Exercises for Stochastic Processes

Tutorial exercises:

T1. Let X be a Feller process on S. Let $n \in \mathbb{N}$, $f_1, \dots, f_n \in C_0(S)$ and $t_1, \dots, t_n > 0$. Show that

$$x \mapsto \mathbb{E}^x \prod_{k=1}^{n} f_k(X_{t_k}) \in C_0(S).$$

T2. Let B be a one-dimensional Brownian motion.

(a) Show that $T_t f(x) := \mathbb{E}^x f(B_t)$ for $f \in C_0(\mathbb{R})$ defines a probability semigroup.

(b) What would go wrong if $C_0(\mathbb{R})$ were replaced by $C_b(\mathbb{R})$?
(The set $C_b(\mathbb{R})$ denotes the set of all continuous and bounded functions on $\mathbb{R}$)

T3. Show that $\mathcal{L}f := f'$ defined on $\mathcal{D}(\mathcal{L}) := \{f \in C_0(\mathbb{R}) \mid f' \in C_0(\mathbb{R})\}$ is a probability generator. What is the corresponding probability semigroup and Feller process?

(Hint for property (G1): consider the Stone-Weierstrass theorem.)

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Homework exercises:

H1. Consider a Markov chain on a state space $S \subset \mathbb{Z}$ with transition function p. Let $T_t f(x) := \sum_{y \in S} p_t(x,y) f(y)$ be the associated semigroup.

(a) Show that, if S is finite, then $\{T_t\}$ is a probability semigroup on $C_0(S)$.

(b) Show that, if S is infinite, then $\{T_t\}$ is a probability semigroup on $C_0(S)$ if and only if

$$\lim_{|x| \to \infty} p_t(x,y) = 0 \text{ for all } y \in S,\ t > 0\,.$$

H2. Show that, for a Q-matrix Q on a finite state space S,

$$\mathcal{L}f(x) := \sum_y q(x,y) f(y)$$

defines a probability generator on $C_0(S)$.

H3. Consider the Q-matrix on $S = \mathbb{N}_0$ for a "pure death process" given by

$$q(0,1) = 1,\ q(0,0) = -1,$$
$$q(k,k-1) = \delta_k,\ q(k,k) = -\delta_k \text{ for } k \in \mathbb{N}\,,$$

with $\delta_k > 0$. Define

$$\mathcal{L}f(x) := \sum_y q(x,y) f(y),$$

on the domain $C_0(S)$. Show that this operator satisfies conditions (G1), (G2) and (G4) of the definition of a probability generator. For what values of the sequence (δ_k) is condition (G3) satisfied?

H4. Show that

$$\mathcal{L}f := f'''$$

defined on

$$\mathcal{D} := \{f \in C_0(\mathbb{R}) \mid f', f'', f''' \in C_0(\mathbb{R})\}$$

is not a probability generator.

(Hint: Consider $f(x) = (-1 + x^2 - x^3)\exp(-\frac{x^2}{2})$.)

Deadline: Monday, 16.12.19