Serre's problem on the density of isotropic fibres in conic bundles

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Let $\pi : X \to \mathbb{P}^1_{\mathbb{Q}}$ be a non-singular conic bundle over $\mathbb{Q}$ having n non-split fibres and denote by $N(\pi, B)$ the cardinality of the fibres of Weil height at most B that possess a rational point. Serre showed in 1990 that a direct application of the large sieve yields

$$N(\pi, B) \ll B^2(\log B)^{-n/2}$$

and raised the problem of proving that this is the true order of magnitude of $N(\pi, B)$ under the necessary assumption that there exists at least one smooth fibre with a rational point. We solve this problem for all non-singular conic bundles of rank at most 3. Our method comprises the use of Hooley neutralisers, estimating divisor sums over values of binary forms, and an application of the Rosser–Iwaniec sieve.