Mathematisches Institut der Universität München
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WS 17/18
Sheet 1

Exercises for Stochastic Processes

Tutorial exercises:

T1. Show that a stochastic process $\mathbb{X}$ on $(\Omega, \mathfrak{F})$ with values in S^T is $\mathfrak{F} - \mathfrak{S}^T$-measurable if and only if all projections X_t are $\mathfrak{F} - \mathfrak{S}$-measurable. ($\mathfrak{S}$ denotes a σ-algebra on S.)

T2. Let $(X_t)_{t\in\mathbb{R}}$ be a real-valued $\mathfrak{F} - \mathfrak{B}^{\mathbb{R}}$-measurable stochastic process with continuous paths. Show that $\sup_{t\in\mathbb{R}} X_t$ is measurable.

T3. Let $\tau_1, \tau_2, \ldots$ be independent and exponentially distributed with parameter $\lambda > 0$. Define

$$N_t := |\{k \geq 1 \mid \tau_1 + \cdots + \tau_k \leq t\}| \ .$$

Show that, if $0 < s < t$, then N_s and $N_t - N_s$ are independently Poisson distributed with parameters λs and $\lambda(t - s)$.

T4. Let M and N be independent Poisson processes with intensities λ and μ. Show that $M + N$ is a Poisson process with intensity $\lambda + \mu$.

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Homework exercises:

H1. (a) Let $(S, \mathfrak{S})$ be a measurable space. Show that, for uncountable $T \subset \mathbb{R}$,

$$\mathfrak{S}^T = \left\{ \{ f \in S^T \mid (f(t_1), f(t_2), \dots) \in A \} \mid t_1, t_2, \dots \in T, A \in \mathfrak{S}^{\{t_1, t_2, \dots\}} \right\}.$$

("All sets in the product σ-algebra are countably determined.")

(b) Conclude that the set of all continuous functions on $T \subset \mathbb{R}$ is no (product-)measurable subset of $\mathbb{R}^T$.

H2. Deduce Kolmogorov's continuity criterion (Theorem 1.3 in the lecture) from Proposition 1.4.

H3. Under which (necessary and sufficient) condition does an i.i.d. family $(X_t)_{t \in \mathbb{R}}$ have a continuous modification?

H4. Let $N, X_1, X_2, \dots$ be independent random variables, N Poisson distributed and X_k uniformly distributed on $[0, 1]$. Show that

$$N_t := \sum_{k=1}^{N} 1_{[0,t]}(X_k) \qquad (t \in [0, 1])$$

is a Poisson process (restricted to $t \in [0, 1]$) in the sense of the "alternative 1" definition from the lecture. How can it be extended to all $t \geq 0$?

Deadline: Tuesday, 24.10.17