1. Addendum to solution of sheet 5 problem H2 (Sketch)

This addendum provides the link between convergence of finite dimensional distribution and the convergence of the maxima in problem H2 of sheet 5.

Put $X_t = B_t - tB_1$. Let $\delta > 0$ and $a \in \mathbb{R}$ be arbitrary. Then,

$$\mathbb{P}(\max_{i<n} X_{i/n} \le a - \delta) - \mathbb{P}(\max_{t\le 1} X_t \le a) \le \sum_{i\le n} \mathbb{P}(\max_{(i-1)/n\le t\le i/n} (X_t - X_{(i-1)/n}) > \delta). \tag{1}$$

Now,

$$\begin{aligned}\mathbb{P}\Big(\max_{(i-1)/n\le t\le i/n} (X_t - X_{(i-1)/n}) > \delta\Big) &\le \mathbb{P}(|B_1| > n\delta/2) + \mathbb{P}(\max_{t\le 1/n} B_t > \delta/2)\\ &= \mathbb{P}(|B_1| > n\delta/2) + \mathbb{P}(|B_{1/n}| > \delta/2),\end{aligned}$$

and both expressions decay exponentially in n. Hence, the right-hand side of (1) tends to 0 as $n \to \infty$. Next, we consider the corresponding difference for the approximating processes.

$$\begin{aligned}&|\mathbb{P}(\max_{t\le 1} B_t \le a \mid |B_1| \le \varepsilon) - \mathbb{P}(\max_{i<n} B_{i/n} \le a - \delta \mid |B_1| \le \varepsilon)| && (2)\\ &\le \sum_{i\le n} \mathbb{P}(\max_{(i-1)/n\le t\le i/n} (B_t - B_{(i-1)/n}) > \delta ||B_1| \le \varepsilon) && (3)\end{aligned}$$

Now,

$$\begin{aligned}\mathbb{P}\Big(\max_{(i-1)/n\le t\le i/n} (B_t - B_{(i-1)/n}) > \delta ||B_1| \le \varepsilon\Big) &\le \mathbb{P}\Big(\max_{(i-1)/n\le t\le i/n} (X_t - X_{(i-1)/n}) > \delta/2 ||B_1| \le \varepsilon\Big)\\ &= \mathbb{P}\Big(\max_{(i-1)/n\le t\le i/n} (X_t - X_{(i-1)/n}) > \delta/2\Big),\end{aligned}$$

where the last equality follows from the independence hint in problem H2 of sheet 5.

Now, for every $n \ge 1$ and $\delta > 0$,

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \mathbb{P}(\max_{t\le 1} B_t \le a \mid |B_1| \le \varepsilon) - \mathbb{P}(\max_{t\le 1} X_t \le a + \delta)\\ &\le \limsup_{\varepsilon\to 0} |\mathbb{P}(\max_{i<n} B_{i/n} \le a \mid |B_1| \le \varepsilon) - \mathbb{P}(\max_{i<n} X_{i/n} \le a)|\\ &\quad + \mathbb{P}(\max_{i<n} X_{i/n} \le a) - \mathbb{P}(\max_{t\le 1} X_t \le a + \delta)\end{aligned}$$

First, by (1) we choose n such that the second expression gets small. For the first summand, we let $\varepsilon \to 0$ and use convergence of finite-dimensional distributions. This yields one inequality Similarly, using

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \mathbb{P}(\max_{t\le 1} X_t \le a + \delta) - \mathbb{P}(\max_{t\le 1} B_t \le a \mid |B_1| \le \varepsilon)\\ &\le \limsup_{\varepsilon\to 0} |\mathbb{P}(\max_{i<n} X_{i/n} \le a) - \mathbb{P}(\max_{i<n} B_{i/n} \le a \mid |B_1| \le \varepsilon)|\\ &\quad + \limsup_{\varepsilon\to 0} |\mathbb{P}(\max_{t\le 1} B_t \le a \mid |B_1| \le \varepsilon) - \mathbb{P}(\max_{i<n} B_{i/n} \le a - \delta \mid |B_1| \le \varepsilon)|\end{aligned}$$

we arrive at the other inequality.