The vanishing theorem (2.7) was proved by Serre [3] for coherent sheaves on algebraic curves and projective algebraic varieties, and later [5] for abstract algebraic varieties. It is analogous to the theorem that singular cohomology on a (real) manifold of dimension n vanishes in degrees $i > n$.

Exercises

2.1. (a) Let $X = \mathbf{A}_k^1$ be the affine line over an infinite field k. Let P,Q be distinct closed points of X, and let $U = X - \{P,Q\}$. Show that $H^1(X,\mathbf{Z}_U) \neq 0$.

*(b) More generally, let $Y \subseteq X = \mathbf{A}_k^n$ be the union of $n+1$ hyperplanes in suitably general position, and let $U = X - Y$. Show that $H^n(X,\mathbf{Z}_U) \neq 0$. Thus the result of (2.7) is the best possible.

2.2. Let $X = \mathbf{P}_k^1$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathcal{O} \to \mathcal{K} \to \mathcal{K}/\mathcal{O} \to 0$ of (II, Ex. 1.21d) is a flasque resolution of $\mathcal{O}$. Conclude from (II, Ex. 1.21e) that $H^i(X,\mathcal{O}) = 0$ for all $i > 0$.

2.3. *Cohomology with Supports* (Grothendieck [7]). Let X be a topological space, let Y be a closed subset, and let $\mathcal{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X,\mathcal{F})$ denote the group of sections of $\mathcal{F}$ with support in Y (II, Ex. 1.20).

(a) Show that $\Gamma_Y(X,\cdot)$ is a left exact functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$.

We denote the right derived functors of $\Gamma_Y(X,\cdot)$ by $H_Y^i(X,\cdot)$. They are the *cohomology groups* of X *with supports* in Y, and coefficients in a given sheaf.

(b) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, with $\mathcal{F}'$ flasque, show that

$$0 \to \Gamma_Y(X,\mathcal{F}') \to \Gamma_Y(X,\mathcal{F}) \to \Gamma_Y(X,\mathcal{F}'') \to 0$$

is exact.

(c) Show that if $\mathcal{F}$ is flasque, then $H_Y^i(X,\mathcal{F}) = 0$ for all $i > 0$.

(d) If $\mathcal{F}$ is flasque, show that the sequence

$$0 \to \Gamma_Y(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}) \to \Gamma(X - Y,\mathcal{F}) \to 0$$

is exact.

(e) Let $U = X - Y$. Show that for any $\mathcal{F}$, there is a long exact sequence of cohomology groups

$$\begin{aligned} 0 &\to H_Y^0(X,\mathcal{F}) \to H^0(X,\mathcal{F}) \to H^0(U,\mathcal{F}|_U) \to \\ &\to H_Y^1(X,\mathcal{F}) \to H^1(X,\mathcal{F}) \to H^1(U,\mathcal{F}|_U) \to \\ &\to H_Y^2(X,\mathcal{F}) \to \dots . \end{aligned}$$

(f) *Excision.* Let V be an open subset of X containing Y. Then there are natural functorial isomorphisms, for all i and $\mathcal{F}$,

$$H_Y^i(X,\mathcal{F}) \cong H_Y^i(V,\mathcal{F}|_V).$$

2.4. *Mayer–Vietoris Sequence.* Let Y_1,Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports

$$\dots \to H^i_{Y_1 \cap Y_2}(X,\mathcal{F}) \to H^i_{Y_1}(X,\mathcal{F}) \oplus H^i_{Y_2}(X,\mathcal{F}) \to H^i_{Y_1 \cup Y_2}(X,\mathcal{F}) \to$$

2.5. Let X be a Zariski space (II, Ex. 3.17). Let $P \in X$ be a closed point, and let X_P be the subset of X consisting of all points $Q \in X$ such that $P \in \{Q\}^-$. We call X_P the *local space* of X at P, and give it the induced topology. Let $j: X_P \to X$ be the inclusion, and for any sheaf $\mathcal{F}$ on X, let $\mathcal{F}_P = j^*\mathcal{F}$. Show that for all i, $\mathcal{F}$, we have

$$H_P^i(X,\mathcal{F}) = H_P^i(X_P,\mathcal{F}_P).$$

2.6. Let X be a noetherian topological space, and let $\{\mathcal{I}_\alpha\}_{\alpha \in A}$ be a direct system of injective sheaves of abelian groups on X. Then $\varinjlim \mathcal{I}_\alpha$ is also injective. [*Hints*: First show that a sheaf $\mathcal{I}$ is injective if and only if for every open set $U \subseteq X$, and for every subsheaf $\mathcal{R} \subseteq \mathbf{Z}_U$, and for every map $f: \mathcal{R} \to \mathcal{I}$, there exists an extension of f to a map of $\mathbf{Z}_U \to \mathcal{I}$. Secondly, show that any such sheaf $\mathcal{R}$ is finitely generated, so any map $\mathcal{R} \to \varinjlim \mathcal{I}_\alpha$ factors through one of the $\mathcal{I}_\alpha$.]

2.7. Let S^1 be the circle (with its usual topology), and let $\mathbf{Z}$ be the constant sheaf $\mathbf{Z}$.

(a) Show that $H^1(S^1,\mathbf{Z}) \cong \mathbf{Z}$, using our definition of cohomology.

(b) Now let $\mathcal{R}$ be the sheaf of germs of continuous real-valued functions on S^1. Show that $H^1(S^1,\mathcal{R}) = 0$.

3 Cohomology of a Noetherian Affine Scheme

In this section we will prove that if $X = \operatorname{Spec} A$ is a noetherian affine scheme, then $H^i(X,\mathcal{F}) = 0$ for all $i > 0$ and all quasi-coherent sheaves $\mathcal{F}$ of $\mathcal{O}_X$-modules. The key point is to show that if I is an injective A-module, then the sheaf $\tilde{I}$ on $\operatorname{Spec} A$ is flasque. We begin with some algebraic preliminaries.

Proposition 3.1A (Krull's Theorem). *Let A be a noetherian ring, let $M \subseteq N$ be finitely generated A-modules, and let $\mathfrak{a}$ be an ideal of A. Then the $\mathfrak{a}$-adic topology on M is induced by the $\mathfrak{a}$-adic topology on N. In particular, for any $n > 0$, there exists an $n' \geqslant n$ such that $\mathfrak{a}^n M \supseteq M \cap \mathfrak{a}^{n'} N$.*

PROOF. Atiyah–Macdonald [1, 10.11] or Zariski–Samuel [1, vol. II, Ch. VIII, Th. 4].

Recall (II, Ex. 5.6) that for any ring A, and any ideal $\mathfrak{a} \subseteq A$, and any A-module M, we have defined the submodule $\Gamma_\mathfrak{a}(M)$ to be $\{m \in M \mid \mathfrak{a}^n m = 0$ for some $n > 0\}$.

Lemma 3.2. *Let A be a noetherian ring, let $\mathfrak{a}$ be an ideal of A, and let I be an injective A-module. Then the submodule $J = \Gamma_\mathfrak{a}(I)$ is also an injective A-module.*

PROOF. To show that J is injective, it will be sufficient to show that for any ideal $\mathfrak{b} \subseteq A$, and for any homomorphism $\varphi: \mathfrak{b} \to J$, there exists a homomorphism $\psi: A \to J$ extending φ. (This is a well-known criterion for an injective module. Godement [1, I, 1.4.1]). Since A is noetherian, $\mathfrak{b}$ is finitely