Mathematisches Institut der Universität München
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WS 17/18
Sheet 4

Exercises for Stochastic Processes

Tutorial exercises:

T1. Show that, for any square-integrable martingale (M_t) and $r < s < t$,

$$\mathbb{E}\left[(M_t - M_s)^2 \mid \mathfrak{F}_r\right] = \mathbb{E}\left[M_t^2 - M_s^2 \mid \mathfrak{F}_r\right].$$

T2. Show that Gaussian processes $(X_t)_{t\geq 0}$ that are martingales have independent increments.

Let B be a standard Brownian motion and $(\mathfrak{F}_t)$ the usual right continuous filtration.

T3. Show that

$$M_t := B_t$$

and

$$N_t := B_t^2 - t$$

define martingales.

T4. Define

$$\tau := \inf\{t \geq 0 \mid B_t \in \{-a, b\}\}$$

for $-a < 0 < b$. Show that $\mathbb{P}(B_\tau = b) = \frac{a}{a+b}$ and $\mathbb{E}(\tau) = ab$.

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Homework exercises:

Let B be a standard Brownian motion and $(\mathfrak{F}_t)$ the usual right continuous filtration.

H1. (a) Show that, for $\sigma \geq 0$, the process

$$\left(e^{\sigma B(t) - \frac{\sigma^2 t}{2}}\right)_{t \geq 0}$$

is a martingale.

(b) Show that the following processes are martingales:

- $(B_t^2 - t)$
- $(B_t^3 - 3tB_t)$
- $(B_t^4 - 6tB_t^2 + 3t^2)$
- ...

and find the general formula for the above sequence.

(Hint: use (a))

H2. For $x \neq 0$ let $\mathbb{P}^x$ denote the distribution of $X_t := x + B$, where $B = (B_t)_{t \geq 0}$ is standard Brownian motion. For $x = 0$ let $\mathbb{P}^0$ denote the distribution of the process $X_t \equiv 0$. Show that the $\{\mathbb{P}^x\}_{x \in \mathbb{R}}$ satisfies the Markov property, but not the strong Markov property, i.e., show that for all $x \in \mathbb{R}$ and $t \geq 0$

$$\mathbb{E}^x[Y \circ \theta_t | \mathscr{F}_t] = \mathbb{E}^{X_t} Y \quad \text{a.s.},$$

for any bounded random variable Y, but

$$\mathbb{E}^x[Y_\tau \circ \theta_\tau | \mathscr{F}_\tau] \neq \mathbb{E}^{X_\tau} Y_\tau \quad \text{a.s. on } \{\tau < \infty\},$$

for some stopping time τ, some $x \in \mathbb{R}$ and some bounded random variable Y_τ.

H3. With $\tau := \inf\{t \geq 0 \mid B_t = a + bt\}$ for $a, b > 0$, show that $\mathbb{P}(\tau < \infty) = e^{-2ab}$.
(Hint: First show using H1(a) that

$$\mathbb{E}[e^{-\lambda\tau} \mathbb{1}_{\{\tau < \infty\}}] = \exp\Big(-a\big(b + \sqrt{b^2 + 2\lambda}\big)\Big),$$

for all $\lambda > 0$.)

H4. Find $\mathbb{E}(\tau^2)$ for $\tau := \inf\{t \geq 0 \mid B(t) \in \{a, b\}\}$ and $a < 0 < b$.

Deadline: Tuesday, 14.11.17