Another reduction of classical ID_ν to constructive ID^i_ν

Wilfried Buchholz

Dedicated to Wolfram Pohlers on his retirement

Introduction.

One of the major problems in reductive proof theory in the early 1970s was to give a proof-theoretic reduction of classical theories of iterated arithmetical inductive definitions to corresponding constructive systems. This problem was solved in [BFPS] in various ways which all where based on the method of cut-elimination (normalization, reps.) for infinitary Tait-style sequent calculi (infinitary systems of natural deduction, resp.). Only quite recently Avigad and Towsner [AT09] succeeded in giving a reduction of classical iterated ID theories to constructive ones by the method of functional interpretation. For a thorough exposition and discussion of all this cf. [Fef].

In the present paper we give yet another reduction of classical ID_ν to $\mathrm{ID}^i_\nu(\mathcal{W})$ based on cut-elimination arguments. $\mathcal{W}$ is a particularly simple accessibility ID; its corresponding operator form $\mathcal{W}(P,Q,y,x)$ (cf. [BFPS]) has the shape $A(x,y)\wedge\forall z(\widetilde{Q}(t(x),z)\to Pq(x,z))$ with primitive recursive A,t,q, and $\widetilde{Q}(u,z):\equiv u\geq 1\wedge(u\geq 2\to Q(u\dot{-}2,z))$. There are two reasons which, as we hope, justify a publication of this additional proof. First, it is considerably more direct then all the existing ones. Second, the method used here stems to a great extent from [Ge36] and therefore may be interesting for historical reasons too. Actually I have already used a variant of this method under the label "notations for infinitary derivations" in several papers (e.g. [Bu91], [Bu97], [Bu01]) without mentioning its close relationship to [Ge36]. When writing [Bu91] I was definitely not aware of this connection; but cf. [Bu95]. The method from [Ge36] can be roughly described as follows: By (primitive) recursion on the build-up of h, for each derivation h in a suitably designed finitary proof system Z of first order arithmetic a family $(h[i])_{i\in I_h}$ of Z-derivations is defined such that $\dfrac{\ldots\Gamma(d[i])\ldots(i\in I_h)}{\Gamma(h)}$ (where $\Gamma(h)$ denotes the endsequent of h) forms an inference in cutfree ω-arithmetic (with repetition-rule). Then the consistency of Z is obtained by quantifierfree transfinite induction over the relation $\prec:=\{(h[i],h): h\in Z\ \&\ i\in I_h\}$. In the present paper we proceed similarly. Let ID_ν be the finitary Tait-style system of ν-fold iterated inductive definitions as introduced in [Bu02]. We extend ID_ν by certain inferences E, D_σ, $\mathsf{S}^{\Pi}_{\mathcal{P},\mathcal{F}}$ (which do not alter the set of derivable sequents) to a finitary system ID^*_ν. This step corresponds very much to the passage from BI^-_1 to BI^*_1 in [Bu01]. Then by primitive recursion on the height of h, for each closed ID^*_ν-derivation h we define a family $(h[\iota])_{\iota\in I_h}$ of closed ID^*_ν-derivations such that $\dfrac{\ldots\Gamma(h[\iota])\ldots(\iota\in I_h)}{\Gamma(h)}$ is an inference in the infinitary system ID^∞_ν. Formulated more technical, we assign to h an inference symbol $\mathsf{tp}(h)$ of ID^∞_ν, and for each $\iota\in|\mathsf{tp}(h)|$ a closed ID^*_ν-derivation $h[\iota]$ such that $\dfrac{\ldots\Gamma(h[\iota])\ldots(\iota\in|\mathsf{tp}(h)|)}{\Gamma(h)}$ is a $\mathsf{tp}(h)$-inference (Lemma 1). On first sight the present system ID^∞_ν looks exactly like the system ID^∞_ν in [Bu02] (which itself is the Tait-style version of the natural deduction system ID^∞_ν from [Bu81]), but there is some subtle difference concerning the index sets $|\widetilde{\Omega}_P|$ of instances of the Ω-rule. In [Bu02], $|\widetilde{\Omega}_P|$ is a set of infinitary derivations while in the present paper $|\widetilde{\Omega}_P|$ is a set of finite derivations, namely $|\widetilde{\Omega}_P|=\mathbf{I}_\mu=$ set of all closed ID^*_ν-derivations h with $\deg(h)=0$ and $\Gamma(h)\subseteq\mathrm{Pos}_\mu$, where $\mu:=\mathrm{lev}(P)$. Now let $\mathcal{W}_\sigma$ be the accessible part of the relation $\{(h[\iota],h): h\in\mathbf{I}_\sigma\ \&\ \iota\in|\mathsf{tp}(h)|_{\mathcal{W}}\}$, where $|\mathcal{I}|_{\mathcal{W}}:=\mathcal{W}_\mu$ if $\mathcal{I}=\widetilde{\Omega}_P$ with $\mu:=\mathrm{lev}(P)<\sigma$, and $|\mathcal{I}|_{\mathcal{W}}:=|\mathcal{I}|$ otherwise. The proof-theoretic reduction of ID_ν to $\mathrm{ID}^i_\nu(\mathcal{W})$ will be established by a proof of transfinite induction over the relation $\{(h[i],h): h\in\mathbf{I}_0\ \&\ i\in|\mathsf{tp}(h)|\}$ which can be locally formalized in $\mathrm{ID}^i_\nu(\mathcal{W})$. The difficulty here is to come along without the uppermost set $\mathcal{W}_\nu$, which would be available in $\mathrm{ID}^i_{\nu+1}(\mathcal{W})$ but not in $\mathrm{ID}^i_\nu(\mathcal{W})$. We overcome this difficulty by using (a generalization of) Gentzen's technique (cf. [Ge43]) for proving transfinite induction up to ordinals $<\varepsilon_0$ within Z.

In order to avoid some annoying but inessential technicalities we restrict our treatment to $\nu < \omega$. So in the whole paper ν is a fixed natural number > 0.

Preliminaries. For the reader's convenience we repeat some basic definitions and abbreviations from [Bu02] (with some minor deviations). Let $\mathcal{L}$ be an arbitrary first order language (i.e. set of function and predicate symbols). Atomic $\mathcal{L}$-formulas are $Rt_1...t_n$ where R is an n-ary predicate symbol (of $\mathcal{L}$), and $t_1,...,t_n$ are $\mathcal{L}$-terms. Expressions of the shape A or $\neg A$, where A is an atomic $\mathcal{L}$-formula, are called *literals*. $\mathcal{L}$-formulas are built up from literals by means of $\wedge, \vee, \forall x, \exists x$. $\mathrm{FV}(A)$ denotes the set of free variables of A. A formula or term A is called *closed* if $\mathrm{FV}(A) = \emptyset$. The *negation* $\neg A$ of a non-atomic formula A is defined via de Morgan's laws. The *rank* $\mathrm{rk}(A)$ of a formula A is defined by: $\mathrm{rk}(A) := 0$ if A is a literal, $\mathrm{rk}(A \wedge B) := \mathrm{rk}(A \vee B) := \max\{\mathrm{rk}(A), \mathrm{rk}(B)\} + 1$, $\mathrm{rk}(\forall x A) := \mathrm{rk}(\exists x A) := \mathrm{rk}(A) + 1$. By $A(x/t)$ we denote the result of substituting t for (every free occurrence of) x in A (renaming bound variables if necessary). Expressions $\lambda x.F$ (where F is a formula) are called *predicates* and denoted by $\mathcal{F}$. For $\mathcal{F} = \lambda x.F$ we set $\mathcal{F}(t) := F(x/t)$. If $\mathcal{P}$ is a unary predicate symbol then $B(\mathcal{P}/\mathcal{F})$ denotes the result of substituting $\mathcal{F}$ for $\mathcal{P}$ in B, i.e. the formula resulting from B be replacing every atom $\mathcal{P}t$ by $\mathcal{F}(t)$. Let X be unary predicate symbol not in $\mathcal{L}$. A *positive operator form in* $\mathcal{L}$ is an $\mathcal{L} \cup \{X\}$-formula $\mathfrak{A}$ in which X occurs only positively (i.e. $\mathfrak{A}$ has no subformula $\neg Xt$) and which has at most one free variable x. We use the following abbreviations: $\mathfrak{A}(\mathcal{F}, t) := \mathfrak{A}(x/t)(X/\mathcal{F})$, $\mathfrak{A}(\mathcal{F}) \subseteq \mathcal{F} := \forall x(\mathfrak{A}(\mathcal{F}, x) \to \mathcal{F}(x))$. For each positive operator form $\mathfrak{A}$ we introduce a new unary predicate symbol $\mathcal{P}_{\mathfrak{A}}$. Finite sets of formulas are called *sequents*. They are denoted by Γ, Δ, Π. We mostly write $A_1,...,A_n$ for $\{A_1,...,A_n\}$, and A, Γ, Δ for $\{A\} \cup \Gamma \cup \Delta$, etc.

Definition ($\mathcal{L}_\sigma$, Pos_σ, level).
Let $\mathcal{L}_0$ be a language consisting of the constant 0 (*zero*), the unary function symbol S (*successor*), and some predicate symbols R for primitive recursive relations, such that the set TRUE_0 of all true closed $\mathcal{L}_0$-literals is itself primitive recursive (under some canonincal arithmetization of syntax). The only closed $\mathcal{L}_0$-terms are the *numerals* $0, S0, SS0, ...$ which we identify with the corresponding natural numbers (elements of $\mathbb{N}$). Arbitrary $\mathcal{L}_0$-terms will be denoted by $t, t_1, ...,$ and (number) variables by x, y.
$\mathcal{L}_{\sigma+1} := \mathcal{L}_0 \cup \{\mathcal{P}_{\mathfrak{A}} : \mathfrak{A} \text{ positive operator form in } \mathcal{L}_\sigma\} \quad (\sigma < \omega)$
$\mathrm{Pos}_\sigma :=$ set of all $\mathcal{L}_{\sigma+1}$-formulas C such every $\mathcal{P}_{\mathfrak{A}}$ occurring negatively in C belongs to $\mathcal{L}_\sigma$.
$\mathrm{lev}(\mathcal{P}_{\mathfrak{A}}) := \mathrm{lev}(\mathcal{P}_{\mathfrak{A}}t) := \min\{\sigma : \mathcal{P}_{\mathfrak{A}}t \in \mathrm{Pos}_\sigma\}$ (level)
Note that this "level" is not exactly the same as "level" in [Bu02].

Proposition.
(1) $\mathcal{L}_\sigma$-formulas $\subseteq \mathrm{Pos}_\sigma \subseteq \mathcal{L}_{\sigma+1}$-formulas
(2) $\mathcal{P}_{\mathfrak{A}}t \in \mathrm{Pos}_\sigma \;\Rightarrow\; \mathfrak{A}(\mathcal{P}_{\mathfrak{A}}, t) \in \mathrm{Pos}_\sigma$.

Abbreviations.
$\mathcal{L}_0$-lit := set of all $\mathcal{L}_0$-literals.
$\bigwedge$-for := set of all formulas of the shape $A \wedge B$ or $\forall x A$.
$C \in \bigwedge^+\text{-for} \;:\Leftrightarrow\; C \in \bigwedge\text{-for} \quad \text{or} \quad C$ has the shape $\mathcal{P}_{\mathfrak{A}}t$

$$C[k] := \begin{cases} C_k & \text{if } C = C_0 \overset{\vee}{\wedge} C_1 \text{ and } k \in \{0,1\} \\ A(x/k) & \text{if } C = \overset{\exists}{\forall} x A \text{ and } k \in \mathbb{N} \end{cases}$$

Definition (Inference symbols).
An *inference symbol* is a formal expression $\mathcal{I}$ for which the following entities are given
- a set $|\mathcal{I}|$ (the *arity* of $\mathcal{I}$),
- a sequent $\Delta(\mathcal{I})$ *(principal formula(s))*,
- for each $\iota \in |\mathcal{I}|$ a sequent $\Delta_\iota(\mathcal{I})$ *(minor formula(s))*,

An inference symbol is called *(in)finitary* if its arity is (in)finite.

Notation. By writing $(\mathcal{I})\quad \dfrac{\ldots\Delta_\iota\ldots(\iota{\in}I)}{\Delta}$
we declare $\mathcal{I}$ as an inference symbol with $|\mathcal{I}| = I$, $\Delta(\mathcal{I}) = \Delta$, $\Delta_\iota(\mathcal{I}) = \Delta_\iota$.
If $I = \{0, ..., n{-}1\}$ we write $\dfrac{\Delta_0 \;\; \Delta_1 \;\; \ldots \;\; \Delta_{n-1}}{\Delta}$, instead of $\dfrac{\ldots\Delta_\iota\ldots(\iota{\in}I)}{\Delta}$.
Inference symbols $\mathcal{I}$ with $|\mathcal{I}| = \emptyset$ are called *axioms*.

Definition (Proof systems).
A *proof system* is given by a language $\mathcal{L}$ and a set of inference symbols in this language, where "$\mathcal{I}$ in $\mathcal{L}$" means that all elements of $\Delta(\mathcal{I}) \cup \bigcup_{\iota\in|\mathcal{I}|}\Delta_\iota(\mathcal{I})$ are $\mathcal{L}$-formulas. A proof system is called *finitary* if all its inference symbols are finitary; otherwise it is called *infinitary.*
From now on the letters A, B, C always denote $\mathcal{L}_\nu$-formulas, and $\mathcal{P}$ ranges over predicate symbols $\mathcal{P}_\mathfrak{A} \in \mathcal{L}_\nu$.

Definition (The finitary proof systems ID_ν and ID^*_ν).
The language of ID_ν is $\mathcal{L}_\nu$, and the inference symbols of ID_ν are

$(\mathsf{Ax}_\Gamma)\ \dfrac{}{\Gamma}$ if $\Gamma \in \mathsf{Ax}(\nu)$
where $\mathsf{Ax}(\nu)$ is a set of $\mathcal{L}_\nu$-sequents such that
(i) $\Gamma \in \mathsf{Ax}(\nu) \Longrightarrow \Gamma(\vec{x}/\vec{t}) \in \mathsf{Ax}(\nu)$
(ii) $\Gamma \in \mathsf{Ax}(\nu)$ & $\mathrm{FV}(\Gamma) = \emptyset \Longrightarrow \Gamma \cap \mathsf{TRUE}_0 \neq \emptyset$ or $\Gamma = \{\neg\mathcal{P}n, \mathcal{P}n\}$ or $\Gamma = \{n \neq n, \neg\mathcal{P}n, \mathcal{P}n\}$
(iii) $\{\neg A, A\} \in \mathsf{Ax}(\nu)$ for each atomic $\mathcal{L}_\nu$-formula A

$(\bigwedge_{A_0\wedge A_1})\ \dfrac{A_0 \qquad A_1}{A_0{\wedge}A_1}$, $\quad (\bigvee^k_{A_0\vee A_1})\ \dfrac{A_k}{A_0{\vee}A_1}\ (k \in \{0,1\})$, $\quad (\bigwedge^y_{\forall xA})\ \dfrac{A(x/y)}{\forall xA}$, $\quad (\bigvee^t_{\exists xA})\ \dfrac{A(x/t)}{\exists xA}$,

$(\mathsf{Cut}_C)\ \dfrac{C \qquad \neg C}{\emptyset}\ (C \in \bigwedge^{+}\text{-for} \cup \mathcal{L}_0\text{-lit})$, $\quad (\mathsf{Ind}^t_\mathcal{F})\ \dfrac{}{\neg\mathcal{F}(0), \neg\forall x(\mathcal{F}(x) \to \mathcal{F}(Sx)), \mathcal{F}(t)}$,

$(\mathsf{Cl}_{\mathcal{P}_\mathfrak{A} t})\ \dfrac{\mathfrak{A}(\mathcal{P}_\mathfrak{A}, t)}{\mathcal{P}_\mathfrak{A} t}$, $\quad (\mathsf{Ind}^{\mathcal{P}_\mathfrak{A} t}_\mathcal{F})\ \dfrac{}{\neg(\mathfrak{A}(\mathcal{F}){\subseteq}\mathcal{F}), \neg\mathcal{P}_\mathfrak{A} t, \mathcal{F}(t)}$.

The inference symbols Ax_Γ, $\bigwedge_{A\wedge B}$, $\bigvee^k_{A\vee B}$, $\bigvee^t_{\exists xA}$, $\mathsf{Cl}_{\mathcal{P}_\mathfrak{A} t}$, and Cut_C with $C \in \mathcal{L}_0$-lit are called *simple.*

The proof system ID^*_ν is obtained from ID_ν by adding the following inference symbols
$(\mathsf{J}^t_{\forall xA})\ \dfrac{\forall xA}{A(x/t)}$, $\quad (\mathsf{J}^k_{A_0\wedge A_1})\ \dfrac{A_0 \wedge A_1}{A_k}$,

$(\mathsf{S}^\Pi_{\mathcal{P},\mathcal{F}})\ \dfrac{\Pi}{\neg(\mathfrak{A}(\mathcal{F}) \subseteq \mathcal{F}), \Pi(\mathcal{P}/\mathcal{F})}$ with $\mathcal{P} = \mathcal{P}_\mathfrak{A}$ and $\Pi \subseteq \mathrm{Pos}_{\mathrm{lev}(\mathcal{P})}$,

$(\mathsf{E})\ \dfrac{\emptyset}{\emptyset}$, $\quad (\mathsf{D}_\sigma)\ \dfrac{\emptyset}{\emptyset}\ (\sigma < \nu)$.

The role of E and D_σ will become clear in the definition of h^+ below.

Inductive Definition of ID^*_ν-derivations
If $\mathcal{I}$ is an inference symbol of ID^*_ν of arity l and $h_0, \ldots, h_{l-1}$ are ID^*_ν-derivations such that for $\Gamma := \Delta(\mathcal{I}) \cup \bigcup_{i<l}(\Gamma(h_i) \setminus \Delta_i(\mathcal{I}))$ we have

- $\mathcal{I} = \bigwedge^y_{\forall xA} \Rightarrow y \notin \mathrm{FV}(\Gamma)$,
- $\mathcal{I} = \mathsf{Cut}_C \Rightarrow \mathrm{FV}(C) \subseteq \mathrm{FV}(\Gamma)$,
- $\mathcal{I} = \bigvee^t_C \Rightarrow \mathrm{FV}(t) \subseteq \mathrm{FV}(\Gamma)$,
- $\mathcal{I} = \mathsf{S}^\Pi_{\mathcal{P},\mathcal{F}} \Rightarrow \mathrm{FV}(\Pi) \subseteq \mathrm{FV}(\Gamma)$ and $h_0 = \mathsf{D}_\sigma h_{00}$ with $\sigma := \mathrm{lev}(\mathcal{P})$,
- $\mathcal{I} = \mathsf{D}_\sigma \Rightarrow \Gamma(h_0) \subseteq \mathrm{Pos}_\sigma$ & $\deg(h_0) = 0$,

then $h := \mathcal{I}h_0 \ldots h_{l-1}$ is an ID^*_ν-derivation and

$$\Gamma(h) := \Gamma \quad \textit{(endsequent of } h\textit{)}, \quad \deg(h) := \begin{cases} \deg(h_0) \dot{-} 1 & \text{if } \mathcal{I} = \mathsf{E} \\ \max\{\mathrm{rk}(C), \deg(h_0), \deg(h_1)\} & \text{if } \mathcal{I} = \mathsf{Cut}_C \\ \sup_{i<l} \deg(h_i) & \text{otherwise} \end{cases}$$

An ID_ν-derivation h is called *closed* if its endsequent $\Gamma(h)$ is closed, i.e. if $\mathrm{FV}(\Gamma(h)) = \emptyset$.

Abbreviations.

ID^*_ν := set of all closed ID^*_ν-derivations.

$h \vdash_m \Gamma \;:\Leftrightarrow\; h \in \mathsf{ID}^*_\nu$ with $\Gamma(h) \subseteq \Gamma$ and $\deg(h) \le m$.

$h \vdash^\sigma_m \Gamma \;:\Leftrightarrow\; h \vdash_m \Gamma$ and $\Gamma \subseteq \mathrm{Pos}_\sigma$.

$\mathbf{I}_\sigma := \{\mathsf{D}_\sigma h : h \vdash^\sigma_0 \Gamma(h)\}$ $(= \{\mathsf{D}_\sigma h : h \in \mathsf{ID}^*_\nu \;\&\; \deg(h) = 0 \;\&\; \Gamma(h) \subseteq \mathrm{Pos}_\sigma\})$ $(\sigma < \nu)$

Definition of $h(y/k)$ (Substitution of numerals).

For $h = \mathcal{I}h_0 \ldots h_{n-1}$ let

$$h(y/k) := \begin{cases} h & \text{if } \mathcal{I} = \bigwedge^y_{\forall xA} \\ \mathcal{I}(y/k)h_0(y/k)\ldots h_{l-1}(y/k) & \text{otherwise} \end{cases}$$

where $\mathcal{I}(y/k)$ is defined as expected, i.e., in such a way that the following holds:

$h \vdash_m \Gamma \;\Rightarrow\; h(y/k) \vdash_m \Gamma(y/k)$.

Convention.

From now on we use h as syntactic variable for closed ID^*_ν-derivations (i.e., elements of ID^*_ν).

Definition (The infinitary proof system ID^∞_ν).

The language of ID^∞_ν consists of all *closed* $\mathcal{L}_\nu$-formulas.

We use P as syntactic variable for formulas of the form $\mathcal{P}_\mathfrak{A} n$ with $\mathcal{P}_\mathfrak{A} \in \mathcal{L}_\nu$.

The inference symbols of ID^∞_ν are

- All simple inference symbols of ID_ν (restricted to closed formulas) where $\Delta(\mathsf{Ax}_\Gamma)$ is slightly modified, namely $\Delta(\mathsf{Ax}_\Gamma) := \begin{cases} \Gamma \cap \mathsf{TRUE}_0 & \text{if } \Gamma \cap \mathsf{TRUE}_0 \neq \emptyset \\ \Gamma & \text{otherwise} \end{cases}$.

- $(\bigwedge_{\forall xA})$ $\dfrac{\ldots A(x/i)\ldots (i \in \mathbb{N})}{\forall xA}$, $\quad(\mathsf{Cut}_C)$ $\dfrac{C \qquad \neg C}{\emptyset}$ $(C \in \bigwedge^+\text{-for})$, $\quad(\mathsf{Rep})$ $\dfrac{\emptyset}{\emptyset}$,

- $(\widetilde{\Omega}_P)$ $\dfrac{P \qquad \ldots \Gamma(q) \setminus \{P\} \ldots (q \in \mathbf{I}_\mu)}{\emptyset}$ with $\mu := \mathrm{lev}(P)$.

Definition of h^+, $\mathsf{tp}(h)$, $h[\iota]$

To each $h \in \mathsf{ID}^*_\nu$ we assign

- an inference symbol $\mathsf{tp}(h)$ of ID^∞_ν,
- for each $\iota \in |\mathsf{tp}(h)|$, a derivation $h[\iota] \in \mathsf{ID}^*_\nu$.

For the sake of conciseness we write

$h^+ = \mathcal{I}(h_\iota)_{\iota \in I}$ for $\mathsf{tp}(h) = \mathcal{I} \;\&\; |\mathcal{I}| = I \;\&\; \forall \iota \in I(h[\iota] = h_\iota)$.

The definition proceeds by (primitive) recursion on the height of h.

In clause 3. we make use of the following abbreviation:

$$\mathsf{Cut}^\circ_C(h_0, h_1) := \begin{cases} \mathsf{Cut}_C(h_0, h_1) & \text{if } C \in \bigwedge^+\text{-for} \cup \mathcal{L}_0\text{-lit} \\ \mathsf{Cut}_{\neg C}(h_1, h_0) & \text{otherwise} \end{cases}$$

Further we denote by $\mathbf{d}_A$ the canonical cutfree ID_ν-derivation of $\{\neg A, A\}$.

1.1. $(\mathcal{I}h_0 \ldots h_{l-1})^+ := \mathcal{I}(h_i)_{i<l}$ if $\mathcal{I}$ is simple.

1.2. $(\bigwedge^y_{\forall xA} \tilde{h})^+ := \bigwedge_{\forall xA} \big(\tilde{h}(y/i)\big)_{i \in \mathbb{N}}$

1.3. $(\mathsf{Ind}^{\mathcal{P}n}_\mathcal{F})^+ := \widetilde{\Omega}_{\mathcal{P}n} \mathsf{Ax}_{\{\neg\mathcal{P}n, \mathcal{P}n\}} \big(\mathsf{S}^{\{\mathcal{P}n\}}_{\mathcal{P},\mathcal{F}}\, q\big)_{q \in \mathbf{I}_\mu}$ with $\mu := \mathrm{lev}(\mathcal{P})$.

2. $(\mathsf{Ind}^n_\mathcal{F})^+ := \mathsf{Rep}(d_n)$ with $d_0 := \mathbf{d}_{\mathcal{F}(0)}$, $d_{i+1} := \bigvee^i_{\exists x(\mathcal{F}(x) \wedge \neg\mathcal{F}(Sx))} \bigwedge_{\mathcal{F}(i) \wedge \neg\mathcal{F}(Si)} d_i \mathbf{d}_{\mathcal{F}(Si)}$

3. If $C \in \bigwedge^{+}$-for and $h_1^+ = \mathcal{I}(h_{1\iota})_{\iota\in I}$ then:
$$(\mathsf{Cut}_C h_0 h_1)^+ := \begin{cases} \mathcal{I}(\mathsf{Cut}_C h_0 h_{1\iota})_{\iota\in I} & \text{if } \neg C \notin \Delta(\mathcal{I}) \\ \mathsf{Cut}^{\circ}_{C[k]}(\mathsf{J}^k_C h_0, \mathsf{Cut}_C h_0 h_{10}) & \text{if } \mathcal{I} = \bigvee^k_{\neg C} \\ \mathsf{Rep}(h_0) & \text{if } \neg C \in \Delta(\mathcal{I}) \text{ and } C = \mathcal{P}n \end{cases}$$

4. If $h^+ = \mathcal{I}(h_\iota)_{\iota\in I}$ then $(\mathsf{E}h)^+ := \begin{cases} \mathsf{Rep}(\mathsf{Cut}_C \mathsf{E}h_0 \mathsf{E}h_1) & \text{if } \mathcal{I} = \mathsf{Cut}_C \text{ with } C \in \bigwedge^{+}\text{-for} \\ \mathcal{I}(\mathsf{E}h_\iota)_{\iota\in I} & \text{otherwise} \end{cases}$

5. If $C \in \bigwedge$-for and $h^+ = \mathcal{I}(h_\iota)_{\iota\in I}$ then $(\mathsf{J}^k_C h)^+ := \begin{cases} \mathsf{Rep}(\mathsf{J}^k_C h_k) & \text{if } \mathcal{I} = \bigwedge_C \\ \mathcal{I}(\mathsf{J}^k_C h_\iota)_{\iota\in I} & \text{otherwise} \end{cases}$.

6. If $\mathcal{P} = \mathcal{P}_{\mathfrak{A}}$, $\mu := \mathrm{lev}(\mathcal{P})$ $(< \nu)$, and $d \in \mathbf{I}_\mu$ with $d^+ = \mathcal{I}(d_\iota)_{\iota\in I}$ then
$$(\mathsf{S}^{\Pi}_{\mathcal{P},\mathcal{F}} d)^+ := \begin{cases} \bigvee^n_{\neg(\mathfrak{A}(\mathcal{F})\subseteq\mathcal{F})}\big(\bigwedge_{\mathfrak{A}(\mathcal{F},n)\wedge\neg\mathcal{F}(n)}(\mathsf{S}^{\Pi\cup\Delta_0(\mathcal{I})}_{\mathcal{P},\mathcal{F}} d_0)\mathbf{d}_{\mathcal{F}(n)}\big) & \text{if } \mathcal{I} = \mathsf{Cl}_{\mathcal{P}n} \text{ with } \mathcal{P}n \in \Pi \\ \mathcal{I}^*(\mathsf{S}^{\Pi\cup\Delta_\iota(\mathcal{I})}_{\mathcal{P},\mathcal{F}} d_\iota)_{\iota\in I} & \text{if } \mathcal{I} = \bigwedge_A, \bigvee^k_A \text{ with } A \in \Pi \\ \mathcal{I}(\mathsf{S}^{\Pi}_{\mathcal{P},\mathcal{F}} d_\iota)_{\iota\in I} & \text{otherwise} \end{cases}$$
where $(\bigwedge_A)^* := \bigwedge_{A(\mathcal{P}/\mathcal{F})}$, $(\bigvee^k_A)^* := \bigvee^k_{A(\mathcal{P}/\mathcal{F})}$.

7. If $h^+ = \mathcal{I}(h_\iota)_{\iota\in I}$ then $(\mathsf{D}_\sigma h)^+ := \begin{cases} \mathsf{Rep}(\mathsf{D}_\sigma h_{\mathsf{D}_\mu h_0}) & \text{if } \mathcal{I} = \widetilde{\Omega}_P \text{ with } \mu := \mathrm{lev}(P) \geq \sigma \\ \mathcal{I}(\mathsf{D}_\sigma h_\iota)_{\iota\in I} & \text{otherwise} \end{cases}$

Definitions.
$\mathrm{ID}^\infty_\nu\lceil\sigma := \mathrm{ID}^\infty_\nu \setminus \{\widetilde{\Omega}_P : \mathrm{lev}(P) \geq \sigma\}$
$\deg(\mathcal{I}) := \begin{cases} \mathrm{rk}(C)+1 & \text{if } \mathcal{I} = \mathsf{Cut}_C \text{ with } C \in \bigwedge^{+}\text{-for} \\ 0 & \text{otherwise} \end{cases}$

Lemma 1.
If $h \vdash^\sigma_m \Gamma$ & $h^+ = \mathcal{I}(h_\iota)_{\iota\in I}$ then
$\mathcal{I} \in \mathrm{ID}^\infty_\nu\lceil\sigma$ & $\Delta(\mathcal{I}) \subseteq \Gamma$ & $\deg(\mathcal{I}) \leq m$ & $\forall\iota \in I(h_\iota \vdash^\sigma_m \Gamma, \Delta_\iota(\mathcal{I}))$.
The proof of this lemma is routine and can be left to the reader (cf. Theorem 3 in [Bu97] and Theorem 5 in [Bu01]).

Iterated Inductive Definition of $\mathcal{W}_\sigma$ $(\sigma < \nu)$
1. If $h \in \mathbf{I}_\sigma$ with $|\mathsf{tp}(h)| \subseteq \mathbb{N}$ and $\forall i \in |\mathsf{tp}(h)|(h[i] \in \mathcal{W}_\sigma)$ then $h \in \mathcal{W}_\sigma$.
2. If $h \in \mathbf{I}_\sigma$ with $\mathsf{tp}(h) = \widetilde{\Omega}_P$, $\mathrm{lev}(P) < \sigma$ and $\forall\iota \in \mathcal{W}_{\mathrm{lev}(P)}(h[\iota] \in \mathcal{W}_\sigma)$ then $h \in \mathcal{W}_\sigma$.

Note that (according to Lemma 1) if $h \in \mathbf{I}_\sigma$ and $\mathsf{tp}(h) = \widetilde{\Omega}_P$ then $\mathrm{lev}(P) < \sigma$.
Note further that $\mathcal{W}_\sigma$ is by definition a subset of $\mathbf{I}_\sigma$.

Our goal is now to show that ID_ν is Π^0_2-conservative over $\mathrm{ID}^i_\nu(\mathcal{W})$ (where $\mathcal{W}$ denotes the operator form corresponding to the iterated inductive definition of $(\mathcal{W}_\sigma)_{\sigma<\nu}$). We will achieve this goal by giving an informal proof of
(1) *"If h is an ID_ν-derivation of a Π^0_2-sentence A and if h has height and degree $\leq m$ then A holds."*
which for each fixed $m \in \mathbb{N}$ can be formalized in $\mathrm{ID}^i_\nu(\mathcal{W})$.

Abbreviations.
$\mathcal{W}^* := \{h : \forall\sigma < \nu(h \vdash^\sigma_0 \Gamma(h) \Rightarrow \mathsf{D}_\sigma h \in \mathcal{W}_\sigma)\}$,
$\mathsf{FALSE}_0 := \{\neg A : A \in \mathsf{TRUE}_0\}$,
$\mathsf{E}^m h := \underbrace{\mathsf{E}\ldots\mathsf{E}}_{m\text{ times}} h$.

Lemma 2.
Let R be a binary relation symbol of $\mathcal{L}_0$.

(a) If $\tilde{h}$ is an ID_ν-derivation of $\exists y R(x,y)$ with $\deg(\tilde{h}) = m$, then for all n we have:
$\mathsf{E}^m \tilde{h}(x/n) \in \mathcal{W}^* \ \Rightarrow\ \mathcal{W}_0 \ni \mathsf{D}_0 \mathsf{E}^m \tilde{h}(x/n) \vdash \exists y R(n,y)$.

(b) $\mathcal{W}_0 \ni h \vdash \Gamma, \exists y R(n,y)$ with $\Gamma \subseteq \mathsf{FALSE}_0 \ \Rightarrow$ there exists k with $R(n,k)$.

Proof:
(a) Obviously $\mathsf{E}^m h(x/n) \vdash^0_0 \exists y R(n,y)$ which yields the claim.
(b) Induction over $\mathcal{W}_0$: We have $h^+ = \mathcal{I}(h_i)_{i \in I}$ with $h_i \in \mathcal{W}_0$ for all $i \in I$.
By Lemma 1 one of the following cases holds:
1. $\mathcal{I} = \mathsf{Rep}$ and $h_0 \vdash \Gamma, \exists y R(n,y)$.
2. $\mathcal{I} = \mathsf{Cut}_C$ with $C \in \mathsf{FALSE}_0$ and $h_0 \vdash \Gamma, C, \exists y R(n,y)$.
3. $\mathcal{I} = \mathsf{Cut}_C$ with $\neg C \in \mathsf{FALSE}_0$ and $h_1 \vdash \Gamma, \neg C, \exists y R(n,y)$.
4. $\mathcal{I} = \bigvee^k_{\exists y R(n,y)}$ with $R(n,k) \in \mathsf{FALSE}_0$ and $h_0 \vdash \Gamma, R(n,k), \exists y R(n,y)$.
5. $\mathcal{I} = \bigvee^k_{\exists y R(n,y)}$ and $R(n,k) \in \mathsf{TRUE}_0$.
In cases 1-4 the claim follows immediately from the IH (induction hypothesis).
In case 5 we are done.

Now for establishing **(1)** it remains to prove
(2) $\mathsf{E}^m h \in \mathcal{W}^*$ holds for each closed ID_ν-derivation h and each $m \in \mathbb{N}$.

Definitions.
For $\mathcal{I} \in \mathrm{ID}^\infty_\nu$ let $|\mathcal{I}|_{\mathcal{W}} := \begin{cases} \{0\} \cup \mathcal{W}_\mu & \text{if } \mathcal{I} = \widetilde{\Omega}_P \text{ and } \mu = \mathrm{lev}(P) \\ |\mathcal{I}| & \text{if } \mathcal{I} \text{ is not of the form } \widetilde{\Omega}_P \end{cases}$.
Note that $|\mathcal{I}|_{\mathcal{W}} \subseteq |\mathcal{I}|$ (since $\mathcal{W}_\mu \subseteq \mathbf{I}_\mu$).
$\Phi(\mathcal{X}) := \{h : \forall \iota \in |\mathsf{tp}(h)|_{\mathcal{W}} (h[\iota] \in \mathcal{X})\}$ and $\mathrm{Prog}(\mathcal{X}) \ :\Leftrightarrow\ \Phi(\mathcal{X}) \subseteq \mathcal{X}$,
where $\mathcal{X}$ ranges over subsets of ID^*_ν.

Then $\mathcal{W}_\sigma$ (for $\sigma < \nu$) satisfies the following "axioms":
$(W_\sigma.1)$ $\mathbf{I}_\sigma \cap \Phi(\mathcal{W}_\sigma) \subseteq \mathcal{W}_\sigma$,
$(W_\sigma.2)$ $\mathbf{I}_\sigma \cap \Phi(\mathcal{X}) \subseteq \mathcal{X} \ \Rightarrow\ \mathcal{W}_\sigma \subseteq \mathcal{X}$.

Lemma 3. $\mathrm{Prog}(\mathcal{W}^*)$.
Proof:
Let $\mathrm{H}_\sigma := \{h : \deg(h) = 0 \ \&\ \Gamma(h) \subseteq \mathrm{Pos}_\sigma\}$. Then $\mathcal{W}^* = \{h : \forall \sigma < \nu (h \in \mathrm{H}_\sigma \Rightarrow \mathsf{D}_\sigma h \in \mathcal{W}_\sigma)\}$.
Suppose $h \in \Phi(\mathcal{W}^*) \ \&\ \sigma < \nu \ \&\ h \in \mathrm{H}_\sigma$. To prove: $\mathsf{D}_\sigma h \in \mathcal{W}_\sigma$. Trivially $\mathsf{D}_\sigma h \in \mathbf{I}_\sigma$.
1. $\mathsf{tp}(h) = \widetilde{\Omega}_P$ with $\sigma \le \mu := \mathrm{lev}(P)$:
From $h \in \mathrm{H}_\sigma$ by Lemma 1 we get $h[0] \in \mathrm{H}_\sigma \subseteq \mathrm{H}_\mu$. Together with $h \in \Phi(\mathcal{W}^*)$ this yields $q := \mathsf{D}_\mu h[0] \in \mathcal{W}_\mu$.
From $q \in \mathcal{W}_\mu$ and $h \in \mathrm{H}_\sigma \cap \Phi(\mathcal{W}^*)$ we conclude $h[q] \in \mathrm{H}_\sigma \cap \mathcal{W}^*$. Hence $\mathsf{D}_\sigma h[q] \in \mathcal{W}_\sigma$ which yields $\mathsf{D}_\sigma h \in \mathcal{W}_\sigma$, since $(\mathsf{D}_\sigma h)^+ = \mathsf{Rep}(\mathsf{D}_\sigma h[q])$.
2. Otherwise: Then $\mathsf{tp}(\mathsf{D}_\sigma h) = \mathsf{tp}(h)$, $|\mathsf{tp}(h)|_{\mathcal{W}} \subseteq |\mathsf{tp}(h)|$ and $(\mathsf{D}_\sigma h)[\iota] = \mathsf{D}_\sigma h[\iota]$ for all $\iota \in |\mathsf{tp}(h)|$ $(*)$.
From $h \in \mathrm{H}_\sigma \cap \Phi(\mathcal{W}^*)$ by L.1 we get $\forall \iota \in |\mathsf{tp}(h)|_{\mathcal{W}} (h[\iota] \in \mathrm{H}_\sigma \cap \mathcal{W}^*)$, and then $\forall \iota \in |\mathsf{tp}(h)|_{\mathcal{W}} (\mathsf{D}_\sigma h[\iota] \in \mathcal{W}_\sigma)$.
Together with $(*)$ this yields $\mathsf{D}_\sigma h \in \mathcal{W}_\sigma$.

Remark.
Now for establishing **(2)** it remains to prove
(3) $\mathrm{Prog}(\mathcal{X}) \Rightarrow h \in \mathcal{X}$, for each closed ID_ν-derivation h and each $\mathcal{X}$,
and to find a *jump* operation $\mathcal{X} \mapsto \overline{\mathcal{X}}$ (á la [Ge43]) such that
(4) $h \in \overline{\mathcal{X}} \Rightarrow \mathsf{E} h \in \mathcal{X}$ and $\mathrm{Prog}(\mathcal{X}) \Rightarrow \mathrm{Prog}(\overline{\mathcal{X}})$.

Lemma 4. $\mathrm{Prog}(\mathcal{X})$ & $\mathrm{lev}(\mathcal{P})=\sigma<\nu$ & $d\in\mathcal{W}_\sigma \;\Rightarrow\; \mathsf{S}^{\Pi}_{\mathcal{P},\mathcal{F}}d\in\mathcal{X}$.
Proof by induction on "$d\in\mathcal{W}_\sigma$":
Assume $d\in\mathcal{W}_\sigma$ with $d^+=\mathcal{I}(d_\iota)_{\iota\in|\mathcal{I}|}$. Then $d\in\mathbf{I}_\sigma$ and $\forall\iota\in|\mathcal{I}|_{\mathcal{W}}(d_\iota\in\mathcal{W}_\sigma)$.
We have to prove: $h:=\mathsf{S}^{\Pi}_{\mathcal{P},\mathcal{F}}d\in\mathcal{X}$.
1.1. $\mathcal{I}=\mathsf{Cl}_{\mathcal{P}n}$ with $\mathcal{P}n\in\Pi$: Then $h^+=\bigvee^n_{\neg(\mathfrak{A}(\mathcal{F})\subseteq\mathcal{F})}\big(\bigwedge_{\mathfrak{A}(\mathcal{F},n)\wedge\neg\mathcal{F}(n)}(\mathsf{S}^{\Pi\cup\Delta_0(\mathcal{I})}_{\mathcal{P},\mathcal{F}}d_0)\mathbf{d}_{\mathcal{F}(n)}\big)$ $(*)$.
By IH from $d_0\in\mathcal{W}_\sigma$ we get $\mathsf{S}^{\Pi\cup\Delta_0(\mathcal{I})}_{\mathcal{P},\mathcal{F}}d_0\in\mathcal{X}$. Further, the premise $\mathrm{Prog}(\mathcal{X})$ yields $\mathbf{d}_{\mathcal{F}(n)}\in\mathcal{X}$.
From $\mathsf{S}^{\Pi\cup\Delta_0(\mathcal{I})}_{\mathcal{P},\mathcal{F}}d_0\in\mathcal{X}$ & $\mathbf{d}_{\mathcal{F}(n)}\in\mathcal{X}$ by $(*)$ and $\mathrm{Prog}(\mathcal{X})$ we get $h\in\mathcal{X}$.
1.2. $\mathcal{I}=\bigwedge_A,\bigvee^k_A$ with $A\in\Pi$: Then $h^+=\mathcal{I}^*\big(\mathsf{S}^{\Pi\cup\Delta_i(\mathcal{I})}_{\mathcal{P},\mathcal{F}}d_i\big)_{i\in|\mathcal{I}|}$ $(*)$.
By IH we get $\forall i\in|\mathcal{I}|(\mathsf{S}^{\Pi\cup\Delta_i(\mathcal{I})}_{\mathcal{P},\mathcal{F}}d_i\in\mathcal{X})$, and then $h\in\mathcal{X}$ by $(*)$ and $\mathrm{Prog}(\mathcal{X})$.
1.3. otherwise: Then $h^+=\mathcal{I}\big(\mathsf{S}^{\Pi}_{\mathcal{P},\mathcal{F}}d_\iota\big)_{i\in|\mathcal{I}|}$ $(*)$.
By IH we get $\forall\iota\in|\mathcal{I}|_{\mathcal{W}}(\mathsf{S}^{\Pi}_{\mathcal{P},\mathcal{F}}d_\iota\in\mathcal{X})$, and then $h\in\mathcal{X}$ by $(*)$ and $\mathrm{Prog}(\mathcal{X})$.

Lemma 5. $\mathrm{Prog}(\mathcal{X})$ & $C\in\bigwedge$-for $\Rightarrow$ $\mathrm{Prog}(\{h_0:\mathsf{J}^k_C h_0\in\mathcal{X}\})$.
Proof: Left to the reader.

Definition. $\mathcal{X}^{C,h_0}:=\{h_1:\mathsf{Cut}_C h_0 h_1\in\mathcal{X}\}$
Lemma 6. Assume $\mathrm{Prog}(\mathcal{X})$.
(a) $C\in\bigwedge$-for & $\forall k(\mathsf{J}^k_C h_0\in\mathcal{X})$ $\Rightarrow$ $\mathrm{Prog}(\mathcal{X}^{C,h_0})$
(b) $h_0\in\mathcal{X}$ $\Rightarrow$ $\mathrm{Prog}(\mathcal{X}^{P,h_0})$.
Proof:
(a) Assume $C\in\bigwedge$-for & $\forall k(\mathsf{J}^k_C h_0\in\mathcal{X})$ & $h_1\in\Phi(\mathcal{X}^{C,h_0})$.
To prove: $h_1\in\mathcal{X}^{C,h_0}$, i.e. $h:=\mathsf{Cut}_C h_0 h_1\in\mathcal{X}$.
Assume $h_1^+=\mathcal{I}(h_{1\iota})_{\iota\in I}$. Then $\forall\iota\in|\mathcal{I}|_{\mathcal{W}}(h_{1\iota}\in\mathcal{X}^{C,h_0})$ and thus $\forall\iota\in|\mathcal{I}|_{\mathcal{W}}(\mathsf{Cut}_C h_0 h_{1\iota}\in\mathcal{X})$.
1. $\neg C\notin\Delta(\mathcal{I})$: From $h^+=\mathcal{I}(\mathsf{Cut}_C h_0 h_{1\iota})_{\iota\in I}$ and $\forall\iota\in|\mathcal{I}|_{\mathcal{W}}(\mathsf{Cut}_C h_0 h_{1\iota}\in\mathcal{X})$ we get $h\in\Phi(\mathcal{X})\subseteq\mathcal{X}$.
2. $\mathcal{I}=\bigvee^k_{\neg C}$: Then $h^+=\mathsf{Cut}^{\circ}_{C[k]}(\mathsf{J}^k_C h_0,\mathsf{Cut}_C h_0 h_{10})$ with $\mathsf{J}^k_C h_0\in\mathcal{X}$ (by assumption) and $\mathsf{Cut}_C h_0 h_{10}\in\mathcal{X}$ as shown above. Hence $h\in\Phi(\mathcal{X})\subseteq\mathcal{X}$.
(b) is proved in the same way as (a).

Lemma 7. For each closed ID_ν-derivation h and each $\mathcal{X}$ we have: $\mathrm{Prog}(\mathcal{X})\Rightarrow h\in\mathcal{X}$.
Proof by induction on the height of h:
Assume $\mathrm{Prog}(X)$.
1. $h=\mathcal{I}h_0...h_{l-1}$ with simple $\mathcal{I}$: Then $h^+=\mathcal{I}\big(h_i\big)_{i<l}$ and, by IH, $h_0,\dots,h_{l-1}\in\mathcal{X}$. Hence $h\in\Phi(\mathcal{X})\subseteq\mathcal{X}$.
2. $h=\bigwedge^y_{\forall xA}\tilde{h}$: Then $h^+=\bigwedge_{\forall xA}(\tilde{h}(y/i))_{i\in\mathbb{N}}$ and, by IH, $\forall i\in\mathbb{N}(\tilde{h}(y/i)\in\mathcal{X})$, i.e. $h\in\Phi(\mathcal{X})\subseteq\mathcal{X}$.
3. $h=\mathsf{Cut}_C h_0 h_1$ with $C\in\bigwedge^+$-for: By IH we get $h_0\in\mathcal{X}$.
3.1. $C\in\bigwedge$-for: By Lemma 5 we get $\forall k.\mathrm{Prog}(\{d:\mathsf{J}^k_C d\in\mathcal{X}\})$ and then, by IH, $\forall k(h_0\in\{d:\mathsf{J}^k_C d\in\mathcal{X}\})$, i.e. $\forall k(\mathsf{J}^k_C h_0\in\mathcal{X})$. From $\mathrm{Prog}(\mathcal{X})$ & $h_0\in\mathcal{X}$ & $\forall k(\mathsf{J}^k_C h_0\in\mathcal{X})$ by Lemma 6a we conclude $\mathrm{Prog}(\mathcal{X}^{C,h_0})$ and then, by IH, $h_1\in\mathcal{X}^{C,h_0}$, i.e. $h\in\mathcal{X}$.
3.2. $C=P$:
From $\mathrm{Prog}(\mathcal{X})$ & $h_0\in\mathcal{X}$ by Lemma 6b we conclude $\mathrm{Prog}(\mathcal{X}^{P,h_0})$ and then, by IH, $h_1\in\mathcal{X}^{P,h_0}$, i.e. $h\in\mathcal{X}$.
4. $h=\mathsf{Ind}^n_{\mathcal{F}}$: Then $h^+=\mathsf{Rep}(d_n)$ with $d_0:=\mathbf{d}_{\mathcal{F}(0)}$, $d_{i+1}:=\bigvee^i_{\exists x(\mathcal{F}(x)\wedge\neg\mathcal{F}(Sx))}\bigwedge_{\mathcal{F}(i)\wedge\neg\mathcal{F}(Si)}d_i\mathbf{d}_{\mathcal{F}(Si)}$.
Using $\mathrm{Prog}(\mathcal{X})$ one easily shows $d_i\in\mathcal{X}$ by induction on i.
5. $h=\mathsf{Ind}^{\mathcal{P}n}_{\mathcal{F}}$: Then $h^+=\widetilde{\Omega}_P\mathsf{Ax}_{\{\neg P,P\}}\big(\mathsf{S}^{\{P\}}_{\mathcal{P},\mathcal{F}}d\big)_{d\in\mathbf{I}_\sigma}$ with $\sigma:=\mathrm{lev}(\mathcal{P})$ and $P:=\mathcal{P}n$.
$\mathrm{Prog}(\mathcal{X})$ yields $\mathsf{Ax}_{\{\neg P,P\}}\in\mathcal{X}$, and by Lemma 4 we have $\forall d\in\mathcal{W}_\sigma(\mathsf{S}^{\{P\}}_{\mathcal{P},\mathcal{F}}d\in\mathcal{X})$. Hence $h\in\Phi(\mathcal{X})\subseteq\mathcal{X}$.

Now we come to the last part of our proof, which begins with the definition of the jump operation $\mathcal{X}\mapsto\overline{\mathcal{X}}$ mentioned in **(4)** above.
Preliminary remark.
ID^*_ν-derivations have been introduced as terms in polish (prefix) notation build up from inference symbols each of which as a fixed finite arity. So every ID^*_ν-derivation is finite sequence of inference symbols.

In the following we use $\mathfrak{a}$, $\mathfrak{a}'$ as syntactic variables for arbitrary finite sequences of inference symbols – including the empty sequence ε. Concatination is expressed by juxtaposition. Example: If $\mathfrak{a} = \mathsf{Cut}_C h_0 \mathsf{J}^k_D \mathsf{Cut}_B h_1$ then $\mathfrak{a}h_2$ is the derivation $\mathsf{Cut}_C h_0 h$ with $h := \mathsf{J}^k_D \mathsf{Cut}_B h_1 h_2$.

Finitary Inductive Definition of $\mathbf{Q}(\mathcal{X})$

(Q1) $\varepsilon \in \mathbf{Q}(\mathcal{X})$.

(Q2) $\mathfrak{a} \in \mathbf{Q}(\mathcal{X})$ & $C \in \bigwedge$-for $\Rightarrow$ $\mathfrak{a}\, \mathsf{J}^k_C \in \mathbf{Q}(\mathcal{X})$.

(Q3) $\mathfrak{a} \in \mathbf{Q}(\mathcal{X})$ & $C \in \bigwedge$-for & $\forall k(\mathfrak{a}\, \mathsf{J}^k_C h \in \mathcal{X})$ $\Rightarrow$ $\mathfrak{a}\, \mathsf{Cut}_C h \in \mathbf{Q}(\mathcal{X})$.

(Q4) $\mathfrak{a} \in \mathbf{Q}(\mathcal{X})$ & $\mathfrak{a}h \in \mathcal{X}$ $\Rightarrow$ $\mathfrak{a}\, \mathsf{Cut}_P h \in \mathbf{Q}(\mathcal{X})$.

Note that $\mathbf{Q}(\mathcal{X})$ is arithmetical in $\mathcal{X}$.

Definition. $\overline{\mathcal{X}} := \{h : \forall \mathfrak{a} \in \mathbf{Q}(\mathcal{X})(\mathfrak{a}\mathsf{E}h \in \mathcal{X})\}$.

Remark.

(i) $h \in \overline{\mathcal{X}}$ $\Rightarrow$ $\mathsf{E}h \in \mathcal{X}$.

(ii) $h \in \overline{\mathcal{X}}$ & $\mathfrak{a} \in \mathbf{Q}(\mathcal{X})$ & $C \in \bigwedge^+$-for $\Rightarrow$ $\mathfrak{a}\mathsf{Cut}_C \mathsf{E}h \in \mathbf{Q}(\mathcal{X})$.

Lemma 8. Let $\mathfrak{a} \in \mathbf{Q}(\mathcal{X})$.

(a) $h^+ = \mathsf{Cut}_A(h_0, h_1)$ $\Rightarrow$ $(\mathfrak{a}h)^+ = \mathsf{Cut}_A(\mathfrak{a}h_0, \mathfrak{a}h_1)$.

(b) $h^+ = \mathsf{Rep}(h_0)$ $\Rightarrow$ $(\mathfrak{a}h)^+ = \mathsf{Rep}(\mathfrak{a}h_0)$.

Proof by induction on "$\mathfrak{a} \in \mathbf{Q}(\mathcal{X})$".

Lemma 9. $\mathrm{Prog}(\mathcal{X})$ & $\mathfrak{a} \in \mathbf{Q}(\mathcal{X})$ & $h^+ = \mathcal{I}(h_\iota)_{\iota \in I}$ & $\forall \iota \in |\mathcal{I}|_{\mathcal{W}}(\mathfrak{a}h_\iota \in \mathcal{X})$ $\Rightarrow$ $\mathfrak{a}h \in \mathcal{X}$.

Proof by induction on "$\mathfrak{a} \in \mathbf{Q}(\mathcal{X})$":

1. $\mathfrak{a} = \varepsilon$: In this case the premises immediately yield $h \in \Phi(\mathcal{X}) \subseteq \mathcal{X}$.

2. $\mathfrak{a} = \mathfrak{a}'\mathsf{J}^k_C$ with $\mathfrak{a}' \in \mathbf{Q}(\mathcal{X})$:

2.1. $\mathcal{I} = \bigwedge_C$:

Then $(\mathsf{J}^k_C h)^+ = \mathsf{Rep}\big(\mathsf{J}^k_C h_k\big)$ and $(\mathfrak{a}h)^+ = (\mathfrak{a}'\mathsf{J}^k_C h)^+ \overset{\text{L.8b}}{=} \mathsf{Rep}(\mathfrak{a}'\mathsf{J}^k_C h_k) = \mathsf{Rep}(\mathfrak{a}h_k)$.

From $\mathrm{Prog}(\mathcal{X})$ & $(\mathfrak{a}h)^+ = \mathsf{Rep}(\mathfrak{a}h_k)$ & $\mathfrak{a}h_k \in \mathcal{X}$ we get $\mathfrak{a}h \in \Phi(\mathcal{X}) \subseteq \mathcal{X}$.

2.2. otherwise: Then $(\mathsf{J}^k_C h)^+ = \mathcal{I}\big(\mathsf{J}^k_C h_\iota\big)_{\iota \in I}$ $(*)$.

$\mathrm{Prog}(\mathcal{X})$ & $\mathfrak{a}' \in \mathbf{Q}(\mathcal{X})$ & $(*)$ & $\forall \iota \in |\mathcal{I}|_{\mathcal{W}}(\mathfrak{a}'\mathsf{J}^k_C h_\iota = \mathfrak{a}h_\iota \in \mathcal{X})$ $\overset{\text{IH}}{\Rightarrow}$ $\mathfrak{a}h = \mathfrak{a}'\mathsf{J}^k_C h \in \mathcal{X}$.

3. $\mathfrak{a} = \mathfrak{a}'\mathsf{Cut}_C h'$ with $\mathfrak{a}' \in \mathbf{Q}(\mathcal{X})$ & $C \in \bigwedge$-for & $\forall k(\mathfrak{a}'\mathsf{J}^k_C h' \in \mathcal{X})$:

3.1. $\neg C \notin \Delta(\mathcal{I})$: Then $(\mathsf{Cut}_C h' h)^+ = \mathcal{I}\big(\mathsf{Cut}_C h' h_\iota\big)_{\iota \in I}$ $(*)$.

$\mathrm{Prog}(\mathcal{X})$ & $\mathfrak{a}' \in \mathbf{Q}(\mathcal{X})$ & $(*)$ & $\forall \iota \in |\mathcal{I}|_{\mathcal{W}}(\mathfrak{a}'\mathsf{Cut}_C h' h_\iota = \mathfrak{a}h_\iota \in \mathcal{X})$ $\overset{\text{IH}}{\Rightarrow}$ $\mathfrak{a}h = \mathfrak{a}'\mathsf{Cut}_C h' h \in \mathcal{X}$.

3.2. $\mathcal{I} = \bigvee^k_{\neg C}$: Then $(\mathsf{Cut}_C h' h)^+ = \mathsf{Cut}^\circ_{C[k]}(\mathsf{J}^k_C h', \mathsf{Cut}_C h' h_0)$ and

$(\mathfrak{a}h)^+ = (\mathfrak{a}'\mathsf{Cut}_C h' h)^+ \overset{\text{L.8a}}{=} \mathsf{Cut}^\circ_{C[k]}(\mathfrak{a}'\mathsf{J}^k_C h', \mathfrak{a}'\mathsf{Cut}_C h' h_0) = \mathsf{Cut}^\circ_{C[k]}(\mathfrak{a}'\mathsf{J}^k_C h', \mathfrak{a}h_0)$.

Further $\mathfrak{a}'\mathsf{J}^k_C h' \in \mathcal{X}$ and $\mathfrak{a}h_0 \in \mathcal{X}$. Hence $\mathfrak{a}h \in \Phi(\mathcal{X}) \subseteq \mathcal{X}$.

4. $\mathfrak{a} = \mathfrak{a}'\mathsf{Cut}_P h'$ with $\mathfrak{a}' \in \mathbf{Q}(\mathcal{X})$ & $\mathfrak{a}'h' \in \mathcal{X}$:

4.1. $\neg P \notin \Delta(\mathcal{I})$: As 3.1.

4.2. $\neg P \in \Delta(\mathcal{I})$: Then $(\mathsf{Cut}_P h' h)^+ = \mathsf{Rep}(h')$ and thus $(\mathfrak{a}h)^+ = (\mathfrak{a}'\mathsf{Cut}_C h' h)^+ \overset{\text{L.8b}}{=} \mathsf{Rep}(\mathfrak{a}'h')$.

Together with $\mathfrak{a}'h' \in \mathcal{X}$ this yields $\mathfrak{a}h \in \Phi(\mathcal{X}) \subseteq \mathcal{X}$.

Lemma 10. $\mathrm{Prog}(\mathcal{X})$ $\Rightarrow$ $\mathrm{Prog}(\overline{\mathcal{X}})$.

Proof:

Assume $\mathrm{Prog}(\mathcal{X})$ & $h \in \Phi(\overline{\mathcal{X}})$ & $\mathfrak{a} \in \mathbf{Q}(\mathcal{X})$. To prove $\mathfrak{a}\mathsf{E}h \in \mathcal{X}$.

For this it suffices to prove $\mathfrak{a}\mathsf{E}h \in \Phi(\mathcal{X})$.

Let $h^+ = \mathcal{I}(h_\iota)_{\iota \in I}$. Then $\forall \iota \in |\mathcal{I}|_{\mathcal{W}}(h_\iota \in \overline{\mathcal{X}})$ and thus $\forall \iota \in |\mathcal{I}|_{\mathcal{W}}(\mathfrak{a}\mathsf{E}h_\iota \in \mathcal{X})$.

1. $\mathcal{I} = \mathsf{Cut}_C$ with $C \in \bigwedge^+$-for: Then $(\mathsf{E}h)^+ = \mathsf{Rep}(\mathsf{Cut}_C \mathsf{E}h_0 \mathsf{E}h_1)$ and therefore, by Lemma 8b, $(\mathfrak{a}\mathsf{E}h)^+ = \mathsf{Rep}(\mathfrak{a}\mathsf{Cut}_C \mathsf{E}h_0 \mathsf{E}h_1)$. From $h_0, h_1 \in \overline{\mathcal{X}}$ & $\mathfrak{a} \in \mathbf{Q}(\mathcal{X})$ we get (by Remark (ii)) $\mathfrak{a}\mathsf{Cut}_C \mathsf{E}h_0 \in \mathbf{Q}(\mathcal{X})$ & $h_1 \in \overline{\mathcal{X}}$, and then $\mathfrak{a}\mathsf{Cut}_C \mathsf{E}h_0 \mathsf{E}h_1 \in \mathcal{X}$. Hence $\mathfrak{a}\mathsf{E}h \in \Phi(\mathcal{X}) \subseteq \mathcal{X}$.

2. otherwise: From $(\mathsf{E}h)^+ = \mathcal{I}(\mathsf{E}h_\iota)_{\iota \in I}$ & $\forall \iota \in |\mathcal{I}|_{\mathcal{W}}(\mathfrak{a}\mathsf{E}h_\iota \in \mathcal{X})$ we conclude $\mathfrak{a}\mathsf{E}h \in \mathcal{X}$ by Lemma 9.

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