Solutions to tutorial exercises for stochastic processes

T1. Consider $\sigma \equiv 1$ and $\tau := \inf\{t \geq 1 \,:\, B_t = 0\}$. Then $\sigma \leq \tau$ and

$$\mathbb{E}[B_\sigma^2] = 1 > 0 = \mathbb{E}[B_\tau^2].$$

T2. We use the Markov property to find

$$\begin{aligned}\mathbb{E}^0[X_t \mid \mathcal{F}_s] &= -\frac{1}{2}\int_0^s f''(B_u)\mathrm{d}u + \mathbb{E}^0\left[f(B_t) - \frac{1}{2}\int_s^t f''(B_u)\mathrm{d}u \,\Big|\, \mathcal{F}_s\right] \\ &= -\frac{1}{2}\int_0^s f''(B_u)\mathrm{d}u + \mathbb{E}^0\left[\left(f(B_{t-s}) - \frac{1}{2}\int_0^{t-s} f''(B_u)\mathrm{d}u\right)\circ\theta_s \,\Big|\, \mathcal{F}_s\right] \\ &= -\frac{1}{2}\int_0^s f''(B_u)\mathrm{d}u + \mathbb{E}^{B_s}\left[f(B_{t-s}) - \frac{1}{2}\int_0^{t-s} f''(B_u)\mathrm{d}u\right].\end{aligned}$$

Since $f'' \in L^1$ we can apply Fubini's theorem to find

$$\mathbb{E}^0[X_t \mid \mathcal{F}_s] = -\frac{1}{2}\int_0^s f''(B_u)\mathrm{d}u + \mathbb{E}^{B_s}[f(B_{t-s})] - \frac{1}{2}\int_0^{t-s}\mathbb{E}^{B_s}[f''(B_u)]\mathrm{d}u. \tag{1}$$

We now focus on the integrand of the last term. Let $p(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\big(-(x-y)^2/2t\big)$ be the normal density, then by using integration by parts twice we find

$$\begin{aligned}\mathbb{E}^{B_s}[f''(B_u)] &= \int_{\mathbb{R}} p(u,B_s,y)f''(y)\mathrm{d}y \\ &= \int_{\mathbb{R}} \frac{\partial^2 p(u,B_s,y)}{\partial y^2} f(y)\mathrm{d}y.\end{aligned}$$

The normal density $p(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\big(-(x-y)^2/2t\big)$ satisfies the differential equation $\frac{\partial}{\partial t}p = \frac{1}{2}\frac{\partial^2}{\partial y^2}p$, so that

$$\mathbb{E}^{B_s}[f''(B_u)] = 2\int_{\mathbb{R}} \frac{\partial p(u,B_s,y)}{\partial u} f(y)\mathrm{d}y.$$

We again use Fubini's theorem:

$$\begin{aligned}\frac{1}{2}\int_0^{t-s}\mathbb{E}^{B_s}[f''(B_u)]\mathrm{d}u &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_\varepsilon^{t-s}\frac{\partial p(u,B_s,y)}{\partial u}\mathrm{d}u f(y)\mathrm{d}y \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\big(p(t-s,B_s,y) - p(\varepsilon,B_s,y)\big)f(y)\mathrm{d}y \\ &= \mathbb{E}^{B_s}[f(B_{t-s})] - \lim_{\varepsilon\to 0}\mathbb{E}^{B_s}[f(B_\varepsilon)] \\ &= \mathbb{E}^{B_s}[f(B_{t-s})] - f(B_s),\end{aligned}$$

where we use the dominated convergence theorem in the last step. Combining this with (1) gives

$$\mathbb{E}^0[X_t \mid \mathcal{F}_s] = f(B_s) - \frac{1}{2}\int_0^s f''(B_u)\mathrm{d}u.$$

T3. We define the stopping time $\tau := \inf\{t \geq 0 : X_t \neq x\}$. Let $0 \leq s \leq t$. The tower property gives

$$\mathbb{P}^x(\tau > t) = \mathbb{P}^x(\tau > t, \tau > s) = \mathbb{E}^x\left[\mathbb{E}^x[\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s]\right] = \mathbb{E}^x\left[\mathbb{1}_{\{\tau>s\}}\mathbb{E}^x[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_s]\right].$$

The Markov property gives

$$\begin{aligned}\mathbb{P}^x(\tau > t) &= \mathbb{E}^x\left[\mathbb{1}_{\{\tau>s\}}\mathbb{E}^x[\mathbb{1}_{\{\tau>t-s\}} \circ \theta_s|\mathcal{F}_s]\right] \\ &= \mathbb{E}^x\left[\mathbb{1}_{\{\tau>s\}}\mathbb{E}^{X_s}[\mathbb{1}_{\{\tau>t-s\}}]\right] \\ &= \mathbb{E}^x\left[\mathbb{1}_{\{\tau>s\}}\mathbb{E}^x[\mathbb{1}_{\{\tau>t-s\}}]\right] = \mathbb{P}^x(\tau > s)\mathbb{P}^x(\tau > t-s).\end{aligned}$$

Since the above functional equation holds for all $0 \leq s \leq t$, it follows that

$$\mathbb{P}^x(\tau > t) = \exp(-ct),$$

for some $c \geq 0$, since $\mathbb{P}^x(\tau > t) \leq 1$.