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by

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Introduction

In [2], [3], [4] I carried out an ordinal analysis of (classical) ID_ν adapting Howard's [7] method for uncountably long terms to a system of infinitary logic with certain new $\Omega_{\mu+1}$-rules. This work involved some special and rather complicated properties of the ordinal notation system $\bar{\Theta}(\Omega)$, a fact which was frequently criticized by S. Feferman. Motivated by Feferman's criticism I developed a proof theoretical treatment of ID_ν by means of wellfounded abstract trees (or constructive ordinals) where no special facts about ordinal notation systems are needed. This approach which provides an easy characterization of the proof theoretic ordinal of ID_ν in terms of some simple tree operations is presented (for $\nu=1$) in part I of these notes. The main feature in our use of wellfounded trees is a relation $t \triangleleft \mathbf{a}$ ("$\mathbf{a}$ majorizes t") between infinite derivations t and trees $\mathbf{a}$ which can be considered as a refined version of the traditional relation $|t| \leq \alpha$, where $|t|$ denotes the ordinary length of t.

In part II we give a new proof of the wellknown fact that the subrecursive hierarchy up to ε_0 exhausts all recursive functions definable in Gödel's system T of prim. rec. functionals of finite types. This is done by a careful reexamination of Tait's [9] normalization procedure and ordinal assignment replacing ordinal calculations by majorization arguments similar to those used in part I. The main step in our proof is the following lemma:

If $t[v^0]$ is a normal[1] term of type 0 with at most one free variable v^0, and if $t[v^0]$ is majorized by the tree $\mathbf{a}$, then for every $n \in \mathbb{N}$ the value of $t[n]$ is less or equal than $F_{\mathbf{a}}(n)$; where $(F_a)_{a \in O_1}$ is a version of the (fast growing) subrecursive hierarchy.

Part III contains an elementary proof of a version of Girard's [6] famous result connecting the slow and fast growing hierarchies of number theoretic functions. This proof was achieved independently from Cichon and Wainer [10] in March 1980. It is very similar to P. Aczel's [1] treatment of the problem which had directly evolved from a study of [10].

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1) A term is called <u>normal</u> if it contains no subterms of shape $(\lambda v t)s$ or $(\ldots t_i \ldots)t_0 s$.

I. Ordinal analysis of $ID_1(\mathfrak{A})$

Formal language $\mathcal{L}$:

$x, y, z, \ldots$ (number variables)
$0, S$ (zero, successor)
$R, R', \ldots$ (constants for prim. rec. predicates)
$\to, \forall$ (implication, universal quantifier)

Let P be a 1-place predicate symbol and $\mathfrak{A}(P,x)$ a formula which is positive in P and has no free variables besides x.

The <u>formal theory $ID_1(\mathfrak{A})$</u> is HA (in the language $\mathcal{L}[P]$) together with the defining axioms for P:

(ID.1) $\forall x(\mathfrak{A}(P,x) \to Px)$

(ID.2) $\forall x(\mathfrak{A}(F,x) \to F(x)) \to \forall x(Px \to F(x))$, for each $F \in \mathcal{L}[P]$.

<u>Remark</u>: If we choose $\mathfrak{A}(P,x)$ such that every occurrence of P in $\mathfrak{A}(P,x)$ is double negated, then $\neg\neg F \leftrightarrow F$ is provable in $ID_1(\mathfrak{A})$ for arbitrary $F \in \mathcal{L}[P]$, and therefore $ID_1(\mathfrak{A})$ contains classical logic.

Abbreviation: A formula of the form $Ra_1 \ldots a_n$ is called an <u>arithmetic prime formula</u> (apf).

<u>Definition of the infinitary system ID_0^∞</u>

We define ID_0^∞ to be the set of all natural deductions which are generated by the following axioms and inference rules:

(Ax 1) A, if A is a true apf.

(Ax 2) $A \to B$, if A is a false apf.

(ID.1) $\forall x(\mathfrak{A}(P,x) \to Px)$.

$$(\to I)\ \frac{\begin{matrix}[A]\\ \vdots\\ B\end{matrix}}{A\to B} \qquad (\to E)\ \frac{A \quad A\to B}{B} \qquad (\forall I)\ \frac{\ldots A(n)\ldots (n\in\mathbb{N})}{\forall x A(x)} \qquad (\forall E)\ \frac{\forall x A(x)}{A(n)}$$.

We only consider deductions without free variables.

A deduction which has no free assumptions is called a <u>closed deduction</u> or <u>proof</u>.

<u>Definition</u>

$\mathcal{P}n := \{ X \in ID_0^\infty : X$ is a normal proof of $Pn \}$

<u>The Ω_1-rule</u>

According to the intuitionistic interpretation of implication a proof of $Pn \to C$ consists of a construction Π which transforms any proof X of Pn into a proof Π_X of C.

The Ω_1-rule can be viewed as a formal version of this explanation:

$$(\Omega_1) \qquad \frac{\begin{matrix} & \vdots \\ & \Pi_X \\ & \vdots \\ \ldots & C & \ldots (X\in\mathcal{P}n)\end{matrix}}{Pn \to C}$$

(that means, if for every $X \in \mathcal{P}n$ Π_X is a deduction of C, then $\frac{\begin{matrix}\vdots\\ \Pi_X \\ \ldots C \ldots (X\in\mathcal{P}n)\end{matrix}}{Pn\to C}$ is a deduction of $Pn \to C$.)

Using this rule (Ω_1) a closed deduction of the induction axioms (ID.2) is obtained as follows: For each deduction $X \in ID_0^\infty$ let Π_X denote the result of substituting F for all occurrences of P in X. Then for every $X \in \mathcal{P}n$ Π_X is a deduction of $F(n)$ from the assumption $\forall x(\mathfrak{A}(F,x) \to F(x))$.

Applying (Ω_1), $(\forall I)$ and $(\to I)$ we get:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[\forall x(\mathfrak{A}(F,x)\to F(x))] \\ \vdots \\ \Pi_X \\ \vdots \\ \ldots\ F(n)\ \ldots\ (X\in \mathcal{P}n)\end{matrix}}{\ldots\ Pn\to F(n)\ \ldots\ (n\in N)}}{\forall x(Px\to F(x))}}{\forall x(\mathfrak{A}(F,x)\to F(x))\to\forall x(Px\to F(x))}$$

Definition of the infinitary system ID_1^∞

We define ID_1^∞ to be the set of all natural deductions which are generated by the axioms and rules of ID_0^∞ and the Ω_1-rule:

$ID_1^\infty := ID_0^\infty + (\Omega_1)$.

The elements of ID_1^∞ are denoted by so called deduction terms; that means we are using Howard's formulae-as-types notation;

Formation rules for deduction terms:

(where $t \vdash A$ means that t denotes a deduction with endformula A)

(Ass) $v^A \vdash A$ for each formula A (assumptions or variables)

(Ax) $0^A \vdash A$ for each axiom of ID_0^∞

$(\to I)$ $t \vdash B \Rightarrow \lambda v^A t \vdash A\to B$

$(\to E)$ $r \vdash A\to B,\ s\vdash A \Rightarrow (rs)\vdash B$

$(\forall I)$ $\forall n\in N(t_n \vdash A(n)) \Rightarrow (t_n)_{n\in N} \vdash \forall x A(x)$

$(\forall E)$ $r\vdash \forall x A(x) \Rightarrow (rn)\vdash A(n)$

(Ω_1) $\forall X\in\mathcal{P}n(t_X\vdash C) \Rightarrow (t_X)_{X\in\mathcal{P}n} \vdash Pn\to C$

We use r, s, t as syntactic variables for deduction terms and numerals. As usual $t[v^A/s]$ denotes the result of substituting the deduction s (of A) for all free occurrences of v^A in t.

Abbreviations

$rs_1 \ldots s_k := (\ldots((rs_1)s_2)\ldots s_k)$

$(\ldots t_i \ldots) := (t_i)_{i\in N}$

$(\ldots t_\chi \ldots) := (t_\chi)_{\chi\in \mathfrak{P}n}$

Definition of $rk(A)$

1. $rk(A) := 0$, if A is a prime formula
2. $rk(A \to B) := \max\{rk(A), rk(B)\} + 1$
3. $rk(\forall x A) := rk(A) + 1$

Definition of $rk(t)$

$rk(t) := rk(A)$, where A is the endformula of t

Definition of the degree $d(t)$

A term of shape $\underline{(\lambda v t)s}$ or $\underline{(\ldots t_i \ldots)n}$ or $\underline{(\ldots t_\chi \ldots) r_0 s}$ is called a redex.

$d(t) := \sup\{rk(r)+1 : \ rs \text{ is a redex occurring in } t\}$

If $d(t) = 0$, i.e. if t contains no redex, then t is called normal.

Reduction steps

(R1) $(\lambda v t)s \ \triangleright \ t[v/s]$

(R2) $(\ldots t_i \ldots)n \ \triangleright \ t_n$

(R3) $(\ldots t_\chi \ldots) r_0 s \ \triangleright \ (\ldots (t_\chi s) \ldots) r_0$

From these reductions steps we now define an operation $*$ on deductions which lowers the degree of a deduction by 1. To obtain r^* we first have to introduce a more general operation $(r, s_1, \ldots, s_k)^*$ which is defined for arbitrary finite sequences $(r, s_1, \ldots, s_k)$ $(k \geq 0)$ such that $rs_1 \ldots s_k$ is a term.

Then we set $r^* := (r)^*$, where $k=0$.

<u>Definition of $(r, s_1, \dots, s_k)^*$</u>

$$(r, s_1, \dots, s_k)^* := \begin{cases} r s_1 \dots s_k & \text{, if } r \text{ is a prime term} \\ \lambda v\, t^* & \text{, if } r = \lambda v t \text{ and } k=0 \\ t^*[v/s_1] s_2 \dots s_k & \text{, if } r = \lambda v t \text{ and } k>0 \\ (t_i^*)_{i \in \mathbb{N}} & \text{, if } r = (t_i)_{i\in\mathbb{N}} \text{ and } k=0 \\ t_{s_1}^* s_2 \dots s_k & \text{, if } r = (t_i)_{i\in\mathbb{N}} \text{ and } k>0 \\ (t_X^*)_{X \in \mathcal{P}_n} & \text{, if } r = (t_X)_{X\in\mathcal{P}_n} \text{ and } k=0 \\ ((t_X, s_2, \dots, s_k)^*)_{X\in\mathcal{P}_n} s_1, & \text{if } r = (t_X)_{X\in\mathcal{P}_n} \text{ and } k>0 \\ (r_1, r_0^*, s_1, \dots, s_k)^* & \text{, if } r = (r_1 r_0). \end{cases}$$

<u>Proposition</u>

If $r s_1 \dots s_k$ is a deduction of C with degree $\leq m+1$ and $d(s_1), \dots, d(s_k) \leq m$, then $(r, s_1, \dots, s_k)^*$ is a deduction of C with degree $\leq m$, and every free assumption of $(r, s_1, \dots, s_k)^*$ also occurs free in $r s_1 \dots s_k$

<u>Corollary</u>

If r is a proof of C with $d(r) = m < \omega$, then $r^{\overbrace{* \dots *}^{m\text{-times}}}$ is a <u>normal</u> proof of C.

From now on we only consider the special case that P is the accessible part of a binary prim. rec. relation R, i.e.
<u>$\mathfrak{A}(P, x) \equiv \forall y (Ryx \to Py)$</u>.

The general case is treated afterwards (pp. 16 - 19)

Collapsing

We define an operation $'$ which transforms every normal proof $t \in ID_1^\infty$ of Pn into a normal proof $t' \in ID_0^\infty$ of the same formula by eliminating all applications of $(\mathcal{I}_1)$ in t. Since t may be (constructively) uncountable but t' always is countable, this operation is called a <u>collapsing function</u>. The definition of t' proceeds by induction on the formation rules for t. Therefore t' has to be defined for all normal proofs t with an endformula Pn or $\mathfrak{A}(P,n)$ or $Rmn \to Pm$:

<u>Definition of t'</u>

1. $(0^A n s)' := 0^A n s'$ (where $A \equiv \forall x(\mathfrak{A}(P,x) \to Px)$)
2. $(t_m)_{m\in\mathbb{N}}' := (t'_m)_{m\in\mathbb{N}}$
3. $(\lambda v^A t)' := \begin{cases} \lambda v^A t[v^A/0^A]' , & \text{if } A \text{ is true} \\ 0^{A\to B} , & \text{if } A \text{ is false}\end{cases}$ (where $A \equiv Rmn$ and $B \equiv Pm$)
4. $((t_\chi)_{\chi\in\mathcal{J}_{n_0}} r_0)' := t'_{r_0'}$ (we have $r_0' \in ID_0^\infty$ and r_0' is a normal proof of the formula Pn_0, i.e. $r_0' \in \mathcal{J}_{n_0}$)

To make things clear we repeat this definition now writing deductions as formula trees:

1. $$\left(\dfrac{\begin{array}{c}\vdots\\ s\\ \mathfrak{A}(P,n)\end{array} \qquad \dfrac{\forall x(\mathfrak{A}(P,x)\to Px)}{\mathfrak{A}(P,n)\to Pn}}{Pn}\right)' := \dfrac{\begin{array}{c}\vdots\\ s'\\ \mathfrak{A}(P,n)\end{array} \qquad \dfrac{\forall x(\mathfrak{A}(P,x)\to Px)}{\mathfrak{A}(P,n)\to Pn}}{Pn}$$

2. $$\left(\dfrac{\begin{array}{c}\vdots\\ t_m\\ \ldots\ Rmn\to Pm \ \ldots (m\in\mathbb{N})\end{array}}{\forall y(Ryn\to Py)}\right)' := \dfrac{\begin{array}{c}\vdots\\ t'_m\\ \ldots\ Rmn\to Pm \ \ldots\ (m\in\mathbb{N})\end{array}}{\forall y(Ryn\to Py)}$$

3. $$\left(\dfrac{\begin{array}{c}\vdots\\ t\\ \vdots\\ Pm\end{array}}{Rmn\to Pm}\right)' := \begin{cases} \dfrac{\begin{array}{c}\vdots\\ t'\\ \vdots\\ Pm\end{array}}{Rmn\to Pm} , & \text{if } Rmn \text{ is true} \\[2ex] Rmn\to Pm , & \text{if } Rmn \text{ is false}\end{cases}$$

$$4.\quad \left(\dfrac{\begin{array}{c} \vdots \\ r_0 \\ \vdots \\ Pn_0 \end{array} \qquad \dfrac{\cdots \begin{array}{c} \vdots \\ t_X \\ \vdots \\ C \end{array} \cdots (X \in \mathfrak{P}n_0)}{Pn_0 \to C}}{C} \right)' := \begin{array}{c} \vdots \\ t'_{r'_0} \\ \vdots \\ C \end{array}$$

The operations $*$ and $'$ together with the canonical embedding[1] of $ID_1(\mathfrak{A})$ into ID_1^∞ provide a reduction procedure which transforms any finite proof of Pn in $ID_1(\mathfrak{A})$ into a normal proof of Pn in ID_0^∞. As we will see below this can be used to obtain an upper bound for $|ID_1(\mathfrak{A})|$, the <u>proof theoretic ordinal of $ID_1(\mathfrak{A})$</u>.

<u>Definition of $|ID_1(\mathfrak{A})|$</u>

$$|ID_1(\mathfrak{A})| := \sup\{ |n|_{\mathfrak{A}} : ID_1(\mathfrak{A}) \vdash Pn \},$$

where $\quad |n|_{\mathfrak{A}} := \inf\{\alpha \in On : n \in \mathbb{P}^\alpha\} \quad (\text{for } n \in \bigcup_{\alpha \in On} \mathbb{P}^\alpha)$

and $\quad \mathbb{P}^\alpha := \{n \in N : \mathbb{N} \models \mathfrak{A}(\bigcup_{\xi<\alpha} \mathbb{P}^\xi, n)\}$.

<u>Lemma 1</u>

If $t \in ID_0^\infty$ is a normal proof of a formula $A(P)$ in which P occurs only positive, then $A(\mathbb{P}^\alpha)$ is true for every ordinal $\alpha \geq |t|$, where $|t|$ denotes the length of t.[2]

Proof by induction on $|t|$.

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1) As we showed on pp. 4,5 the axioms $(ID.2)$ are provable in ID_1^∞ (using (Ω_1)). The other embedding steps are trivial.

2) Definition of $|t|$ for $t \in ID_0^\infty$: $|vA| := |0^A| := 0$, $|\lambda v t| := |t|+1$, $|(\ldots t_i \ldots)| := \sup_{i \in N}(|t_i|+1)$, $|t_0 \ldots t_{k+1}| := \max\{|t_0|, \ldots, |t_{k+1}|\}+1$ if t_0 is not (rs).

Abstract trees

Inductive definition of the tree classes $\mathcal{O}_k$

1. $0 \in \mathcal{O}_k$
2. $a \in \mathcal{O}_k \Rightarrow a+1 \in \mathcal{O}_k$
3. $\ell < k \wedge \forall \xi \in \mathcal{O}_\ell (0 \neq a_\xi \in \mathcal{O}_k) \Rightarrow (a_\xi)_{\xi \in \mathcal{O}_\ell} \in \mathcal{O}_k$

We identify $\mathcal{O}_0$ and $\mathbb{N}$.

Definition of the length $|a|$ for $a \in \mathcal{O}_1$

$|0| := 0$, $|a+1| := |a|+1$, $|(a_i)_{i\in\mathbb{N}}| := \sup\{|a_i|+1 : i \in \mathbb{N}\}$.

Definition of $a+b$ and 2^b for $a, b \in \mathcal{O}_k$

$a+0 := a$	$2^0 := 1$
$a+(b+1) := (a+b)+1$	$2^{b+1} := 2^b + 2^b$
$a+(b_\xi)_{\xi\in\mathcal{O}_\ell} := (a+b_\xi)_{\xi\in\mathcal{O}_\ell}$	$2^{(b_\xi)_{\xi\in\mathcal{O}_\ell}} := (2^{b_\xi})_{\xi\in\mathcal{O}_\ell}$

Definition of $\dot\omega$, $\dot\Omega$ and $\dot\varepsilon_{\Omega+1}$:

$\dot\omega := \dot\Omega_0$, $\dot\Omega := \dot\Omega_1$, $\dot\varepsilon_{\Omega+1} := (2^{\cdot^{\cdot^{\cdot^{2^{\dot\Omega+\dot\omega+\dot\omega}}}}}\}i)_{i\in\mathbb{N}}$, where $\dot\Omega_\ell := (1+\xi)_{\xi\in\mathcal{O}_\ell}$

In the following a, b, c always denote elements of $\mathcal{O}_2$.

Definition of the collapsing function $D : \mathcal{O}_2 \to \mathcal{O}_1$

1. $D0 := \dot\omega$
2. $D(a+1) := (Da)+1$
3. $D(a_i)_{i\in\mathbb{N}} := (Da_i)_{i\in\mathbb{N}}$
4. $D(a_\xi)_{\xi\in\mathcal{O}_1} := Da_{Da_0}$

Remarks

(i) D corresponds to the function $\lambda\alpha\varphi_\alpha(0)$, where $(\varphi_\alpha)_{\alpha<\varepsilon_{\Omega+1}}$ is the usual Bachmann hierarchy

(ii) $|D\dot\varepsilon_{\Omega+1}| = \varphi_{\varepsilon_{\Omega+1}}(0)$ (= Howard ordinal)

In the rest of this part we will prove the following theorem:

Theorem 1

$|ID_1(\mathcal{O})| \leq |D\dot{\varepsilon}_{\Omega+1}|$.

To prove this theorem we introduce a new relation $\triangleleft$ between deductions $t \in ID_1^\infty$ and trees $a \in \mathcal{O}_2$:

Inductive definition of $t \triangleleft a$ (a majorizes t) for $t \in ID_1^\infty$ and $a \in \mathcal{O}_2$

($\triangleleft$1) $t \triangleleft a$, if t is a prime term. (numerals are also considered as prime terms)

($\triangleleft$2) $t \triangleleft a \Rightarrow \lambda v t \triangleleft a+1$

($\triangleleft$3) $t_0 \triangleleft a \wedge \dots \wedge t_n \triangleleft a \Rightarrow t_0 \dots t_n \triangleleft a+1$

($\triangleleft$4) $\forall i \in \mathbb{N}(t_i \triangleleft a) \Rightarrow (t_i)_{i \in \mathbb{N}} \triangleleft a+1$

($\triangleleft$5) $\forall X \in \mathcal{P}_n \forall \xi \in \mathcal{O}_1 (X \triangleleft \xi \to t_X \triangleleft a_\xi) \wedge r_0 \triangleleft a_0 \Rightarrow (\dots t_X \dots) r_0 \triangleleft (a_\xi)_{\xi \in \mathcal{O}_1}$

($\triangleleft$6) $\exists i \in \mathbb{N}(t \triangleleft a_i) \Rightarrow t \triangleleft (a_i)_{i \in \mathbb{N}}$

($\triangleleft$7) $\forall \xi \in \mathcal{O}_1 (t \triangleleft a_\xi) \Rightarrow t \triangleleft (a_\xi)_{\xi \in \mathcal{O}_1}$

Remark In fact this is an iterated inductive definition of two relations $\triangleleft_0, \triangleleft_1$ (where $\triangleleft_1 = \triangleleft$ and $\triangleleft_0 = \triangleleft \restriction ID_0^\infty \times \mathcal{O}_1$), and rule ($\triangleleft$5) should be written more precisely as follows:

$\forall X \in \mathcal{P}_n \forall \xi \in \mathcal{O}_1 (X \triangleleft_0 \xi \to t_X \triangleleft_1 a_\xi) \wedge r_0 \triangleleft_1 a_0 \Rightarrow (\dots t_X \dots) r_0 \triangleleft_1 (a_\xi)_{\xi \in \mathcal{O}_1}$.

Proposition: $t \in ID_0^\infty \wedge a \in \mathcal{O}_1 \wedge t \triangleleft a \Rightarrow |t| \leq |a|$.

Lemma 2

a) $t \triangleleft a+1 \wedge b \neq 0 \Rightarrow t \triangleleft a+b$

b) $t \triangleleft a \Rightarrow t \triangleleft a+b$

c) $t \triangleleft b \Rightarrow t \triangleleft a+b$

Proof of lemma 2:

a) Induction on b:

1. $b = b_0 + 1$: If $b_0 = 0$, then $a+b = a+1$ and the assertion is trivial. Otherwise by I.H. we have $t \lhd a+b_0$ and therefore $t \lhd a+b$ by ($\lhd$3) for $n=0$.
2. $b = (b_i)_{i\in\mathbb{N}}$: Then $b_0 \neq 0$ and therefore by I.H. $t \lhd a+b_0$. Hence $t \lhd (a+b_i)_{i\in\mathbb{N}} = a+b$ by ($\lhd$6).
3. $b = (b_\xi)_{\xi\in\mathcal{O}_1}$: Then $\forall \xi \in \mathcal{O}_1 (b_\xi \neq 0)$ and therefore by I.H. $\forall \xi \in \mathcal{O}_1$ $t \lhd a+b_\xi$. Hence $t \lhd (a+b_\xi)_{\xi\in\mathcal{O}_1} = a+b$ by ($\lhd$7).

b) For $b=0$ the assertion is trivial. Suppose now $b \neq 0$. From $t \lhd a$ by ($\lhd$3) we obtain $t \lhd a+1$ and then $t \lhd a+b$ by a).

c) is proved by induction on b.

Lemma 3

$s \lhd a$ and $t \lhd b \Rightarrow t[{}^v/_s] \lhd a+b$.

Proof by induction on b using lemma 2.

Lemma 4

If $rs_1\ldots s_k$ ($k \geq 0$) is a deduction term with $r \lhd b$ and $s_i \lhd a$ ($1 \leq i \leq k$), then $(r, s_1, \ldots, s_k)^* \lhd a + 2^b$.

Proof by induction on b: We write $()^*$ for $(r, s_1, \ldots, s_k)^*$.

($\lhd$1) r prime term. Then $r \lhd a$, $s_1 \lhd a, \ldots, s_k \lhd a$. Hence $()^* = rs_1\ldots s_k \lhd a+1$ by ($\lhd$3). From this we obtain $()^* \lhd a+2^b$ by lemma 2a.

($\lhd$2) $r = \lambda v t$ and $b = b_0+1$ with $t \lhd b_0$.

$k=0$: By I.H. we have $t^* \lhd a+2^{b_0}$. From this by ($\lhd$2) and lemma 2a we obtain $\lambda v t^* \lhd a+2^{b_0}+2^{b_0} = a+2^b$.

$k>0$: By I.H. we have $t^* \lhd 2^{b_0}$. From this and $s_1 \lhd a$ by lemma 3 we obtain $t^*[{}^v/_{s_1}] \lhd a+2^{b_0}$. From $s_2 \lhd a, \ldots, s_k \lhd a$ by lemma 2 b) we obtain $s_2 \lhd a+2^{b_0}, \ldots, s_k \lhd a+2^{b_0}$. Hence

$()^* = t^*[{}^v/_{s_1}]s_2 \ldots s_k \lhd a+2^{b_0}+2^{b_0} = a+2^b$ by lemma 2a) and $(\lhd 3)$.

<u>$(\lhd 3)$</u> $r = t_0 \ldots t_n$ and $b = b_0+1$ with $t_0 \lhd b_0, \ldots, t_n \lhd b_0$.

Then $()^* = (t_0, t_1^*, \ldots, t_n^*, s_1, \ldots, s_k)^*$. By I.H. and lemma 2b) we have $\forall s \in \{t_1^*, \ldots, t_n^*, s_1, \ldots, s_k\}(s \lhd a+2^{b_0})$. From this and I.H. (applied to t_0) we obtain $()^* \lhd a+2^{b_0}+2^{b_0} = a+2^b$.

<u>$(\lhd 4)$</u> $r = (t_i)_{i\in N}$ and $b = b_0+1$ with $\forall i \in N(t_i \lhd b_0)$.

$k=0$: By I.H. we have $\forall i \in N(t_i^* \lhd a+2^{b_0})$. From this by $(\lhd 4)$ and lemma 2a we obtain $()^* = (t_i^*)_{i\in N} \lhd a+2^{b_0}+2^{b_0} = a+2^b$.

$k>0$: By I.H. we have $t_{s_1}^* \lhd a+2^{b_0}$. From $s_2 \lhd a, \ldots, s_k \lhd a$ by lemma 2b) we obtain $s_2 \lhd a+2^{b_0}, \ldots, s_k \lhd a+2^{b_0}$. Hence

$()^* = t_{s_1}^* s_2 \ldots s_k \lhd a+2^{b_0}+2^{b_0} = a+2^b$ by lemma 2a) and $(\lhd 3)$.

<u>$(\lhd 5)$</u> $r = (\ldots t_\chi \ldots)r_0$ and $b = (b_\xi)_{\xi\in O_1}$ with $\forall \chi,\xi(\chi \lhd \xi \to t_\chi \lhd b_\xi) \wedge r_0 \lhd b_0$.

Then $()^* = ((\ldots t_\chi \ldots), r_0^*, s_1, \ldots, s_k)^* = (\ldots(t_\chi, s_1, \ldots, s_k)^* \ldots)r_0^*$.

By I.H. we have $\forall \chi,\xi(\chi \lhd \xi \to (t_\chi, s_1, \ldots, s_k)^* \lhd a+2^{b_\xi}) \wedge r_0^* \lhd a+2^{b_0}$.

Hence $()^* \lhd (a+2^{b_\xi})_{\xi\in O_1} = a+2^b$ by $(\lhd 5)$.

<u>$(\lhd 6)$</u> $b = (b_i)_{i\in N}$ and $r \lhd b_i$ for some $i \in N$. Then by I.H. $()^* \lhd a+2^{b_i}$, from which $()^* \lhd (a+2^{b_i})_{i\in N} = a+2^b$ follows by $(\lhd 6)$.

<u>$(\lhd 7)$</u> $b = (b_\xi)_{\xi\in O_1}$ and $\forall \xi \in O_1(r \lhd b_\xi)$. Then by I.H. $\forall \xi \in O_1$ $()^* \lhd a+2^{b_\xi}$, from which $()^* \lhd (a+2^{b_\xi})_{\xi\in O_1} = a+2^b$ follows by $(\lhd 7)$.

<u>Lemma 5</u>

If $t \in ID_1^\infty$ is a normal proof of Pn or $\mathfrak{A}(P,n)$ or $Rmn \to Pm$, then:

$t \lhd a \Rightarrow t' \lhd Da$.

Proof by induction on a:

<u>1.</u> $t = (\ldots t_\chi \ldots)r_0$ and $a = (a_\xi)_{\xi\in O_1}$ with (1) $r_0 \lhd a_0$ and

(2) $\forall \chi,\xi(\chi \lhd \xi \to t_\chi \lhd a_\xi)$.

Then $t' = t'_{r_0'}$. From (1) by I.H. follows $r_0' \lhd Da_0$.

From $r_0' \triangleleft D\alpha_0$ and (2) follows $t_{r_0'} \triangleleft \alpha_{Dr_0}$, and from this by I.H. $t'_{r_0'} \triangleleft D\alpha_{D\alpha_0} = D\alpha$.

2. $\alpha = (\alpha_i)_{i \in \mathbb{N}}$ and $t \triangleleft \alpha_i$ for some $i \in \mathbb{N}$. Then by I.H. we have $t' \triangleleft D\alpha_i$. Hence $t' \triangleleft (D\alpha_i)_{i\in\mathbb{N}} = D\alpha$ by $(\triangleleft 6)$.

3. $\alpha = (\alpha_\xi)_{\xi \in \mathcal{O}_1}$ and $\forall \xi \in \mathcal{O}_1 (t \triangleleft \alpha_\xi)$. Then we have $t \triangleleft \alpha_{Dr_0}$ and therefore by I.H. $t' \triangleleft D\alpha_{D\alpha_0} = D\alpha$.

4. In all other cases the assertion follows immediately from I.H.

Proof of theorem 1

(1) For each closed axiom C of $ID_1(\mathfrak{A})$ there exists a normal proof $t \in ID_1^\infty$ with $t \triangleleft \dot{\Omega} + \dot{\omega} + 3$.

Proof:

1. $C \equiv \forall x(\mathfrak{A}(F,x) \to F(x)) \to \forall x(Px \to F(x))$. Let $A :\equiv \forall x(\mathfrak{A}(F,x) \to F(x))$. For each $X \in ID_0^\infty$ let Π_X denote the result of substituting F for all occurrences of P in X. Obviously $X \triangleleft \xi$ implies $\Pi_X \triangleleft \xi$ and also $\Pi_X \triangleleft 1+\xi$ (by lemma 2c). We therefore have $\forall X \in \mathcal{P}n\, \forall \xi \in \mathcal{O}_1 (X \triangleleft \xi \to \Pi_X \triangleleft 1+\xi) \wedge v^{Pn} \triangleleft 1+0$. Hence $t_n := \lambda v^{Pn} (\Pi_X)_{X \in \mathcal{P}n} v^{Pn} \triangleleft \dot{\Omega}+1$ by $(\triangleleft 5)$ and $(\triangleleft 2)$. Using $(\triangleleft 4)$ and $(\triangleleft 6)$ we obtain $(t_n)_{n\in\mathbb{N}} \triangleleft \dot{\Omega}+\omega$ and then $\lambda v^A (t_n)_{n\in\mathbb{N}} \triangleleft \dot{\Omega}+\dot{\omega}+3$ by $(\triangleleft 2)$ and lemma 2d). Obviously $\lambda v^A (t_n)_{n\in\mathbb{N}}$ is a normal proof of C.

2. $C \equiv F(0) \to \forall x(F(x) \to F(Sx)) \to \forall x F(x)$. Let $A :\equiv \forall x(F(x) \to F(Sx))$. Let $t_0 := v^{F(0)}$ and $t_{n+1} := v^A n t_n$. Then $t_n \vdash F(n)$ and therefore $t := \lambda v^{F(0)} \lambda v^A (t_n)_{n\in\mathbb{N}}$ is a proof of C. Obviously t is normal. By induction on n follows $t_n \triangleleft 1+n$ and therefore $t_n \triangleleft \dot{\omega}$ by $(\triangleleft 6)$. Using $(\triangleleft 4)$ and lemma 2c) we obtain $(t_n)_{n\in\mathbb{N}} \triangleleft \dot{\Omega}+\dot{\omega}+1$. Hence $t \triangleleft \dot{\Omega}+\dot{\omega}+3$ by $(\triangleleft 2)$.

3. For each other axiom of $ID_1(\mathfrak{A})$ there exists a normal proof t with $t \lhd l$ for some $l \in \mathbb{N}$. Using ($\lhd$6) and lemma 2 b), c) we obtain $t \lhd \dot{\Omega} + \dot{\omega} + 3$.

(2) $ID_1(\mathfrak{A}) \vdash Pn \;\Rightarrow\; |n|_{\mathfrak{A}} < |D\dot{\varepsilon}_{\Omega+1}|$

Proof.

Let q be a proof of Pn in $ID_1(\mathfrak{A})$. By (1) q can be transformed into a proof $t \in ID_1^\infty$ of Pn with $m := d(t) < \omega$ and $t \lhd \dot{\Omega} + \dot{\omega} + k$ for some $k \in \mathbb{N} \setminus \{0\}$.

Let $\alpha := \dot{\Omega} + \dot{\omega} + \dot{\omega}$.

By ($\lhd$6) follows $t \lhd \alpha$. Using lemmas 4, 5 we now obtain a normal proof $\hat{t} := t^{\overbrace{*\cdots *}^{m}} \in ID_0^\infty$ of Pn with $\hat{t} \lhd D\underbrace{2^{\cdot^{\cdot^{2^\alpha}}}}_{m}$. From $\hat{t} \lhd D2^{\cdot^{\cdot^{2^\alpha}}}$ it follows that $|\hat{t}| \le |D2^{\cdot^{\cdot^{2^\alpha}}}|$. Since $D\dot{\varepsilon}_{\Omega+1} = (D\underbrace{2^{\cdot^{\cdot^{2^\alpha}}}}_{i})_{i \in \mathbb{N}}$, we have $|D2^{\cdot^{\cdot^{2^\alpha}}}| < |D\dot{\varepsilon}_{\Omega+1}|$. Since $\hat{t} \in ID_0^\infty$ is a normal proof of Pn, we have $n \in P^{|\hat{t}|}$, i.e. $|n|_{\mathfrak{A}} \le |\hat{t}|$ (cf. lemma 1). Hence $|n|_{\mathfrak{A}} < |D\dot{\varepsilon}_{\Omega+1}|$.

Treatment of $ID_1(\mathfrak{A})$ for arbitrary positive $\mathfrak{A}$

Notations

$\Gamma, \Delta, \ldots$ denote finite sets of formulas.

Δ_t denotes the set of all free assumptions of the deduction t.

Expressions $\Gamma \Rightarrow C$ are called sequents.

t is called a deduction of $\Gamma \Rightarrow C$ iff $\Delta_t \subseteq \Gamma$ and C is the endformula of t.

A formula of $\mathcal{L}[P]$ is called positive (negative) if it is an element of the set Pos (Neg):

Inductive definition of Pos and Neg:

1. All atomic formulas of $\mathcal{L}[P]$ are in Pos. All atomic formulas of $\mathcal{L}$ are in Neg.
2. $A \in Pos\,(Neg) \implies \forall x A \in Pos(Neg)$
3. $A \in Neg\,(Pos)$ and $B \in Pos\,(Neg) \implies (A \to B) \in Pos(Neg)$.

From now on $\mathfrak{A}(P,x)$ (or $\mathfrak{A}$) denotes an arbitrary (fixed) positive formula containing at most x free.
Under this general assumption the definition of the collapsing function ' as given on pg. 8, does not work in any case:

Consider $\mathfrak{A}(P,x) \equiv \forall y(A(P,x,y) \to B(P,x,y)) \to C(P,x)$, and assume that

$$t = \left\{ \begin{array}{c} t_0 \left\{ \begin{array}{c} [\forall y(A(P,n,y) \to B(P,n,y))] \\ | \\ C(P,n) \end{array} \right. \\ \dfrac{\dfrac{C(P,n)}{\mathfrak{A}(P,n)} \qquad \dfrac{\forall x(\mathfrak{A}(P,x) \to Px)}{\mathfrak{A}(P,n) \to Pn}}{Pn} \end{array} \right.$$

is a normal proof of Pn in ID_1^∞. Then to define t' by induction on the formation rules for t one also has

to define t_0', where t_0 is a deduction with the free assumption $\forall y(A(P,u,y) \to B(P,u,y))$.
Speaking more generally we have to define t' for all normal deductions $t \in ID_1^\infty$ with positive endformula and $\Delta_t \subseteq Neg$.
Consider now a deduction

$$t = \left\{ \frac{\begin{array}{c} \vdots \\ r_0 \\ \vdots \\ Pn_0 \end{array} \qquad \dfrac{\begin{array}{c} \vdots \\ t_X \\ \vdots \\ \ldots\ C\ \ldots\ (X \in \mathcal{P}n_0) \end{array}}{Pn_0 \to C}}{C} \right. \qquad (*)$$

where $\Delta_{r_0} \neq \phi$.

Applying our collapsing function $'$ to r_0 we obtain a normal deduction $r_0' \in ID_0^\infty$ of Pn_0. But since r_0' may contain free assumptions, it is not necessarily an element of $\mathcal{P}n_0$, and therefore we cannot define $t' := t_{r_0'}$, as we did in the special case on pg. 8.

To overcome this difficulty we modify the Ω_1-rule by ~~extendi~~ enlarging the index set $\mathcal{P}n_0$ such that for every normal deduction $r_0 \in ID_1^\infty$ of Pn_0 with $\Delta_{r_0} \subset Neg$ the collapsed deduction r_0' is an element of $\mathcal{P}n_0$ and thus there exists a subdeduction $t_{r_0'}$ of $(t_X)_{X \in \mathcal{P}n_0}^{r_0}$ which can be used to define t' in the above example $(*)$.

<u>Definition of $\mathcal{P}n$</u>

$\mathcal{P}n := \{ X \in ID_0^\infty : X$ is a normal deduction of Pn, and $\Delta_X \subseteq Neg \}$.

Ω_1-rule

$$\frac{\begin{array}{c}[\Delta_X] \\ | \\ t_X \\ | \\ \ldots\ C\ \ldots\ (X \in \mathcal{P}n)\end{array}}{\mathcal{P}n \to C}$$

where a free assumption A of t_X is discharged by the application of (Ω_1) if, and only if, A is an element of Δ_X.

Using sequents the Ω_1-rule can be formulated as follows:

(Ω_1) If for every $X \in \mathcal{P}n$ t_X is a deduction of $\Delta_X \cup \Gamma \Rightarrow C$, then $(t_X)_{X \in \mathcal{P}n}$ is a deduction of $\Gamma \Rightarrow (\mathcal{P}n \to C)$.

On the basis of this refined version of the Ω_1-rule we can now define the collapsing function $'$ for all normal deductions $t \in ID_1^\infty$ of sequents $\Gamma \Rightarrow C$ where $C \in Pos$ and $\Gamma \subseteq Neg$;

1. $(r s_1 \ldots s_k)' := r s_1' \ldots s_k'$, if r is a prime term
2. $(\lambda v t)' := \lambda v t'$
3. $(\ldots t_i \ldots)' := (\ldots t_i' \ldots)$
4. $((t_X)_{X \in \mathcal{P}n_0} r_0)' := t'_{r_0'}$

Lemma 6

If $t \in ID_1^\infty$ is a normal deduction of $\Gamma \Rightarrow C$ with $C \in Pos$ and $\Gamma \subseteq Neg$, then t' is a normal deduction of $\Gamma \Rightarrow C$ in ID_0^∞, and moreover we have $\forall a \in O_2 (t \lhd a \Rightarrow t' \lhd Da)$ as in the special case.

The refined version of the Ω_1-rule is still strong enough for deriving the induction axioms (ID.2):

For each deduction $\mathcal{X} \in \mathcal{ID}_0^\infty$ let $\Pi_\mathcal{X}$ denote the result of substituting F for all occurrences of P in $\mathcal{X}$ which (in some suitable sense) are linked to the endformula of $\mathcal{X}$. Now the following proposition holds:

If $\mathcal{X} \in \mathcal{ID}_0^\infty$ is a normal deduction of $\Delta_\mathcal{X} \Rightarrow Pn$, where $\Delta_\mathcal{X} \subseteq Neg$, then $\Pi_\mathcal{X}$ is a normal deduction of $\Delta_\mathcal{X} \cup \{\forall x(\mathfrak{A}(F,x) \to F(x))\} \Rightarrow F(n)$.

Hence $t_n := (\Pi_\mathcal{X})_{\mathcal{X} \in \mathcal{P}_n}$ is a deduction of $\{\forall x(\mathfrak{A}(F,x) \to F(x))\} \Rightarrow (Pn \to F(n))$, and therefore $\lambda v^A (t_n)_{n \in \mathbb{N}}$ is a proof of $\underbrace{\forall x(\mathfrak{A}(F,x) \to F(x))}_{\equiv: A} \to \forall x(Px \to F(x))$.

All definitions and proofs concerning the relation $\triangleleft$ between (deduction) terms and trees are not affected by the above modification of (Ω_1).

II. Ordinal analysis of Gödel's T

Definition of a fast-growing hierarchy $(F_\alpha)_{\alpha\in O_1}$

1. $F_0(n) := n+1$
2. $F_{\alpha+1}(n) := F_\alpha(n)+1$
3. $F_{(\alpha_i)_{i\in\mathbb{N}}}(n) := F_{\alpha_k}(n)$, where $k := F_{\alpha_0}(n)$

We give a direct proof of the following theorem.

Theorem 2

If $f: \mathbb{N}\to\mathbb{N}$ is definable in Gödel's system T of prim. rec. functionals of finite types, then there are $m, l \in \mathbb{N}$ such that

$\forall n\in\mathbb{N}(f(n) \le F_\alpha(n))$, where $\alpha := 2^{2^{\cdot^{\cdot^{2^{\omega+l}}}}}\}m$.

Infinite terms (TAIT [9])

Types

0 is a type, and if σ, τ are types then also $(\sigma\to\tau)$ is a type.

Formation rules for infinite terms

(Tm 1) $v_0^\sigma, v_1^\sigma, \ldots$ are terms of type σ (variables)
0 is a term of type 0 (zero)
S is a term of type $(0\to 0)$ (successor)
} prime terms

(Tm 2) If t is a term of type τ, then $\lambda v^\sigma t$ is a term of type $(\sigma\to\tau)$.

(Tm 3) If r, s are terms of types $(\sigma\to\tau), \sigma$, then (rs) is a term of type τ.

(Tm 4) If for every $i\in\mathbb{N}$ t_i is a term of type τ, then $(t_i)_{i\in\mathbb{N}}$ is a term of type $(0\to\tau)$.

We use r, s, t as syntactic variables for terms generated by (Tm1)-(Tm4). As usual $t[v^\sigma/s]$ denotes the result of substituting s for all free occurrences of v^σ in t (changing bound variables if necessary).

Abbreviations

$\sigma_1 \dots \sigma_k \tau := (\sigma_1 \to (\sigma_2 \to \dots (\sigma_n \to \tau) \dots))$

$r s_1 \dots s_k := (\dots((r s_1) s_2) \dots s_k)$

$(\dots t_i \dots) := (t_i)_{i \in \mathbb{N}}$,

Definition of $rk(\tau)$

$rk(0) := 0$, $rk(\sigma \to \tau) := \max\{rk(\sigma), rk(\tau)\} + 1$.

Definition of $rk(t)$

$rk(t) := rk(\tau)$, where τ is the type of t.

Definition of the degree $d(t)$

A term of shape $\underline{(\lambda v t) s}$ or $\underline{(\dots t_i \dots) r_0 s}$ is called a redex.

$d(t) := \sup\{rk(r)+1 : rs \text{ is a redex occurring in } t\}$

If $d(t)=0$, i.e. if t contains no redex, then t is called normal.

Reduction steps

(R1) $(\lambda v t) s \quad \triangleright \quad t[v/s]$

(R2) $(\dots t_i \dots) r_0 s \quad \triangleright \quad (\dots (t_i s) \dots) r_0$

From these reduction steps we now define an operation * on terms which lowers the degree of a term r (at least) by 1 and does not change the meaning of r. To obtain r^* we first have to introduce a more general operation $(r, s_1, \dots, s_k)^*$ which will be defined for arbitrary finite sequences $(r, s_1, \dots, s_k)$ such that $k \geq 0$ and $r s_1 \dots s_k$ is a term. Then we set $r^* := (r)^*$.

Definition of $(r, s_1, \dots, s_k)^*$

$$(r, s_1, \dots, s_k)^* := \begin{cases} r s_1 \dots s_k & \text{, if } r \text{ is a prime term} \\ \lambda v t^* & \text{, if } r = \lambda v t \text{ and } k = 0 \\ t^*[v/s_1] s_2 \dots s_k & \text{, if } r = \lambda v t \text{ and } k > 0 \\ (\dots t_i^* \dots) & \text{, if } r = (\dots t_i \dots) \text{ and } k = 0 \\ (\dots (t_i, s_2, \dots, s_k)^* \dots) s_1 & \text{, if } r = (\dots t_i \dots) \text{ and } k > 0 \\ (r_1, r_0^*, s_1, \dots, s_k)^* & \text{, if } r = (r_1 r_0) \end{cases}$$

Proposition

If $r s_1 \dots s_k$ $(k \geq 0)$ is a term such $d(r s_1 \dots s_k) \leq m+1$ and $d(s_1), \dots, d(s_k) \leq m$, then $d((r, s_1, \dots, s_k)^*) \leq m$.

Corollary

If $d(r) = m < \omega$, then $r^{\overbrace{* \dots *}^{m}}$ is normal.

Majorization of terms by abstract trees

Inductive definition of $t \lhd \alpha$ (α majorizes t) for $\alpha \in \mathcal{O}_1$

($\lhd$1) $t \lhd \alpha$, if t is a prime term.

($\lhd$2) $t \lhd \alpha \Rightarrow \lambda v t \lhd \alpha + 1$

($\lhd$3) $t_0 \lhd \alpha \wedge \dots \wedge t_n \lhd \alpha \Rightarrow t_0 \dots t_n \lhd \alpha + 1$

($\lhd$4) $\forall i, j \in \mathbb{N} (i \leq j \to t_i \lhd \alpha_j) \wedge r_0 \lhd \alpha_0 \Rightarrow (\dots t_i \dots) r_0 \lhd (\alpha_i)_{i \in \mathbb{N}}$

($\lhd$5) $\forall i \in \mathbb{N} (t \lhd \alpha_i) \Rightarrow t \lhd (\alpha_i)_{i \in \mathbb{N}}$

Remark

This definition differs from that of $t \lhd \alpha$ (on pg. 11) in those cases which concern terms $(t_i)_{i \in \mathbb{N}}$ and trees $(\alpha_i)_{i \in \mathbb{N}}$;

where $\lhd$ was defined more liberal than $\lhd$. The terms $(t_i)_{i\in\mathbb{N}}$ and trees $(d_i)_{i\in\mathbb{N}}$ here are treated in the same way as the terms $(t_\chi)_{\chi\in\mathfrak{I}_n}$ and trees $(a_\xi)_{\xi\in O_1}$ were treated in part I. This correspondence also holds in the definition of $(r, s_1, \ldots, s_k)^*$.

Lemma 1

a) $t \lhd a+1 \wedge b \neq 0 \Rightarrow t \lhd a+b$.

b) $t \lhd a \Rightarrow t \lhd a+b$.

c) $t \lhd b \Rightarrow t \lhd a+b$.

Lemma 2

$t \lhd b$ and $s \lhd a \Rightarrow t[{}^v/_s] \lhd a+b$

Lemma 3

If $rs_1 \ldots s_k$ $(k \geq 0)$ is a term with $r \lhd b$ and $s_i \lhd a$ (for $i=1,\ldots,k$), then $(r, s_1, \ldots, s_k)^* \lhd a+2^b$.

These lemmata are proved in the same way as the corresponding lemmata in part I.

Lemma 4

Let t be a normal term of type 0 with at most one free variable v^0. Then we have

$$\forall \alpha \in O_1\,(t \lhd \alpha \Rightarrow \forall n \in \mathbb{N}\,(V_n t \leq F_\alpha(n))),$$

where $V_n t$ denotes the value of $t[{}^{v^0}/_n]$.

Proof by induction on α:

($\lhd$1) If t is a prime term, then $t=0$ or $t=v^0$. In both cases we have $V_n t \leq n \leq F_\alpha(n)$.

($\lhd$3) $t = St_0$ and $\alpha = \alpha_0+1$ with $t_0 \lhd \alpha_0$. Then by I.H. we have

$V_n t_0 \leq F_{\alpha_0}(n)$. Hence $V_n t = V_n t_0 + 1 \leq F_{\alpha_0}(n) + 1 = F_\alpha(n)$.

($\lhd 4$) $t = (\dots t_i \dots) r_0$ and $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ with $\forall i j (i \leq j \to t_i \lhd \alpha_j) \wedge r_0 \lhd \alpha_0$.
Then $V_n t = V_n t_i$, where $i := V_n r_0$, and $F_\alpha(n) = F_{\alpha_j}(n)$, where $j := F_{\alpha_0}(n)$.
From $r_0 \lhd \alpha_0$ by I.H. follows $i = V_n r_0 \leq F_{\alpha_0}(n) = j$ and therefore $t_i \lhd \alpha_j$. Again by I.H. we now obtain $V_n t_i \leq F_{\alpha_j}(n)$.

($\lhd 5$) $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ and $\forall i \in \mathbb{N} (t \lhd \alpha_i)$. Then we have $F_\alpha(n) = F_{\alpha_i}(n)$ with $i := F_{\alpha_0}(n)$, and by I.H. $V_n t \leq F_{\alpha_i}(n)$. Hence $V_n t \leq F_\alpha(n)$.

Proof of theorem 2:

Gödel's system T can be described as the set all terms which are generated by the rules (Tm1), (Tm2), (Tm3), (Tm4)$^-$

(Tm4)$^-$ If for every $i \in \mathbb{N}$ t_i is a term of type τ such that $t_0 = v^\tau$ and $t_{i+1} = v^{o\tau\tau}\,\bar{i}\,t_i$, then $(t_i)_{i\in\mathbb{N}} u$ is a term of type τ.[1] (Here u denotes any variable of type o; and $\bar{i}$ denotes the term $\underbrace{(S(\dots(S0)\dots))}_{i\text{ times}}$, i.e. $\bar{0} := 0$, $\overline{i+1} := (S\bar{i})$.)

Proposition: $t \in T \Rightarrow d(t) < \omega$ and $t \lhd \bar{\omega} + l$ for some $l \in \mathbb{N}$.
Proof by induction on the rules (Tm1)–(Tm4)$^-$:
We only consider the rule (Tm4)$^-$. In all other cases the assertion follows immediately from the I.H.
Let $t = (t_i)_{i\in\mathbb{N}} u$, where $t_0 = v^\tau$, $t_{i+1} = v^{o\tau\tau}\,\bar{i}\,t_i$.
By induction on i we obtain $\bar{i} \lhd i$ and $t_i \lhd i$.
Hence $\forall i \in \mathbb{N} (t_i \lhd i+1) \wedge u \lhd 1$ and therefore
$t = (t_i)_{i\in\mathbb{N}} u \lhd (i+1)_{i\in\mathbb{N}} = \bar{\omega}$ by ($\lhd 4$).

---

1) The term $\lambda v^\tau \lambda v^{o\tau\tau} \lambda u ((t_i)_{i\in\mathbb{N}} u)$ represents the functional R_τ of prim. recursion: $R_\tau g h 0 = g$, $R_\tau g h (n+1) = h n (R_\tau g h n)$.

Now let $f : \mathbb{N} \to \mathbb{N}$ be definable in T. That means, there exists a term $t \in T$ of type o with at most one free variable v^o such that $\forall n \in \mathbb{N} (f(n) = V_n t)$. (As before $V_n t$ denotes the value of $t[v^o/n]$.)
By the above proposition we have $m := d(t) < \omega$ and $t \lhd \dot{\omega} + l$ for some $l \in \mathbb{N}$. From this by lemma 3 we obtain a normal term $\hat{t} := t^{\underbrace{*,\dots,*}_{m}}$ with $\hat{t} \lhd \underbrace{2^{\cdot^{\cdot^{2^{\dot{\omega}+l}}}}}_{m} =: \alpha$ and $\forall n \in \mathbb{N} (f(n) = V_n \hat{t})$. Now the assertion $\forall n (f(n) \leq F_\alpha(n))$ follows by lemma 4 (applied to $\hat{t}$).

III. A connection between the slow and fast growing hierarchies

Definition of the slow growing hierarchy $(g_\alpha)_{\alpha\in O_2}$

1. $g_0(n) := 0$
2. $g_{\alpha+1}(n) := g_\alpha(n)+1$
3. $g_{(\alpha_i)_{i\in\mathbb{N}}}(n) := g_{\alpha_n}(n)$
4. $g_{(\alpha_\xi)_{\xi\in O_1}}(n) := (g_{\alpha_\xi}(n))_{\xi\in\mathbb{N}}$

We repeat the definitions of $(F_\alpha)_{\alpha\in O_1}$ and $D: O_2\to O_1$

$F_0(n) := n+1$	$D0 := \dot{\omega}$
$F_{\alpha+1}(n) := F_\alpha(n)+1$	$D(\alpha+1) := D\alpha+1$
$F_{(\alpha_i)_{i\in\mathbb{N}}}(n) := F_{\alpha_k}(n)$, where $k := F_{\alpha_0}(n)$.	$D(\alpha_i)_{i\in\mathbb{N}} := (D\alpha_i)_{i\in\mathbb{N}}$
	$D(\alpha_\xi)_{\xi\in O_1} := D\alpha_{D\alpha_0}$

Corollary (1) $\alpha\in O_2 \Rightarrow g_\alpha(n)\in O_1$.

(2) $\alpha\in O_1 \Rightarrow g_\alpha(n)\in\mathbb{N}$.

Inductive definition of a set E of expressions

(E1) $x\in E$.

(E2) $1\in E$.

(E3) $\mathfrak{u},\mathfrak{b}\in E \Rightarrow (\mathfrak{u}+\mathfrak{b})\in E$ and $2^{\mathfrak{b}}\in E$.

For $\mathfrak{u}\in E$ and $\xi\in O_k$ let $\mathfrak{u}(\xi)$ denote the tree given by $\mathfrak{u}$ when x is interpreted as ξ.

Theorem 3

$$F_{\mathfrak{u}(\dot{\omega})} = g_{D\mathfrak{u}(\dot{\Omega})} \quad \text{for every } \mathfrak{u}\in E.$$

Lemma 1 For $a, b\in O_2$ we have

a) $g_{a+b}(n) = g_a(n)+g_b(n)$.

b) $g_{2^b}(n) = 2^{g_b(n)}$.

Proof by induction on b:

a) 1. $g_{a+0}(n) = g_a(n) + 0 = g_a(n) + g_0(n)$.

2. $g_{a+(b+1)}(n) = g_{a+b}(n) + 1 \underset{I.H.}{=} g_a(n) + g_b(n) + 1 = g_a(n) + g_{b+1}(n)$.

3. $g_{a+(b_i)_{i\in\mathbb{N}}}(n) = g_{(a+b_i)_{i\in\mathbb{N}}}(n) = g_{a+b_n}(n) \underset{I.H.}{=} g_a(n) + g_{b_n}(n) = g_a(n) + g_b(n)$.

4. $g_{a+(b_\xi)_{\xi\in\theta_1}}(n) = g_{(a+b_\xi)_{\xi\in\theta_1}}(n) = (g_{a+b_\xi}(n))_{\xi\in\mathbb{N}} \underset{I.H.}{=} (g_a(n) + g_{b_\xi}(n))_{\xi\in\mathbb{N}} = g_a(n) + (g_{b_\xi}(n))_{\xi\in\mathbb{N}}$.

b) 1. $g_{2^0}(n) = g_1(n) = 1 = 2^0 = 2^{g_0(n)}$.

2. $g_{2^{b+1}}(n) = g_{2^b+2^b}(n) \underset{a)}{=} g_{2^b}(n) + g_{2^b}(n) \underset{I.H.}{=} 2^{g_b(n)} + 2^{g_b(n)} = 2^{g_b(n)+1}$.

3. $g_{2^{(b_i)_{i\in\mathbb{N}}}}(n) = g_{2^{b_n}}(n) \underset{I.H.}{=} 2^{g_{b_n}(n)} = 2^{g_b(n)}$.

4. $g_{2^{(b_\xi)_{\xi\in\theta_1}}}(n) = (g_{2^{b_\xi}}(n))_{\xi\in\mathbb{N}} \underset{I.H.}{=} (2^{g_{b_\xi}(n)})_{\xi\in\mathbb{N}} = 2^{(g_{b_\xi}(n))_{\xi\in\mathbb{N}}} = 2^{g_{(b_\xi)_{\xi\in\theta_1}}(n)}$.

<u>Inductive definition of a subset $E(\Omega)$ of θ_2</u>

1. $a \in \theta_1 \cup \{\Omega\} \Rightarrow a \in E(\Omega)$

2. $a, b \in E(\Omega) \Rightarrow (a+b) \in E(\Omega)$

3. $(b_\xi)_{\xi\in\theta_\ell} \in E(\Omega) \Rightarrow (2^{b_\xi})_{\xi\in\theta_\ell} \in E(\Omega)$

<u>Lemma 2</u>

a) $c+1 \in E(\Omega) \Rightarrow c \in E(\Omega)$

b) $b \in E(\Omega) \Rightarrow 2^b \in E(\Omega)$

c) $(c_\xi)_{\xi\in\theta_\ell} \in E(\Omega)$ $(\ell = 0,1) \Rightarrow \forall \xi \in \theta_\ell\, (c_\xi \in E(\Omega))$

Proof by induction over $E(\Omega)$ (for a) and c)).

<u>a)</u> 1. $c+1 \in \theta_1 \Rightarrow c \in \theta_1$.

2. $c+1 = a+b$ with $a, b \in E(\Omega)$. Then one of the following two cases holds:

2.1. $c+1 = a$: Then $c \in E(\Omega)$ by I.H.

2.2. $c = a+b_0$ with $b = b_0+1$: From $b_0+1 \in E(\Omega)$ by I.H. follows $b_0 \in E(\Omega)$.
From $a, b_0 \in E(\Omega)$ we obtain $c = a+b_0 \in E(\Omega)$.

<u>b)</u> Proof by induction on b:

1. $b = 0$: Then $2^b = 1 \in \theta_1 \subset E(\Omega)$.

2. $b = b_0+1$: Then $2^b = 2^{b_0}+2^{b_0}$ and $b_0 \in E(\Omega)$ by a).
Applying I.H. we obtain $2^{b_0} \in E(\Omega)$ and then $2^{b_0}+2^{b_0} \in E(\Omega)$.

3. $b = (b_\xi)_{\xi\in O_\ell}$: Then $2^b = (2^{b_\xi})_{\xi\in O_\ell} \in E(\Omega)$ by definition of $E(\Omega)$.

c) Let $c := (c_\xi)_{\xi\in O_\ell}$.

1.1. $c \in O_1$: Then $c_\xi \in O_1 \subset E(\Omega)$.

1.2. $c = \dot\Omega$: Then $c_\xi = 1+\xi \in O_1 \subset E(\Omega)$.

2. $c = a+b$ with $a, b \in E(\Omega)$. Then we have $b = (b_\xi)_{\xi\in O_\ell}$ with $c_\xi = a + b_\xi$. From $b \in E(\Omega)$ by I.H. we obtain $b_\xi \in E(\Omega)$. From $a, b_\xi \in E(\Omega)$ follows $a + b_\xi \in E(\Omega)$.

3. $c = (2^{b_\xi})_{\xi\in O_\ell}$ with $(b_\xi)_{\xi\in O_\ell} \in E(\Omega)$. Then we have $c_\xi = 2^{b_\xi}$ and by I.H. $b_\xi \in E(\Omega)$. Using b) we obtain $c_\xi \in E(\Omega)$.

Lemma 3

$$(c_\xi)_{\xi\in O_1} \in E(\Omega) \;\Rightarrow\; \forall \xi \in O_1 \,\big(g_{c_\xi}(n) = g_{c_{g_\xi(n)}}(n) \big)$$

Proof by induction over $E(\Omega)$: Let $c := (c_\xi)_{\xi\in O_1}$

1. $c = \dot\Omega = (1+\xi)_{\xi\in O_1}$: $g_{c_\xi}(n) = g_{1+\xi}(n) \underset{1a)}{=} 1 + g_\xi(n) = g_{1+g_\xi(n)}(n) = g_{c_{g_\xi(n)}}(n)$.

2. $c = a+b$: Then $b = (b_\xi)_{\xi\in O_1}$ and $c_\xi = a + b_\xi$.

Hence $g_{c_\xi}(n) = g_{a+b_\xi}(n) \underset{1a)}{=} g_a(n) + g_{b_\xi}(n) \underset{I.H.}{=} g_a(n) + g_{b_{g_\xi(n)}}(n) \underset{1a)}{=} g_{a+b_{g_\xi(n)}}(n)$
$= g_{c_{g_\xi(n)}}(n)$.

3. $c = (2^{b_\xi})_{\xi\in O_1}$: $g_{c_\xi}(n) = g_{2^{b_\xi}}(n) \underset{1b)}{=} 2^{g_{b_\xi}(n)} \underset{I.H.}{=} 2^{g_{b_k}(n)} \underset{1b)}{=} g_{2^{b_k}}(n)$
$= g_{c_k}(n)$, where $k := g_\xi(n)$.

Lemma 4

$$a \in E(\Omega) \Rightarrow F_{g_a(n)}(n) = g_{Da}(n).$$

Proof by induction over a:

1. $a = 0$: $F_{g_0(n)}(n) = F_0(n) = n+1 = g_{n+1}(n) = g_{\dot\omega}(n) = g_{D0}(n)$.

2. $a = a_0 + 1$: $F_{g_a(n)}(n) = F_{g_{a_0}(n)+1}(n) = F_{g_{a_0}(n)}(n) + 1 \underset{I.H.}{=} g_{Da_0}(n) + 1 = g_{Da}(n)$.
Note that by lemma 2a) $a \in E(\Omega)$ implies $a_0 \in E(\Omega)$.

3. $a = (a_i)_{i\in N}$: $F_{g_a(n)}(n) = F_{g_{a_n}(n)}(n) \underset{I.H.}{=} g_{Da_n}(n) = g_{Da}(n)$, since $Da = (Da_i)_{i\in N}$.

Note that by lemma 2c, $a \in E(\Omega)$ implies $a_n \in E(\Omega)$.

4. $a = (a_\xi)_{\xi \in \theta_1}$: Then we have

(1) $Da = Da_\xi$, where $\xi := Da_0$

(2) $F_{g_{a(n)}}(n) = F_{(g_{a_i}(n))_{i\in N}}(n) = F_{g_{a_k}(n)}(n)$, where $k := F_{g_{a_0}(n)}(n)$.

By lemma 2c we have $a_0, a_\xi \in E(\Omega)$ and therefore by I.H.

(3) $k = F_{g_{a_0}(n)}(n) = g_{Da_0}(n) = g_\xi(n)$,

(4) $F_{g_{a_\xi}(n)}(n) = g_{Da_\xi}(n)$.

By lemma 3 we have

(5) $g_{a_\xi}(n) = g_{a_{g_\xi(n)}}(n)$.

Hence $F_{g_{a(n)}}(n) \underset{(2)}{=} F_{g_{a_k}(n)}(n) \underset{(3)}{=} F_{g_{a_{g_\xi(n)}}(n)}(n) \underset{(5)}{=} F_{g_{a_\xi}(n)}(n) \underset{(4)}{=} g_{Da_\xi}(n)$
$\underset{(1)}{=} g_{Da}(n)$.

Lemma 5

$\mathfrak{A} \in E \Rightarrow g_{\mathfrak{A}(\dot\Omega)}(n) = \mathfrak{A}(\dot\omega)$.

Proof by induction over E:

(E1) $g_{\dot\Omega}(n) = (g_{1+\xi}(n))_{\xi \in N} = (1+\xi)_{\xi \in N} = \dot\omega$.

(E2) $g_1(n) = 1$.

(E3) $g_{(\mathfrak{A}+\mathfrak{B})(\dot\Omega)}(n) = g_{\mathfrak{A}(\dot\Omega)+\mathfrak{B}(\dot\Omega)}(n) \underset{1a)}{=} g_{\mathfrak{A}(\dot\Omega)}(n) + g_{\mathfrak{B}(\dot\Omega)}(n) \underset{I.H.}{=} \mathfrak{A}(\dot\omega) + \mathfrak{B}(\dot\omega) = (\mathfrak{A}+\mathfrak{B})(\dot\omega)$.

$g_{2^{\mathfrak{B}}(\dot\Omega)}(n) = g_{2^{\mathfrak{B}(\dot\Omega)}}(n) \underset{1b)}{=} 2^{g_{\mathfrak{B}(\dot\Omega)}(n)} \underset{I.H.}{=} 2^{\mathfrak{B}(\dot\omega)} = 2^{\mathfrak{B}}(\dot\omega)$.

Proof of theorem 3:

By lemma 5 we have $F_{\mathfrak{A}(\dot\omega)}(n) = F_{g_{\mathfrak{A}(\dot\Omega)}(n)}(n)$.

By lemma 4 we have $F_{g_{\mathfrak{A}(\dot\Omega)}(n)}(n) = g_{D\mathfrak{A}(\dot\Omega)}(n)$, since $\mathfrak{A}(\dot\Omega) \in E(\Omega)$.
(cf. lemma 2b!)

Hence $F_{\mathfrak{A}(\dot\omega)}(n) = g_{D\mathfrak{A}(\dot\Omega)}(n)$.

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