Solutions to tutorial exercises for stochastic processes

T1. Let $Z \sim \text{Exp}(1)$. Then Z has characteristic function

$$\mathbb{E}[e^{i\theta Z}] = \frac{1}{1-i\theta} = e^{-\psi(\theta)},$$

where

$$\psi(\theta) := \log(1-i\theta).$$

If we show that $\psi(\theta)$ satisfies the Lévy-Khinchin formula, then there exists a Lévy process with $\mathbb{E}[\exp(i\theta X_1)] = \exp(-\psi(\theta))$, so that $X_1 \sim \text{Exp}(1)$. The derivative of ψ satisfies

$$\psi'(\theta) = -i\frac{1}{1-i\theta} = -\int_0^\infty ie^{i\theta x}e^{-x}\mathrm{d}x.$$

We now have

$$\psi(\theta) = \psi(0) + \int_0^\theta \psi'(s)\mathrm{d}s = -\int_0^\theta \int_0^\infty ie^{isx}e^{-x}\mathrm{d}x\mathrm{d}s = -\int_0^\infty e^{-x}\int_0^\theta ie^{isx}\mathrm{d}s\mathrm{d}x,$$

where we used Fubini's theorem to switch the integrals. It follows that

$$\psi(\theta) = -\int_0^\infty \frac{e^{-x}}{x}(e^{i\theta x}-1)\mathrm{d}x = \int_0^\infty \left(1-e^{i\theta x} - i\theta x\mathbb{1}_{\{x<1\}}\right)\pi(\mathrm{d}x) + i\theta\int_0^1 x\frac{e^{-x}}{x}\mathrm{d}x,$$

where $\pi(\mathrm{d}x) = \mathbb{1}_{\{x>0\}}\frac{e^{-x}}{x}\mathrm{d}x$. Finally we have

$$\psi(\theta) = i\theta\left(1-\frac{1}{e}\right) + \int_0^\infty \left(1-e^{i\theta x} - i\theta x\mathbb{1}_{\{x<1\}}\right)\pi(\mathrm{d}x).$$

Moreover, π is a Lévy-measure:

$$\int_{\mathbb{R}\setminus\{0\}} (x^2\wedge 1)\pi(\mathrm{d}x) = \int_0^1 xe^{-x}\mathrm{d}x + \int_1^\infty \frac{e^{-x}}{x}\mathrm{d}x < 2 < \infty.$$

So $\psi(\theta)$ satisfies the Lévy-Khinchin formula with triplet $(1-1/e, 0, \pi)$.

T2. Let $X_t = \sum_{n=1}^{N_t} Y_n$, where N_t is a Poisson process with intensity λ and $Y_1, Y_2, \ldots$ are i.i.d. random variables and independent of N. Let $f(u) := \mathbb{E}[e^{iuY_1}]$. We have

$$\begin{aligned}\mathbb{E}\left[e^{iuX_t}\right] &= \mathbb{E}\left[\exp\left(iu\sum_{n=1}^{N_t}Y_n\right)\right] = \sum_{k=0}^\infty \mathbb{E}\left[\exp\left(iu\sum_{n=1}^{k}Y_n\right)\mathbb{1}_{\{N_t=k\}}\right] \\ &= \sum_{k=0}^\infty f(u)^k\mathbb{P}(N_t=k) = \sum_{k=0}^\infty f(u)^k e^{-\lambda t}\frac{(\lambda t)^k}{k!} = \exp(-\lambda t + \lambda t f(u)),\end{aligned}$$

where we used Fubini's theorem to switch the expectation and the summation. It follows that the characteristic exponent ψ of X_t satisfies

$$\psi(u) = -\lambda + \lambda f(u).$$

Define the measure π on $\mathbb{R}$ by $A \mapsto \lambda\mathbb{P}(Y_1 \in A)$. Then

$$\int_{\mathbb{R}\setminus\{0\}} (x^2 \wedge 1)\pi(\mathrm{d}x) = \lambda\mathbb{E}[Y_1^2 \mathbb{1}_{\{|Y_1|\leq 1\}}] + \lambda\mathbb{P}(|Y_1| > 1) \leq 2\lambda < \infty.$$

Furthermore we have

$$\begin{aligned}\int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - iux\mathbb{1}_{\{|x|<1\}}\right) \pi(\mathrm{d}x) &= \lambda\mathbb{E}[e^{iuY_1}\mathbb{1}_{\{Y_1\neq 0\}}] - \lambda\mathbb{P}(Y_1 \neq 0) - iu\lambda\mathbb{E}[Y_1\mathbb{1}_{\{|Y_1|<1\}}] \\ &= -\lambda + \lambda\mathbb{P}(Y_1 = 0) + \lambda f(u) - \lambda\mathbb{P}(Y_1 = 0) - iu\lambda\mathbb{E}[Y_1\mathbb{1}_{\{|Y_1|<1\}}].\end{aligned}$$

We now choose $a = \lambda\mathbb{E}[Y_1\mathbb{1}_{\{|Y_1<1|\}}]$, and $\sigma = 0$. It follows that

$$\begin{aligned}\psi(u) &= iau - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - iux\mathbb{1}_{\{|x|<1\}}\right) \pi(\mathrm{d}x) \\ &= iu\lambda\mathbb{E}[Y_1\mathbb{1}_{\{|Y_1|<1\}}] - \lambda + \lambda f(u) - iu\lambda\mathbb{E}[Y_1\mathbb{1}_{\{|Y_1|<1\}}] = -\lambda + \lambda f(u),\end{aligned}$$

so that X_t has Lévy-khinchin triple $(a, 0, \pi)$.

T3. Suppose we have two Lévy-Khinchin triples (a, σ^2, π) and $(\tilde{a}, \tilde{\sigma}^2, \tilde{\pi})$ with $\psi(\theta) = \tilde{\psi}(\theta)$. If we can show that $\lim_{\theta\to\infty} \mathrm{Re}\left(\frac{\psi(\theta)}{\theta^2}\right) = -\frac{\sigma^2}{2}$ and $\lim_{\theta\to\infty} \mathrm{Re}\left(\frac{\tilde{\psi}(\theta)}{\theta^2}\right) = -\frac{\tilde{\sigma}^2}{2}$ then it follows that $\sigma^2 = \tilde{\sigma}^2$. We have

$$\mathrm{Re}\frac{\psi(\theta)}{\theta^2} = -\frac{\sigma^2}{2} + \int_{\mathbb{R}} \frac{\cos(\theta x) - 1}{\theta^2}\pi(\mathrm{d}x).$$

Since $1 - \cos(x) \leq x^2$ we can bound

$$\left|\frac{\cos(\theta x) - 1}{\theta^2}\right| \leq x^2\mathbb{1}_{\{|x|\leq 1\}} + \frac{2}{\theta^2}\mathbb{1}_{\{|x|>1\}}.$$

Suppose $\theta > \sqrt{2}$, then

$$\left|\frac{\cos(\theta x) - 1}{\theta^2}\right| \leq x^2 \wedge 1,$$

which is integrable with respect to π. So we can apply the dominated convergence theorem to find

$$\lim_{\theta\to\infty} \mathrm{Re}\left(\frac{\psi(\theta)}{\theta^2}\right) = -\frac{\sigma^2}{2}.$$

Similarly we can compute the limit for $\tilde{\psi}(\theta)$.