Exercises for Stochastic Processes

Tutorial exercises:

T1. (Mass-transport principle) Let $\{T_{u,v}\}_{u,v\in\mathbb{Z}^d}$ be a collection of non-negative random variables such that $\{T_{u+z,v+z}\}_{u,v\in\mathbb{Z}^d}$ has the same distribution as $\{T_{u,v}\}_{u,v\in\mathbb{Z}^d}$ for every $z\in\mathbb{Z}^d$. Show that $\mathbb{E}[\sum_{z\in\mathbb{Z}^d} T_{0,z}] = \mathbb{E}[\sum_{z\in\mathbb{Z}^d} T_{z,0}]$.

T2. (Matchings) Let $X = \{X_i\}_{i\ge 1}$ and $Y = \{Y_j\}_{j\ge 1}$ be stationary point processes in $\mathbb{R}^d$ with intensities $\lambda_x, \lambda_y > 0$. A *stationary perfect matching* between X and Y is a measurable function $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{N} \times \mathcal{N} \to \{0,1\}$ such that

(a) For every $X_i \in X$ there exists exactly one $Y_j \in Y$ with $f(X_i, Y_j, X, Y) = 1$.

(b) For every $Y_j \in Y$ there exists exactly one $X_i \in X$ with $f(X_i, Y_j, X, Y) = 1$.

(c) $f(x+z, y+z, \varphi+z, \psi+z) = f(x, y, \varphi, \psi)$ holds for all $x, y, z \in \mathbb{R}^d$ and $\varphi, \psi \in \mathcal{N}$.

Show that if there is a perfect matching between X and Y, then $\lambda_x = \lambda_y$

T3. Determine the Palm distribution of the Poisson cluster process from problem H1b

T4. (Poisson line process)

(a) Let $X = \{X_i\}_{i\ge 1}$ be a homogeneous Poisson point process in $\mathbb{R}^2$ with intensity $\lambda > 0$ and $\{\ell_i\}_{i\ge 1}$ be an iid family of lines through the origin with uniformly distributed orientation that are also independent of X. Show that with probability 1 there exist infinitely many $i \ge 1$ such that $X_i + \ell_i$ intersects $[0,1]^2$.

(b) Let $R = \{R_i\}_{i\ge 1}$ be a homogeneous Poisson point process in $\mathbb{R}$ with intensity $\lambda > 0$ and $\{U_i\}_{i\ge 1}$ be an iid family of uniform $[0, 2\pi]$-distributed random variables that are also independent of R. Define $l_i = \{(x,y) \in \mathbb{R}^2 : x\cos(U_i) + y\sin(U_i) = R_i\}$. Let $B \subset \mathbb{R}^2$ be a bounded Borel set. Compute $\mathbb{E}[\sum_{i\ge 1} |l_i \cap B|]$, where $|l_i \cap B|$ denotes the length of the part of l_i inside B.

Homework exercises:

H1. (Cluster process) Let $X = \{X_i\}_{i\geq 1}$ be a Poisson point process in $\mathbb{R}^d$ with locally finite intensity measure λ. Let $Y = \{Y_j\}_{j\leq N}$ be a point process with $\mathbb{E}[N] < \infty$. Let $\{Y^{(i)}\}_{i\geq 1}$ be iid copies of Y that are also independent of X. Then,

$$Z = \bigcup_{i\geq 1}(X_i + Y^{(i)}) = \{X_i + Y_j^{(i)}\}_{i,j\geq 1}$$

is called *Poisson cluster process.*

(a) Show that the intensity measure λ' of Z is locally finite if $Y \subset A$ almost surely for some bounded set A.

(b) Show that the intensity measure λ' of Z is given by

$$\lambda'(B) = \int \mathbb{E}[\#(Y \cap (B - x))]\lambda(\mathrm{d}x)$$

(c) Show that Z is a stationary point process with intensity $\lambda_0\mathbb{E}[N]$ if X is a homogeneous Poisson point process with intensity $\lambda_0 > 0$.

H2. Let $d \geq 2$ and $X = \{X_i\}_{i\geq 1}$ be a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda > 0$. Let X_o denote the closest point of X to the origin. Show that $(X - X_o) \setminus \{o\} = \{X_i - X_o\}_{i\geq 1} \setminus \{o\}$ is not a homogeneous Poisson point process with intensity λ.

Deadline: Tuesday, 06.02.18