Solutions to tutorial exercises for stochastic processes

T1. Let f be a non-negative harmonic function for X. Then

$$|f(x)| = f(x) = \mathbb{E}[f(x)] = \mathbb{E}\big[|f(x)|\big] < \infty,$$

so that f is bounded. Since X is irreducible and recurrent it follows that every bounded harmonic function for X is constant.

T2. We will prove the statement by induction on n. The induction base is exactly the Feller property. Now suppose

$$x \mapsto \mathbb{E}^x \prod_{k=1}^{n} f_k(X_{t_k})$$

is continuous. For $n+1$ we can use the tower property and the Markov property to obtain

$$\begin{aligned} E^x\left[\prod_{k=1}^{n+1} f_k(X_{t_k})\right] &= E^x\left[\prod_{k=1}^{n} f_k(X_{t_k})\mathbb{E}^x\left[f_{n+1}(X_{t_{n+1}}) \mid \mathfrak{F}_{t_n}\right]\right] \\ &= E^x\left[\prod_{k=1}^{n} f_k(X_{t_k})\mathbb{E}^{X_{t_n}}\left[f_{n+1}(X_{t_{n+1}-t_n})\right]\right]. \end{aligned}$$

Let

$$g(x) = \mathbb{E}^x\big[f_{n+1}(X_{t_{n+1}-t_n})\big],$$

then $g(x)$ is continuous by the Feller property. Therefore $(f_n g)(x)$ is continuous. We conclude that

$$E^x\left[\prod_{k=1}^{n+1} f_k(X_{t_k})\right] = E^x\left[\prod_{k=1}^{n-1} f_k(X_{t_k})(f_n g)(X_{t_n})\right]$$

is continuous by the induction hypothesis.

T3. (a) We only prove property (S2)of the probability semigroup, the other properties were proven in the lecture in the proof of Theorem 4.1. We have

$$\begin{aligned} \|T_t f - f\| = \sup_{x\in S}\big|(T_t f)(x) - f(x)\big| &= \sup_{x\in S}\big|\mathbb{E}^0\left[f(x+B_t)\right] - f(x)]\big| \\ &\le \mathbb{E}^0\left[\sup_{x\in S}\big|f(x+B_t) - f(x)\big|\right]. \end{aligned}$$

Since $f \in C_0(S)$, it vanishes at infinity, and it is therefore uniformly continuous. This implies that

$$\sup_{x\in S} \left| f(x+B_t) - f(x) \right| \to 0 \quad \text{as} \quad t \downarrow 0 \quad \text{a.s.},$$

since $B_t \to 0$ almost surely. It follows by the dominated convergence that $\|T_t f - f\| \to 0$ as $t \downarrow 0$.

(b) If $f \in C_b(S)$ it is not necessarily uniformly continuous, so that the above argument does not hold. For example consider $f(x) = \max\{\cos(x^2), 0\}$. Then it can be shown that for every $t > 0$ there exists an $x \in S$ such that $\mathbb{E}^0 \left[f(x+B_t) \right] - f(x)]$ is bounded away from zero independent of t, so that $T_t f$ does not converge to f.

T4. (G1): Firstly $\mathcal{D}(\mathcal{L})$ is vector space. To use the Stone-Weierstrass theorem we further need to show that $\mathcal{D}(\mathcal{L})$ separates points and vanishes nowhere. Consider the functions $f_a(x) = \exp(-(x-a)^2) \in \mathcal{D}(\mathcal{L})$. Then for all pairs $x \neq y$ in S we have $f_x(x) = 1$ and $f_x(y) < 1$, so that $\mathcal{D}(\mathcal{L})$ separates points. Furthermore since $f_x(x) = 1$, the space vanishes nowhere. The theorem now states that $\mathcal{D}(\mathcal{L})$ is dense in $C_0(S)$.

(G2): Let $\lambda > 0$ and $g = f - \lambda f'$. Since $f \in C_0(\mathbb{R})$ we have $\inf_x f(x) \leq 0$. Similarly $\inf_x g(x) \leq 0$. If $\inf_x f(x) = 0$ we immediately have $\inf_x g(x) \leq \inf_x f(x)$. Now suppose $\inf_x f(x) < 0$, then since f is continuous there exists $x_0 \in S$ with $f(x_0) = \inf_x f(x)$ and $f'(x_0) = 0$. We now get

$$\inf_x f(x) = f(x_0) = f(x_0) - \lambda f'(x_0) \geq \inf_x g(x).$$

(G3): Let $g \in C_0(s)$. We need to show that there exists an $f \in C_0(\mathbb{R})$ with $f - \lambda f' = g$. This differential equation is solved by

$$f(x) = Ce^{\frac{1}{\lambda}x} - \int_0^x \frac{1}{\lambda} g(y) e^{\frac{1}{\lambda}(x-y)} \mathrm{d}y.$$

To make the computations easier we take

$$C = \int_0^\infty \frac{1}{\lambda} g(y) e^{-\frac{1}{\lambda}y},$$

so that

$$f(x) = \int_x^\infty \frac{1}{\lambda} g(y) e^{\frac{1}{\lambda}(x-y)} \mathrm{d}y.$$

We need to show that $f \in C_0(\mathbb{R})$. Continuity follows immediately, so it remains to show that f vanishes at infinity. We have

$$|f(x)| \leq \frac{1}{\lambda} \sup_{y\in[x,\infty)} |g(y)| \lambda \to 0 \quad \text{as} \quad x \to \infty.$$

For the other limit we can write

$$f(x) = -\int_{-\infty}^{0} \frac{1}{\lambda} g(x-y) e^{\frac{1}{\lambda}y} \mathrm{d}y.$$

The integrand is bounded by $\frac{1}{\lambda}\|g\|$, so that by dominated convergence

$$\lim_{x\to-\infty} |f(x)| = \left| \int_{-\infty}^{0} \lim_{x\to-\infty} \frac{1}{\lambda} g(x-y) e^{\frac{1}{\lambda}y} \mathrm{d}y \right| = 0.$$

(G4): Let $\lambda > 0$. Consider $f_n(x) = \exp\left(\frac{-x^2}{n}\right)$, and

$$g_n(x) = f_n(x) - \lambda f_n'(x) = \exp\left(\frac{-x^2}{n}\right) + \frac{2\lambda x}{n} \exp\left(\frac{-x^2}{n}\right).$$

Then $\sup_n \|g_n\| < \infty$, $f_n \to 1$ and $g_n \to 1$ pointwise as $n \to \infty$.

This belongs to the process that moves deterministically to the right at unit speed: $X_t = X_0 + t$. The semigroup of this process is given by

$$(T_t f)(x) = \mathbb{E}^x[f(X_t)] = f(x+t).$$

This process indeed has generator f':

$$\lim_{t\downarrow 0} \frac{(T_t f)(x) - f(x)}{t} = \lim_{t\downarrow 0} \frac{f(x+t) - f(x)}{t} = f'(x).$$