Solutions to tutorial exercises for stochastic processes

T1. (a) (G1): $\mathcal{D}(\mathcal{L})$ is a vector space closed under multiplication. We can use the Stone-Weierstrass theorem to show that $\mathcal{D}(\mathcal{L})$ is dense in $C_0(\mathbb{R})$.

(G2):Let $\lambda > 0$, $f \in \mathcal{D}(\mathcal{L})$ and $g = f - \frac{\lambda}{2} f''$. If $\inf_x f(x) = 0$ we have

$$\inf_x g(x) \leq \lim_{x\to\infty} f(x) - \frac{\lambda}{2} f''(x) = 0 = \inf_x f(x).$$

Now suppose $\inf_x f(x) < 0$. Then there exists an x_0 with $f(x_0) = \inf_x f(x)$ and $f''(x_0) > 0$. We find

$$\inf_x g(x) = f(x_0) - \frac{\lambda}{2} f''(x_0) < f(x_0) = \inf_x f(x).$$

(G3):Let $\lambda > 0$ and let $g \in C_0(\mathbb{R})$. Let $\mu = \frac{1}{\lambda}$. Consider the resolvent of the Brownian motion semigroup $f := U_\mu g$ given by (see (c))

$$\begin{aligned} f(x) &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\mu}} e^{-\sqrt{2\mu}|x-y|} g(y)\mathrm{d}y \\ &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\mu}} e^{-\sqrt{2\mu}(x-y)} g(y)\mathrm{d}y + \int_{x}^{\infty} \frac{1}{\sqrt{2\mu}} e^{-\sqrt{2\mu}(y-x)} g(y)\mathrm{d}y. \end{aligned}$$

We can compute the derivative of f:

$$\begin{aligned} f'(x) &= -\int_{-\infty}^{x} e^{-\sqrt{2\mu}(x-y)} g(y)\mathrm{d}y + \frac{1}{\sqrt{2\mu}} g(x) + \int_{x}^{\infty} e^{-\sqrt{2\mu}(y-x)} g(y)\mathrm{d}y - \frac{1}{\sqrt{2\mu}} g(x) \\ &= -\int_{-\infty}^{x} e^{-\sqrt{2\mu}(x-y)} g(y)\mathrm{d}y + \int_{x}^{\infty} e^{-\sqrt{2\mu}(y-x)} g(y)\mathrm{d}y. \end{aligned}$$

And the second derivative:

$$\begin{aligned} f''(x) &= \sqrt{2\mu} \int_{-\infty}^{x} e^{-\sqrt{2\mu}(x-y)} g(y)\mathrm{d}y - g(x) + \int_{x}^{\infty} e^{-\sqrt{2\mu}(y-x)} g(y)\mathrm{d}y - g(x) \\ &= 2\mu f(x) - 2g(x). \end{aligned}$$

So that

$$\frac{1}{\mu} g = f - \frac{1}{2\mu} f'' = f - \lambda \mathcal{L} f.$$

So $\frac{1}{\mu} g \in \mathcal{R}(I - \lambda\mathcal{L})$. It follows that $\mathcal{R}(I - \lambda\mathcal{L}) = C_0(\mathbb{R})$.

(G4): Let $\lambda > 0$ and consider $f_n(x) = \exp\left(-\frac{x^2}{n}\right)$. Define $g_n = f_n - \frac{\lambda}{2} f_n''$. Then $f_n \in \mathcal{D}(\mathcal{L})$ for all $n \in \mathbb{N}$, $f_n(x) \to 1$ pointwise for all $x \in \mathbb{R}$ and $g_n(x) \to 1$ pointwise for all $x \in \mathbb{R}$. Furthermore $\sup_n \|g_n\| < \infty$.

(b) Let $T_tf(x) := \mathbb{E}^x f(B_t)$ be the Brownian motion semigroup. We will show that

$$\left\| \frac{T_tf - f}{t} - \frac{1}{2}f'' \right\| \to 0 \quad \text{as} \quad t \downarrow 0.$$

We can write

$$T_tf(x) = \mathbb{E}^x f(B_t) = \mathbb{E}^0\big[f(x + \sqrt{t}B_1)\big].$$

We use a Taylor approximation of f around x so that for some ξ dependent on x, t and B_1 we have

$$\begin{aligned} T_tf(x) - f(x) &= \mathbb{E}^0\big[f(x + \sqrt{t}B_1) - f(x)\big] \\ &= f'(x)\mathbb{E}^0\big[\sqrt{t}B_1\big] + \frac{1}{2}\mathbb{E}^0\left[f''(\xi)tB_1^2\right] \\ &= \frac{1}{2}t\int_{\mathbb{R}} f''(\xi)y^2\phi(y)\mathrm{d}y, \end{aligned}$$

where $\phi(y) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)$. We have $|\xi - x| \le |\sqrt{bt}y + at|$. Therefore by uniform continuity of f'' it holds that

$$\sup_x |f''(\xi) - f''(x)| \to 0 \quad \text{as} \quad t \to 0.$$

We conclude

$$\begin{aligned} \sup_x \left| \frac{T_tf(x) - f(x)}{t} - \frac{1}{2}f''(x) \right| &= \frac{1}{2}\sup_x \left| \int_{\mathrm{R}} (f''(\xi) - f(x))\phi(y)\mathrm{d}y \right| \\ &\le \int_{\mathrm{R}} \sup_x \big|f''(\xi) - f(x)\big|\phi(y)\mathrm{d}y \to 0 \quad \text{as} \quad t \downarrow 0, \end{aligned}$$

by the dominated convergence theorem.

(c) Let $f \in C_0(\mathbb{R})$, we can write

$$\begin{aligned} T_tf(x) = \mathbb{E}^x[f(B_t)] = \mathbb{E}^0[f(x + B_t)] &= \int_{\mathbb{R}} f(x+z)\frac{1}{\sqrt{2\pi t}}e^{-\frac{z^2}{2t}}\mathrm{d}z \\ &= \int_{\mathbb{R}} f(y)\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}\mathrm{d}y. \end{aligned}$$

Hence, for $\alpha > 0$:

$$(U_\alpha f)(x) = \int_0^\infty e^{-\alpha t}\int_{\mathbb{R}} f(y)\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}\mathrm{d}y\mathrm{d}t = \int_{\mathbb{R}} f(y)\int_0^\infty \frac{1}{\sqrt{2\pi t}}e^{-\alpha t - \frac{(x-y)^2}{2t}}\mathrm{d}t\mathrm{d}y,$$

where we used Fubini's theorem to exchange the integrals. We have

$$-\alpha t - \frac{(x-y)^2}{2t} = \left(\sqrt{\alpha t} - \frac{|x-y|}{\sqrt{2t}}\right)^2 + \sqrt{2\alpha}|x-y|,$$

so that

$$\int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-\alpha t - \frac{(x-y)^2}{2t}} \mathrm{d}t = \frac{e^{-\sqrt{2\alpha}|x-y|}}{\sqrt{2\alpha}} \int_0^\infty \sqrt{\frac{\alpha}{\pi t}} \exp\left(-\left(\sqrt{\alpha t} - \frac{|x-y|}{\sqrt{2t}}\right)^2\right) \mathrm{d}t.$$

It remains to show that

$$\int_0^\infty \sqrt{\frac{\alpha}{\pi t}} \exp\left(-\left(\sqrt{\alpha t} - \frac{|x-y|}{\sqrt{2t}}\right)^2\right) \mathrm{d}t = 1.$$

We use the change of variables

$$s := \frac{|x-y|^2}{2\alpha t},$$

so that

$$\int_0^\infty \sqrt{\frac{\alpha}{\pi t}} \exp\left(-\left(\sqrt{\alpha t} - \frac{|x-y|}{\sqrt{2t}}\right)^2\right) \mathrm{d}t = \\ \int_0^\infty -\frac{|x-y|^2}{2\alpha s^2} \alpha \sqrt{\frac{2s}{\pi |x-y|^2}} \exp\left(-\left(\frac{|x-y|}{\sqrt{2s}} - \sqrt{\alpha s}\right)^2\right) \mathrm{d}s.$$

The exponential on the left and right hand site of the above expression are the same. Therefore

$$\begin{aligned} 2\int_0^\infty \sqrt{\frac{\alpha}{\pi t}} \exp\left(-\left(\sqrt{\alpha t} - \frac{|x-y|}{\sqrt{2t}}\right)^2\right) \mathrm{d}t &= \int_0^\infty \sqrt{\frac{\alpha}{\pi t}} \exp\left(-\left(\sqrt{\alpha t} - \frac{|x-y|}{\sqrt{2t}}\right)^2\right) \mathrm{d}t \\ &\quad - \int_0^\infty \frac{|x-y|^2}{2s^2} \sqrt{\frac{2s}{\pi|x-y|^2}} \exp\left(-\left(\frac{|x-y|}{\sqrt{2s}} - \sqrt{\alpha s}\right)^2\right) \mathrm{d}s \\ &= \int_0^\infty \left(\sqrt{\frac{\alpha}{\pi t}} - \frac{|x-y|^2}{2t^2}\sqrt{\frac{2t}{\pi|x-y|^2}}\right) \exp\left(-\left(\sqrt{\alpha t} - \frac{|x-y|}{\sqrt{2t}}\right)^2\right) \mathrm{d}t \\ &= \int_0^\infty \left(\sqrt{\frac{\alpha}{\pi t}} - \frac{|x-y|}{\sqrt{2\pi t^3}}\right) \exp\left(-\left(\sqrt{\alpha t} - \frac{|x-y|}{\sqrt{2t}}\right)^2\right) \mathrm{d}t. \end{aligned}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\alpha t} - \frac{|x-y|}{\sqrt{2t}}\right) = \frac{\sqrt{\alpha}}{2\sqrt{t}} + \frac{|x-y|}{2\sqrt{2t^3}},$$

so by another change of variables

$$\int_0^\infty \sqrt{\frac{\alpha}{\pi t}} \exp\left(-\left(\sqrt{\alpha t} - \frac{|x-y|}{\sqrt{2t}}\right)^2\right) \mathrm{d}t = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left(-\sigma^2\right) \mathrm{d}\sigma = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1.$$

T2. Define the Feller process $X_t = at + \sqrt{b}B_t$. Let $\phi(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$. Then for all $f \in \{f \in C_0(\mathbb{R}) \mid f', f'' \in C_0(\mathbb{R})\}$ we have

$$\begin{aligned} T_t f(x) - f(x) &= \mathbb{E}^x\left[f\left(at + \sqrt{b}B_t\right) - f(x)\right] \\ &= \mathbb{E}^0\left[f\left(x + at + \sqrt{bt}B_1\right) - f(x)\right]. \end{aligned}$$

We can use a Taylor approximation of f around x to get

$$\begin{aligned} T_t f(x) - f(x) &= f'(x)\mathbb{E}^0[at + \sqrt{bt}B_1] + \frac{1}{2}\mathbb{E}^0\left[f''(\xi)(at + \sqrt{bt}B_1)^2\right] \\ &= f'(x)at + \frac{1}{2}\int_{\mathbb{R}} f''(\xi)(at + \sqrt{bt}y)^2\phi(y)\mathrm{d}y \end{aligned}$$

for some ξ dependant on x, y and t with $|\xi - x| \leq |\sqrt{bt}y + at|$. Therefore by uniform continuity of f'' we have

$$\sup_x |f''(\xi) - f''(x)| \to 0 \quad \text{as} \quad t \to 0.$$

We will now show that X_t has generator $af' + \frac{b}{2}f''$:

$$\begin{aligned} &\sup_x \left|\frac{1}{t}\big(T_t f(x) - f(x)\big) - af'(x) - \frac{b}{2}f''(x)\right| = \\ &\sup_x \left|\frac{1}{2}\int_{\mathbb{R}} f''(\xi)a^2 t\phi(y)\mathrm{d}y + a\sqrt{bt}\int_{\mathbb{R}} f''(\xi)y\phi(y)\mathrm{d}y + \frac{b}{2}\int_{\mathbb{R}}(f''(\xi) - f(x))y^2\phi(y)\mathrm{d}y\right| \\ &\leq \frac{1}{2}a^2 t\|f''\| + |a|\sqrt{b}\sqrt{t}\|f''\|\int_{\mathbb{R}}|y|\phi(y)\mathrm{d}y + \frac{b}{2}\int_{\mathbb{R}}\sup_x |f''(\xi) - f''(x)|y^2\phi(y)\mathrm{d}y \\ &\to 0 \quad \text{as} \quad t \to 0, \end{aligned}$$

where we use the dominated convergence theorem for the convergence of the last term. So it follows that the generator of X_t is $\mathcal{L}$ and that $\mathcal{L}$ is a probability generator. The operators T_t form the corresponding probability semigroup.

T3. (a) We can use Taylor's theorem to write for some ξ dependent on x, t and B_1^i:

$$\begin{aligned} T_t f(x) - f(x) &= \mathbb{E}^0\big[f(x + \sqrt{t}B_1) - f(x)\big] \\ &= \sum_{i=1}^n f_i(x)\mathbb{E}^0[\sqrt{t}B_1^i] + \frac{1}{2}\sum_{i,j}\mathbb{E}^0[f_{ij}(\xi)tB_1^i B_1^j] \\ &\leq \frac{t}{2}\sum_{i=1}^n \mathbb{E}^0[f_{ii}(\xi){B_1^i}^2] + \frac{t}{2}\sum_{i,j}\|f_{ij}\|\mathbb{E}^0[tB_1^i B_1^j] \\ &= \frac{t}{2}\sum_{i=1}^n \mathbb{E}^0[f_{ii}(\xi){B_1^i}^2]. \end{aligned}$$

Similarly we have

$$T_t f(x) - f(x) \geq \frac{t}{2} \sum_{i=1}^{n} \mathbb{E}^0[f_{ii}(\xi) {B_1^i}^2].$$

If $t \downarrow 0$ then $\xi \to x$. Therefore by uniform continuity of f_{ii} and the dominated convergence theorem:

$$\begin{aligned}
\sup_x \left| \frac{T_t f(x) - f(x)}{t} - \frac{1}{2}\Delta f(x) \right| &\leq \frac{1}{2} \sup_x \left| \int_{\mathbb{R}} \big(\Delta f(\xi) - \Delta f(x)\big) y^2 \phi(y) \mathrm{d}y \right| \\
&\leq \frac{1}{2} \int_{\mathbb{R}} \sup_x |\Delta f(\xi) - \Delta f(x)| \, y^2 \phi(y) \mathrm{d}y \to 0 \quad \text{as} \quad t \downarrow 0.
\end{aligned}$$

(b) Let $\varepsilon > 0$, we can write

$$\begin{aligned}
\frac{u(t+\varepsilon, x) - u(t,x)}{\varepsilon} &= \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \frac{\mathrm{d}}{\mathrm{d}s} T_s g(x) \mathrm{d}s \\
&= \frac{1}{\varepsilon} \int_0^{\varepsilon} \frac{\mathrm{d}}{\mathrm{d}s} T_{t+s} g(x) \mathrm{d}s \\
&= \frac{1}{\varepsilon} \int_0^{\varepsilon} \frac{\mathrm{d}}{\mathrm{d}s} T_t T_s g(x) \mathrm{d}s \\
&= \frac{1}{\varepsilon} \int_0^{\varepsilon} T_t \frac{\mathrm{d}}{\mathrm{d}s} T_s g(x) \mathrm{d}s,
\end{aligned}$$

since T_t is a linear operator. We obtain

$$\begin{aligned}
\frac{u(t+\varepsilon, x) - u(t,x)}{\varepsilon} &= \frac{1}{\varepsilon} \int_0^{\varepsilon} T_t \mathcal{L} g(x) \mathrm{d}s \\
&\to T_t \mathcal{L} g(x) \quad \text{as} \quad \varepsilon \to 0,
\end{aligned}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t,x) = T_t \mathcal{L} g(x) = \mathbb{E}^x \left[\frac{1}{2} \Delta g(X_t) \right] = \frac{1}{2} \Delta \mathbb{E}^x \left[g(X_t) \right] = \frac{1}{2} \Delta u(t,x),$$

where we used Leibniz's rule to switch the differentiation and integration.

T4. (a) We say that a process X is stationary if $X_t \sim \mu$ (under $\mathbb{P}^x$) implies $X_{t+s} \sim \mu$ for all t, s. Suppose $X_t \sim \mu$, then

$$\mathbb{E}^x f(X_s) = \int f \mathrm{d}\mu.$$

The new definition of stationary now says that

$$\int T_s f \mathrm{d}\mu = \int f \mathrm{d}\mu = \mathbb{E}^x f(X_t).$$

On the other hand

$$\int T_s f \mathrm{d}\mu = \int \mathbb{E}^x f(X_s) \mu(\mathrm{d}x) = \mathbb{E}^x \mathbb{E}^{X_t} f(X_s) = \mathbb{E}^x f(X_{t+s}).$$

So

$$\mathbb{E}^x f(X_t) = \mathbb{E}^x f(X_{t+s}),$$

which is the old definition of stationary if we take indicator functions for f.

(b) Suppose μ is stationary, so $\int T_t f - f \mathrm{d}\mu = 0$. Then for all $t > 0$ we have

$$\left| \int \mathcal{L} f \mathrm{d}\mu \right| = \left| \int \mathcal{L} f - \frac{T_t f - f}{t} \mathrm{d}\mu \right| \leq \left\| \mathcal{L} f - \frac{T_t f - f}{t} \right\|_\infty \to 0 \quad \text{as} \quad t \downarrow 0.$$

Now suppose $\int \mathcal{L} f \mathrm{d}\mu = 0$. We now have

$$\int T_t f - f \mathrm{d}\mu = \int \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} T_s f \mathrm{d}s \mathrm{d}\mu = \int \int_0^t \mathcal{L} T_s f \mathrm{d}s \mathrm{d}\mu = \int_0^t \int \mathcal{L} T_s f \mathrm{d}\mu \mathrm{d}s = 0,$$

where we used Fubini's theorem in the last step.