Solutions to tutorial exercises for stochastic processes

T1. Let $G(x,y)$ be the Green function of the Markov chain. Let x be the recurrent state, so that $G(x,x)=\infty$. We will show that state y is recurrent by showing that $G(y,y)=\infty$. Let $t,s\geq 0$. We use the Chapman-Kolmogorov equation twice to obtain

$$p_{2t+s}(y,y)\geq p_t(y,x)p_{t+s}(x,y)\geq p_t(y,x)p_s(x,x)p_t(x,y).$$

Therefore,

$$G(y,y)=\int_0^\infty p_s(y,y)\mathrm{d}s\geq\int_{2t}^\infty p_s(y,y)\mathrm{d}s=\int_0^\infty p_{2t+s}(y,y)\mathrm{d}s\geq p_t(y,x)G(x,x)p_t(x,y).$$

Since the Markov chain is irreducible we have $p_t(y,x),p_t(x,y)>0$, we conclude that $G(y,y)=\infty$.

T2. '$\Rightarrow$': Using strict stationarity of the Markov chain we find

$$\pi(x)=\mathbb{P}(X_0=x)=\mathbb{P}(X_t=x)=\sum_{x_0\in S}\mathbb{P}(X_t=x,X_0=x_0)=\sum_{x_0\in S}\pi(x_0)p_t(x_0,x),$$

so $\pi(\cdot)$ is invariant.

'$\Leftarrow$': Let $n\in\mathbb{N}$, $0\leq t_1<t_2<\cdots<t_n$ and $s>0$. Let $x_1,\ldots,x_n\in S$. We use the invariance property of $\pi(\cdot)$ to obtain

$$\begin{aligned}\mathbb{P}(X_{t_1+s}=x_1,\ldots,X_{t_n+s}=x_n)&=\sum_{x_0\in S}\pi(x_0)p_{t_1+s}(x_0,x_1)p_{t_2-t_1}(x_1,x_2)\ldots p_{t_n-t_{n-1}}(x_{n-1},x_n)\\&=\pi(x_1)p_{t_2-t_1}(x_1,x_2)\ldots p_{t_n-t_{n-1}}(x_{n-1},x_n),\end{aligned}$$

which is independent of s, so the Markov chain is strictly stationary.

T3. '$\Rightarrow$': By reversibility we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\pi(x)p_t(x,y)\Big|_{t=0}&=\frac{\mathrm{d}}{\mathrm{d}t}\pi(y)p_t(y,x)\Big|_{t=0}\qquad&\forall x,y\in S\\ \pi(x)q(x,y)&=\pi(y)q(y,x)&\forall x,y\in S.\end{aligned}$$

'$\Leftarrow$': Assume that $\pi(x)q(x,y)=\pi(y)q(y,x)$ for all $x,y\in S$. This can be stated as

$$\begin{pmatrix}\pi(1)&&\\&\ddots&\\&&\pi(n)\end{pmatrix}Q=Q^T\begin{pmatrix}\pi(1)&&\\&\ddots&\\&&\pi(n)\end{pmatrix}.$$

Since S is finite the transition probabilities P_t are given by $\exp(tQ)$. Therefore

$$\begin{aligned}
\begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} P_t &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} Q^k \\
&= \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} + \sum_{k=1}^{\infty} \frac{t^k}{k!} Q^T \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} Q^{k-1} \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(Q^T\right)^k \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} \\
&= e^{tQ^T} \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} \\
&= P_t^T \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix}.
\end{aligned}$$

It follows that $\pi(\cdot)$ is reversible.

T4. Let $\varepsilon > 0$. Choose $K \in \mathbb{N}$ such that

$$\left| \frac{a_K}{K} - \inf \left\{ \frac{a_n}{n}, n \in \mathbb{N} \right\} \right| < \frac{\varepsilon}{2},$$

this is possible by the definition of infimum. Now choose $M \in \mathbb{N}$ such that $\frac{a_r}{KM} < \frac{\varepsilon}{2}$ for all $r = 0, 1, \ldots, K-1$. Let $N = KM$, and let $n \geq N$. We can find $r, s \in \mathbb{N}$ such that $n = sK + r$ and $r < K$. Now, using the subadditivity property:

$$\frac{a_n}{n} \leq \frac{sa_K}{sK + r} + \frac{a_r}{sK + r} \leq \frac{sa_K}{sK} + \frac{a_r}{KM} \leq \frac{sa_K}{sK} + \frac{a_r}{KM} < \inf \left\{ \frac{a_n}{n}, n \in \mathbb{N} \right\} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Since we also have $\frac{a_n}{n} \geq \inf \left\{ \frac{a_n}{n}, n \in \mathbb{N} \right\}$, the result follows.