Solutions to tutorial exercises for stochastic processes

T1. If $\beta+\delta=0$ then by the Kolmogorov backward-equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(x,y)=0,$$

so that $p_t(x,y)$ is constant. Since $p_0(x,x)=1$, $p_t(x,y)=\mathbb{1}_{\{x=y\}}$. Now suppose that $\beta+\delta>0$. The Kolmogorov backward-equations are solved uniquely by $P_t=\exp(tQ)$. The matrix Q has eigenvalues 0 and $-\beta-\delta$. We can find a matrix U such that

$$Q=U\begin{pmatrix}0 & 0\\ 0 & -\beta-\delta\end{pmatrix}U^{-1}.$$

This is solved by

$$U=\begin{pmatrix}1 & -\frac{\beta}{\beta+\delta}\\ 1 & \frac{\delta}{\beta+\delta}\end{pmatrix}.$$

We conclude

$$\begin{aligned}P_t&=U\begin{pmatrix}1 & 0\\ 0 & \exp(-(\beta+\delta)t)\end{pmatrix}U^{-1}\\ &=\begin{pmatrix}\frac{\delta}{\beta+\delta}+\frac{\beta}{\beta+\delta}e^{-t(\beta+\delta)} & \frac{\beta}{\beta+\delta}\left(1-e^{-t(\beta+\delta)}\right)\\ \frac{\delta}{\beta+\delta}\left(1-e^{-t(\beta+\delta)}\right) & \frac{\beta}{\beta+\delta}+\frac{\delta}{\beta+\delta}e^{-t(\beta+\delta)}\end{pmatrix}.\end{aligned}$$

T2. (a) Let t^* be the time of the first jump. If $X_0=0$ then this t^* is exponentially distributed with parameter β. We now have

$$\begin{aligned}p_t(0,1)&=\mathbb{P}^0(X_t=1)\\ &=\sum_{k=0}^{n-1}\mathbb{P}^0\left(X_t=1,t^*\in\left[\frac{k}{n}t,\frac{k+1}{n}t\right]\right)\\ &=\sum_{k=0}^{n-1}\mathbb{P}^0\left(X_t=1\;\middle|\;t^*\in\left[\frac{k}{n}t,\frac{k+1}{n}t\right]\right)\mathbb{P}^0\left(t^*\in\left[\frac{k}{n}t,\frac{k+1}{n}t\right]\right).\qquad(1)\end{aligned}$$

Since

$$\sigma\left(t^*\in\left[\frac{k}{n}t,\frac{k+1}{n}t\right]\right)\subseteq\mathfrak{F}_{t^*\wedge\frac{k+1}{n}t},$$

we have by the tower property

$$
\begin{aligned}
\mathbb{P}^0\left(X_t = 1 \;\middle|\; t^* \in \left[\frac{k}{n}t, \frac{k+1}{n}t\right]\right) &= \mathbb{E}^0\left[\mathbb{1}_{\{X_t=1\}} \;\middle|\; t^* \in \left[\frac{k}{n}t, \frac{k+1}{n}t\right]\right] \\
&= \mathbb{E}^0\left[\mathbb{E}^0\left[\mathbb{1}_{\{X_t=1\}} \middle| \mathfrak{F}_{t^* \wedge \frac{k+1}{n}t}\right] \;\middle|\; t^* \in \left[\frac{k}{n}t, \frac{k+1}{n}t\right]\right].
\end{aligned}
$$

We now use the strong Markov property to get

$$
\begin{aligned}
\mathbb{P}^0\left(X_t = 1 \;\middle|\; t^* \in \left[\frac{k}{n}t, \frac{k+1}{n}t\right]\right) &= \mathbb{E}^0\left[\mathbb{P}^{X_{t^* \wedge \frac{k+1}{n}t}}\left(X_{t - t^* \wedge \frac{k+1}{n}t} = 1\right) \;\middle|\; t^* \in \left[\frac{k}{n}t, \frac{k+1}{n}t\right]\right] \\
&= \mathbb{E}^0\left[\mathbb{P}^1\left(X_{t-t^*} = 1\right) \;\middle|\; t^* \in \left[\frac{k}{n}t, \frac{k+1}{n}t\right]\right] \\
&\leq \sup_{s \in \left[\frac{k}{n}t, \frac{k+1}{n}t\right]} \mathbb{P}^1\left(X_{t-s} = 1\right),
\end{aligned}
$$

and similarly

$$
\mathbb{P}^0\left(X_t = 1 \;\middle|\; t^* \in \left[\frac{k}{n}t, \frac{k+1}{n}t\right]\right) \geq \inf_{s \in \left[\frac{k}{n}t, \frac{k+1}{n}t\right]} \mathbb{P}^1\left(X_{t-s} = 1\right).
$$

So the sum in (1) converges to the Riemann-Stieltjes integral as $n \to \infty$:

$$
\begin{aligned}
p_t(0,1) &= \int_0^t \mathbb{P}^1(X_{t-s} = 1)\mathrm{d}F(s) \\
&= \int_0^t \beta e^{-\beta s} p_{t-s}(1,1)\mathrm{d}s,
\end{aligned}
$$

where $F(s) = 1 - \exp(-\beta s)$. By using the same argumentation we find

$$
p_t(1,0) = \int_0^t \delta e^{-\delta s} p_{t-s}(0,0)\mathrm{d}s
$$

(b) By definition of the Q-matrix we have

$$
\begin{aligned}
q(0,1) &= \frac{\mathrm{d}}{\mathrm{d}t} p_t(0,1)\bigg|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \beta e^{-\beta s} p_{t-s}(1,1)\mathrm{d}s\bigg|_{t=0} \\
&= \beta e^{-\beta t} p_0(1,1)\bigg|_{t=0} \\
&= \beta.
\end{aligned}
$$

Similarly we can find $q(1,0) = \delta$. Since the row sums of the Q matrix are zero we conclude $q(0,0) = -\beta$ and $q(1,1) = -\delta$.

T3. Items (a) and (b) are equivalent since

$$\{N(t) < \infty\} = \left\{ \sum_{n=0}^{\infty} \tau_n > t \right\}.$$

We will now show the equivalence of (b) and (c). Let $\lambda > 0$. Conditioned on $Z_0, Z_1, \ldots, \tau_n$ are independent exponential random variables. Therefore, using the moment generating function of the exponential distribution:

$$\mathbb{E}\left[\exp\left(-\lambda \sum_{k=0}^{n} \tau_k \right) \,\middle|\, Z_0, Z_1, \ldots \right] = \prod_{k=0}^{n} \frac{c(Z_k)}{c(Z_k) + \lambda}$$

We now take expectations on both sides and let $n \to \infty$ to find

$$\mathbb{E}\left[\exp\left(-\lambda \sum_{k=0}^{\infty} \tau_k \right) \right] = \mathbb{E}\left[\prod_{k=0}^{\infty} \frac{c(Z_k)}{c(Z_k) + \lambda} \right],$$

where we use dominated convergence to switch limit and expectation. The product $\prod_{k=0}^{\infty} \frac{c(Z_k)}{c(Z_k)+\lambda} > 0$ if and only if $\prod_{k=0}^{\infty} \left(1 + \frac{\lambda}{c(Z_k)}\right) < \infty$ and this is equivalent to $\sum_{k=0}^{\infty} \frac{\lambda}{c(Z_k)} < \infty$. We conclude by taking $\lambda \downarrow 0$:

$$\mathbb{P}\left(\sum_{k=0}^{\infty} \tau_k < \infty \right) = \mathbb{P}\left(\sum_{k=0}^{\infty} \frac{1}{c(Z_k)} < \infty \right).$$