Exercises for Stochastic Processes

Tutorial exercises:

T1. Is there a Lévy process X with $X_1 \sim \text{Exp}(1)$?

T2. Let X be a compound Poisson process (as in T11.3). Compute the characteristic function of X_t and find the corresponding Lévy-Khinchin triple.

T3. Show that the same characteristic exponent ψ in the Lévy-Khinchin formula cannot be represented via several different choices of σ (i.e. that the "Brownian part" of a Lévy process is uniquely determined).

$\left(\text{Hint: Show that } \lim_{\theta\to\infty} \operatorname{Re}\left(\frac{\psi(\theta)}{\theta^2}\right) = -\frac{\sigma^2}{2}\right)$

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Homework exercises:
Define

$$C_c^2(\mathbb{R}) := \{f \in C_0(\mathbb{R}) \,:\, f \text{ has compact support and } f', f'' \in C_0(\mathbb{R})\}.$$

H1. Consider a Feller process with continuous paths whose generator is given by

$$\mathcal{L}f := \frac{c(x)}{2} f'' ,$$

defined on $C_c^2(\mathbb{R})$, and where $c(x)$ is strictly positive and continuous.

(a) Let $\tau_{a,b}$ be the first hitting time of $\{a, b\}$, with $a < b$ and $a < x < b$. Show that

$$\mathbb{E}^x \tau_{a,b} = \int_a^b \frac{2}{c(u)} \frac{(x \wedge u - a)(b - x \vee u)}{b - a} \mathrm{d}u \,.$$

(Apply Theorem 4.7 to a function $f \in C_c^2(\mathbb{R})$, with $f(x) = \int_a^x \int_a^z 2/c(u)\mathrm{d}u\mathrm{d}z$ to find a suitable martingale.)

(b) Let τ_a be its first hitting time of $a \in \mathbb{R}$. Show that, for $x > a$, $\mathbb{E}^x \tau_a$ is finite if and only if $\int_0^\infty \frac{\mathrm{d}x}{c(x)} < \infty$.

H2. Consider a Feller process on $\mathbb{R}$ whose generator, restricted to C_c^2 functions, is given by $\mathcal{L}f = cf''$, where $c \in C_b(\mathbb{R})$ is a nonnegative function. Show that such a process has a continuous modification.
(The ansatz you have seen for the Fisher-Wright diffusion goes through with some modifications.)

H3. Let (a, σ, π) be a Lévy-Khinchin triple. Assume $\int_{\mathbb{R}\setminus\{0\}} \pi(\mathrm{d}x) < \infty$. Find a Lévy process with the corresponding characteristic function.

Deadline: Monday, 20.01.19