Exercises for Stochastic Processes

Tutorial exercises:

Let B be a standard Brownian motion and $(\mathcal{F}_t)$ the usual right continuous filtration.

T1. Show that

$$M_t := B_t$$

and

$$N_t := B_t^2 - t$$

are martingales.

T2. Show that Gaussian processes $(X_t)_{t\geq 0}$ that are martingales have independent increments.

T3. Show that, for any square-integrable martingale (M_t) and $r < s < t$,

$$\mathbb{E}\left[(M_t - M_s)^2 \mid \mathcal{F}_r\right] = \mathbb{E}\left[M_t^2 - M_s^2 \mid \mathcal{F}_r\right].$$

T4. For $-a < 0 < b$ we define

$$\tau := \inf\{t \geq 0 \mid B_t \in \{-a, b\}\}$$

. Show that $\mathbb{P}(B_\tau = b) = \frac{a}{a+b}$ and $\mathbb{E}(\tau) = ab$.

T5. Let $v \leq u \leq w$. Show that there is a unique probability distribution $\mathbb{P}$ on $\mathbb{R}$, such that $\mathbb{P}(\{v, w\}) = 1$ and the distribution has mean u:

$$\int x \, \mathbb{P}(dx) = u.$$

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Homework exercises:

Let B be a standard Brownian motion and $(\mathcal{F}_t)$ the usual right continuous filtration.

H1. (a) Show that, for any $\sigma \geq 0$, the process

$$\left(e^{\sigma B_t - \frac{\sigma^2 t}{2}}\right)_{t\geq 0}$$

is a martingale.

(b) Show that the following processes are martingales:

- $(B_t^2 - t)$
- $(B_t^3 - 3tB_t)$
- $(B_t^4 - 6tB_t^2 + 3t^2)$
- ...

and find the general formula for the above sequence.

(Hint: use (a))

H2. For $a, b > 0$, we define $\tau := \inf\{t \geq 0 \mid B_t = a + bt\}$. Show that $\mathbb{P}(\tau < \infty) = e^{-2ab}$.
(Hint: First show using H1(a) that

$$\mathbb{E}[e^{-\lambda\tau}\mathbb{1}_{\{\tau<\infty\}}] = \exp\Big(-a\big(b + \sqrt{b^2 + 2\lambda}\big)\Big),$$

for all $\lambda > 0$.)

H3. Compute $\mathbb{E}(\tau^2)$ for $\tau := \inf\{t \geq 0 \mid B(t) \in \{-a, b\}\}$ and $-a < 0 < b$.

Deadline: Monday, 18.11.19