mathsf{Cut}_C h_0 h_1$ with $\mathsf{tp}(h_0) = \mathcal{I}$ and $C \notin \Delta(\mathcal{I})$: Let $\gamma := \mathrm{o}(h_0)$, $\beta := \mathrm{o}(h_1)$.
Then $h_0 \overset{\gamma}{\underset{m}{\vdash}} \Gamma, C$, $h_1 \overset{\beta}{\underset{m}{\vdash}} \Gamma, \neg C$.
By IH we obtain $h_0[\iota] \overset{\gamma_\iota}{\underset{m}{\vdash}} \Gamma, C, \Delta_\iota(\mathcal{I})$ with $\gamma_\iota < \gamma$, for all $\iota \in |\mathcal{I}|$.
Hence $h[\iota] = \mathsf{Cut}_C h_0[\iota] h_1 \overset{\gamma_\iota \# \beta}{\underset{m}{\vdash}} \Gamma, \Delta_\iota(\mathcal{I})$ with $\gamma_\iota \# \beta < \gamma \# \beta = \mathrm{o}(h) \le \alpha$.

$h_0 \overset{\gamma}{\underset{m}{\vdash}} \Gamma, C$ & $\mathsf{tp}(h_0) = \mathcal{I}$ & $C \notin \Delta(\mathcal{I}) \overset{\mathrm{IH}}{\Rightarrow} \Delta(\mathcal{I}) \subseteq \Gamma$ & $(\mathcal{I} = \mathsf{Cut}_A \Rightarrow \mathrm{rk}(A) < m)$.

4.3. $h = \mathsf{Cut}_C h_0 h_1$ with $\mathrm{rk}(C) = 0$ and $C \in \Delta^0 := \Delta(\mathsf{tp}(h_0))$, $\neg C \in \Delta^1 := \Delta(\mathsf{tp}(h_1))$:
Then $\mathcal{I} = \mathsf{Ax}_\Delta$ with $\Delta := (\Delta^0 \setminus \{C\}) \cup (\Delta^0 \setminus \{\neg C\})$, and by IH $\Delta^i \subseteq \Gamma(h_i)$. Hence $\Delta(\mathcal{I}) = \Delta \subseteq \Gamma(h) \subseteq \Gamma$.

5. $h = \mathsf{E} h_0$ with $\mathsf{tp}(h_0) = \mathsf{Cut}_C$: Then $\mathsf{tp}(h) = \mathsf{Rep}$, $h[0] = \mathsf{Cut}_C \mathsf{E} h_0[0] \mathsf{E} h_0[1]$ and $\mathsf{deg}(h_0) \le m{+}1$.

Let $\gamma := \mathrm{o}(h_0)$. Then $h_0 \overset{\gamma}{\underset{m+1}{\vdash}} \Gamma$.

By IH we have $\mathrm{rk}(C) < m{+}1$ and $h_0[0] \overset{\gamma_0}{\underset{m+1}{\vdash}} \Gamma, C$, $h_0[1] \overset{\gamma_1}{\underset{m+1}{\vdash}} \Gamma, \neg C$ with $\gamma_0, \gamma_1 < \gamma$.
Hence $\mathsf{E} h_0[0] \overset{3^{\gamma_0}}{\underset{m}{\vdash}} \Gamma, C$ and $\mathsf{E} h_0[1] \overset{3^{\gamma_1}}{\underset{m}{\vdash}} \Gamma, \neg C$.

From this together with $\mathrm{rk}(C) \le m$ we get $h[0] = \mathsf{Cut}_C \mathsf{E} h_0[0] \mathsf{E} h_0[1] \overset{3^{\gamma_1} \# 3^{\gamma_1}}{\underset{m}{\vdash}} \Gamma$ and $3^{\gamma_0} \# 3^{\gamma_1} < 3^\gamma = \mathrm{o}(h) \le \alpha$.

6. $h = \mathsf{Ind}^{x,t}_F h_0$: Then $\mathcal{I} = \mathsf{Rep}$, $\mathrm{rk}(F) \le m$, and $h_0 \overset{\gamma}{\underset{m}{\vdash}} \Gamma, \neg F(x), F(\mathsf{S}\, x)$ with $\gamma := \mathrm{o}(h_0)$, $\gamma + \omega \le \alpha$.

$$h[0] = \cfrac{\cfrac{\cfrac{\cfrac{h_0(y/0) \quad h_0(y/1)}{\mathsf{Cut}_{F(1)}} \qquad h_0(y/2)}{\vdots} \qquad h_0(y/n{-}1)}{\mathsf{Cut}_{F(n-1)}} \qquad c^{y,t}_F}{\mathsf{Cut}_{F(n)}} \qquad \text{with } \mathsf{Z} \ni c^{y,t}_F \vdash_1 \neg F(n), F(t).$$

Lemma 6.3 (Consistency of Z)

Let $\mathsf{Z}^*_\perp$ be the set of all closed Z^*-derivations h with $\Gamma(h) = \emptyset$ & $\mathsf{deg}(h) = 0$.

(a) $h \in \mathsf{Z}^*_\perp \implies h[0] \in \mathsf{Z}^*_\perp$ & $\mathrm{o}(h[0]) < \mathrm{o}(h)$,

(b) There is no Z-derivation d with $\Gamma(d) = \emptyset$.

Proof:

(a) $h \in \mathsf{Z}^*_\perp \overset{6.2a}{\Rightarrow} \Delta(\mathsf{tp}(h)) \subseteq \Gamma(h) = \emptyset$ & $\mathsf{deg}(h) = 0 \overset{6.2b}{\Rightarrow} \mathsf{tp}(h) = \mathsf{Rep}$.

$h \in \mathsf{Z}^*_\perp$ & $\mathsf{tp}(h) = \mathsf{Rep} \overset{6.2c}{\Rightarrow} h[0] \in \mathsf{Z}^*_\perp$ & $\mathrm{o}(h[0]) < \mathrm{o}(h)$.

(b) By transfinite induction up to ε_0 from (a) we get $\mathsf{Z}^*_\perp = \emptyset$.

Now assume that d is a Z-derivation with $\Gamma(d) = \emptyset$. W.l.o.g. we may assume that d is closed.

Let $m := \mathsf{deg}(d)$. Then $\mathsf{E}^m d = \mathsf{E}...\mathsf{E}d \in \mathsf{Z}^*_\perp$. *Contradiction.*

Theorem 6.4. $\quad \mathrm{PRA} + \mathsf{QF}\text{-}\mathrm{TI}(\varepsilon_0) \vdash \mathrm{Con}(\mathsf{Z})$.

Proof: The (transfinite) induction formula in 6.3b is the Π_1-formula $F(\alpha) := \forall h(\mathrm{o}(h) = \alpha \to h \notin \mathsf{Z}^*_\perp)$:

$\forall \beta \prec \alpha F(\beta)$ & $\mathrm{o}(h) = \alpha$ & $h \in \mathsf{Z}^*_\perp \overset{6.3a}{\Rightarrow} (\mathrm{o}(h[0]) = \mathrm{o}(h[0]) \to h[0] \notin \mathsf{Z}^*_\perp)$ & $h[0] \in \mathsf{Z}^*_\perp \Rightarrow \perp$.

It remains to prove that Π_1-$\mathrm{TI}(\varepsilon_0)$ can be derived from QF-$\mathrm{TI}(\varepsilon_0)$.

Let $F(x) = \forall y A(x,y)$ with $A \in \mathsf{QF}$.

There are primitive recursive functions q, r, p such that $p(\ulcorner\alpha\urcorner, k) = \ulcorner\omega\cdot\alpha + k\urcorner$ for all $\alpha < \varepsilon_0$, $k < \omega$, and

$\mathsf{PRA} \vdash x \in \mathrm{OT} \to p(x,y) \in \mathrm{OT} \wedge q(p(x,y)) = x \wedge r(p(x,y)) = y$;

$\mathsf{PRA} \vdash \forall x, y(q(y) \prec q(x) \to y \prec x)$.

Abbreviation: $G(z) :\equiv A(q(z), r(z))$

(*) $\mathrm{Prog}_\prec(F) \to \mathrm{Prog}_\prec(G)$.

Proof: Assume (1) $\mathrm{Prog}_\prec(F)$, (2) $\forall z \prec c G(z)$.

To prove: $G(c)$, i.e. $A(q(c), r(c))$. By (1) this can be obtained from $\forall x \prec q(c) \forall y A(x,y)$.

$a \prec q(c)$ & $b := p(a,k) \Rightarrow q(b) = a \prec q(c) \wedge r(b) = k \Rightarrow b \prec c \overset{(2)}{\Rightarrow} G(b) \Rightarrow A(a,k)$.

Now we conclude as follows: $\mathrm{Prog}_\prec(F) \overset{(*)}{\Rightarrow} \mathrm{Prog}_\prec(G) \overset{\mathsf{QF}\text{-}\mathrm{TI}}{\Longrightarrow} \forall z G(z) \Rightarrow \forall z A(q(z), r(z)) \Rightarrow$

$\Rightarrow \forall x \in \mathrm{OT} \forall y A(x,y) \Rightarrow \forall x \in \mathrm{OT}\, F(x) \overset{\mathrm{Prog(F)}}{\Longrightarrow} \forall x F(x)$, since $(x \notin \mathrm{OT} \to \forall y(y \not\prec x))$.

Definition

Let $\mathsf{deg}_{\mathsf{QF}}(h)$ be defined in the same way as $\mathsf{deg}(h)$ only that $\mathrm{rk}(C)$ is replaced by $\mathrm{rk}_{\mathsf{QF}}(C)$:

$$\mathsf{deg}_{\mathsf{QF}}(\mathcal{I}h_0...h_{n-1}) := \begin{cases} \mathsf{deg}_{\mathsf{QF}}(h_0) \dot{-} 1 & \text{if } \mathcal{I} = \mathsf{E} \\ \max(\{\mathsf{deg}_{\mathsf{QF}}(\mathcal{I})\} \cup \{\mathsf{deg}_{\mathsf{QF}}(h_i) : i < n\}) & \text{otherwise} \end{cases}$$

where $\mathsf{deg}_{\mathsf{QF}}(\mathcal{I}) := \begin{cases} \mathrm{rk}_{\mathsf{QF}}(C) & \text{if } \mathcal{I} = \mathsf{Cut}_C \text{ or } \mathsf{Ind}_C^{x,t} \\ 0 & \text{otherwise} \end{cases}$.

Lemma 6.5 (Refinement of 6.2)

For every closed Z^*-derivation h we have:

(a) $\mathsf{tp}(h) = \mathsf{Cut}_C \implies \mathrm{rk}_{\mathsf{QF}}(C) < \mathsf{deg}_{\mathsf{QF}}(h)$.

(b) $\mathsf{deg}_{\mathsf{QF}}(h[\iota]) \leq \mathsf{deg}_{\mathsf{QF}}(h)$ for each $\iota \in |\mathsf{tp}(h)|$.

Characterization of the provably recursive functions of Z

Let $(\mathrm{OT}, \prec)$ be the wellordering of ordertype ε_0 as introduced in §5.

Theorem 6.6

If Π_{m+1}-IA $\vdash \forall x \exists y A(x,y)$ (A a quantifierfree $\mathcal{L}_0$-formula) then there are primitive recursive functions $g : \mathbb{N}^2 \to \mathbb{N}$, $\theta : \mathbb{N}^2 \to \mathbb{N}$ and an $\alpha < \omega_{m+2}$ such that $\forall n(\theta(n,0) = \ulcorner \alpha \urcorner)$ and $\mathbb{N} \models \forall x A(x, f(x))$,
where $f(n) := g(n, \min\{k : \theta(n,k{+}1) \not\prec \theta(n,k)\})$.

Proof:

We assume a canonical arithmetization (coding) $q \mapsto \ulcorner q \urcorner$ of syntax (terms, formulas, sequents, finite derivations, derivation terms etc.). A set M of syntactical objects is called primitive recursive if the set $\{\ulcorner q \urcorner : q \in M\}$ is primitive recursive. An operation (function) Φ on syntactical objects or ordinals $< \varepsilon_0$ is called primitive recursive if there is a primitive recursive function $f : \mathbb{N}^n \to \mathbb{N}$ such that $f(\ulcorner q_1 \urcorner, ..., \ulcorner q_n \urcorner) = \ulcorner \Phi(q_1, ..., q_n) \urcorner$ for all $(q_1, ..., q_n)$ in the domain of Φ.

Assume Π_{m+1}-IA $\vdash \forall x \exists y A(x,y)$. By partial cut-elimination there exists a Z-derivation $\mathbf{d}$ of $\exists y A(x,y)$ with $\mathrm{rk}_{\mathsf{QF}}(C) \le m$ for all cut or induction formulas C. Then $\mathsf{deg}_{\mathsf{QF}}(\mathbf{d}) \le m$ and $\mathrm{o}(\mathbf{d}) < \omega^\omega = \omega_2$. This $\mathbf{d}$ will be fixed for the whole proof. W.l.o.g. we may assume that $\mathrm{FV}(\mathbf{d}) \subseteq \{x\}$. Hence $\mathbf{d}(n) := \mathbf{d}(x/\underline{n})$ is closed for any $n \in \mathbb{N}$. We may also assume that no set variable X occurs in $\mathbf{d}$.

Let $\mathcal{L}_0^-$ be a *finite* subset of $\mathcal{L}_0$ (= PR), such that

(i) $0, \mathsf{S} \in \mathcal{L}_0^-$ and each function symbol occurring in d belongs to $\mathcal{L}_0^-$,

(ii) with $p \in \mathcal{L}_0^-$ also each function symbol occurring in (the definition of) p belongs to $\mathcal{L}_0^-$.

Let $\widetilde{\mathsf{Z}}^*$ be the restriction of Z^* to $\mathcal{L}_0^-$.

Let $\mathrm{TRUE}_{\mathsf{QF}}^-$ ($\mathrm{FALSE}_{\mathsf{QF}}^-$) be the set of all quantifierfree true (false) $\mathcal{L}_0^-$-sentences. It is wellknown that $\mathrm{TRUE}_{\mathsf{QF}}^-$ (and $\mathrm{FALSE}_{\mathsf{QF}}^-$) is primitive recursive, and that there is a primitive recursive function which for any two $\mathcal{L}_0^-$-terms s,t of equal value computes a $\widetilde{\mathsf{Z}}^*$-derivation c of $s{=}t$ with $\mathrm{o}(c) < \omega$ and $\mathsf{deg}(c) \le 1$. Obviously the functions $\Gamma(\cdot)$, $\mathrm{o}(\cdot)$, $\mathsf{deg}(\cdot)$, $\mathsf{tp}(\cdot)$ are primitive recursive, and $\cdot[\cdot]$ restricted to $\widetilde{\mathsf{Z}}^*$ is primitive recursive too.

Let $\mathbf{D}$ be the set of all closed $\widetilde{\mathsf{Z}}^*$-derivations.

Definition of the primitive recursive function $\mathrm{red} : \mathbf{D} \cup \{0\} \to \mathbf{D} \cup \{0\}$

$$\mathrm{red}(h) := \begin{cases} 0 & \text{if } h = 0 \text{ or } \mathsf{tp}(h) = \mathsf{Ax}_\Delta \text{ or } \mathsf{tp}(h) = \bigvee^s_{\exists y B} \text{ with } B_y(s) \in \mathrm{TRUE}_{\mathsf{QF}}^- \\ h[1] & \text{if } \mathsf{tp}(h) = \mathsf{Cut}_C \text{ with } C \in \mathrm{TRUE}_{\mathsf{QF}}^- \\ h[1] & \text{if } \mathsf{tp}(h) = \bigwedge_{A_0 \wedge A_1} \text{ with } A_0 \in \mathrm{TRUE}_{\mathsf{QF}}^- \\ h[0] & \text{otherwise} \end{cases}$$

Definition. $h(n,k) := \mathrm{red}^{(k)}(\underbrace{\mathsf{E}...\mathsf{E}}_{m}\mathbf{d}(n))$, $\alpha := 3_m(\mathrm{o}(\mathbf{d}))$.

$\mathsf{deg}_{\mathsf{QF}}(h(n,0)) = \mathsf{deg}_{\mathsf{QF}}(\mathbf{d}) \dot{-} m \le m \dot{-} m = 0$, and $\mathrm{o}(h(n,0)) = \alpha < 3_m(\omega^\omega) = \omega_{m+2}$.

Explanation. $h(n,0)^{\boldsymbol{\omega}}$ is a Z^∞-derivation of $\exists y A(n,y)$ with all cut-formulas in QF. The definition of $h(n,k)$ captures the following informal procedure. One goes upwards in the derivation $h(n,0)^{\boldsymbol{\omega}}$ searching for a true instance $A(n,t)$. At Cut- and $\wedge$-inferences one chooses that branch where the minor formula is false. The search stops if one arrives at an inference $\bigvee^t_{\exists y A(n,y)}$ with $A(n,t) \in \mathrm{TRUE}^-_{\mathsf{QF}}$.

Proposition. If $h(n,k) \neq 0$ then

(a) $\mathrm{o}(h(n,k)) \leq \alpha$,

(b) $\mathsf{deg}_{\mathsf{QF}}(h(n,k)) = 0$,

(c) $\Gamma(h(n,k)) \subseteq \{\exists y A(n,y)\} \cup \mathrm{FALSE}^-_{\mathsf{QF}}$

Proof by induction on k:

(a), (b) are obvious, since $\mathrm{o}(h[i]) < \mathrm{o}(h)$, $\mathsf{deg}_{\mathsf{QF}}(h[i]) \leq \mathsf{deg}_{\mathsf{QF}}(h)$, $\mathrm{o}(h(n,0)) = \alpha$, $\mathsf{deg}_{\mathsf{QF}}(h(n,0)) = 0$.

(c) $k = 0$: $\Gamma(h(n,0)) = \Gamma(\mathbf{d}(n)) = \{\exists y A(n,y)\}$.

$k > 0$: IH $\Rightarrow$ $\Gamma(h(n,k{-}1)) \subseteq \{\exists y A(n,y)\} \cup \mathrm{FALSE}^-_{\mathsf{QF}} \overset{h(n,k)\neq 0 \;\&\; 6.2a \;\&\; 6.5a \;\&\; (b)}{\Longrightarrow}$

$\Rightarrow$ $\mathsf{tp}(h(n,k{-}1)) = \bigvee^t_{\exists y A(n,y)}$ with $A(n,t) \in \mathrm{FALSE}^-_{\mathsf{QF}}$ or Rep or Cut_C with $C \in \mathsf{QF}$ or $\bigwedge_B$ or $\bigvee^i_B$ with $B \in \mathrm{FALSE}^-_{\mathsf{QF}}$ $\overset{6.2c}{\Rightarrow}$ $\Gamma(h(n,k)) \subseteq \Gamma(h(n,k{-}1)) \cup \mathrm{FALSE}^-_{\mathsf{QF}} \subseteq \{\exists y A(n,y)\} \cup \mathrm{FALSE}^-_{\mathsf{QF}}$.

Definition.

$\theta(n,k) := \ulcorner \mathrm{o}(h(n,k)) \urcorner$ (where $\mathrm{o}(0) := 0$)

$$g(n,k) := \begin{cases} t^{\mathcal{N}} & \text{if } k > 0 \text{ and } \mathsf{tp}(h(n,k{-}1)) = (\bigvee^t_{\exists y A(n,y)}) \\ 0 & \text{otherwise} \end{cases}$$

$f(n) := g(n, \min\{k : \theta(n,k{+}1) \not\prec \theta(n,k)\})$.

Now let k be the least number such that $\theta(n,k{+}1) \not\prec \theta(n,k)$.

Assumption: $h(n,k) \neq 0$. Then [by Prop.(c)] $\mathsf{tp}(h(n,k)) \neq \mathsf{Ax}$ and thus

$\theta(n,k{+}1) = \mathrm{o}(h(n,k{+}1)) \overset{6.2c}{\prec} \mathrm{o}(h(n,k)) = \theta(n,k)$. Contradiction.

Hence $h(n,k) = 0$ and thus $k > 0$ and [by Definition of red] $\mathsf{tp}(h(n,k{-}1)) = \mathsf{Ax}$ or $\mathsf{tp}(h(n,k{-}1)) = \bigvee^t_{\exists y B(y)}$ with $B(t) \in \mathrm{TRUE}^-_0$. By Proposition (c) and Theorem 6.2a from this we get $\mathsf{tp}(h(n,k{-}1)) = \bigvee^t_{\exists y A(n,y)}$ with $A(n,t) \in \mathrm{TRUE}^-_0$. Hence $f(n) = g(n,k) = t^{\mathcal{N}}$ and $\mathbb{N} \models A(n, f(n))$.

The Hardy-Hierarchy

Definition (Fundamental sequences for ordinals $< \varepsilon_0$)

1. $0[n] := 1[n] := 0$.
2. $\omega^{\alpha+1}[n] := \omega^\alpha \cdot (n{+}1)$.
3. $\omega^\lambda[n] := \omega^{\lambda[n]}$, for $\lambda \in \mathit{Lim}$.
4. $\alpha[n] := \alpha_0 + ... + \alpha_{k-1} + \alpha_k[n]$, if $\alpha =_{NF} \alpha_0 + ... + \alpha_k$.

Proposition. $(\alpha{+}1)[n] = \alpha$.

Definition. $N\alpha := N\alpha_1 + \ldots + N\alpha_k + k$ if $\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$ with $k \geq 0$ and $\alpha_k \leq \ldots \leq \alpha_1 < \varepsilon_0$

Lemma 6.7

(a) $\alpha \in \mathit{Lim} \implies \forall n(\alpha[n] < \alpha[n{+}1]) \;\&\; \alpha = \sup\{\alpha[n] : n \in \mathbb{N}\}$;

(b) $\alpha > 0 \implies N\alpha[0] < N\alpha$

(c) $\alpha[n] < \beta < \alpha \implies \alpha[n] \leq \beta[0]$

(d) $\alpha[n] < \beta < \alpha \implies N\alpha[n] < N\beta$

(e) $\beta < \alpha \implies \beta \le \alpha[N\beta]$

Proof:

(a),(b) obvious.

(c) Induction on β. Let $\beta =_{NF} \beta_0 + ... + \beta_k$.

1. Assume $\omega^\alpha \cdot (n{+}1) < \beta < \omega^{\alpha+1}$. Then $k > n$ and $\beta_0 = ... = \beta_n = \omega^\alpha$.

From this we get $\omega^\alpha \cdot (n{+}1) \le \beta_0 + ... + \beta_{k-1} + \beta_k[0] = \beta[0]$.

2. Assume $\omega^{\lambda[n]} < \beta < \omega^\lambda$ und $\lambda \in Lim$. Then $\omega^{\lambda[n]} \le \beta_0 = \omega^\gamma < \omega^\lambda$.

If $k = 0$ then $\lambda[n] < \gamma < \lambda$ and therefore (by IH) $\lambda[n] \le \gamma[0]$. Hence $\omega^{\lambda[n]} \le \omega^{\gamma[0]} = \omega^\gamma[0] = \beta[0]$.

If $k > 0$ then $\omega^{\lambda[n]} \le \beta_0 + ... + \beta_{k-1} + \beta_k[0] = \beta[0]$.

3. Assume $\alpha =_{NF} \alpha_0 + ... + \alpha_m$, $m > 0$ and $\alpha[n] = \alpha_0 + ... + \alpha_{m-1} + \alpha_m[n] < \beta < \alpha$. Then $m \le k$, $\alpha_m[n] < \beta_m + ... + \beta_k < \alpha_m$ and $\alpha_i = \beta_i$ for $i < m$. By IH we get $\alpha_m[n] \le (\beta_m + ... + \beta_k)[0] = \beta_m + ... + \beta_{k-1} + \beta_k[0]$ and then $\alpha[n] \le \beta_0 + ... + \beta_{k-1} + \beta_k[0] = \beta[0]$.

(d) By (c) we have $\alpha[n] = \beta[0]...[0]$. Hence $N\alpha[n] \le N\beta[0] < N\beta$.

(e) Let $\alpha \in Lim$. According to (a),(d) we then have $\forall n(N\alpha[n] < N\alpha[n{+}1])$, and therefore $\forall n(n \le N\alpha[n])$. Now the claim is obtained as follows: $\alpha[N\beta] < \beta < \alpha \;\Rightarrow\; N\alpha[N\beta] \overset{(d)}{<} N\beta \le N\alpha[N\beta]$. Contradiction.

Definition of $H_\alpha : \mathbb{N} \to \mathbb{N}$ for $\alpha < \varepsilon_0$

$H_0(n) := n, \quad H_\alpha(n) := H_{\alpha[n]}(n{+}1)$ for $\alpha > 0$.

Lemma 6.8

(a) $H_\alpha(n) < H_\alpha(n+1)$,

(b) $\alpha[m] < \beta < \alpha \;\Rightarrow\; H_{\alpha[m]}(n{+}1) \le H_\beta(n)$,

(c) $\beta < \alpha \;\&\; N\beta \le n \;\Rightarrow\; H_\beta(n) < H_\alpha(n)$,

(d) $\alpha > 0 \;\Rightarrow\; H_\alpha(n) = \min\{k \ge n : \alpha[n]...[k{-}1] = 0\} = n + \min\{l : \alpha[n][n{+}1]\dots[n{+}l{-}1] = 0\}$.

Proof:

(a),(b) simultaneous induction on α: Let $\alpha > 0$.

(a) 1. $\alpha \in Lim$: $H_\alpha(n) = H_{\alpha[n]}(n{+}1) \overset{\text{IHa}}{<} H_{\alpha[n]}(n{+}3) \overset{\text{IHb}}{\le} H_{\alpha[n+1]}(n{+}2) = H_\alpha(n{+}1)$.

2. $\alpha = \alpha_0{+}1$: $H_\alpha(n) = H_{\alpha_0}(n{+}1) \overset{\text{IHa}}{<} H_{\alpha_0}(n{+}2) = H_\alpha(n{+}1)$.

(b) From $\alpha[m] < \beta < \alpha$ we obtain $\alpha[m] \le \beta[n] < \alpha$ by Lemma 6.7. If $\alpha[m] = \beta[n]$ then $H_{\alpha[m]}(n{+}1) = H_{\beta[n]}(n{+}1) = H_\beta(n)$. Otherwise $H_{\alpha[m]}(n{+}1) \overset{\text{IHb}}{\le} H_{\beta[n]}(n) \overset{\text{IHa}}{<} H_{\beta[n]}(n{+}1) = H_\beta(n)$.

(c) Induction on α: $\beta < \alpha \overset{\text{L.6.7e}}{\Rightarrow} \beta \le \alpha[N\beta] \le \alpha[n] \overset{\text{(a)+IH}}{\Rightarrow} H_\beta(n) < H_\beta(n{+}1) \le H_{\alpha[n]}(n{+}1) = H_\alpha(n)$.

(d) Let $k \ge n$ minimal such that $\alpha[n]...[k{-}1] = 0$.

Then $H_\alpha(n) = H_{\alpha[n]}(n{+}1) = ... = H_{\alpha[n]\dots[k-1]}(k) = H_0(k) = k$.

Abbreviation.

$$NF(\alpha,\beta) \;:\Leftrightarrow \begin{cases} \alpha = 0 \text{ or } \beta = 0 \text{ or} \\ [\,\alpha = \omega^{\alpha_0} + ... + \omega^{\alpha_n} \;\&\; \beta = \omega^{\beta_0} + ... + \omega^{\beta_m} \text{ with } \alpha_0 \ge ... \ge \alpha_n \ge \beta_0 \ge ... \ge \beta_m\,] \end{cases}.$$

Proposition. $NF(\alpha,\beta) \;\&\; \beta > 0 \;\Rightarrow\; (\alpha+\beta)[n] = \alpha + \beta[n] \;\&\; NF(\alpha, \beta[n])$.

Lemma 6.9

(a) $NF(\alpha,\beta) \;\Rightarrow\; H_{\alpha+\beta} = H_\alpha \circ H_\beta$.

(b) $H_{\omega^{\alpha+1}}(n) = H_{\omega^\alpha}^{(n+1)}(n{+}1)$ and $H_{\omega^\lambda}(n) = H_{\omega^{\lambda[n]}}(n{+}1)$ for $\lambda \in Lim$.

(c) For each primitive recursive function f there exists a $k \in \mathbb{N}$ such that $\forall\vec{x}(\, f(\vec{x}) < H_{\omega^k}(\max\{\vec{x}\})\,)$.

Proof:

(a) Induction on β: 1. $H_{\alpha+0}(n) = H_\alpha(n) = H_\alpha(H_0(n))$.

2. $\beta > 0$: $H_{\alpha+\beta}(n) = H_{(\alpha+\beta)[n]}(n{+}1) = H_{\alpha+\beta[n]}(n{+}1) \overset{\text{IH}}{=} H_\alpha(H_{\beta[n]}(n{+}1)) = H_\alpha(H_\beta(n))$.

(b) $H_{\omega^{\alpha+1}}(n) = H_{\omega^\alpha\cdot(n+1)}(n{+}1) \overset{\text{(a)}}{=} H_{\omega^\alpha}^{(n+1)}(n{+}1)$.

(c) From (b) it follows that $(k,n) \mapsto H_{\omega^k}(n)$ is (a variant of) the Ackermann function.

Theorem 6.10

Let $\theta : \mathbb{N}\times\mathbb{N} \to \mathrm{OT}$ be primitive recursive, and $\alpha < \omega_{m+2}$ such that $\forall n(\,\theta(n,0) \preceq \ulcorner\alpha\urcorner\,)$.

Then there is an $\tilde{\alpha} < \omega_{m+2}$ such that $\min\{l : \theta(n,l{+}1) \not\prec \theta(n,l)\} < H_{\tilde{\alpha}}(n)$ $(\forall n)$.

Proof:

Abbreviation: $\hat{\theta}(n,l) := o(\theta(n,l))$, where $o(\cdot)$ is the isomorphism from $(\mathrm{OT},\prec)$ onto $(\varepsilon_0,<)$.

W.l.o.g. $\forall n(\,\hat{\theta}(n,0) = \alpha\,)$.

Let $w(i,n,l) := N(\omega^i \cdot (\hat{\theta}(n,l{+}1){+}1))$. One easily sees that w is primitive recursive.

Let $g(i,n,l) := \max\{w(i,n,l), i, n, l\}$.

There exists a $k \geq 1$ such that $g(i,n,l{+}1) < H_{\omega^k}(\max\{i,n,l\})$ and $g(i,n,0) < H_{\omega^k}(\max\{i,n\})$ $(\forall i,n,l)$.

Then we have

(1) $g(k,n,l{+}1) < H_{\omega^k}(g(k,n,l))$ $(\forall n,l)$.

Abbreviation: $\varphi(n,l) := H_{\omega^k\cdot\hat{\theta}(n,l)}(g(k,n,l))$.

(2) $\hat{\theta}(n,l{+}1) < \hat{\theta}(n,l) \;\Rightarrow\; \varphi(n,l{+}1) < \varphi(n,l)$.

Proof: $H_{\omega^k\cdot\hat{\theta}(n,l+1)}(g(k,n,l{+}1)) \overset{(1)}{<} H_{\omega^k\cdot\hat{\theta}(n,l+1)}H_{\omega^k}(g(k,n,l)) = H_{\omega^k\cdot(\hat{\theta}(n,l+1)+1)}(g(k,n,l)) \overset{(*)}{\leq}$
$\leq H_{\omega^k\cdot\hat{\theta}(n,l)}(g(k,n,l)) = \varphi(n,l)$.

$(*)$ $\omega^k \cdot (\hat{\theta}(n,l{+}1){+}1) \leq \omega^k \cdot \hat{\theta}(n,l)$ and $N(\omega^k \cdot (\hat{\theta}(n,l{+}1){+}1)) = w(k,n,l) \leq g(k,n,l)$.

(3) $\exists l \leq \varphi(n,0)(\,\varphi(n,l{+}1) \not< \varphi(n,l)\,)$.

Proof: $[\forall l \leq j(\,\varphi(n,l{+}1) < \varphi(n,l)\,) \Rightarrow j < \varphi(n,0)]$ and therefore *not* $\forall l \leq \varphi(n,0)(\,\varphi(n,l{+}1) < \varphi(n,l)\,)$.

(2) & (3) & $\alpha = \hat{\theta}(n,0) \;\Rightarrow\; \exists\, l \leq H_{\omega^k\cdot\alpha}(g(k,n,0))[\theta(n,l{+}1) \not\prec \theta(n,l)\,]$.

$H_{\omega^k\cdot\alpha}(g(k,n,0)) < H_{\omega^k\cdot\alpha}H_{\omega^k}(\max\{k,n\}) \leq H_{\omega^k\cdot(\alpha+1)+k}(n)$.

$\alpha < \omega_{m+2} = \omega^{\omega_{m+1}} \;\Rightarrow\; \omega^k{\cdot}(\alpha{+}1) + k < \omega^k \cdot (\alpha{+}2) < \omega^k \cdot \omega^{\omega_{m+1}} = \omega^{\omega_{m+1}} = \omega_{m+2}$.

Theorem 6.11

If Π_{m+1}-IA $\vdash \forall x\exists y A(x,y)$ (A a quantifierfree $\mathcal{L}_0$-formula) then there is an $\alpha < \omega_{m+2}$ such that $\forall n \exists\, l < H_\alpha(n)\, \mathbb{N} \models A(n,l)$.

Proof:

By Theorem 6.6 there are primitive recursive functions g, θ and an $\alpha_0 < \omega_{m+2}$ such that $\forall n(\theta(n,0) = \ulcorner\alpha_0\urcorner)$ and $\mathbb{N} \models \forall x A(x, f(x))$, where $f(n) := g(n, f^*(n))$, $f^*(n) := \min\{l : \theta(n,l+1) \not\prec \theta(n,l)\}$).
By 6.10 there exists $\beta < \omega_{m+2}$ with $\forall n(f^*(n) < H_\beta(n))$. Further there exists $k < \omega$ with $\forall n, i(\, g(n,i) < H_{\omega^k}(\max\{n,i\})\,)$, hence $f(n) < H_{\omega^k}(\max\{n, f^*(n)\}) \le H_{\omega^k}H_\beta(n)$.
Since $\omega_{m+2} = \sup_{i\in\mathbb{N}} \omega_{m+1}(i)$, there is $\gamma := \omega_{m+1}(i)$ such that $\omega^k, \beta < \gamma$.
It follows that there is an n_0 such that $H_{\omega^k}H_\beta(n) \le H_\gamma H_\gamma(n) = H_{\gamma+\gamma}(n)$ for all $n \ge n_0$.
Hence $f(n) < H_{\omega^k}H_\beta(n_0{+}n) \le H_{\gamma+\gamma}(n_0{+}n) = H_{\gamma+\gamma+n_0}(n)$ (and $\gamma{+}\gamma{+}n_0 < \omega_{m+2}$).

Corollary. Π_{m+1}-IA $\nvdash$ '$\forall n \exists l(\, \omega_{m+2}[n][n{+}1]...[l] = 0\,)$'.

Proof: Assume Π_{m+1}-IA $\vdash$ '$\forall n \exists l(....)$'.
Then there is an $\alpha < \omega_{m+2}$ such that $\forall n \exists l < H_\alpha(n)(\, \omega_{m+2}[n][n{+}1]...[l] = 0\,)$, i.e. $\forall n \exists l < H_\alpha(n)(\, H_{\omega_{m+2}}(n) \le l{+}1\,)$. This implies $\forall n(\, H_{\omega_{m+2}}(n) \le H_\alpha(n)\,)$.
But by L.6.8c we have $\forall n \ge N(\alpha)(\, H_\alpha(n) < H_{\omega_{m+2}}(n)\,)$. Contradiction.

Below we will show Π_{m+1}-IA $\vdash$ '$\forall n \exists l(\, \alpha[n][n{+}1]...[l] = 0\,)$' *for each* $\alpha < \omega_{m+2}$.

Definition. $F_\alpha := H_{\omega^\alpha}$ *(Fast-Growing Hierarchy)*

Corollary. $F_0(n) = n{+}1$, $F_{\alpha+1}(n) = F_\alpha^{(n+1)}(n{+}1)$, $F_\lambda(n) = F_{\lambda[n]}(n{+}1)$ for $\lambda \in Lim$.

Remark.
In the literature the Hardy- and the Fast-Growing Hierarchy occur in several variants.
The most common of these are:

$h_0(n) := n$, $h_{\alpha+1}(n) := h_\alpha(n{+}1)$, $h_\lambda(n) := h_{\lambda[n]}(n)$;
$f_0(n) := n{+}1$, $f_{\alpha+1}(n) := f_\alpha^{(n+1)}(n)$, $f_\lambda(n) := f_{\lambda[n]}(n)$.

One easily sees that (i) $h_{\omega^\alpha} = f_\alpha$, and (ii) $h_\alpha(n) \le H_\alpha(n) < h_\alpha(n+1)$.

Proof of (ii) by induction on α:
1. $h_{\alpha+1}(n) = h_\alpha(n{+}1) \overset{\text{IH}}{\le} H_\alpha(n{+}1) = H_{\alpha+1}(n) = H_\alpha(n{+}1) \overset{\text{IH}}{<} h_\alpha(n{+}2) = h_{\alpha+1}(n{+}1)$.
2. $\alpha \in Lim$: $h_\alpha(n) = h_{\alpha[n]}(n) \overset{\text{IH}}{\le} H_{\alpha[n]}(n) < H_{\alpha[n]}(n+1) = H_\alpha(n)$, and
$H_\alpha(n) = H_{\alpha[n]}(n{+}1) \overset{\text{L.6.8b}}{\le} H_{\alpha[n+1]}(n) \overset{\text{IH}}{<} h_{\alpha[n+1]}(n{+}1) = h_\alpha(n{+}1)$.

Definition. By PRWO(α) we denote the axiom scheme
$\forall\vec{x}(f(\vec{x},0) \preceq \ulcorner\alpha\urcorner \to \exists y(f(\vec{x},y{+}1) \not\prec f(\vec{x},y)))$ (f primitive recursive).

Theorem 6.12.
(a) $\mathsf{PRA} + \mathrm{PRWO}(\alpha) \vdash$ '$\forall n \exists l(\alpha[n]...[l] = 0)$'.
(b) Σ_1-IA $+ \Pi_2$-TI(α) $\vdash$ PRWO(ω^α).

Corollary.
Π_{m+1}-IA $\vdash$ PRWO(α) and '$\forall n \exists l(\alpha[n]...[l] = 0)$' for each $\alpha < \omega_{m+2}$.

Proof:
Let $p(\langle a,n\rangle) := \langle a[n], n{+}1\rangle$, and $f(a,n,k) := \left(p^{(k)}(\langle a,n\rangle)\right)_0$ where $\cdot[\cdot] : \mathbb{N}^2 \to \mathbb{N}$ is the canonical primitive recursive function such that $\ulcorner\beta\urcorner[n] = \ulcorner\beta[n]\urcorner$ for all $\beta < \varepsilon_0$, $n \in \mathbb{N}$. We now argue in PRA.
PRWO(α) $\Rightarrow$ $\forall n\exists k(f(\ulcorner\alpha\urcorner,n,k{+}1) \not\prec f(\ulcorner\alpha\urcorner,n,k))$ $\Rightarrow$ $\forall n\exists k(f(\ulcorner\alpha\urcorner,n,k) = 0)$ $\Rightarrow$ $\forall n\exists k(\ulcorner\alpha\urcorner[n]\dots[k] = 0)$.

(b) Abb.: $D(x,n) := \forall k(f(x,n{+}k{+}1) \prec f(x,n{+}k))$.
Assumptions:
(1) $\forall b \prec a\, \forall c,n(D(x,n) \to (f(x,n) \prec c \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^b \to \exists k(f(x,n{+}k) \prec c)))$,
(2) $D(x,n)$,
(3) $f(x,n) \prec c \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^a$.

We prove: $\exists k(f(x,n{+}k) \prec c)$.
1. $a = 0$: $f(x,n) \prec c \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^0 \overset{(2)}{\Rightarrow} f(x,n{+}1) \prec f(x,n) \preceq c$.
2. $a = \langle a_0, ..., a_m\rangle$ with $a_m \neq 0$ (i.e. $o(a) \in \mathit{Lim}$): immediate by (1).
3. $a = b \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^0$: Then by (3), $f(x,n) \prec c \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^b \cdot l$ for some l.
By Σ_1-IA on i we prove: $\exists k(f(x,n{+}k) \prec c \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^b \cdot (l \dot{-} i))$.
3.1. $i = 0$: trivial.
3.2. $i \to i{+}1$: Let $i < l$. From (2) we get (2') $\forall k.D(x,n{+}k)$.
$\exists k(f(x,n{+}k) \prec c \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^b \cdot (l \dot{-} i)) \overset{(1),(2')}{\Rightarrow} \exists k \exists k'(f(x,n{+}k{+}k') \prec c \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^b \cdot ((l \dot{-} i) \dot{-} 1) = c \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^b \cdot (l \dot{-} (i{+}1)))$.

So within Σ_1-IA we have shown: (1) $\Rightarrow \forall c,n(D(x,n) \to (f(x,n) \prec c \mathbin{\widehat{\oplus}} \boldsymbol{\omega}^a \to \exists k(f(x,n{+}k) \prec c)))$.
By Π_2-TI(α) this yields $\forall n(D(x,n) \to (f(x,n) \preceq \ulcorner\omega^\alpha\urcorner \to \exists k(f(x,n{+}k) \prec 0)))$.
Hence $\forall n(D(x,n) \to f(x,n) \preceq \ulcorner\omega^\alpha\urcorner \to \bot)$, and then $\forall n(f(x,n) \preceq \ulcorner\omega^\alpha\urcorner \to \neg D(x,n))$.

Proof of the Corollary: Lemma 5.13(Corollary) $\Longrightarrow$ Π_{m+1}-IA $\vdash \Pi_2$-TI(α) for each $\alpha < \omega_{m+1}$ $\overset{6.12b}{\Longrightarrow}$
$\Longrightarrow$ Π_{m+1}-IA $\vdash$ PRWO(α) for each $\alpha < \omega_{m+2}$.

§**7 Combinatorial independence results** (To be revised)

Goodstein's Theorem

For $1 \le n \in \mathbb{N}$ and $x \in \mathbb{N}$ let $\mathcal{S}_n(x)$ be the number obtained by writing x in complete Cantor normal form at base $n{+}1$ and then replacing the base $n{+}1$ by $n{+}2$.

The *Goodstein sequence* for $a \in \mathbb{N}$ $(\mathrm{GS}(a,n))_{n\in\mathbb{N}}$ is defined by
$\mathrm{GS}(a,0) := \mathrm{GS}(a,1) := a,\ \ \mathrm{GS}(a,n+1) := \mathcal{S}_n(\mathrm{GS}(a,n)) \dot{-} 1$ for $n \ge 1$.

We will prove:
(i) Every Goodstein sequence terminates (i.e. $\forall a \exists n.\mathrm{GS}(a,n) = 0$);
(ii) $\mathsf{Z} \nvdash$ '$\forall a \exists n.\mathrm{GS}(a,n) = 0$'.

Definition von $\Phi_b^\alpha : \omega \to \varepsilon_0$ für $2 \le b < \omega$ und $b < \alpha < \varepsilon_0$
$\Phi_b^\alpha(x) := \alpha^{\Phi_b^\alpha(x_1)} \cdot n_1 + \ldots + \alpha^{\Phi_b^\alpha(x_k)} \cdot n_k$, falls $x = \sum_{i=1}^k b^{x_i} \cdot n_i$ mit $x_1 > \ldots > x_k$ und $n_1, ..., n_k \in \{1, ..., b{-}1\}$.

$\mathcal{S}_n(x) := \Phi_{n+1}^{n+2}(x)\ \ (n \ge 1,\, x \in \omega)$. Abk.: $\theta_n(x) := \Phi_{n+1}^{\omega}(x)\ \ (n \ge 1,\, x \in \omega)$.

Lemma 7.1 Sei $n \ge 1$.
(a) $x < y < \omega \ \Rightarrow\ \mathcal{S}_n(x) < \mathcal{S}_n(y)\ \&\ \theta_n(x) < \theta_n(y)$,
(b) $\theta_n(x) = \theta_{n+1}\mathcal{S}_n(x)$.

Sei nun $a_n := \mathrm{GS}(a,n)$. Dann gilt:
$a_n > 0 \ \Rightarrow\ \mathcal{S}_n(a_n) > 0 \ \Rightarrow\ \theta_{n+1}(a_{n+1}) = \theta_{n+1}(\mathcal{S}_n(a_n) - 1) \overset{7.1a}{<} \theta_{n+1}(\mathcal{S}_n(a_n)) \overset{7.1b}{=} \theta_n(a_n)$.

Definition $\mathrm{P}_n(0) := 0,\ \mathrm{P}_n(\alpha+1) := \alpha,\ \ \mathrm{P}_n(\lambda) := \mathrm{P}_n(\lambda[n])$.

Lemma 7.2 Sei $n > 0$.
(a) $\alpha > 0 \Rightarrow \mathrm{P}_n(\alpha) < \alpha$,
(b) $\mathrm{P}_n(\alpha+\beta) = \alpha + \mathrm{P}_n(\beta)$, falls $NF(\alpha,\beta)$,
(c) $\mathrm{P}_n(\omega^\alpha) = \mathrm{P}_n(\omega^{\mathrm{P}_n(\alpha)} \cdot (n{+}1))$, falls $\alpha > 0$,
(d) $\mathrm{P}_n(\theta_n(x)) = \theta_n(x \dot{-} 1)$,
(e) $n \ge 1 \ \Rightarrow\ \theta_n \mathrm{GS}(a,n) = \mathrm{P}_n ... \mathrm{P}_2\theta_1(a)$.

Lemma 7.3
(a) $\alpha > 0 \ \Rightarrow\ h_\alpha(n) = h_{\mathrm{P}_n(\alpha)}(n+1)$.
(b) $h_\alpha(n) = \min\{k \ge n : \mathrm{P}_{k-1}...\mathrm{P}_n(\alpha) = 0\}$.

Satz 7.4
$\mathcal{Z} \nvdash \forall x \exists y[\,\mathrm{GS}(x,y) = 0\,]$.

Proof
Sei $e(0) := 1,\ e(m+1) := 2^{e(m)}$.
Annahme: $\mathcal{Z} \vdash \forall x \exists y[\,\mathrm{GS}(x,y) = 0\,]$.
$\Rightarrow\ \mathcal{Z} \vdash \forall x \exists y[\,\mathrm{GS}(e(x)+x,y) = 0\,] \overset{6.12}{\Rightarrow} \exists \alpha < \varepsilon_0 \forall m \exists n < H_\alpha(m)[\,\mathrm{GS}(e(m)+m,n) = 0\,] \ \Rightarrow$
$\Rightarrow\ \exists p,n[n < H_{\omega_p}(p)\ \&\ \mathrm{GS}(e(p)+p,n) = 0\,]$. ($p \ge N(\alpha)$ mit $\alpha < \omega_p$; vergl. 6.9c)
$\mathrm{GS}(e(p){+}p,n) = 0 \ \Rightarrow\ n \ge 2\ \&\ \mathrm{P}_n...\mathrm{P}_2(\omega_p{+}p) = \theta_n\mathrm{GS}(e(p){+}p,n) = 0 \ \Rightarrow\ H_{\omega_p+p}(1) < h_{\omega_p+p}(2) \le n{+}1 \ \Rightarrow$
$H_{\omega_p}(p) = H_{\omega_p+p}(0) < H_{\omega_p+p}(1) \le n$. *Widerspruch.*

The Paris-Harrington Result

Abbreviations:

Let $k, m, n, r \in \mathbb{N} (= \omega)$, κ a cardinal, N a set. $[N]^m := \{X \subseteq N : card(X) = m\}$

Let f be a function with $\text{dom}(f) = [N]^m$:

X is f-homogeneous $\iff$ $\emptyset \neq X \subseteq N$ & $f{\restriction}[X]^m$ constant.

$N \longrightarrow (\kappa)^m_r$ $:\Leftrightarrow$ $\forall f : [N]^m \to r \exists X (X$ f-homogeneous & $card(X) \geq \kappa)$

$N \overset{*}{\longrightarrow} (\kappa)^m_r$ $:\Leftrightarrow$ $\forall f : [N]^m \to r \exists X (X$ f-homogeneous & $card(X) \geq \max\{\kappa, \min(X)\})$ (for $N \subseteq \mathbb{N}$)

Ramsey Theorem $\forall m, r \in \omega (\omega \longrightarrow (\omega)^m_r)$

Finite Ramsey Theorem $\forall m, r, \kappa \in \omega \exists N \in \omega (N \longrightarrow (\kappa)^m_r)$

PH $\forall m, r, \kappa \in \omega \exists N \in \omega (N \overset{*}{\longrightarrow} (\kappa)^m_r)$

(Proof of the Finite Ramsey Theorem in PA: cf. Hajek, Pudlak Ch.2, Sec.1)

Proof of PH: Let $m, r, \kappa \in \omega$ be fixed. To prove: $\exists n \in \omega \forall f \neg\Phi(f, n)$
where $\Phi(f, n)$ $:\Leftrightarrow$ $f : [n]^m \to r$ & $\forall X (X$ f-hom $\Rightarrow$ $card(X) < \max\{\kappa, \min(X)\})$.

Assumption: $\forall n \exists f\, \Phi(f, n)$. By König's Lemma there is a function $f^*[\omega]^m \to r$ such that
(+) $\forall n\, \Phi(f^*{\restriction}[n]^m, n)$.

By Ramsey's Theorem there exists an infinite f^*-homogeneous set $X \subseteq \omega$. We choose $N < \omega$ such that $card(X \cap N) > \max\{\kappa, \min(X)\}$. For $f := f^*{\restriction}[N]^m$ we then have $f{\restriction}[X \cap N]^m = f^*{\restriction}[X \cap N]^m = constant$. Hence $X \cap N$ is f-homogeneous and $card(X \cap N) > \max\{\kappa, \min(X)\} = \max\{\kappa, \min(X \cap N)\}$, i.e., $\neg\Phi(f, N)$. *Contradiction* to (+).

[[Construction of f^*: Let $\Phi_n := \{f : \Phi(f, n)\}$ and $M(f) := \{i : \exists g \in \Phi_i (f \subseteq g)\}$. — Starting with $f_0 := \emptyset$ we define a sequence $(f_n)_{n \in \omega}$ such that $f_n \in \Phi_n$ & $f_n \subseteq f_{n+1}$ & $card(M(f_n)) \geq \omega$, and we set $f^* := \bigcup_{n \in \omega} f_n$.
Definition of f_{n+1}: Let $E := \{f \in \Phi_{n+1} : f_n \subseteq f\}$. Then $M(f_n) = \{n\} \cup \bigcup_{f \in E} M(f)$, and E is finite. This together with $\forall i > n \forall f \in \Phi_i (f{\restriction}[n{+}1]^m \in \Phi_{n+1})$ implies the existence of an $f_{n+1} \in E$ such that $card(M(f_{n+1})) \geq \omega$.]]

Theorem 7.5 $\forall m \geq 1 \forall k (H_{\omega_m(k)}(k{+}1) < R_m(k))$ with $R_m(k) := \min\{N : N \overset{*}{\longrightarrow} (2m{+}k{+}4)^{m+1}_{k+\Sigma_{i<m} 3^i}\}$

Corollary

a) $\mathcal{Z}_m \nvdash \forall \kappa, r \exists N (N \overset{*}{\longrightarrow} (\kappa)^{m+1}_r)$ $(m \geq 1)$

b) $\mathcal{Z} \nvdash \forall m, \kappa, r \exists N (N \overset{*}{\longrightarrow} (\kappa)^{m+1}_r)$

Proof of the Corollary:

a) Assumption: $\mathcal{Z}_m \vdash \forall \kappa, r \exists N (N \overset{*}{\longrightarrow} (\kappa)^{m+1}_r)$. Then $\mathcal{Z}_m \vdash \forall \kappa, r \exists N (N \overset{*}{\longrightarrow} (2m{+}k{+}4)^{m+1}_{k+\Sigma_{i<m} 3^i}\})$.

By 6.12 there is an $\alpha < \omega_{m+1}$ such that $\forall k (R_m(k) < H_\alpha(k))$.

Let $k \in \omega$ such that $\alpha < \omega_m(k)$ and $N(\alpha) \leq k$.

Then $H_{\omega_m(k)}(k{+}1) \overset{7.5}{<} R_m(k) < H_\alpha(k) < H_{\omega_m(k)}(k)$. *Contradiction.*

b) $\mathcal{Z} \vdash \forall n, \kappa, r \exists N (N \overset{*}{\longrightarrow} (\kappa)^{n+1}_r)$

$\Longrightarrow \mathcal{Z}_m \vdash \forall n, \kappa, r \exists N (N \overset{*}{\longrightarrow} (\kappa)^{n+1}_r)$ for suitable m

$\Longrightarrow \mathcal{Z}_m \vdash \forall \kappa, r \exists N (N \overset{*}{\longrightarrow} (\kappa)^{m+1}_r)$.

§8 The collapsing functions ψ_σ

In this section we introduce certain ordinal functions ψ_σ, and a primitive recursive ordinal notation system $(\mathrm{OT}, \prec)$ based on these functions, which will later be used to establish an ordinal analysis of the theories ID_n of finitely iterated inductive definitions.

Here we are working in ZFC.

In particular we assume the Axiom of Choice, so that every infinite successor cardinal $\aleph_{\alpha+1}$ is regular.

$\alpha, \beta, \gamma, \delta, \xi, \eta, \zeta, \sigma, \mu, \rho$ denote ordinals.

Notation. $\Omega_\sigma := \aleph_\sigma$; $\Omega^-_{\sigma+1} := \begin{cases} 1 & \text{if } \sigma = 0 \\ \Omega_\sigma & \text{if } \sigma > 0 \end{cases}$; $\beta^+ := \min\{\Omega_{\sigma+1} : \beta < \Omega_{\sigma+1}\}$.

$\Re :=$ the class of all uncountable regular cardinals.

Definition.

Given a countable set $\mathcal{F}$ of ordinal functions and an ordinal β let

$\mathrm{Cl}(\mathcal{F};\beta) :=$ the closure of β under $+$ and all functions in $\mathcal{F}$.

Lemma 8.1.

If $\mathcal{F}$ is a countable set of ordinal functions, then for each $\kappa \in \Re$

the set $\{\beta < \kappa : \mathrm{Cl}(\mathcal{F};\beta) \cap \kappa \subseteq \beta\}$ is closed unbounded in κ.

Proof:

unbounded: Let $\beta_0 < \kappa$ be given. We set $\beta_{n+1} := \min\{\eta : \mathrm{Cl}(\mathcal{F};\beta_n) \cap \kappa \subseteq \eta\}$, $\beta := \sup_{n<\omega} \beta_n$.

Since $\forall \eta < \kappa(\mathit{card}(\mathrm{Cl}(\mathcal{F};\eta)) < \kappa)$, we obtain (by induction on n) $\beta_n < \kappa$, and then $\beta_0 \le \beta < \kappa$ and $\mathrm{Cl}(\mathcal{F};\beta) \cap \kappa \subseteq \bigcup_{n<\omega} \mathrm{Cl}(\mathcal{F};\beta_n) \cap \kappa \subseteq \bigcup_{n<\omega} \beta_{n+1} \subseteq \beta$.

closed: If $\beta = \sup(X)\ \&\ \forall \eta{\in}X(\mathrm{Cl}(\mathcal{F};\eta) \cap \kappa \subseteq \eta)$ then $\mathrm{Cl}(\mathcal{F};\beta) \cap \kappa \subseteq \bigcup_{\eta\in X} \mathrm{Cl}(\mathcal{F};\eta) \cap \kappa \subseteq \bigcup_{\eta \in X} \eta = \beta$.

Definition.

Let ν_0 be a fixed countable ordinal.

By transfinite recursion on α we define ordinals $\psi_\sigma(\alpha)$ $(\sigma < \nu_0)$ by

$\psi_\sigma(\alpha) := \min\{\beta \ge \Omega^-_{\sigma+1} : C(\alpha,\beta) \cap \Omega_{\sigma+1} \subseteq \beta\}$ with $C(\alpha,\beta) := \mathrm{Cl}(\{\widetilde{\psi}_\sigma{\restriction}\alpha : \sigma < \nu_0\};\beta)$,

where $\widetilde{\psi}_\sigma{\restriction}\alpha$ is the restriction of ψ_σ to $\{\xi < \alpha : \xi \in C(\xi, \psi_\sigma(\xi))\}$.

Abbreviation. $C_\sigma(\alpha) := C(\alpha, \psi_\sigma(\alpha))$

Remark. $C_\sigma(\alpha) \cap \Omega_{\sigma+1} = \psi_\sigma(\alpha)$

Theorem 8.2 (Basic properties of ψ_σ)

(a) $\psi_\sigma(\alpha) < \Omega_{\sigma+1}$ *(collapsing property)*;

(b) $\psi_\sigma(0) = \Omega^-_{\sigma+1}$, and each $\psi_\sigma(\alpha)$ is an additive principal number;

(c) $\alpha_0 < \alpha_1 \implies \psi_\sigma(\alpha_0) \le \psi_\sigma(\alpha_1)\ \&\ C_\sigma(\alpha_0) \subseteq C_\sigma(\alpha_1)$;

(d) $\alpha_0 < \alpha_1\ \&\ \alpha_0 \in C_\sigma(\alpha_0) \implies \psi_\sigma(\alpha_0) < \psi_\sigma(\alpha_1)$;

(e) $\psi_\sigma(\alpha_0) = \psi_\sigma(\alpha_1)\ \&\ \underbrace{\alpha_0 \in C_\sigma(\alpha_0)\ \&\ \alpha_1 \in C_\sigma(\alpha_1)}_{\textit{normalform condition}} \implies \alpha_0 = \alpha_1$.

Proof:

(a) By L.8.1 $\{\beta < \Omega_{\sigma+1} : \beta \geq \Omega^-_{\sigma+1}$ & $C(\alpha,\beta) \cap \Omega_{\sigma+1} \subseteq \beta\} \neq \emptyset$,
and therefore $\psi_\sigma(\alpha) = \min\{\beta \geq \Omega^-_{\sigma+1} : C(\alpha,\beta) \cap \Omega_{\sigma+1} \subseteq \beta\} < \Omega_{\sigma+1}$.

(b) Obviously $C(0,\Omega^-_{\sigma+1}) = \Omega^-_{\sigma+1}$, and therefore $\psi_\sigma(0) = \Omega^-_{\sigma+1}$.
$\psi_\sigma(\alpha)$ is closed under $+$, since $\psi_\sigma(\alpha) = C_\sigma(\alpha) \cap \Omega_{\sigma+1}$.

(c) $C(\alpha_0, \psi_\sigma(\alpha_1)) \cap \Omega_{\sigma+1} \subseteq C(\alpha_1, \psi_\sigma(\alpha_1)) \cap \Omega_{\sigma+1} = \psi_\sigma(\alpha_1) \;\Rightarrow\; \psi_\sigma(\alpha_0) \leq \psi_\sigma(\alpha_1) \;\Rightarrow\; C_\sigma(\alpha_0) \subseteq C_\sigma(\alpha_1)$.

(d) $\alpha_0 < \alpha_1$ & $\alpha_0 \in C_\sigma(\alpha_0) \overset{(c)}{\Rightarrow} \alpha_0 < \alpha_1$ & $\alpha_0 \in C_\sigma(\alpha_0) \subseteq C_\sigma(\alpha_1) \overset{\text{Def.}C_\sigma + (a)}{\Longrightarrow} \psi_\sigma(\alpha_0) \in C_\sigma(\alpha_1) \cap \Omega_{\sigma+1} = \psi_\sigma(\alpha_1)$.

(e) If $\alpha_0 < \alpha_1$ then the assumption $\alpha_0 \in C_\sigma(\alpha_0)$ together with (d) yields $\psi_\sigma(\alpha_0) < \psi_\sigma(\alpha_1)$.

Lemma 8.3

$C_0(\alpha) = C(\alpha, 1)$

Proof by induction on α:

Let us assume that $C_0(\xi) = C(\xi, 1)$, for all $\xi < \alpha$ (IH). We have to prove $\psi_0(\alpha) \subseteq C(\alpha, 1)$. Let $\alpha > 0$ (otherwise $\psi_0(\alpha) = 1 \subseteq C(\alpha, 1)$). As we will show below, the IH implies that $\beta := C(\alpha, 1) \cap \Omega_1$ is in fact an ordinal. Then $C(\alpha,\beta) \cap \Omega_1 \subseteq C(\alpha, 1) \cap \Omega_1 = \beta$ and thus $\psi_0(\alpha) \leq \beta$, since $\beta \geq 1 = \Omega^-_1$.

Claim: $\gamma \in C(\alpha, 1) \cap \Omega_1 \;\Rightarrow\; \gamma \subseteq C(\alpha, 1)$.

Proof by side induction on the definition of $C(\alpha, 1)$:

1. $\gamma = \psi_0(\xi)$ with $\xi < \alpha$:

By the above IH we have $C_0(\xi) = C(\xi, 1)$. Hence $\gamma = \psi_0(\xi) \subseteq C(\xi, 1) \subseteq C(\alpha, 1)$.

2. $\gamma = \gamma_0 + \gamma_1$ with $\gamma_0, \gamma_1 \in C(\alpha, 1) \cap \Omega_1$:

Then by SIH $\gamma_0, \gamma_1 \subseteq C(\alpha, 1)$ which (together with $\gamma_0 \in C(\alpha, 1)$) yields $\gamma_0 + \gamma_1 \subseteq C(\alpha, 1)$.

Corollary. $\psi_0(\alpha) = C(\alpha, 1) \cap \Omega_1$

Lemma 8.4.

If $\gamma = \gamma_0 + ... + \gamma_n$ with additive principal numbers $\gamma_0 \geq ... \geq \gamma_n$ then: $\gamma \in C(\alpha,\beta) \;\Leftrightarrow\; \gamma_0, ..., \gamma_n \in C(\alpha,\beta)$.

Proof of "$\Rightarrow$" by induction on the definition of $C(\alpha,\beta)$:

For $\gamma < \beta$ or $n = 0$ the claim is trivial. Otherwise $\gamma = \xi + \eta$ with $\xi, \eta \in C(\alpha,\beta)$.

Then $\xi = \gamma_0 + ... + \gamma_{k-1} + \xi_1 + ... + \xi_l$ and $\eta = \gamma_k + ... + \gamma_n$, and the claim follows by I.H.

In order to avoid some technical complications we now assume $\nu_0 = \omega$.

From now on the letters σ, ρ, μ, ν range over numbers $< \omega$.

Below we will introduce a system of ordinal notations based on the ordinal functions ψ_σ. The canonical way for that is to consider the set T of all terms which are generated from the constant 0 by means of function symbols $\oplus, D_0, D_1, ...$ for the ordinal functions $+, \psi_0, \psi_1, ...$. Then one looks for a (primitive) recursive characterization of the relation $<_o := \{(a,b) \in \mathrm{T}\times\mathrm{T} : \mathrm{o}(a) < \mathrm{o}(b)\}$, where $\mathrm{o}(a) \in On$ is the canonical interpretation of $a \in \mathrm{T}$. It turns out that the relation $<_o$ has a particularly simple characterization when it is restricted to the subset $\mathrm{OT} \subseteq \mathrm{T}$ of those terms $a \in \mathrm{T}$ which are in "normalform" (i.e. $\mathrm{o}(b) \in C_\sigma(\mathrm{o}(b))$ for each subterm $D_\sigma b$ of a, and $\mathrm{o}(a_n) \leq \ldots \leq \mathrm{o}(a_0)$ for each subterm $a_0 \oplus \ldots \oplus a_n$ of a).

Now we define the set T of terms, a linear ordering $\prec$ on T, for any $a \in \mathrm{T}$ and $\sigma < \omega$ a set $G_\sigma a$ of subterms of a, and the set OT of *ordinal terms* (i.e. terms in normalform) in such a way that, for all $a, c \in \mathrm{OT}$,
(a) $c \prec a \;\Leftrightarrow\; \mathrm{o}(c) < \mathrm{o}(a)$ and (b) $G_\sigma c \prec a \;\Leftrightarrow\; \mathrm{o}(c) \in C_\sigma(\mathrm{o}(a))$.

(Here $G_\sigma a \prec c$ abbreviates $\forall x \in G_\sigma a (x \prec c)$.)

Inductive definition of T

1. $0 \in \mathrm{T}$.
2. If $a \in \mathrm{T}$ and $\sigma < \omega$, then $D_\sigma a \in \mathrm{T}$; we call $D_\sigma a$ a *principal term.*
3. If $a_0, ..., a_n \in \mathrm{T}$ are principal terms and $n \geq 1$, then $(a_0 \oplus ... \oplus a_n) \in \mathrm{T}$.

Definition. For $a \in \mathrm{T}$ let $\ell(a)$ be the *length* (number of symbols) of a.

Notation. For principal terms $a_0, ..., a_{n-1}$ and $n \geq 0$ we set $a_0 \oplus ... \oplus a_{n-1} := \begin{cases} 0 & \text{if } n = 0 \\ a_0 & \text{if } n = 1 \\ (a_0 \oplus ... \oplus a_{n-1}) & \text{if } n > 1 \end{cases}$

So every $a \in \mathrm{T}$ can be uniquely written as $a = D_{\sigma_0} a_0 \oplus \ldots \oplus D_{\sigma_{n-1}} a_{n-1}$ with $n \geq 0$ and $a_0, ..., a_{n-1} \in \mathrm{T}$.

Further we define: $(a_0 \oplus ... \oplus a_{n-1}) \oplus (b_0 \oplus ... \oplus b_{m-1}) := a_0 \oplus ... \oplus a_{n-1} \oplus b_0 \oplus ... \oplus b_{m-1}$,
and $a \cdot n := \underbrace{a \oplus ... \oplus a}_{n}$ for principal terms a_i, b_i, a.

Definition of $\mathrm{o} : \mathrm{T} \longrightarrow On$

$\mathrm{o}(D_{\sigma_0} a_0 \oplus \ldots \oplus D_{\sigma_{n-1}} a_{n-1}) := \psi_{\sigma_0} \mathrm{o}(a_0) + \ldots + \psi_{\sigma_{n-1}} \mathrm{o}(a_{n-1})$

Definition of $a \prec b$ **for** $a, b \in \mathrm{T}$

1. $0 \prec b \;:\Longleftrightarrow\; b \neq 0$
2. $D_\sigma a \oplus \tilde{a} \prec D_\rho b \oplus \tilde{b} \;:\Longleftrightarrow\; \sigma < \rho$ or ($\sigma = \rho$ & $a \prec b$) or ($\sigma = \rho$ & $a = b$ & $\tilde{a} \prec \tilde{b}$)

Remark. $\prec$ is a linear ordering on T, but it's not a wellordering (e.g. $\ldots \prec D_0 D_0 D_1 0 \prec D_0 D_1 0 \prec D_1 0$).

Abbreviations. For $X, Y \subseteq \mathrm{T}$ and $a \in \mathrm{T}$ let
$X \preceq Y \;:\Leftrightarrow\; \forall x \in X \exists y \in Y (x \preceq y)$;
$X \prec a \;:\Leftrightarrow\; \forall x \in X (x \prec a)$;
$a \preceq X \;:\Leftrightarrow\; \neg(X \prec a)$ $[\Leftrightarrow\; \exists x \in X (a \preceq x)]$.

Definition of $G_\sigma a$

1. $G_\sigma(a_0 \oplus ... \oplus a_{n-1}) := \bigcup_{i<n} G_\sigma a_i$, 2. $G_\sigma D_\mu a := \begin{cases} \{a\} \cup G_\sigma a & \text{if } \sigma \leq \mu \\ \emptyset & \text{if } \mu < \sigma \end{cases}$

Inductive definition of OT

1. $0 \in \mathrm{OT}$.
2. $a \in \mathrm{OT}$ & $G_\sigma a \prec a \;\Rightarrow\; D_\sigma a \in \mathrm{OT}$.
3. $a_0, ..., a_n \in \mathrm{OT}$ $(n \geq 1)$ principal terms with $a_n \preceq \ldots \preceq a_0 \;\Longrightarrow\; (a_0 \oplus ... \oplus a_n) \in \mathrm{OT}$.

The elements of OT are called *ordinal terms.* We identify $n \in \mathbb{N}$ with the ordinal term $\underbrace{D_0 0 \oplus ... \oplus D_0 0}_{n}$.

Abbreviation. $\Omega_0 := \omega := D_0 1$, $\Omega_\sigma := D_\sigma 0$ for $\sigma > 0$.

Theorem 8.5. For $a, c \in \mathrm{OT}$ we have

(a) $c \prec a \;\Leftrightarrow\; \mathrm{o}(c) < \mathrm{o}(a)$;

(b) $G_\sigma c \prec a \;\Leftrightarrow\; \mathrm{o}(c) \in C_\sigma(\mathrm{o}(a))$.

Proof by induction on the length of c simultaneous for (a),(b):

(a) We only prove "$\Rightarrow$". The reverse implication follows from "$\Rightarrow$", since $\prec$ is total.

Let $c = D_\sigma c_0 \oplus c_1 \oplus ... \oplus c_m$, $a = D_\rho a_0 \oplus a_1 \oplus ... \oplus a_n$ with principal terms $c_1, ..., c_m, a_1, ..., a_n$.

1. $\sigma < \rho$: From $c_m \preceq \ldots \preceq c_1 \preceq D_\sigma c_0$ we get by IH $\mathrm{o}(c_m) \leq \ldots \leq \mathrm{o}(c_1) \leq \mathrm{o}(D_\sigma c_0) = \psi_\sigma \mathrm{o}(c_0) < \Omega_{\sigma+1}$ and thus $\mathrm{o}(c) < \Omega_{\sigma+1} \leq \Omega_\rho \leq \mathrm{o}(D_\rho a_0) \leq \mathrm{o}(a)$.

2. $\sigma = \rho$ and $c_0 \prec a_0$: By IH $\mathrm{o}(c_0) < \mathrm{o}(a_0)$. Since $D_\sigma c_0 \in \mathrm{OT}$, we have $G_\sigma c_0 \prec c_0$ and thus by IH $\mathrm{o}(c_0) \in C(\mathrm{o}(c_0), \psi_\sigma \mathrm{o}(c_0))$. Hence $\psi_\sigma \mathrm{o}(c_0) < \psi_\sigma \mathrm{o}(a_0)$ by Theorem 8.2d. Now $\mathrm{o}(c) \prec \mathrm{o}(a)$ follows as in 1. (using that $\psi_\sigma \mathrm{o}(a)$ is additively closed).

3. $\sigma = \rho \;\&\; c_0 = a_0 \;\&\; c_1 \oplus ... \oplus c_m \prec a_1 \oplus ... \oplus a_n$: Immediate by IH.

(b) Abbreviation: $C := C_\sigma(\mathrm{o}(a))$.

1. $c = c_0 \oplus ... \oplus c_n$: Then $\mathrm{o}(c_0), ..., \mathrm{o}(c_n)$ are additive principal numbers with $\mathrm{o}(c_0) \geq ... \geq \mathrm{o}(c_n)$ by I.H.(a). $G_\sigma c \prec a \;\Leftrightarrow\; \bigwedge_{i \leq n} G_\sigma c_i \prec a \overset{\mathrm{IH}}{\Leftrightarrow} \bigwedge_{i \leq n} \mathrm{o}(c_i) \in C \overset{\mathrm{L.8.4}}{\Leftrightarrow} \mathrm{o}(c) \in C$.

2. $c = D_\mu c_0$ with $\mu < \sigma$: Then $G_\sigma c = \emptyset$ and $\mathrm{o}(c) \in \Omega_{\mu+1} \subseteq \Omega_\sigma \subseteq C$.

3. $c = D_\mu c_0$ with $\sigma \leq \mu$: Then $G_\mu c_0 \prec c_0$ and therefore by I.H. $\mathrm{o}(c_0) \in C_\mu(\mathrm{o}(c_0))$ $(*)$.

"$\Rightarrow$": $\{c_0\} \cup G_\sigma c_0 = G_\sigma c \prec a \overset{\mathrm{IH}}{\Rightarrow} \mathrm{o}(c_0) < \mathrm{o}(a) \;\&\; \mathrm{o}(c_0) \in C \overset{(*)}{\Rightarrow} \mathrm{o}(c) = \psi_\mu \mathrm{o}(c_0) \in C$.

"$\Leftarrow$": 1. $\sigma = \mu$: $\psi_\sigma \mathrm{o}(c_0) = \mathrm{o}(c) \in C \cap \Omega_{\sigma+1} = \psi_\sigma \mathrm{o}(a) \;\Rightarrow\; \mathrm{o}(c_0) < \mathrm{o}(a) \overset{\mathrm{IH}}{\Rightarrow} c_0 \prec a \overset{G_\mu c_0 \prec c_0}{\Longrightarrow} G_\sigma c \prec a$.

2. $\sigma < \mu$: $\psi_\sigma \mathrm{o}(a) < \Omega_{\sigma+1} \leq \mathrm{o}(c) \in C \;\Rightarrow\; \psi_\mu(\mathrm{o}(c_0)) = \mathrm{o}(c) = \psi_\mu(\xi)$ for some $\xi \in C$ with $\xi < \mathrm{o}(a)$ and $\xi \in C_\mu(\xi) \overset{(*)+\mathrm{Th.8.2e}}{\Longrightarrow} \mathrm{o}(c_0) = \xi < \mathrm{o}(a) \;\&\; \mathrm{o}(c_0) = \xi \in C \overset{\mathrm{IH}}{\Rightarrow} G_\sigma(c) = \{c_0\} \cup G_\sigma c_0 \prec a$.

Corollary. $(\mathrm{OT}, \prec)$ is a wellordering.

Definition. $\mathrm{OT}_0 := \{a \in \mathrm{OT} : a \prec D_1 0\}$

Theorem 8.6.

(a) $\{\mathrm{o}(a) : a \in \mathrm{OT}\} = C(\Omega_\omega, 1)$, and the mapping $\mathrm{o}{\restriction}\mathrm{OT}$ is injective.

(b) $\{\mathrm{o}(a) : a \in \mathrm{OT}_0\} = \psi_0(\Omega_\omega) = \|\mathrm{OT}_0, \prec\|$ (= order type of the wellordering $(\mathrm{OT}_0, \prec)$).

Proof:

(a) 1. Obviously $C(\Omega_\omega, 1) \subseteq \Omega_\omega$ (+). 2. From 8.5a it follows that $\mathrm{o}{\restriction}\mathrm{OT}$ is injective.

3. By induction on the definition of $C(\Omega_\omega, 1)$ we prove: $\gamma \in C(\Omega_\omega, 1) \Rightarrow \exists a \in \mathrm{OT}(\gamma = \mathrm{o}(a))$.

3.1. $\gamma = 0$: trivial.

3.2. $\gamma = \xi + \eta$: $\xi = \mathrm{o}(a_0 \oplus ... \oplus a_n)$, $\eta = \mathrm{o}(b_0 \oplus ... \oplus b_m))$ with $a_i, b_i \in \mathrm{OT}$ and $a_0 \succeq \ldots \succeq a_n$, $b_0 \succeq \ldots \succeq b_m$. Then, for some $k \leq n{+}1$, $a := a_0 \oplus ... \oplus a_{k-1} \oplus b_0 \oplus ... \oplus b_m \in \mathrm{OT}$ and $\gamma = \mathrm{o}(a)$.

3.3. $\gamma = \psi_\sigma(\xi)$ with $\xi < \Omega_\omega \;\&\; \xi \in C(\Omega_\omega, 1) \;\&\; \xi \in C_\sigma(\xi)$:
By IH we have $b \in \mathrm{OT}$ with $\xi = \mathrm{o}(b)$. $\mathrm{o}(b) \in C_\sigma(\mathrm{o}(b)) \overset{8.5}{\Rightarrow} G_\sigma b \prec b \Rightarrow D_\sigma b \in \mathrm{OT}$ and $\gamma = \mathrm{o}(D_\sigma b)$.

4. $a \in \mathrm{OT} \;\Rightarrow\; \mathrm{o}(a) \in C(\Omega_\omega, 1)$: Proof by induction on $\ell(a)$, using 8.2a and 8.5b.
$a = D_\sigma b \in \mathrm{OT} \Rightarrow b \in \mathrm{OT} \;\&\; G_\sigma b \prec b \overset{\mathrm{IH}}{\Rightarrow} \mathrm{o}(b) \in C_\sigma(\Omega_\omega, 1) \;\&\; \mathrm{o}(b) \in C_\sigma(\mathrm{o}(b)) \overset{(+)}{\Rightarrow} \mathrm{o}(a) = \psi_\sigma(\mathrm{o}(b)) \in C(\Omega_\omega, 1)$.

(b) Note that $\mathrm{o}(D_1 0) = \psi_1(0) \overset{8.2b}{=} \Omega_1$. Therefore (and by 8.5a, 8.6a), $a \mapsto \mathrm{o}(a)$ maps OT_0 order preserving onto $C(\Omega_\omega, 1) \cap \Omega_1 \overset{8.3}{=} C_0(\Omega_\omega) \cap \Omega_1 = \psi_0(\Omega_\omega)$. This implies $\|\mathrm{OT}_0, \prec\| = \psi_0(\Omega_\omega)$.

Fundamental sequences

In order to get a better insight into the structure $(\mathrm{T}, \prec)$ and a better understanding of the collapsing functions ψ_σ we now present an assignment of (fundamental) sequences to the elements of T. For each term $a \in \mathrm{T}$ we define its *(cofinality) type* $\mathsf{tp}(a) \in \{0, 1, \omega\} \cup \{\Omega_{\mu+1} : \mu < \omega\}$ and a family $(a[x])_{x\in|\mathsf{tp}(a)|}$ of terms $a[x] \in \mathrm{T}$, such that the following holds, where $|0| := \emptyset$, $|1| := \{0\}$, $|\omega| := \mathbb{N}$, $|\Omega_{\mu+1}| := \{D_\mu b : b \in \mathrm{T}\}$:

Theorem 8.7.

(a) $x \in |\mathsf{tp}(a)| \implies a[x] \prec a$

(b) $x, x' \in |\mathsf{tp}(a)| \;\&\; x \prec x' \implies a[x] \prec a[x']$

(c) $\mathsf{tp}(a) = 1 \implies a = a[0] \oplus 1$

(d) $a, c \in \mathrm{OT} \;\&\; c \prec a \;\&\; \mathsf{tp}(a) \neq 1 \implies \exists x \in \mathrm{OT} \cap |\mathsf{tp}(a)|(\, c \prec a[x]\,)$

(e) $a, x \in \mathrm{OT} \;\&\; x \in |\mathsf{tp}(a)| \implies a[x] \in \mathrm{OT}$

Note that, according to Theorem 8.7, only for $a \in \mathrm{OT}$ and only relative to $(\mathrm{OT}, \prec)$ is the family $(a[x])_{x\in|\mathsf{tp}(a)|}$ a fundamental sequence of a in the proper sense. But later we will give a natural interpretation of the terms $a \in \mathrm{T}$ as wellfounded trees (so-called *tree ordinals*) which harmonizes with the assignment $(a, x) \mapsto a[x]$.

Definition of $\mathsf{tp}(a)$ and $a[x]$ for $a \in \mathrm{T}$, $x \in |\mathsf{tp}(a)|$

1. $\mathsf{tp}(0) := 0$.
2. $\mathsf{tp}(D_0 0) := 1,\;\; (D_0 0)[0] := 0$.
3. $\mathsf{tp}(D_{\mu+1} 0) := \Omega_{\mu+1},\;\; (D_{\mu+1}0)[x] := x$.
4. $\mathsf{tp}(a) = 1 \;\Rightarrow\; \mathsf{tp}(D_\sigma a) := \omega,\;\; (D_\sigma a)[i] := (D_\sigma a[0]) \cdot (i{+}1)$.
5. $\mathsf{tp}(a) \in \{\omega\} \cup \{\Omega_{\mu+1} : \mu < \sigma\} \;\Rightarrow\; \mathsf{tp}(D_\sigma a) := \mathsf{tp}(a),\;\; (D_\sigma a)[x] := D_\sigma a[x]$.
6. $\mathsf{tp}(a) = \Omega_{\mu+1} \;\&\; \mu \geq \sigma \;\Rightarrow\; \mathsf{tp}(D_\sigma a) := \omega,\;\; (D_\sigma a)[i] := D_\sigma a[x_i]$ with $x_0 := \Omega_\mu$, $x_{n+1} := D_\mu a[x_n]$.
7. $\mathsf{tp}(a_0 {\oplus} \ldots {\oplus} a_n) := \mathsf{tp}(a_n),\;\; (a_0 {\oplus} \ldots {\oplus} a_n)[x] := (a_0 {\oplus} \ldots {\oplus} a_{n-1}) \oplus a_n[x] \qquad (n \geq 1)$.

For technical reasons we also set $a[n] := a[0]$, if $\mathsf{tp}(a) = 1$.

Proof of Theorem 8.7:

(a),(b),(c) are easily verified by induction on $\ell(a)$.

(d) is also proved by induction on $\ell(a)$. Here all cases except 5., 6. are straightforward. So let us assume $D_\sigma c \oplus \tilde{c} \prec D_\sigma a$. Then $c \prec a$, and by IH $c \prec a[x]$ for some $x \in \mathrm{OT} \cap |\mathsf{tp}(a)|$. Hence $D_\sigma c \oplus \tilde{c} \prec D_\sigma a[x]$.
If $\mathsf{tp}(a) \in \{\omega\} \cup \{\Omega_{\mu+1} : \mu < \sigma\}$, we have $(D_\sigma a)[x] = D_\sigma a[x]$ and we are done.

Now let us assume that $\mathsf{tp}(a) = \Omega_{\mu+1}$ with $\mu \geq \sigma$.

By induction on $\ell(a)$ one can prove

(1) $\mathsf{tp}(a) = \Omega_{\mu+1} \;\&\; c \in \mathrm{OT} \;\&\; a[\Omega_\mu] \preceq c \prec a \implies \exists x = D_\mu(b{\oplus}1) \in \mathrm{OT}(\, b \in G_\mu c \;\&\; c \prec a[x]\,)$

from which we conclude

(2) $\mathsf{tp}(a) = \Omega_{\mu+1} \;\&\; c \in \mathrm{OT} \;\&\; a[\Omega_\mu] \preceq c \;\&\; \{c\} \cup G_\mu c \prec a \implies$

$$\implies \exists b \in \mathrm{OT}(\, \ell(b) < \ell(c) \;\&\; \{b\} \cup G_\mu b \prec a \;\&\; c \prec a[D_\mu(b{\oplus}1)])$$

Obviously (2) suggests to define $x_0 := \Omega_\mu$, $x_{n+1} := D_\mu a[x_n]$ in order to obtain by induction on $\ell(c)$

(3) $\mathsf{tp}(a) = \Omega_{\mu+1}$ & $c \in \mathrm{OT}$ & $\{c\} \cup G_\mu c \prec a \implies \exists n(c \prec a[x_n])$.

(Induction step: premise & $a[x_0] \preceq c \overset{(2)}{\Rightarrow} \exists b \in \mathrm{OT}(\ \ell(b) < \ell(c)$ & $\{b\} \cup G_\mu b \prec a$ & $c \prec a[D_\mu(b\oplus 1)]) \overset{\mathrm{IH}}{\Rightarrow}$
$\Rightarrow \exists b \in \mathrm{OT}\exists n(\ b \prec a[x_n]$ & $c \prec a[D_\mu(b\oplus 1)]\,) \Rightarrow \exists b \in \mathrm{OT}\exists n(c \prec a[D_\mu(b\oplus 1)] \preceq a[D_\mu a[x_n]] = a[x_{n+1}])\ )$

Now $\exists n(D_\sigma c \oplus \tilde{c} \prec (D_\sigma a)[n])$ is obtained as follows:
$\mathrm{OT} \ni D_\sigma c \oplus \tilde{c} \prec D_\sigma a \overset{\sigma\le\mu}{\Longrightarrow} G_\mu c \subseteq G_\sigma c \prec c \prec a \overset{(3)}{\Rightarrow} \exists n(D_\sigma c \oplus \tilde{c} \prec D_\sigma a[x_n] = (D_\sigma a)[n])$.

Proof of (1):
1. $a = \Omega_{\mu+1}$: Then $c = D_\mu c_0 \oplus \tilde{c} \prec D_\mu(c_0\oplus 1)$. Let $b := c_0$.
2. $a = a_0 \oplus a_1$ with $\mathsf{tp}(a_1) = \Omega_{\mu+1}$:
Then $c = a_0 \oplus c_1$ with $a_1[\Omega_\mu] \preceq c_1 \prec a_1$, and the claim follows immediately from the IH.
3. $a = D_\rho a_0$ with $\mu < \rho$: Then $\mathsf{tp}(a_0) = \Omega_{\mu+1}$ and $a[x] = D_\rho a_0[x]$.
Further $c = D_\rho c_0 \oplus \tilde{c}$ with $a_0[\Omega_\mu] \preceq c_0 \prec a_0$.
By IH we get $c_0 \prec a_0[x]$ for some $x = D_\mu(b\oplus 1) \in \mathrm{OT}$ with $b \in G_\mu c_0$.
Since $\mu < \rho$, $G_\mu c_0 \subseteq G_\mu c$. From $c_0 \prec a_0[x]$ we get $c \prec D_\rho a_0[x] = a[x]$.

For the proof of 8.7e we need some preparations.

Definition.
$b \prec\!\!\prec_x a \ :\Longleftrightarrow\ \ b \prec a$ & $\forall\sigma\forall c(b \preceq c \preceq a \Rightarrow G_\sigma b \preceq G_\sigma c \cup G_\sigma x)$.

Lemma 8.8.
$b \prec\!\!\prec_x a$ & $G_\sigma a \prec a$ & $G_\sigma x \prec b \implies G_\sigma b \prec b$.

Proof: We have $G_\sigma b \preceq G_\sigma a \cup G_\sigma x \prec a$.
Assumption: $b \preceq G_\sigma b$. Then there exists a subterm d of b with minimal length such that $b \preceq G_\sigma d \prec a$. By the minimality of d we have $d = D_\mu c$ with $G_\sigma c \prec b \preceq c \prec a$. Using $b \prec\!\!\prec_x a$ and $G_\sigma x \prec b$ we obtain $G_\sigma b \preceq G_\sigma c \cup G_\sigma x \prec b$. Contradiction.

Lemma 8.9.
$b_0 \prec\!\!\prec_x b \implies a \oplus b_0 \prec\!\!\prec_x a \oplus b$ and $D_\mu b_0 \prec\!\!\prec_x D_\mu b$.

Proof:
1. Suppose $a \oplus b_0 \preceq c \preceq a \oplus b$. Then $c = a \oplus c_0$ with $b_0 \preceq c_0 \preceq b$.
Hence $G_\sigma(a \oplus b_0) = G_\sigma a \cup G_\sigma b_0 \preceq G_\sigma a \cup G_\sigma c \cup G_\sigma x = G_\sigma c \cup G_\sigma x$.
2. Suppose $D_\mu b_0 \preceq c \preceq D_\mu b$. Then $c = (D_\mu c_0) \oplus c_1$ with $b_0 \preceq c_0 \preceq b$. If $\mu < \sigma$, then $G_\sigma(D_\mu b_0) = \emptyset$.
Now let $\mu \ge \sigma$. Using the premise $b_0 \prec\!\!\prec_x b$ we obtain $G_\sigma b_0 \preceq G_\sigma c_0 \cup G_\sigma x$, and then
$G_\sigma(D_\mu b_0) = \{b_0\} \cup G_\sigma b_0 \preceq \{c_0\} \cup G_\sigma c_0 \cup G_\sigma x \subseteq G_\sigma c \cup G_\sigma x$.

Lemma 8.10.
(a) $a \in \mathrm{T}$ & $\mathsf{tp}(a) = \Omega_{\mu+1}$ & $x \in |\mathsf{tp}(a)| \implies a[x] \prec\!\!\prec_x a$.
(b) $a \in \mathrm{T}$ & $\mathsf{tp}(a) \in \{1, \omega\} \implies a[j] \prec\!\!\prec_0 a$

Proof by induction on $\ell(a)$:

(a) By 8.7a we have $a[x] \prec a$. — Suppose $a[x] \preceq c \preceq a$. We have to prove $G_\rho a[x] \preceq G_\rho c \cup G_\rho x$.

1. $a = D_0 0$ or $a = D_{\mu+1} 0$: trivial.

2. $a = D_\sigma b$ with $\mathsf{tp}(b) = \Omega_{\mu+1}$ & $\mu < \sigma$:

By I.H. we get $b[x] \prec\!\!\prec_x b$ and then $a[x] = D_\sigma b[x] \prec\!\!\prec_x D_\sigma b = a$ by 8.9.

3. $a = (a_0 \oplus ... \oplus a_n)$ $(n \geq 1)$:

By I.H. we get $a_n[x] \prec\!\!\prec_x a_n$ and then $a[x] = (a_0 \oplus ... \oplus a_{n-1}) \oplus a_n[x] \prec\!\!\prec_x (a_0 \oplus ... \oplus a_{n-1}) \oplus a_n = a$ by 8.9.

(b) By 8.7a we have $a[j] \prec a$. — Suppose $a[j] \preceq c \preceq a$. We have to prove $G_\rho a[j] \preceq G_\rho c$.

1. $a = D_\sigma b$ with $\mathsf{tp}(b) = 1$: Then $a[j] = (D_\sigma b[0]) \cdot (j{+}1)$ and $G_\rho a[j] = G_\rho(D_\sigma b[0])$.

By I.H. and 3.5 we get $D_\sigma b[0] \prec\!\!\prec_0 D_\sigma b = a$.

We also have $D_\sigma b[0] \preceq c \preceq a$ and therefore $G_\rho(D_\sigma b[0]) \preceq G_\rho c$.

2. $a = D_\sigma b$ and $\mathsf{tp}(b) = \Omega_{\mu+1}$ with $\sigma \leq \mu$: Then $a[j] = D_\sigma b[x_j]$ with $x_0 = \Omega_\mu$, $x_{i+1} = D_\mu b[x_i]$.

Suppose that $\rho \leq \sigma$, since otherwise $G_\rho a[j] = \emptyset$.

From $a[j] \preceq c \preceq a$ it follows that $c = (D_\sigma c_0) \oplus c_1$ with $b[x_j] \preceq c_0 \preceq b$.

By (a) we have $\forall i (b[x_i] \prec\!\!\prec_{x_i} b)$. By side induction on i we prove $G_\rho b[x_i] \preceq G_\rho c_0 \cup \{1\}$ for $i \leq j$:

$$b[x_i] \prec\!\!\prec_{x_i} b \;\&\; b[x_i] \preceq c_0 \preceq b \;\Rightarrow\; G_\rho b[x_i] \preceq G_\rho c_0 \cup G_\rho x_i \overset{(*)}{\preceq} \{c_0\} \cup G_\rho c_0 \cup \{1\}.$$

$$(*) \quad G_\rho x_i \begin{cases} \subseteq \{0,1\} & \text{if } i = 0 \\ = \{b[x_{i-1}]\} \cup G_\rho b[x_{i-1}] \overset{\text{SIH}}{\preceq} \{c_0\} \cup G_\rho c_0 \cup \{1\} & \text{if } i > 0 \end{cases}$$

Now we obtain $G_\rho a[j] = \{b[x_j]\} \cup G_\rho b[x_j] \preceq \{c_0\} \cup G_\rho c_0 \cup \{1\} \overset{(+)}{\preceq} G_\rho c$. [(+) $b[0] \prec b[x_0] \preceq c_0 \Rightarrow 1 \preceq c_0$.]

3. $a = D_\sigma b$ with $\mathsf{tp}(b) = \omega$ or $a = (a_0 \oplus ... \oplus a_n)$ $(n \geq 1)$ with $\mathsf{tp}(a_n) = \omega$: as in (a).

Proof of 8.7e by induction on $\ell(a)$:

1. $a = (a_0 \oplus ... \oplus a_n) \in \mathrm{OT}$ $(n \geq 1)$: Then $a_0, ..., a_n \in \mathrm{OT}$ and $a_n[x] \prec a_n \preceq \ldots \preceq a_0$.

By I.H. we have $a_n[x] \in \mathrm{OT}$. Hence $a[x] = (a_0 \oplus ... \oplus a_{n-1}) \oplus a_n[x] \in \mathrm{OT}$.

2. $a = D_\sigma b \in \mathrm{OT}$: Then $b \in \mathrm{OT}$ and $G_\sigma b \prec b$.

2.1. $\mathsf{tp}(b) = 1$: By I.H. and 8.10 we obtain $b[0] \in \mathrm{OT}$ and $b[0] \prec\!\!\prec_0 b$.

From $b[0] \prec\!\!\prec_0 b$ and $G_\sigma b \prec b$ we get $G_\sigma b[0] \prec b[0]$ by 8.8. Hence $a[x] = (D_\sigma b[0]) \cdot (x{+}1) \in \mathrm{OT}$.

2.2. $a = D_\sigma b$ with $\mathsf{tp}(b) \in \{\omega\} \cup \{\Omega_{\mu+1} : \mu < \sigma\}$: By I.H. and 8.10 we have $b[x] \in \mathrm{OT}$ and $b[x] \prec\!\!\prec_x b$.

Since $x \in |\mathsf{tp}(b)|$ with $\mathsf{tp}(b) \in \{\omega\} \cup \{\Omega_{\mu+1} : \mu < \sigma\}$, we have $G_\sigma x \prec b[x]$.

By 8.8 from $b[x] \prec\!\!\prec_x b \;\&\; G_\sigma b \prec b \;\&\; G_\sigma x \prec b[x]$ we get $G_\sigma b[x] \prec b[x]$. Hence $a[x] = D_\sigma b[x] \in \mathrm{OT}$.

2.3. $\mathsf{tp}(b) = \Omega_{\mu+1}$ with $\sigma \leq \mu$: Then $a[x] = D_\sigma b[x_j]$ with $x_0 = \Omega_\mu$, $x_{i+1} = D_\mu b[x_i]$.

We have to show $D_\sigma b[x_j] \in \mathrm{OT}$.

(1) $G_\mu b[x_i] \subseteq G_\sigma b[x_i]$ [since $\sigma \leq \mu$]

(2) $\forall i (x_i \in \mathrm{OT} \Rightarrow b[x_i] \in \mathrm{OT})$ [by I.H.]

(3) $G_\sigma x_i \prec b[x_i] \;\Rightarrow\; G_\sigma b[x_i] \prec b[x_i]$ [$b[x_i] \prec\!\!\prec_{x_i} b \;\&\; G_\sigma b \prec b \;\&\; G_\sigma x_i \prec b[x_i] \overset{8.8}{\Rightarrow} G_\sigma b[x_i] \prec b[x_i]$]

(4) $G_\sigma x_i \prec b[x_i]$

[Side Ind. on i: $G_\sigma x_0 \subseteq \{0,1\} \prec b[x_0]$. $G_\sigma(x_{i+1}) = \{b[x_i]\} \cup G_\sigma b[x_i] \overset{\text{SIH}+(3)}{\preceq} b[x_i] \prec b[x_{i+1}]$]

(5) $x_i \in \mathrm{OT}$ and $a[i] \in \mathrm{OT}$.

Proof by induction on i: 1. $x_0 = \Omega_\mu \in \mathrm{OT}$.

2. $x_i \in \mathrm{OT} \overset{(2)-(4)}{\Rightarrow} b[x_i] \in \mathrm{OT}$ & $G_\sigma b[x_i] \prec b[x_i] \overset{(1)}{\Rightarrow} x_{i+1} = D_\mu b[x_i] \in \mathrm{OT}$ and $a[i] = D_\sigma b[x_i] \in \mathrm{OT}$.

Lemma 8.11

$0 \neq a \in \mathrm{OT}_0 \implies \mathsf{tp}(a) \in \{1, \omega\}$ and $\mathrm{o}(a) = \begin{cases} \mathrm{o}(a[0]) + 1 & \text{if } \mathsf{tp}(a) = 1 \\ \sup_{n\in\mathbb{N}}(\mathrm{o}(a[n])+1) & \text{if } \mathsf{tp}(a) = \omega \end{cases}$

Proof:

We only show that $\mathrm{o}(a) = \sup_{n\in\mathbb{N}}(\mathrm{o}(a[n])+1)$ if $\mathsf{tp}(a) = \omega$. Let $a \in \mathrm{OT}_0$ with $\mathsf{tp}(a) = \omega$.

Then $a[n] \prec a$ & $a[n] \in \mathrm{OT}_0$ (by 8.7a,e) and $\mathrm{o}(a[n]) < \mathrm{o}(a)$ (by 8.5a). Now let $\gamma < \mathrm{o}(a)$. Then (by 8.6b) $\gamma = \mathrm{o}(c)$ for some $c \in \mathrm{OT}_0$ with $c \prec a$. Theorem 8.7d yields $c \prec a[n]$ for some $n \in \mathbb{N}$. Hence $\gamma < \mathrm{o}(a[n])$.

Lemma 8.12.

(a) $\alpha < \varepsilon_0 \Rightarrow \alpha \in C_0(\alpha)$ & $\psi_0(\alpha) = \omega^\alpha$.

(b) $0 < \sigma$ & $\alpha < \varepsilon_{\Omega_\sigma+1} \Rightarrow \alpha \in C_\sigma(\alpha)$ & $\psi_\sigma(\alpha) = \omega^{\Omega_\sigma+\alpha} = \Omega_\sigma \cdot \omega^\alpha$.

Proof: Exercise.

Lemma 8.13.

$a \in \mathrm{OT} \implies (a \prec D_0\Omega_{\nu+1} \Leftrightarrow$ no D_σ with $\sigma > \nu$ occurs in $a)$.

Proof:

"$\Rightarrow$": $a = D_0a_1 \oplus ... \oplus D_0a_n$ with $G_0a_i \prec a_i \prec \Omega_{\nu+1}$.

By induction on $\ell(a)$ one obtains: $a \in \mathrm{T}$ & $\{a\} \cup G_0a \prec \Omega_{\nu+1} \Rightarrow a \in \mathrm{T}(\nu) := \mathrm{T}(0, \oplus, D_0, ..., D_\nu)$.

$[a = D_\sigma a_0$ & $\{a\} \cup G_0a \prec \Omega_{\nu+1} = D_{\nu+1}0 \Rightarrow \sigma < \nu{+}1$ & $\{a_0\} \cup G_0a_0 = G_0a \prec \Omega_{\nu+1}.\ ]$

"$\Leftarrow$": left to the reader.

Tree ordinals

Inductive definition of classes $\mathbb{T}_\sigma$ of tree ordinals

1. $\mathbf{0} := (\,) \in \mathbb{T}_\sigma$
2. $\boldsymbol{\alpha} \in \mathbb{T}_\sigma \Rightarrow \boldsymbol{\alpha}{+}\mathbf{1} := (\boldsymbol{\alpha}) \in \mathbb{T}_\sigma$
3. $\forall n \in \mathbb{N}(\boldsymbol{\alpha}_n \in \mathbb{T}_\sigma) \Rightarrow (\boldsymbol{\alpha}_n)_{n\in\mathbb{N}} \in \mathbb{T}_\sigma$
4. $\mu < \sigma$ & $\forall \boldsymbol{\xi} \in \mathbb{T}_\mu(\boldsymbol{\alpha}_{\boldsymbol{\xi}} \in \mathbb{T}_\sigma) \Rightarrow (\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\mathbb{T}_\mu} \in \mathbb{T}_\sigma$

$\mathbb{T}_{<\omega} := \bigcup_{\sigma<\omega} \mathbb{T}_\sigma$. The elements of $\mathbb{T}_{<\omega}$ are called *tree ordinals* (denoted by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$).

Note

Every $\boldsymbol{\alpha} \in \mathbb{T}_\sigma$ is of the form $(\boldsymbol{\alpha}_\iota)_{\iota\in I}$ with $I \in \{\emptyset, \{0\}, \mathbb{N}\} \cup \{\mathbb{T}_\mu : \mu < \sigma\}$.

We define $\|(\boldsymbol{\alpha}_\iota)_{\iota\in I}\| := \sup_{\iota\in I}(\|\boldsymbol{\alpha}_\iota\| + 1)$.

Abbreviations

$\overline{0} := \mathbf{0}$, $\overline{n{+}1} := \overline{n}{+}\mathbf{1}$, $\mathbf{1} := \overline{1}$, $\boldsymbol{\Omega}_0 := (\overline{n})_{n\in\mathbb{N}}$, $\boldsymbol{\Omega}_{\mu+1} := (\boldsymbol{\xi})_{\boldsymbol{\xi}\in\mathbb{T}_\mu}$

Definition of $\boldsymbol{\alpha} + \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \cdot n$

$\boldsymbol{\alpha} + \mathbf{0} := \boldsymbol{\alpha}$, $\boldsymbol{\alpha} + (\boldsymbol{\beta}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in I} := (\boldsymbol{\alpha} + \boldsymbol{\beta}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in I}$ if $I \neq \emptyset$,

$\boldsymbol{\alpha} \cdot 0 := \mathbf{0}$, $\boldsymbol{\alpha} \cdot (n{+}1) := (\boldsymbol{\alpha} \cdot n) + \boldsymbol{\alpha}$

Proposition. (a) $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{T}_\sigma \Rightarrow \boldsymbol{\alpha} + \boldsymbol{\beta} \in \mathbb{T}_\sigma$, (b) $\boldsymbol{\alpha} + (\boldsymbol{\beta} + \boldsymbol{\gamma}) = (\boldsymbol{\alpha} + \boldsymbol{\beta}) + \boldsymbol{\gamma}$

Definition of $\mathbb{D}_\sigma : \mathbb{T}_{<\omega} \longrightarrow \mathbb{T}_\sigma$

The definition of $\mathbb{D}_\sigma(\boldsymbol{\alpha})$ proceeds by transf. rec. on $\boldsymbol{\alpha}$ simultaneously for all $\sigma < \omega$.

$\mathbb{D}_0(\mathbf{0}) := \mathbf{1}$, $\mathbb{D}_\sigma(\mathbf{0}) := \boldsymbol{\Omega}_\sigma$ if $\sigma \neq 0$

$\mathbb{D}_\sigma(\boldsymbol{\alpha}{+}\mathbf{1}) := \big(\mathbb{D}_\sigma(\boldsymbol{\alpha}) \cdot (n{+}1)\big)_{n\in\mathbb{N}}$

$$\mathbb{D}_\sigma((\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in I}) := \begin{cases} \big(\mathbb{D}_\sigma(\boldsymbol{\alpha}_{\boldsymbol{\xi}})\big)_{\boldsymbol{\xi}\in I} & \text{if } I \in \{\mathbb{N}\} \cup \{\mathbb{T}_\mu : \mu{<}\sigma\} \\ \big(\mathbb{D}_\sigma(\boldsymbol{\alpha}_{\boldsymbol{\xi}_n})\big)_{n\in\mathbb{N}} & \text{if } I = \mathbb{T}_\mu \text{ with } \mu \geq \sigma \\ \text{with } \boldsymbol{\xi}_0 := \boldsymbol{\Omega}_\mu,\ \boldsymbol{\xi}_{n+1} := \mathbb{D}_\mu(\boldsymbol{\alpha}_{\boldsymbol{\xi}_n}) & \end{cases}$$

Remark

For $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in I} \in \mathbb{T}_\sigma \setminus \{0\}$ we have $\mathbb{D}_\sigma(\boldsymbol{\alpha}) = \begin{cases} \big(\mathbb{D}_\sigma(\boldsymbol{\alpha}_0) \cdot (n{+}1)\big)_{n\in\mathbb{N}} & \text{if } I = \{0\} \\ \big(\mathbb{D}_\sigma(\boldsymbol{\alpha}_{\boldsymbol{\xi}})\big)_{\boldsymbol{\xi}\in I} & \text{otherwise} \end{cases}$.

This means that on $\mathbb{T}_\sigma$ the function $\mathbb{D}_\sigma$ behaves like the ordinal function $\alpha \mapsto \omega^{\Omega_\sigma+\alpha}$ (if $\sigma > 0$) or $\alpha \mapsto \omega^\alpha$ (if $\sigma = 0$).

Now we are going to prove that $\|\mathbb{D}_0\mathbb{D}_\nu^{(m)}(\mathbf{0})\|$ equals $\psi_0\psi_\nu^{(m)}(0)$. By comparing the definition of $\mathbb{D}_\sigma$ with the assignment of fundamental sequences above and taking Theorems 8.5, 8.7 into consideration this should be more or less clear. To obtain a rigorous proof we introduce the canonical interpretation $\mathbf{t} : \mathrm{T} \longrightarrow \mathbb{T}_{<\omega}$ and show that this respects the fundamental sequences $(a[x])_{x\in|\mathsf{tp}(a)}$.

Definition of $\mathbf{t} : \mathrm{T} \longrightarrow \mathbb{T}_{<\omega}$**.** $\mathbf{t}(D_{\sigma_0}a_0{\oplus}...{\oplus}D_{\sigma_{n-1}}a_{n-1}) := \mathbb{D}_{\sigma_0}\mathbf{t}(a_1) + \ldots + \mathbb{D}_{\sigma_{n-1}}\mathbf{t}(a_{n-1})$

Theorem 8.14. For each $a \in \mathrm{T}$ we have

(i) $\mathsf{tp}(a) = 1 \Rightarrow \mathbf{t}(a) = \mathbf{t}(a[0]) + \mathbf{1}$,

(ii) $\mathsf{tp}(a) = \omega \Rightarrow \mathbf{t}(a) = (\mathbf{t}(a[n]))_{n\in\mathbb{N}}$,

(iii) $\mathsf{tp}(a) = \Omega_{\mu+1} \Rightarrow \mathbf{t}(a) = (\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\mathbb{T}_\mu}$ with $\forall x \in |\Omega_{\mu+1}|(\mathbf{t}(a[x]) = \boldsymbol{\alpha}_{\mathbf{t}(x)})$

Proof:

Let $\mathcal{FS}(a)$ abbreviate the claim (i)&(ii)&(iii). Then in a straightforward way one proves

(1) $\mathcal{FS}(a)$ & $\mathcal{FS}(b) \implies \mathcal{FS}(b{\oplus}a)$, (2) $\mathcal{FS}(a) \implies \mathcal{FS}(D_\sigma a)$,

from which one obtains $(\forall a \in \mathrm{T})\mathcal{FS}(a)$ by induction on $\ell(a)$.

Theorem 8.15. $a \in \mathrm{OT}_0 \implies \mathrm{o}(a) = \|\mathbf{t}(a)\|$

Proof by induction on $\mathrm{o}(a)$:

Let $a \neq 0$. By L.8.11 $\mathsf{tp}(a) \in \{1, \omega\}$ and $\mathrm{o}(a) = \begin{cases} \mathrm{o}(a[0]) + 1 & \text{if } \mathsf{tp}(a) = 1 \\ \sup_{n\in\mathbb{N}}(\mathrm{o}(a[n]){+}1) & \text{if } \mathsf{tp}(a) = \omega \end{cases}$.

If $\mathsf{tp}(a) = \omega$ then $\mathbf{t}(a) \overset{8.14}{=} (\mathbf{t}(a[n]))_{n\in\mathbb{N}}$ and therefore

$\|\mathbf{t}(a)\| = \sup_{n\in\mathbb{N}}(\|\mathbf{t}(a[n])\|{+}1) \overset{\mathrm{IH}}{=} \sup_{n\in\mathbb{N}}(\mathrm{o}(a[n]) + 1) \overset{\mathrm{L.8.11}}{=} \mathrm{o}(a)$

The case "$\mathsf{tp}(a) = 1$" is treated in the same way.

Corollary. $\|\mathbb{D}_0\mathbb{D}_\nu^{(m)}(\mathbf{0})\| = \psi_0\psi_\nu^{(m)}(0)$.

Proof:

$\|\mathbb{D}_0\mathbb{D}_\nu^{(m)}(\mathbf{0})\| \overset{\mathrm{Def}}{=} \|\mathbf{t}(D_0D_\nu^{(m)}0)\| \overset{8.14}{=} \mathrm{o}(D_0D_\nu^{(m)}0) \overset{\mathrm{Def}}{=} \psi_0\psi_\nu^{(m)}(0)$.

§11 Wellfoundedness proofs in ID_ν

$\mathrm{T}_0 := \{0\} \cup \{D_0 a_0 \oplus ... \oplus D_0 a_n : a_0, ..., a_n \in \mathrm{T},\ n \geq 0\}$.

Let $\nu < \omega$ be fixed.

$\mathfrak{A}_\sigma(X,a)\ :\Leftrightarrow\ \mathsf{tp}(a) \in \{0,1,\omega\} \cup \{\Omega_{\mu+1} : \mu < \sigma\}\ \&\ \forall x \in W(\mathsf{tp}(a))(a[x] \in X)$,
where $W(0) := \emptyset$, $W(1) := \{0\}$, $W(\omega) := \mathbb{N}$, $W(\Omega_{\mu+1}) := |\Omega_{\mu+1}| \cap W_\mu = \{D_\mu b \in W_\mu : b \in \mathrm{T}\}$.

$\mathfrak{A}_\sigma(X) := \{x \in \mathrm{T} : \mathfrak{A}_\sigma(X,x)\}$.

$W_\sigma := \bigcap\{X \subseteq \mathrm{T} : \mathfrak{A}_\sigma(X) \subseteq X\}$

$X^{(a)} := \{y \in \mathrm{T} : a \oplus y \in X\}$

$\overline{X} := \{y \in \mathrm{T} : \forall x \in X(x \oplus D_\nu y \in X)\}$

$W^* := \{x \in \mathrm{T} : \forall \sigma < \nu(D_\sigma x \in W_\sigma\}$

Lemma 11.1.
(a) $\mathfrak{A}_\sigma(X) \subseteq X\ \&\ a \in X \implies \mathfrak{A}_\sigma(X^{(a)}) \subseteq X^{(a)}\ \ (\sigma \leq \nu)$.
(b) $a, b \in W_\sigma \implies a \oplus b \in W_\sigma\ \ (\sigma < \nu)$.

Lemma 11.2. $\mathfrak{A}_\nu(X) \subseteq X \implies \mathfrak{A}_\nu(\overline{X}) \subseteq \overline{X}$.

Lemma 11.3. $\mathfrak{A}_\nu(W^*) \subseteq W^*$.

Lemma 11.4. If $a \in \mathrm{T}$ contains no symbol D_σ with $\sigma > \nu$, then $\mathfrak{A}_\nu(X) \subseteq X \to a \in X$.

Lemma 11.5. If $a \in \mathrm{T}_0$ contains no symbol D_σ with $\sigma > \nu$, then $a \in W_0$.

Theorem 11.6. $|\mathrm{ID}_\nu| = \psi_0(\varepsilon_{\Omega_\nu+1})$.

Proof: 1. By Corollary 10.5 we have $|\mathrm{ID}_\nu| \leq \psi_0(\varepsilon_{\Omega_\nu+1})$.

2. Let $\alpha < \psi_0(\varepsilon_{\Omega_\nu+1})$; then $\alpha < \|\mathbb{D}_0(\mathbb{D}_\nu^{(m)}(\mathbf{0}))\|$ for some m.

Let $a := D_0 D_\nu^{(m)} 0$. As shown above, $\mathrm{ID}_\nu \vdash a \in W_0$; hence $\alpha < \|\mathbb{D}_0(\mathbb{D}_\nu^{(m)}(\mathbf{0}))\| = \|\mathbf{t}(a)\| = |a|_{W_0} \leq |\mathrm{ID}_\nu|$.

§9 Theories for Iterated Inductive Definitions

Definition

Let M be a set, and $\Phi : \mathfrak{P}(M) \to \mathfrak{P}(M)$ *monotone*, i.e. $\forall X, Y \in \mathfrak{P}(M)(X \subseteq Y \Rightarrow \Phi(X) \subseteq \Phi(Y))$.

$\mathrm{I}_\Phi := \bigcap\{X \in \mathfrak{P}(M) : \Phi(X) \subseteq X\}$ *(the intersection of all Φ-closed subsets of M)*

We say that the set I_Φ is *inductively defined by* Φ.

Definitions of this kind are called *(generalized) inductive definitions.*

Lemma 9.1

(a) $\Phi(X) \subseteq X \implies \mathrm{I}_\Phi \subseteq X$, for each set $X \subseteq M$.

(b) $\Phi(\mathrm{I}_\Phi) = \mathrm{I}_\Phi$. So, I_Φ is the least Φ-closed set and also the least fixpoint of Φ.

(c) $\mathrm{I}_\Phi \cap \Phi(X) \subseteq X \implies \mathrm{I}_\Phi \subseteq X$.

Proof:

(a) trivial.

(b) HS: $\Phi(\mathrm{I}_\Phi) \subseteq \mathrm{I}_\Phi$. Proof: $\forall X(\Phi(X) \subseteq X \Rightarrow \mathrm{I}_\Phi \subseteq X) \stackrel{\Phi \text{ mon.}}{\Rightarrow} \forall X(\Phi(X) \subseteq X \Rightarrow \Phi(\mathrm{I}_\Phi) \subseteq \Phi(X) \subseteq X) \Rightarrow \Phi(\mathrm{I}_\Phi) \subseteq \bigcap\{X : \Phi(X) \subseteq X\} = \mathrm{I}_\Phi$. — HS $\Rightarrow$ $Y := \Phi(\mathrm{I}_\Phi) \subseteq \mathrm{I}_\Phi \Rightarrow \Phi(Y) \subseteq \Phi(\mathrm{I}_\Phi) = Y \stackrel{\text{(a)}}{\Rightarrow} \mathrm{I}_\Phi \subseteq Y = \Phi(\mathrm{I}_\Phi)$.

(c) $\mathrm{I}_\Phi \cap \Phi(X) \subseteq X \Rightarrow \Phi(\mathrm{I}_\Phi \cap X) \subseteq \Phi(\mathrm{I}_\Phi) \cap \Phi(X) = \mathrm{I}_\Phi \cap X \stackrel{\text{(a)}}{\Rightarrow} \mathrm{I}_\Phi \subseteq \mathrm{I}_\Phi \cap X \subseteq X$.

Definition. $\mathrm{I}_\Phi^\alpha := \Phi(\mathrm{I}_\Phi^{<\alpha})$ with $\mathrm{I}_\Phi^{<\alpha} := \bigcup_{\xi<\alpha} \mathrm{I}_\Phi^\xi$ $(\alpha \in On)$

Lemma 9.2.

(a) $\alpha < \beta \Rightarrow \mathrm{I}_\Phi^\alpha \subseteq \mathrm{I}_\Phi^\beta$; (b) $\mathrm{I}_\Phi^{\alpha+1} = \Phi(\mathrm{I}_\Phi^\alpha)$;

(c) $\mathrm{I}_\Phi^{<\alpha} = \mathrm{I}_\Phi^\alpha$ for some $\alpha \in On$; (d) If $\mathrm{I}_\Phi^{<\alpha} = \mathrm{I}_\Phi^\alpha$, then $\forall \beta \geq \alpha(\mathrm{I}_\Phi^\beta = \mathrm{I}_\Phi)$.

Proof:

(a) trivial.

(b) $\mathrm{I}_\Phi^{\alpha+1} = \Phi(\mathrm{I}_\Phi^{<\alpha+1}) \stackrel{\text{(a)}}{=} \Phi(\mathrm{I}_\Phi^\alpha)$.

(c) Otherwise $F : On \to \mathfrak{P}(M)$, $\alpha \mapsto \mathrm{I}_\Phi^\alpha$ would be injective. But then $\mathfrak{P}(M)$ would not be a set.

(d) 1. By induction on β we get $\mathrm{I}_\Phi^\beta \subseteq \mathrm{I}_\Phi$ for all β: I.H. $\Rightarrow \mathrm{I}_\Phi^{<\beta} \subseteq \mathrm{I}_\Phi \Rightarrow \mathrm{I}_\Phi^\beta = \Phi(\mathrm{I}_\Phi^{<\beta}) \subseteq \Phi(\mathrm{I}_\Phi) = \mathrm{I}_\Phi$.
2. $\mathrm{I}_\Phi^{<\alpha} = \mathrm{I}_\Phi^\alpha = \Phi(\mathrm{I}_\Phi^{<\alpha}) \Rightarrow \mathrm{I}_\Phi \subseteq \mathrm{I}_\Phi^{<\alpha} \subseteq \mathrm{I}_\Phi^\beta$ for $\beta \geq \alpha$.

Definition.

For each relation $R \subseteq M \times M$ let $\Phi_R : \mathfrak{P}(M) \to \mathfrak{P}(M)$, $\Phi_R(X) := \{x \in M : \forall y R x (y \in X)\}$.

$\mathrm{Acc}(M, R) := \mathrm{I}_{\Phi_R}$ *(the accessible part of (M, R))*

$(\mathrm{Acc}(M, R) = \bigcap\{X \subseteq M : \forall x \in M(\forall y R x(y \in X) \Rightarrow x \in X)\})$

Lemma 9.4.

Let R be a binary relation on M, and $\mathrm{Acc} := \mathrm{Acc}(M, R)$.

(a) $\forall x[x \in M \;\&\; \forall y R x(y \in \mathrm{Acc}) \Leftrightarrow x \in \mathrm{Acc}]$.

(b) $\forall x \in \mathrm{Acc}[\forall y R x(y \in X) \Rightarrow x \in X] \implies \mathrm{Acc} \subseteq X$, for every $X \subseteq M$. ($R{\restriction}\mathrm{Acc}$ is wellfounded)

(c) R wellfounded $\iff$ $M = \mathrm{Acc}$.

Proof:

(a) follows from 9.1b. (b) follows from 9.1c.

(c) "$\Rightarrow$": By (a) we have $\forall x \in M(\forall yRx(y \in \mathrm{Acc}) \Rightarrow x \in \mathrm{Acc})$.

By R-induction from this we get $\forall x \in M(x \in \mathrm{Acc})$.

"$\Leftarrow$": follows from (b).

Definition. For $x \in \mathrm{I}_\Phi$ let $|x|_\Phi := \min\{\alpha : x \in \mathrm{I}_\Phi^\alpha\}$

Lemma 9.5. If $\Phi = \Phi_R$ then $|x|_\Phi = \sup\{|y|_\Phi{+}1 : yRx\}$ for every $x \in \mathrm{Acc}(M, R)$.

Proof:

$x \in \mathrm{I}_\Phi^\alpha \Leftrightarrow x \in \Phi(\mathrm{I}_\Phi^{<\alpha}) \Leftrightarrow \forall yRx(y \in \mathrm{I}_\Phi^{<\alpha}) \Leftrightarrow \forall yRx(|y|_\Phi < \alpha)$.

Hence $|x|_\Phi = \min\{\alpha : x \in \mathrm{I}_\Phi^\alpha\} = \min\{\alpha : \forall yRx(|y|_\Phi < \alpha)\} = \sup\{|y|_\Phi{+}1 : yRx\}$

Syntax

If P is a unary predicate symbol, and A, F are formulas, and $\mathcal{F} = \lambda xF$ then $A(P/\mathcal{F})$ denotes the result of substituting $\mathcal{F}$ for P in A, i.e. the formula resulting from A be replacing every atom Pt by $\mathcal{F}(t)$.

Definition. Let $\mathcal{L}$ be 1st-order language, and X a set variable (unary predicate symbol) not in $\mathcal{L}$. A *positive operator form in* $\mathcal{L}$ is an $\mathcal{L} \cup \{X\}$-formula $\mathfrak{A}$ in which X occurs only positively (i.e. $\mathfrak{A}$ has no subformula $\neg Xt$) and which has at most one free variable x.

We use the following abbreviations: $\mathfrak{A}(\mathcal{F}, t) := \mathfrak{A}(X/\mathcal{F}, x/t)$, $\mathfrak{A}(\mathcal{F}) \subseteq \mathcal{F} := \forall x(\mathfrak{A}(\mathcal{F}, x) \to \mathcal{F}(x))$.

For each positive operator form $\mathfrak{A}$ we introduce a (new) unary predicate symbol $P_\mathfrak{A}$.

Definition of the languages $\mathcal{L}_\sigma$ $(0 \le \sigma < \omega)$

$\mathcal{L}_0 := \mathrm{PR} \cup \{=\}$, the language of arithmetic as so far.

$\mathcal{L}_{\sigma+1} := \mathcal{L}_0 \cup \{P_\mathfrak{A} : \mathfrak{A}$ positive operator form in $\mathcal{L}_\sigma\}$

$\mathcal{L}_{<\omega} := \bigcup_{\sigma<\omega} \mathcal{L}_\sigma$

Remark. $\mathcal{L}_\sigma \subseteq \mathcal{L}_{\sigma+1}$.

Definition of $\mathrm{lev}(A)$

$\mathrm{lev}(A) := 0$ if A is an $\mathcal{L}_0[X]$-literal

$\mathrm{lev}(P_\mathfrak{A}t) := \mathrm{lev}(\mathfrak{A})$, $\mathrm{lev}(\neg P_\mathfrak{A}t) := \mathrm{lev}(\mathfrak{A}) + 1$

$\mathrm{lev}(A \wedge B) := \mathrm{lev}(A \vee B) := \max\{\mathrm{lev}(A), \mathrm{lev}(B)\}$

$\mathrm{lev}(\forall xA) := \mathrm{lev}(\exists xA) := \mathrm{lev}(A)$

$\mathrm{lev}(P_\mathfrak{A}) := \mathrm{lev}(\mathfrak{A})$, $\mathrm{lev}(\Gamma) := \max\{\mathrm{lev}(A) : A \in \Gamma\}$

Remark

$\mathrm{lev}(P_\mathfrak{A}) < \sigma$ for each predicate symbol $P_\mathfrak{A}$ in $\mathcal{L}_\sigma$,

$\mathrm{lev}(A) \le \sigma$ for each $\mathcal{L}_\sigma$-formula A.

From now on A, B, C denote $\mathcal{L}_{<\omega}$-formulas.

The proof system ID_ν

The language of ID_ν is $\mathcal{L}_\nu$.

The inference symbols of ID_ν are those of Z (in the language $\mathcal{L}_\nu$) together with

$(\mathrm{Cl}_{P_{\mathfrak{A}}t})\ \ \dfrac{\mathfrak{A}(P_{\mathfrak{A}},t)}{P_{\mathfrak{A}}t} \qquad\qquad (\mathsf{Ind}^{P_{\mathfrak{A}},t}_{\mathcal{F}})\ \ \dfrac{}{\neg(\mathfrak{A}(\mathcal{F}){\subseteq}\mathcal{F}),\neg P_{\mathfrak{A}}t,\mathcal{F}(t)}$

The infinitary proof systems $\mathrm{ID}^\infty_\sigma$ ($\sigma<\omega$)

The language of $\mathrm{ID}^\infty_\sigma$ consists of all closed $\mathcal{L}_{<\omega}$-formulas A.

We use P [P_μ, resp.] as syntactic variable for the predicate symbols $P_{\mathfrak{A}}$ [with $\mathrm{lev}(\mathfrak{A})=\mu$, resp.].

Definition

$\mathcal{AX}(\mathrm{ID}^\infty_\sigma)$:= set of all sequents Δ such that

– all elements of Δ are closed literals,

– $\Delta\cap\mathsf{TRUE}_0\neq\emptyset$ or Δ contains a subset $\{Ps,\neg Pt\}$ with $s^{\mathcal{N}}=t^{\mathcal{N}}$ and $\mathrm{lev}(P)<\sigma$.

The inference symbols of $\mathrm{ID}^\infty_\sigma$ are the following

(Ax_Δ) Δ if $\Delta\in\mathcal{AX}(\mathrm{ID}^\infty_\sigma)$, and the symbols $(\bigwedge_A)$, $(\bigvee^t_A)$, (Cut_C) of Z in the language $\mathcal{L}_{<\omega}$;

$(\mathrm{Cl}_{P_{\mathfrak{A}}t})\ \ \dfrac{\mathfrak{A}(P_{\mathfrak{A}},t)}{P_{\mathfrak{A}}t}\ \ (\mathrm{lev}(P_{\mathfrak{A}})\leq\sigma) \qquad\qquad (\Omega_{Pt})\ \ \dfrac{Pt\qquad \ldots\Delta^{Pt}_q\ldots(q\in|P|)}{\emptyset}\qquad (\mathrm{lev}(P)<\sigma)$

$|P|$:= set of all cutfree ID^∞_μ-derivations, where $\mu:=\mathrm{lev}(P)$, $\Delta^{Pt}_q:=\Gamma(q)\setminus\{Pt\}$

The set $\mathsf{ID}^\infty_\sigma$ of all $\mathrm{ID}^\infty_\sigma$-derivations is introduced by an inductive definition (as given in §4 for arbitrary proof systems $\mathfrak{S}$) under the assumption that the sets ID^∞_μ for $\mu<\sigma$ are already defined. — $\mathsf{ID}^\infty_{<\omega}:=\bigcup_{\sigma<\omega}\mathsf{ID}^\infty_\sigma$.

As usual we write $\mathsf{ID}^\infty_\sigma\ni d\vdash_m\Gamma$ to express that d is a derivation (in $\mathsf{ID}^\infty_\sigma$) with $\Gamma(d)\subseteq\Gamma$ and $\mathsf{crk}(d)\leq m$. (For the definition of $\mathsf{crk}(d)$ (*cut-rank* of d) cf. §4.)

The (Ω_{Pt})-rule can be motivated as follows (with $\mu:=\mathrm{lev}(P)<\sigma$):

Imitating the constructive interpretation of implication we start by saying:

> *"An $\mathrm{ID}^\infty_\sigma$-derivation of $Pt\to B$ is an operation $q\mapsto d_q$ transforming every cutfree ID^∞_μ-derivation of Pt into an $\mathrm{ID}^\infty_\sigma$-derivation of B ".*

This may be replaced by the stricter version:

> *"An $\mathrm{ID}^\infty_\sigma$-derivation of $Pt\to B$ is an operation $q\mapsto d_q$ transforming every cutfree ID^∞_μ-derivation of $A\to Pt$ into an $\mathrm{ID}^\infty_\sigma$-derivation of $A\to B$ (for any formula A)".*

In terms of the Tait-calculus used here this amounts to the following rule:

$(\widetilde{\Omega}_{Pt})$ *If for each Δ and each cutfree ID^∞_μ-derivation q of Δ,Pt , d_q is an $\mathrm{ID}^\infty_\sigma$-derivation of Δ,Γ, then $(d_q)_{q\in|P|}$ is an $\mathrm{ID}^\infty_\sigma$-derivation of $\neg Pt,\Gamma$ ".*

Now (Ω_{Pt}) is just a combination of $(\widetilde{\Omega}_{Pt})$ and (Cut_{Pt}).

The following definitions and Theorem 9.6 are needed for the embedding of ID_ν into ID_ν^∞, i.e., for deriving $\neg(\mathfrak{A}(\mathcal{F})\subseteq\mathcal{F}), \neg P_{\mathfrak{A}}t, \mathcal{F}(t)$ by means of $(\Omega_{P_{\mathfrak{A}}t})$.

Definitions (Substitution)

For each closed $\mathcal{L}_\sigma$-formula A let $\boldsymbol{c}_A$ be the canonical cutfree $\mathrm{ID}_\sigma^\infty$-derivation of $\neg A, A$.

$$\boldsymbol{e}^t_{\mathfrak{A},\mathcal{F}} := \bigvee\nolimits^t_G \bigwedge\nolimits_{\mathfrak{A}(\mathcal{F},t)\wedge\neg\mathcal{F}(t)} \boldsymbol{c}_{\mathfrak{A}(\mathcal{F},t)} \boldsymbol{c}_{\mathcal{F}(t)} \approx \dfrac{\dfrac{\neg\mathfrak{A}(\mathcal{F},t),\mathfrak{A}(\mathcal{F},t) \qquad \neg\mathcal{F}(t),\mathcal{F}(t)}{\mathfrak{A}(\mathcal{F},t)\wedge\neg\mathcal{F}(t),\neg\mathfrak{A}(\mathcal{F},t),\mathcal{F}(t)}}{G,\neg\mathfrak{A}(\mathcal{F},t),\mathcal{F}(t)} \quad \text{with } G := \neg(\mathfrak{A}(\mathcal{F})\subseteq\mathcal{F}).$$

Given $P = P_{\mathfrak{A}}$, a predicate $\mathcal{F}$, and a sequent Π we define an operation $\mathcal{S}^\Pi_{P,\mathcal{F}} : \mathsf{ID}^\infty_{\mathrm{lev}(P)} \to \mathsf{ID}^\infty_{<\omega}$ which transforms any derivation $d \in \mathsf{ID}^\infty_{\mathrm{lev}(P)}$ of Γ,Π into a derivation $d^* := \mathcal{S}^\Pi_{P,\mathcal{F}}(d)$ of $G,\Gamma,\Pi(P/\mathcal{F})$. Roughly speaking d^* results from d by substituting certain occurrences of P by $\mathcal{F}$. In doing so, some inferences $(\mathrm{Cl}_{Pt})\dfrac{\mathfrak{A}(P,t)}{Pt}$ are turned into $\dfrac{\mathfrak{A}(\mathcal{F},t)}{\mathcal{F}(t)}$ which is not an inference of $\mathrm{ID}^\infty_{<\omega}$.

Therefore those inferences (Cl_{Pt}) are replaced by
$$\dfrac{\overset{d_0^*}{G,\mathfrak{A}(\mathcal{F},t)} \qquad \overset{\boldsymbol{e}^t_{\mathfrak{A},\mathcal{F}}}{G,\neg\mathfrak{A}(\mathcal{F},t),\mathcal{F}(t)}}{G,\mathcal{F}(t)}\,(Cut)$$

The precise definition of $\mathcal{S}^\Pi_{P,\mathcal{F}}(d)$ runs as follows

$$\mathcal{S}^\Pi_{P,\mathcal{F}}(\mathcal{I}(d_\iota)_{\iota\in I}) := \begin{cases} \mathsf{Cut}_{\mathfrak{A}(\mathcal{F},t)}\mathcal{S}^{\Pi\cup\Delta_0(\mathcal{I})}_{P,\mathcal{F}}(d_0)\boldsymbol{e}^t_{\mathfrak{A},\mathcal{F}} & \text{if } \mathcal{I} = \mathrm{Cl}_{Pt} \text{ with } Pt\in\Pi \\ \mathcal{I}^*\big(\mathcal{S}^{\Pi\cup\Delta_\iota(\mathcal{I})}_{P,\mathcal{F}}(d_\iota)\big)_{\iota\in I} & \text{if } \mathcal{I} = \bigwedge_A \text{ or } \bigvee^t_A \text{ with } A\in\Pi \\ \mathcal{I}\big(\mathcal{S}^\Pi_{P,\mathcal{F}}(d_\iota)\big)_{\iota\in I} & \text{otherwise} \end{cases}$$

where $(\bigwedge_A)^* := \bigwedge_{A(P/\mathcal{F})}$, $(\bigvee^t_A)^* := \bigvee^t_{A(P/\mathcal{F})}$.

Abbreviation. $A[\iota] := \begin{cases} A_\iota & \text{if } A = A_0 \overset{\vee}{\wedge} A_1 \text{ and } \iota\in\{0,1\} \\ B(x/\iota) & \text{if } A = \overset{\exists}{\forall}xB \text{ and } \iota\in T \end{cases}$

The following theorem is easily verified. Note that the axioms $\mathsf{Ax}_{\{\neg Pt,Pt\}}$ do not belong to $\mathrm{ID}^\infty_{\mathrm{lev}(P)}$!

Theorem 9.6

$\mathrm{ID}^\infty_{\mathrm{lev}(P)} \ni d \vdash_0 \Gamma,\Pi \;\&\; \mathrm{rk}(\mathfrak{A}(\mathcal{F},x)) < m \implies \mathcal{S}^\Pi_{P,\mathcal{F}}(d) \vdash_m \neg(\mathfrak{A}(\mathcal{F})\subseteq\mathcal{F}),\Gamma,\Pi(P/\mathcal{F})$.

Proof:

Abb.: $G := \neg\mathfrak{A}(\mathcal{F})\subseteq\mathcal{F}$, $A^* := A(P/\mathcal{F})$, $\Pi^* := \Pi(P/\mathcal{F}) = \{A^* : A\in\Pi\}$, $(\bigwedge_A)^* := \bigwedge_{A^*}$, $(\bigvee^t_A)^* := \bigvee^t_{A^*}$

Let $\mathrm{ID}^\infty_{\mathrm{lev}(P)} \ni d = \mathcal{I}(d_\iota)_{\iota\in I} \vdash_0 \Gamma,\Pi$. Then $\Delta(\mathcal{I})\subseteq\Gamma,\Pi$ and $\mathrm{ID}^\infty_{\mathrm{lev}(P)} \ni d_\iota \vdash_0 \Gamma,\Pi,\Delta_\iota(\mathcal{I})$ for all $\iota\in I$.

1. $\mathcal{I} = \mathrm{Cl}_{Pt}$ with $Pt\in\Pi$: By IH $\mathcal{S}^{\Pi\cup\Delta_0(\mathcal{I})}(d_0) \vdash_m G,\Gamma,\Pi^*,\mathfrak{A}(\mathcal{F},t)$.
By definition $\boldsymbol{e}^t_{\mathfrak{A},\mathcal{F}} \vdash_0 G,\neg\mathfrak{A}(\mathcal{F},t),\mathcal{F}(t)$. Hence $\mathcal{S}^\Pi(d)\vdash_m G,\Gamma,\Pi^*$, since $\mathcal{F}(t)\in\Pi^*$.

2. $\mathcal{I} = \bigwedge_A$ or $\bigvee^t_A$ with $A\in\Pi$: Let $\Pi_\iota := \Pi\cup\Delta_\iota(\mathcal{I})$. Then by IH $\mathcal{S}^{\Pi_\iota}(d_\iota)\vdash_m G,\Gamma,\Pi^*_\iota$.
Now $\Delta_\iota(\mathcal{I})^* = \{A[\iota]^*\} = \{A^*[\iota]\} = \Delta_\iota(\mathcal{I}^*)$ and $\Delta(\mathcal{I}^*) = \{A^*\}\subseteq\Pi^*$.
Hence $\mathcal{S}^{\Pi_\iota}(d_\iota)\vdash_m G,\Gamma,\Pi^*,\Delta_\iota(\mathcal{I}^*)$ $(\forall\iota\in I)$ and thus $\mathcal{S}^\Pi(d) = \mathcal{I}^*\big(\mathcal{S}^{\Pi_\iota}(d_\iota)\big)_{\iota\in I}\vdash_m G,\Gamma,\Pi^*$.

3. Otherwise: Then $\Delta(\mathcal{I})\subseteq\Gamma$ or $A = A^*$ for all $A\in\Delta(\mathcal{I})$. Hence $\Delta(\mathcal{I})\subseteq\Gamma,\Pi^*$.
By IH $\mathcal{S}^\Pi(d_\iota)\vdash_m G,\Gamma,\Pi^*,\Delta_\iota(\mathcal{I})$, for all $\iota\in I$. Hence $\mathcal{S}^\Pi(d) = \mathcal{I}\big(\mathcal{S}^\Pi(d_\iota)\big)_{\iota\in I}\vdash_m G,\Gamma,\Pi^*$.

[[If $\Delta(\mathcal{I})\not\subseteq\Gamma$, then $\mathcal{I} = \mathsf{Ax}_\Delta$ with ($\Delta\cap\mathsf{TRUE}_0\neq\emptyset$ or $\{\neg P't,P's\}\subseteq\Delta$ with $\mathrm{lev}(P')<\mathrm{lev}(P)$) or $\mathcal{I} = \mathrm{Cl}_{P't}$ with $\mathrm{lev}(P')\leq\mathrm{lev}(P)$ & $P'\neq P$.]]

Embedding of ID_ν into ID_ν^∞

For each closed ID_ν-derivation h we define an ID_ν^∞-derivation h^∞ such that $h^\infty \vdash_m \Gamma(h)$ for some $m \in \mathbb{N}$.

0. $(\mathsf{Ax}_\Delta)^\infty := \mathsf{Ax}_\Delta$

1. $(\bigwedge^x_{\forall x A} h_0)^\infty := \bigwedge_{\forall x A}\big(h_0(x/t)^\infty\big)_{t\in T_0}$

2. $(\mathsf{Ind}_F^{x,t} h_0)^\infty := \begin{cases} \boldsymbol{c}_F^{x,t} & \text{if } n = 0 \\ \mathsf{Cut}_{F(\underline{n})}\boldsymbol{e}_{n-1}\boldsymbol{c}_F^{x,t} & \text{if } n > 0 \end{cases}$ where

   $n := t^{\mathcal{N}}$, $\boldsymbol{c}_F^{x,t} \vdash_0^{2\mathrm{rk}(F)} \neg F_x(\underline{n}), F_x(t)$, $\boldsymbol{e}_0 := h_0(x/0)^\infty$, $\boldsymbol{e}_i := \mathsf{Cut}_{F(\underline{i})}\boldsymbol{e}_{i-1}h_0(x/\underline{i})^\infty$ for $i > 0$.

3. $(\mathsf{Ind}_{\mathcal{F}}^{P,t})^\infty := \dfrac{\mathsf{Ax}_{\{\neg Pt, Pt\}} \quad \ldots \mathcal{S}^{\{Pt\}}_{P,\mathcal{F}}(q) \ldots (q \in |Pt|)}{\Omega_{Pt}}$

4. Otherwise: $(\mathcal{I}h_0 ... h_{n-1})^\infty := \mathcal{I}h_0^\infty \ldots h_{n-1}^\infty$

Theorem 9.7 (Embedding)
$\mathrm{ID}_\nu \ni h \vdash \Gamma$ & h closed $\implies$ $\mathrm{ID}_\nu^\infty \ni h^\infty \vdash_m \Gamma$ for some $m \in \mathbb{N}$.

Proof: straightforward.

Especially $(\mathsf{Ind}_{\mathcal{F}}^{P,t})^\infty \vdash_m \neg(\mathfrak{A}(\mathcal{F})\subseteq\mathcal{F}), \neg Pt, \mathcal{F}(t)$ (where $P = P_{\mathfrak{A}}$) is obtained from:

$q \in |Pt| \;\Rightarrow\; \mathrm{ID}^\infty_{\mathrm{lev}(P)} \ni q \vdash_0 \Delta_q^{Pt}, Pt \overset{\text{Theorem 9.6}}{\Rightarrow} \mathcal{S}^{\{Pt\}}_{P,\mathcal{F}}(q) \vdash_m \neg(\mathfrak{A}(\mathcal{F})\subseteq\mathcal{F}), \Delta_q^{Pt}, \mathcal{F}(t).$

Abbreviations

$\bigwedge$-For := set of all formulas of the shape $A \wedge B$ or $\forall x A$.

$\bigwedge^+$-For := $\mathsf{TRUE}_0 \cup \bigwedge$-For $\cup$ set of all formulas $P_{\mathfrak{A}}t$.

Theorem 9.8
By tree recursion one can define operations $\mathcal{J}_C^t$, $\mathcal{R}_C$, $\mathcal{E}$, $\mathcal{D}_\sigma$ on $\mathsf{ID}^\infty_{<\omega}$ with the following properties:

($\bigwedge$-Inversion) $d \vdash_m \Gamma, C$ & $C \in \bigwedge$-For $\implies$ $\mathcal{J}_C^t(d) \vdash_m C[t]$.

(Reduction) $e \vdash_m \Gamma, C$ & $d \vdash_m \Gamma, \neg C$ & $C \in \bigwedge^+$-For & $\mathrm{rk}(C) \le m$ $\implies$ $\mathcal{R}_C(e,d) \vdash_m \Gamma$.

(Elimination) $d \vdash_{m+1} \Gamma \implies \mathcal{E}(d) \vdash_m \Gamma$.

(Collapsing) $d \vdash_0 \Gamma$ & $\mathrm{lev}(\Gamma) \le \sigma \Rightarrow \mathrm{ID}^\infty_\sigma \ni \mathcal{D}_\sigma(d) \vdash_0 \Gamma$.

Proof:

For $d = \mathcal{I}(d_\iota)_{\iota\in I} \in \mathsf{ID}^\infty_{<\omega}$ and $e \in \mathsf{ID}^\infty_{<\omega}$ we define

$$\mathcal{J}_C^t(d) := \begin{cases} \mathcal{J}_C^t(d_t) & \text{if } \mathcal{I} = \bigwedge_C \\ \mathcal{I}\big(\mathcal{J}_C^t(d_\iota)\big)_{\iota\in I} & \text{otherwise} \end{cases} \qquad (C \in \textstyle\bigwedge\text{-For})$$

$$\mathcal{R}_C(e,d) := \begin{cases} \mathsf{Cut}_{C[t]}\mathcal{J}_C^t(e)\mathcal{R}_C(e,d_0) & \text{if } \mathcal{I} = \bigvee^t_{\neg C} \\ e & \text{if } \mathcal{I} = \mathsf{Ax}_{\{\neg C, C\}} \\ \mathcal{I}\big(\mathcal{R}_C(e,d_\iota)\big)_{\iota\in I} & \text{otherwise (i.e., if } \neg C \notin \Delta(\mathcal{I})) \end{cases} \qquad (C \in \textstyle\bigwedge^+\text{-For})$$

$$\mathcal{E}(d) := \begin{cases} \mathcal{R}_C(\mathcal{E}(d_0),\mathcal{E}(d_1)) & \text{if } \mathcal{I} = \mathsf{Cut}_C \text{ with } C \in \bigwedge^+\text{-For} \\ \mathcal{R}_{\neg C}(\mathcal{E}(d_1),\mathcal{E}(d_0)) & \text{if } \mathcal{I} = \mathsf{Cut}_C \text{ with } \neg C \in \bigwedge^+\text{-For}\,. \\ \mathcal{I}\big(\mathcal{E}(d_\iota)\big)_{\iota\in I} & \text{otherwise} \end{cases}$$

$$\mathcal{D}_\sigma(d) := \begin{cases} \mathcal{D}_\sigma(d_{\mathcal{D}_\mu(d_0)}) & \text{if } \mathcal{I} = \Omega_{Pt} \text{ with } \mu := \text{lev}(P) \geq \sigma \\ \mathcal{I}\big(\mathcal{D}_\sigma(d_\iota)\big)_{\iota\in I} & \text{otherwise} \end{cases}$$

One easily verifies that the so defined operations have the asserted properties.

Let us look at $\mathcal{D}_\sigma(d)$ for $d = \Omega_{Pt}(d_q)_{q\in\{0\}\cup|P|} \vdash_0 \Gamma$ with $\text{lev}(\Gamma) \leq \sigma \leq \mu := \text{lev}(P)$.

Then $d_0 \vdash_0 \Gamma, Pt$ and $d_q \vdash_0 \Gamma, \Delta_q^{Pt}$ for all $q \in |P|$ (†).

By IH $\text{ID}_\mu^\infty \ni q_0 := \mathcal{D}_\mu(d_0) \vdash_0 \Gamma, Pt$. Hence $q_0 \in |P|$ and $\Delta_{q_0}^{Pt} \subseteq \Gamma$.

Now (†) yields $d_{q_0} \vdash_0 \Gamma$, and by IH we get $\text{ID}_\sigma^\infty \ni \mathcal{D}_\sigma(d_{q_0}) \vdash_0 \Gamma$.

Remark: The definition of $\mathcal{D}_\sigma(d)$ almost automatically arises if one pursues the goal to eliminate from d all Ω_P-inferences with $\text{lev}(P) \geq \sigma$.

Definition

For $\mathfrak{A}$ with $\text{lev}(\mathfrak{A}) = 0$ let $\text{I}_\mathfrak{A}^\alpha := \{n : \mathfrak{A}(\text{I}_\mathfrak{A}^{<\alpha}, n)\}$, where $\text{I}_\mathfrak{A}^{<\alpha} := \bigcup_{\xi<\alpha} \text{I}_\mathfrak{A}^\xi \quad (\alpha \in On)$.

$|n|_\mathfrak{A} := \min\{\alpha : n \in \text{I}_\mathfrak{A}^\alpha\} \quad (\text{if } n \in \bigcup_{\alpha\in On} \text{I}_\mathfrak{A}^\alpha)$

$|\text{ID}_\nu| := \sup\{|n|_\mathfrak{A} : \text{lev}(\mathfrak{A}) = 0 \;\&\; \text{ID}_\nu \vdash P_\mathfrak{A}\underline{n}\}$ *(proof-theoretic ordinal of* ID_ν*)*

Remark

The proof theoretic ordinal of a theory *Th* is commonly defined as the supremum of the ordertypes of primitive recursive wellorderings $\prec$ which are provably wellfounded in Th. In the language $\mathcal{L}_\nu$ the wellfoundedness of $\prec$ is expressed by the formula $\forall x P_{\mathfrak{A}_\prec} x$ where $\mathfrak{A}_\prec(X,x) := \forall y(y \prec x \to Xy)$. Since the ordertype of $\prec$ is equal to $\sup\{|n|_{\mathfrak{A}_\prec} + 1 : n \in \text{I}_{\mathfrak{A}_\prec}^{<\Omega}\}$, it easily follows that the proof theoretic ordinal of ID_ν is less or equal to the ordinal $|\text{ID}_\nu|$ defined above. That actually both ordinals coincide follows from Theorem 1.3.11 in [Po98] where it is shown that the proof theoretic ordinal of a theory $Th \supseteq \text{PA}$ is equal to its Π_1^1-ordinal.

For $d \in \mathsf{ID}_{<\omega}^\infty$ let $\|d\| := \mathsf{hgt}(d)$, i.e. $\|\mathcal{I}(d_\iota)_{\iota\in I}\| := \sup_{\iota\in I}(\|d_\iota\|+1)$ (*length, depth, height* of d).

By $(\mathcal{N}, \text{I}^{<\alpha})$ we denote the expansion of the standard model $\mathcal{N}$ where each predicate constant $P_\mathfrak{A}$ of level 0 is interpreted by $\text{I}_\mathfrak{A}^{<\alpha}$.

Theorem 9.9 (Boundedness)

$\text{ID}_0^\infty \ni d \vdash_0 \Gamma \;\;\&\;\; \text{lev}(\Gamma) = 0 \implies (\mathcal{N}, \text{I}^{<\|d\|}) \models \Gamma$

Proof by induction on $\|d\|$.

Theorem 9.10.

If h is a closed ID_ν-derivation of Γ with $\mathrm{lev}(\Gamma)=0$ then

$(\mathcal{N},\mathrm{I}^{<\alpha}) \models \Gamma$ with $\alpha = \|\mathcal{D}_0(\mathcal{E}^m(h^\infty))\|$ for some $m\in\mathbb{N}$.

Proof:

$$\mathrm{ID}_\nu \ni h \vdash \Gamma \overset{\text{Embedding}}{\Longrightarrow} \mathrm{ID}_\nu^\infty \ni h^\infty \vdash_m \Gamma \text{ for some } m$$
$$\overset{\text{Cutelim}}{\Longrightarrow} \mathrm{ID}_\nu^\infty \ni \mathcal{E}^m(h^\infty) \vdash_0 \Gamma$$
$$\overset{\text{Collapsing}}{\Longrightarrow} \mathrm{ID}_0^\infty \ni \mathcal{D}_0(\mathcal{E}^m(h^\infty)) \vdash_0 \Gamma$$
$$\overset{\text{Boundedness}}{\Longrightarrow} (\mathcal{N},\mathrm{I}^{<\alpha}) \models \Gamma \text{ with } \alpha := \|\mathcal{D}_0(\mathcal{E}^m(h^\infty))\|$$

Definition

$\eta_\nu := \sup\{\|\mathcal{D}_0(\mathcal{E}^m(h^\infty))\| : m\in\mathbb{N}$ and h a closed ID_ν-derivation with endsequent of level 0 $\}$

Then Theorem 9.10 shows that $|\mathrm{ID}_\nu| \le \eta_\nu$. In what follows we will prove $\eta_\nu \le \sup_{m\in\mathbb{N}} \psi_0\psi_\nu^m(0) = \psi_0(\varepsilon_{\Omega_\nu+1})$.

Remark

Note the similarity between

" $\mathcal{D}_\sigma(d) = \mathcal{D}_\sigma(d_{\mathcal{D}_\mu(d_0)})$ if $d = \Omega_P\big(d_\iota\big)_{\iota\in\{0\}\cup|P|}$ with $\mu = \mathrm{lev}(P) \ge \sigma$ "

and

" $(D_\sigma a)[1] = D_\sigma a[D_\mu a[\Omega_\mu]]$ if $a\in\mathrm{T}$ and $\mathsf{tp}(a) = \Omega_{\mu+1}$ with $\mu\ge\sigma$ ".

This observation will be pursued in §10.

§10 Majorization of infinitary derivations by tree ordinals

We are now going to relate infinitary derivations $d\in\mathrm{ID}_\nu^\infty$ to tree ordinals $\boldsymbol{\alpha}$. From every derivation $d\in\mathrm{ID}_\nu^\infty$ one obtains a tree ordinal $\mathfrak{o}(d)$ essentially by deleting all inference symbols (and possibly other data) assigned to the nodes of d (namely $\mathfrak{o}(\mathcal{I}(d_\iota)_{\iota\in I}) := \big(\mathfrak{o}(d_\iota)\big)_{\iota\in I}$). Now the first idea which comes into mind is that $\mathfrak{o}(d)$ should equal $\mathbf{t}(a)$ for suitable $a\in\mathrm{OT}$ (at least if $d=h^\infty$ with $h\in\mathrm{ID}_\nu$). But this doesn't work; instead one can establish a weaker relation between $\mathfrak{o}(d)$ and $\mathbf{t}(a)$, namely that in a certain sense $\mathfrak{o}(d)$ is "embeddable" into $\mathbf{t}(a)$. Below we will define a relation $d \lhd \boldsymbol{\alpha}$ (d is *majorized* by $\boldsymbol{\alpha}$) between infinitary derivations d and tree ordinals $\boldsymbol{\alpha}$, which corresponds to this informal notion of embeddability.

The main properties of $\lhd$ will be:

(i) $d\lhd\boldsymbol{\alpha}$ & $d\in\mathsf{ID}_0^\infty \Rightarrow \|d\|\le\|\boldsymbol{\alpha}\|$,

(ii) $d\lhd\boldsymbol{\alpha}$ & $d\in\mathsf{ID}_\nu^\infty \Rightarrow \mathcal{E}(d)\lhd\mathbb{D}_\nu(\boldsymbol{\alpha})$,

(iii) $d\lhd\boldsymbol{\alpha} \Rightarrow \mathcal{D}_\sigma(d)\lhd\mathbb{D}_\sigma(\boldsymbol{\alpha})$.

Mainly by means of (i)-(iii) we will establish that $\|\mathcal{D}_0(\mathcal{E}^m(h^\infty))\| \le \|\mathbb{D}_0\mathbb{D}_\nu^{m+2}(\mathbf{0})\|$ and thus

$|\mathrm{ID}_\nu| \le \eta_\nu \le \sup_{m\in\mathbb{N}} \|\mathbb{D}_0\mathbb{D}_\nu^m(\mathbf{0})\|$, i.e. $|\mathrm{ID}_\nu| \le \sup_{m\in\mathbb{N}} \psi_0\psi_\nu^m(0) = \psi_0(\varepsilon_{\Omega_\nu+1})$.

The following definition and lemma are auxiliary.

Definition of $\boldsymbol{\alpha}^{\ominus}$, $\ll^0$ and $\ll$

$$\boldsymbol{\alpha}^{\ominus} := \begin{cases} \boldsymbol{\alpha}_0 & \text{if } \boldsymbol{\alpha} = \boldsymbol{\alpha}_0{+}\mathbf{1} \text{ or } \boldsymbol{\alpha} = (\boldsymbol{\alpha}_i)_{i\in\mathbb{N}} \\ \boldsymbol{\alpha}_{\boldsymbol{\Omega}_\mu} & \text{if } \boldsymbol{\alpha} = (\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\mathbb{T}_\mu} \end{cases}$$

$\boldsymbol{\beta} \ll^0 \boldsymbol{\alpha} \ :\Longleftrightarrow\ (\boldsymbol{\alpha} \neq \mathbf{0} \ \&\ \boldsymbol{\beta} = \boldsymbol{\alpha}^{\ominus})$ or $(\boldsymbol{\alpha} = (\boldsymbol{\alpha}_i)_{i\in\mathbb{N}} \ \&\ \exists i{\in}\mathbb{N}(\boldsymbol{\beta} = \boldsymbol{\alpha}_i))$

$\ll$ ($\underline{\ll}$, resp.) is the transitive (transitive and reflexive, resp.) closure of $\ll^0$.

Lemma 10.1

(a) $\boldsymbol{\alpha} \neq \mathbf{0} \ \Rightarrow\ (\boldsymbol{\gamma} \oplus \boldsymbol{\alpha})^{\ominus} = \boldsymbol{\gamma} \oplus \boldsymbol{\alpha}^{\ominus} \ \ \&\ \ \mathbb{D}_\sigma(\boldsymbol{\alpha})^{\ominus} = \mathbb{D}_\sigma(\boldsymbol{\alpha}^{\ominus})$

(b) $\mathbf{1} \underline{\ll} \mathbb{D}_\sigma(\boldsymbol{\alpha})$

(c) $\boldsymbol{\beta} \ll \boldsymbol{\alpha} \ \Rightarrow\ \boldsymbol{\gamma} + \boldsymbol{\beta} \ll \boldsymbol{\gamma} + \boldsymbol{\alpha}$

(d) $\boldsymbol{\beta} \ll \boldsymbol{\alpha} \ \Rightarrow\ \mathbb{D}_\sigma(\boldsymbol{\beta}) \ll \mathbb{D}_\sigma(\boldsymbol{\alpha})$

(e) $n \ll \boldsymbol{\Omega}_\sigma \ll \boldsymbol{\Omega}_{\sigma+1}$

Proof of (a):

1. $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0{+}\mathbf{1}$: $\mathbb{D}_\sigma(\boldsymbol{\alpha})^{\ominus} = \mathbb{D}_\sigma(\boldsymbol{\alpha}_0) \cdot 1 = \mathbb{D}_\sigma(\boldsymbol{\alpha}^{\ominus})$.
2. $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_i)_{i\in\mathbb{N}}$: $\mathbb{D}_\sigma(\boldsymbol{\alpha})^{\ominus} = \mathbb{D}_\sigma(\boldsymbol{\alpha}_0) = \mathbb{D}_\sigma(\boldsymbol{\alpha}^{\ominus})$.
3. $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\mathbb{T}_\mu}$: $\mathbb{D}_\sigma(\boldsymbol{\alpha})^{\ominus} = \mathbb{D}_\sigma(\boldsymbol{\alpha}_{\boldsymbol{\Omega}_\mu}) = \mathbb{D}_\sigma(\boldsymbol{\alpha}^{\ominus})$.

Proof of (c):

1. $\boldsymbol{\alpha} \neq \mathbf{0} \ \&\ \boldsymbol{\beta} = \boldsymbol{\alpha}^{\ominus}$: Then $\mathbb{D}_\sigma(\boldsymbol{\beta}) = \mathbb{D}_\sigma(\boldsymbol{\alpha}^{\ominus}) = \mathbb{D}_\sigma(\boldsymbol{\alpha})^{\ominus}$.
2. $(\boldsymbol{\alpha} = (\boldsymbol{\alpha}_i)_{i\in\mathbb{N}} \ \&\ \boldsymbol{\beta} = \boldsymbol{\alpha}_n)$: Then $\mathbb{D}_\sigma(\boldsymbol{\alpha}) = (\mathbb{D}_\sigma(\boldsymbol{\alpha}_i))_{i\in\mathbb{N}} \ \&\ \mathbb{D}_\sigma(\boldsymbol{\beta}) = \mathbb{D}_\sigma(\boldsymbol{\alpha}_n)$.

Definition of $d \lhd \boldsymbol{\alpha}$ *(Majorization)*

$d \lhd \boldsymbol{\alpha}$ ($\boldsymbol{\alpha}$ *majorizes* d) if, and only if, one of the following clauses holds:

($\lhd$1) $d = \mathcal{I}(d_i)_{i\in|\mathcal{I}|}$ with $\mathcal{I} \neq \Omega_P$ and $\boldsymbol{\alpha} = \boldsymbol{\beta}{+}\mathbf{1}$ with $d_i \lhd \boldsymbol{\beta}$ for all $i \in |\mathcal{I}|$

($\lhd$2) $d = \Omega_{P_\mu}(d_q)_{q\in\{0\}\cup|P_\mu|} \ \&\ \boldsymbol{\alpha} = (\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\mathbb{T}_\mu} \ \ \&\ \ \forall q \in \{0\}\cup|P_\mu|\, \forall \boldsymbol{\xi} \in \mathbb{T}_\mu(q \lhd \boldsymbol{\xi} \Rightarrow d_q \lhd \boldsymbol{\alpha}_{\boldsymbol{\xi}})$

($\lhd$3) $d \lhd \boldsymbol{\beta} \ \&\ \boldsymbol{\beta} \ll \boldsymbol{\alpha}$

(By convention $0 \lhd \boldsymbol{\alpha}$ for any $\boldsymbol{\alpha}$.)

Lemma 10.2. $\quad d \lhd \boldsymbol{\alpha} \ \&\ \boldsymbol{\alpha} \in \mathbb{T}_0 \implies \|d\| \leq \|\boldsymbol{\alpha}\|$.

Theorem 10.3.

(a) $d \lhd \boldsymbol{\alpha} \implies \mathcal{J}_C^k(d) \lhd \boldsymbol{\alpha}$

(b) $d \lhd \boldsymbol{\alpha} \implies \mathcal{S}^{\Pi}_{P,\mathcal{F}}(d) \lhd \boldsymbol{\Omega}_\sigma + \boldsymbol{\alpha}$ for each σ

(c) $e \lhd \boldsymbol{\beta} \ \&\ d \lhd \boldsymbol{\alpha} \implies \mathcal{R}_C(e, d) \lhd \boldsymbol{\beta} + \boldsymbol{\alpha}$

(d) $d \lhd \boldsymbol{\alpha} \in \mathbb{T}_\nu \implies \mathcal{E}(d) \lhd \mathbb{D}_\nu(\boldsymbol{\alpha})$

(e) $d \lhd \boldsymbol{\alpha} \implies \mathcal{D}_\sigma(d) \lhd \mathbb{D}_\sigma(\boldsymbol{\alpha})$

Proof by induction on $\boldsymbol{\alpha}$:

We only carry out the essential cases of (c),(d),(e).

(c) 1. $d = \mathsf{Ax}_{\{\neg C, C\}}$: $\mathcal{R}(e, d) = e \lhd \boldsymbol{\beta} \underline{\ll} \boldsymbol{\beta} + \boldsymbol{\alpha}$.

2. $d = \bigvee_{\neg C}^{k} d_0$ & $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0{+}\mathbf{1}$ & $d_0 \lhd \boldsymbol{\alpha}_0$:
$\mathcal{R}(e,d_0) \overset{\text{IH}}{\lhd} \boldsymbol{\beta}+\boldsymbol{\alpha}_0$ & $\mathcal{J}(e) \overset{\text{(a)}}{\lhd} \boldsymbol{\beta} \underline{\ll} \boldsymbol{\beta}+\boldsymbol{\alpha}_0 \implies \mathcal{R}(e,d) = \mathsf{Cut}\,\mathcal{J}(e)\mathcal{R}(e,d_0) \lhd (\boldsymbol{\beta}+\boldsymbol{\alpha}_0)+\mathbf{1} = \boldsymbol{\beta}+\boldsymbol{\alpha}$.

3. $d = \Omega_{P_\mu}(d_q)_{q\in I}$ & $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\mathbb{T}_\mu}$ & $\forall q \in I \forall \boldsymbol{\xi} \in \mathbb{T}_\mu (q \lhd \boldsymbol{\xi} \Rightarrow d_q \lhd \boldsymbol{\alpha}_{\boldsymbol{\xi}})$:
IH $\Rightarrow$ $\forall q \in I \forall \boldsymbol{\xi} \in \mathbb{T}_\mu (q \lhd \boldsymbol{\xi} \Rightarrow \mathcal{R}(e,d_q) \lhd \boldsymbol{\beta}+\boldsymbol{\alpha}_{\boldsymbol{\xi}}) \Rightarrow \mathcal{R}(e,d) = \Omega_{P_\mu}\big(\mathcal{R}(e,d_q)\big)_{q\in I} \lhd (\boldsymbol{\beta}+\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\mathbb{T}_\mu} = \boldsymbol{\beta}+\boldsymbol{\alpha}$.

(d) 1. $d = \mathsf{Cut}_C d_0 d_1$ with $C \in \bigwedge^{+}$-For, and $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0{+}\mathbf{1}$ & $d_0, d_1 \lhd \boldsymbol{\alpha}_0$: IV $\Rightarrow$ $\mathcal{E}(d_i) \lhd \mathbb{D}_\nu(\boldsymbol{\alpha}_0) \overset{\text{(c)}}{\Rightarrow} \mathcal{E}(d) = \mathcal{R}_C(\mathcal{E}(d_0),\mathcal{E}(d_1)) \lhd \mathbb{D}_\nu(\boldsymbol{\alpha}_0)+\mathbb{D}_\nu(\boldsymbol{\alpha}_0) = \mathbb{D}_\nu(\boldsymbol{\alpha}_0)\cdot 2 \Rightarrow \mathcal{E}(d) \lhd \big(\mathbb{D}_\nu(\boldsymbol{\alpha}_0)\cdot(n{+}1)\big)_{n\in\mathbb{N}} = \mathbb{D}_\nu(\boldsymbol{\alpha})$.

2. $d = \Omega_{P_\mu}(d_q)_{q\in\{0\}\cup|P_\mu|}$ & $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\mathbb{T}_\mu}$ & $\forall q \in \{0\}\cup|P_\mu|\, \forall \boldsymbol{\xi} \in \mathbb{T}_\mu (q \lhd \boldsymbol{\xi} \Rightarrow d_q \lhd \boldsymbol{\alpha}_{\boldsymbol{\xi}})$:
Since $\boldsymbol{\alpha} \in \mathbb{T}_\nu$, we have $\mu < \nu$ and $\mathbb{D}_\nu(\boldsymbol{\alpha}) = \big(\mathbb{D}_\nu(\boldsymbol{\alpha}_{\boldsymbol{\xi}})\big)_{\boldsymbol{\xi}\in\mathbb{T}_\mu}$.
IH $\Rightarrow$ $\forall q \in \{0\}\cup|P_\mu|\, \forall \boldsymbol{\xi} \in \mathbb{T}_\mu (q \lhd \boldsymbol{\xi} \Rightarrow \mathcal{E}(d_q) \lhd \mathbb{D}_\nu(\boldsymbol{\alpha}_{\boldsymbol{\xi}})) \overset{\text{Def}}{\Rightarrow} \mathcal{E}(d) = \Omega_{P_\mu}(\mathcal{E}(d_q))_{q\in\{0\}\cup|P_\mu|} \lhd \big(\mathbb{D}_\nu(\boldsymbol{\alpha}_{\boldsymbol{\xi}})\big)_{\boldsymbol{\xi}\in\mathbb{T}_\mu}$.

(e) 1. $d = \mathcal{I}(d_i)_{i\in I}$ with $\mathcal{I} \neq \Omega_P$ and $\boldsymbol{\alpha} = \boldsymbol{\beta}{+}\mathbf{1}$ with $d_i \lhd \boldsymbol{\beta}$ for all $i \in I$: IH $\Rightarrow$ $\forall i (\mathcal{D}_\sigma(d_i) \lhd \mathbb{D}_\sigma(\boldsymbol{\beta}))$ $\Rightarrow$
$\Rightarrow$ $\mathcal{D}_\sigma(d) = \mathcal{I}(\mathcal{D}_\sigma(d_i))_{i\in I} \lhd \mathbb{D}_\sigma(\boldsymbol{\beta})+\mathbf{1} \ll \mathbb{D}_\sigma(\boldsymbol{\beta})+\boldsymbol{\Omega}_\sigma \underline{\ll} \mathbb{D}_\sigma(\boldsymbol{\beta})+\mathbb{D}_\sigma(\boldsymbol{\beta}) \underline{\ll} \mathbb{D}_\sigma(\boldsymbol{\beta}+\mathbf{1})$.

2. $d = \Omega_{P_\mu}(d_q)_{q\in I}$ & $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\mathbb{T}_\mu}$ & $\forall q \in I \forall \boldsymbol{\xi} \in \mathbb{T}_\mu (q \lhd \boldsymbol{\xi} \Rightarrow d_q \lhd \boldsymbol{\alpha}_{\boldsymbol{\xi}})$:
2.1. $\mu < \sigma$: IH $\Rightarrow$ $\forall q \in I \forall \boldsymbol{\xi} \in \mathbb{T}_\mu (q \lhd \boldsymbol{\xi} \Rightarrow \mathcal{D}_\sigma(d_q) \lhd \mathbb{D}_\sigma(\boldsymbol{\alpha}_{\boldsymbol{\xi}}))$ $\Rightarrow$
$\Rightarrow$ $\mathcal{D}_\sigma(d) = \Omega_{P_\mu}(\mathcal{D}_\sigma(d_q))_{q\in I} \lhd (\mathbb{D}_\sigma(\boldsymbol{\alpha}_{\boldsymbol{\xi}}))_{\boldsymbol{\xi}\in\mathbb{T}_\mu} = \mathbb{D}_\sigma(\boldsymbol{\alpha})$.

2.2. $\mu \geq \sigma$: Then $\mathcal{D}_\sigma(d) = \mathcal{D}_\sigma(d_{\mathcal{D}_\mu(d_0)})$ and $\mathbb{D}_\sigma(\boldsymbol{\alpha}) = (\mathbb{D}_\sigma(\boldsymbol{\alpha}_{\boldsymbol{\xi}_n}))_{n\in\mathbb{N}}$ with $\boldsymbol{\xi}_0 = \boldsymbol{\Omega}_\mu$, $\boldsymbol{\xi}_{n+1} = \mathbb{D}_\mu(\boldsymbol{\alpha}_{\boldsymbol{\xi}_n})$.
$0 \lhd \boldsymbol{\xi}_0 \Rightarrow d_0 \lhd \boldsymbol{\alpha}_{\boldsymbol{\xi}_0} \overset{\text{IH}}{\Longrightarrow} q := \mathcal{D}_\mu(d_0) \lhd \mathbb{D}_\mu(\boldsymbol{\alpha}_{\boldsymbol{\xi}_0}) = \boldsymbol{\xi}_1 \Longrightarrow d_q \lhd \boldsymbol{\alpha}_{\boldsymbol{\xi}_1} \overset{\text{IH}}{\Longrightarrow}$
$\mathcal{D}_\sigma(d) = \mathcal{D}_\sigma(d_q) \lhd \mathbb{D}_\sigma(\boldsymbol{\alpha}_{\boldsymbol{\xi}_1}) \ll \big(\mathbb{D}_\sigma(\boldsymbol{\alpha}_{\boldsymbol{\xi}_i})\big)_{i\in\mathbb{N}} = \mathbb{D}_\sigma(\boldsymbol{\alpha})$.

Theorem 10.4 (Embedding). For each closed ID_ν-derivation h we have $h^\infty \lhd \boldsymbol{\Omega}_\nu{\cdot}(2{+}n(h))$,

where $n(\mathcal{I}h_0...h_{m-1}) := \max\{0, n(h_0), ..., n(h_{m-1})\}+1$.

Proof:
By definition $(\mathsf{Ind}_{\mathcal{F}}^{P,t})^\infty = \dfrac{\mathsf{Ax}_{\{\neg Pt, Pt\}} \quad \ldots \mathcal{S}_{P,\mathcal{F}}^{\{Pt\}}(q) \ldots (q \in |Pt|)}{\Omega_{Pt}}$.

By Theorem 10.3b we have $\forall q \in |P_\mu| \forall \boldsymbol{\xi} \in \mathbb{T}_\mu (q \lhd \boldsymbol{\xi} \Rightarrow \mathcal{S}_{P,\mathcal{F}}^{\{Pn\}}(q) \lhd \boldsymbol{\Omega}_\nu + \boldsymbol{\xi})$ which together with $\forall \boldsymbol{\xi} \in \mathbb{T}_\mu (\mathsf{Ax}_{\{\neg Pt, Pt\}} \lhd \boldsymbol{\Omega}_\nu + \boldsymbol{\xi})$ yields $(\mathsf{Ind}_{\mathcal{F}}^{P,t})^\infty \lhd \boldsymbol{\Omega}_\nu + \boldsymbol{\Omega}_{\mathrm{lev}(P)+1} \underline{\ll} \boldsymbol{\Omega}_\nu + \boldsymbol{\Omega}_\nu = \boldsymbol{\Omega}_\nu{\cdot}2$.

The other cases are easy.

Theorem 10.5. Let $\nu > 0$. If h is a closed ID_ν-derivation of Γ with $\mathrm{lev}(\Gamma) = 0$ then $(\mathcal{N}, \mathrm{I}^{<\alpha}) \models \Gamma$ with $\alpha = \|\mathbb{D}_0(\mathbb{D}_\nu^m(\mathbf{0}))\|$ for some $m \in \mathbb{N}$.

Proof:
Theorem 9.10 $\Rightarrow$ $(\mathcal{N}, \mathrm{I}^{<\alpha}) \models \Gamma$ with $\alpha = \|\mathcal{D}_0(\mathcal{E}^m(h^\infty))\|$ for some $m < \omega$.
$h^\infty \overset{\text{Th.10.4}}{\lhd} \boldsymbol{\Omega}_\nu{\cdot}(2{+}n) \overset{\text{Def}}{\ll} \mathbb{D}_\nu(\mathbf{1}) \overset{\text{L.10.1b,d}}{\underline{\ll}} \mathbb{D}_\nu\mathbb{D}_\nu(\mathbf{0}) \overset{\text{Th.10.3d,e}}{\Longrightarrow}$
$\mathcal{D}_0(\mathcal{E}^m(h^\infty)) \lhd \mathbb{D}_0\mathbb{D}_\nu^{m+2}(\mathbf{0}) \overset{\text{L.10.2}}{\Longrightarrow} \|\mathcal{D}_0(\mathcal{E}^m(h^\infty))\| \leq \|\mathbb{D}_0\mathbb{D}_\nu^{m+2}(\mathbf{0})\|$.

Corollary
$|\mathrm{ID}_\nu| \leq \sup_{m\in\mathbb{N}} \|\mathbb{D}_0(\mathbb{D}_\nu^m(\mathbf{0}))\| = \psi_0(\varepsilon_{\Omega_\nu+1})$.

Two Applications

Let $\widehat{\mathrm{T}} := \{a \in \mathrm{T} : a \text{ principal term}\}$ and $\widehat{\mathrm{OT}} := \mathrm{OT} \cap \widehat{\mathrm{T}}$.

As one easily sees, the set $\widehat{\mathrm{T}}$ can be inductively generated by

$a_0, ..., a_{n-1} \in \widehat{\mathrm{T}}\ (n \geq 0)\ \&\ \sigma < \omega \implies D_\sigma(a_0 \oplus ... \oplus a_{n-1}) \in \widehat{\mathrm{T}}$.

Hence $\widehat{\mathrm{T}}$ is nothing other than the set of all finite, ordered trees with labels $\sigma < \omega$, and each term $a = a_0 \oplus ... \oplus a_{n-1} \in \mathrm{T}$ can be considered as a tree with immediate subtrees $a_0, ..., a_{n-1} \in \widehat{\mathrm{T}}$ and an unlabeled root. The assignment of (fundamental) sequences $(a[x])_{x \in |\mathtt{tp}(a)|}$ can then be seen as the definition of a reduction procedure (or rewriting relation) $a \hookrightarrow_x a[x]$ on T. In [Bu87] this reduction procedure (restricted to $\mathrm{T}_0 := \{D_0 a_0 \oplus ... \oplus D_0 a_{n-1} : a_0, ..., a_{n-1} \in \mathrm{T}\}$) had been cooked up as a so-called hydra game, where in the i^{th} round of the game (or battle) the hydra a transforms itself into a new hydra $a[n_i]$. Using Theorem 8.14 and Theorem 10.5 one easily concludes that the hydra game terminates (i.e., $\forall a \in \mathrm{T}_0 \forall (n_i)_{i \in \mathbb{N}} \exists k(\, a[n_0][n_1] \ldots [n_k] = 0\,)$), and that this fact is not provable in $\mathrm{ID}_{<\omega}$:

Let W_0 be inductively defined by the rule: $a \in \mathrm{T}_0\ \&\ [\, a \neq 0 \Rightarrow \forall n(a[n] \in W_0)\,] \implies a \in W_0$.

Then "$a \in W_0$" says that each $\hookrightarrow$-reduction sequence starting with a terminates. Hence "$\forall a \in \mathrm{T}_0(a \in W_0)$" expresses termination of the hydra game. Now using Theorem 8.14 by induction on $\mathbf{t}(a)$ we get
$\forall a \in \mathrm{T}_0(\, a \in W_0\ \&\ |a|_{W_0} = \|\mathbf{t}(a)\|\,)$.

The unprovability result is obtained as follows
$\mathrm{ID}_\nu \vdash \forall x(D_0 D_\nu^x 0 \in W_0) \overset{\text{Th.10.5}}{\implies} \exists m \forall n(\, |D_0 D_\nu^n 0|_{W_0} < \|\mathbb{D}_0 \mathbb{D}_\nu^m(0)\|\,) \implies$
$\implies \exists m(\, |D_0 D_\nu^m 0|_{W_0} < \|\mathbf{t}(D_0 D_\nu^m 0)\| = |D_0 D_\nu^m 0|_{W_0}\,)$. *Contradiction.*

Another interesting observation about the system $(\mathrm{OT}, \prec)$ is due to Okada [Ok88] and provides a rather short proof of H. Friedman's result that the extended Kruskal Theorem on finite labeled trees implies the wellfoundedness of $(\mathrm{OT}, \prec)$ (provably in ACA_0). This runs as follows.

First we define a binary relation $\sqsubseteq$ on $\widehat{\mathrm{T}}$ such that $a \sqsubseteq b$ is equivalent to "there exists a homeomorphic embedding $f : a \to b$ satisfying Friedman's *gap condition* (including the gap condition for the root)".

Definition of $a \sqsubseteq b$ for $a, b \in \widehat{\mathrm{T}}$
Let $a = D_\rho(a_0 \oplus ... \oplus a_{m-1})$ and $b = D_\sigma(b_0 \oplus ... \oplus b_{n-1})$.

$a \sqsubseteq b$ iff one of the following two clauses holds

(i) $\rho = \sigma$ and $\exists$ injective $q : \{0, ..., m-1\} \to \{0, ..., n-1\}$ such that $a_i \sqsubseteq b_{q(i)}$ for $i < m$,

(ii) $\rho \leq \sigma$ and $\exists j < n(\, a \sqsubseteq b_j\,)$.

Then we define a relation $\prec^* \subseteq \prec$ and prove $\forall a, b \in \widehat{\mathrm{OT}}(\, a \sqsubseteq b \Rightarrow a \preceq^* b\,)$.

Definition
$a \prec^* b :\Leftrightarrow a \prec b\ \&\ \forall \rho(G_\rho a \preceq G_\rho b)$ (with $X \preceq Y :\Leftrightarrow \forall x \in X \exists y \in Y(x \preceq y)$)

Lemma 4.1

(a) $a \prec^* b \implies D_\sigma a \prec^* D_\sigma b$

(b) $D_\rho a \preceq^* b \;\&\; \rho \leq \sigma \;\&\; G_\sigma b \prec b \implies D_\rho a \prec^* D_\sigma b$

Proof:

(a) $a \prec^* b \;\&\; G_\rho D_\sigma a \neq \emptyset \Rightarrow G_\rho D_\sigma a = \{a\} \cup G_\rho a \preceq \{b\} \cup G_\rho b = G_\rho D_\sigma b$.

(b) 1. $D_\rho a \preceq^* b \;\&\; \mu \leq \rho \leq \sigma \Rightarrow G_\mu D_\rho a \preceq G_\mu b \subseteq G_\mu D_\sigma b$. Hence $\forall \mu (G_\mu D_\rho a \preceq G_\mu D_\sigma b)$.

2. Proof of $D_\rho a \prec D_\sigma b$: Let $\rho = \sigma$ (otherwise the claim is trivial). Then $a \in G_\sigma(D_\rho a) \preceq G_\sigma b \prec b$.

Theorem 4.2

$a, b \in \widehat{\mathrm{OT}} \;\&\; a \sqsubseteq b \implies a \preceq b$

Proof: By induction on $\ell(b)$ we prove the stronger statement $a \preceq^* b$.

Let $a = D_\rho(a_0 \oplus ... \oplus a_{m-1})$ and $b = D_\sigma(b_0 \oplus ... \oplus b_{n-1})$.

(i) $\rho = \sigma \;\&\; \forall i < m (a_i \sqsubseteq b_{q(i)}) \;\&\; \forall i, j < m (i \neq j \Rightarrow q(i) \neq q(j))$: By IH we have $a_i \preceq^* b_{q(i)}$ for $i < m$. From this we get $(a_0 \oplus ... \oplus a_{m-1}) \preceq^* (b_0 \oplus ... \oplus b_{n-1})$ and then by L.4.1a $a = D_\sigma(a_0 \oplus ... \oplus a_{m-1}) \preceq^* D_\sigma(b_0 \oplus ... \oplus b_{n-1}) = b$.

(ii) $\rho \leq \sigma$ and $\exists j < n (\, a \sqsubseteq b_j \,)$: By IH we have $a \preceq^* b_j \preceq^* (b_0 \oplus ... \oplus b_{n-1}) =: c$. Since $b = D_\sigma c \in \mathrm{OT}$, we also have $G_\sigma c \prec c$. By L.4.1b this yields $a = D_\rho(a_0 \oplus ... \oplus a_{m-1}) \prec^* D_\sigma c = b$.