Solutions to tutorial exercises for stochastic processes

T1. Consider $\sigma \equiv 1$ and $\tau := \inf\{t \geq 1 \ : \ B_t = 0\}$. Then $\sigma \leq \tau$ and

$$\mathbb{E}[B_\sigma^2] = 1 > 0 = \mathbb{E}[B_\tau^2].$$

T2. (a) Consider $\tau := \inf\{t \geq 0 \ : \ B_t = 1\}$. Then $\mathbb{E}B_\tau^2 = 1 < \infty$. Suppose $\mathbb{E}[\tau] < \infty$, then we can apply the optional stopping theorem to find

$$\mathbb{E}[B_\tau] = \mathbb{E}[B_0] = 0.$$

However, by the definition of τ, we have that $B_\tau = 1$. So we conclude that $\mathbb{E}[\tau] = \infty$.

(b) Let τ be a stopping time with $\mathbb{E}[\tau] = \infty$ and $\mathbb{E}[\sqrt{\tau}] < \infty$. Since $\mathbb{E}[\sqrt{\tau}]$ is finite we conclude that τ is finite almost surely. Let $t > 0$, then by the optional stopping theorem:

$$\mathbb{E}[B_{\tau\wedge t}^2] = \mathbb{E}[\tau \wedge t].$$

Since $\tau \wedge t \to \tau$ almost surely as $t \to \infty$ we find by the monotone convergence theorem that $\mathbb{E}[\tau \wedge t] \to \mathbb{E}[\tau] = \infty$. We will now prove that $\mathbb{E}[B_\tau^2] \geq \mathbb{E}[B_{\tau\wedge t}^2]$ for all $t > 0$, which will complete the proof of the statement. From the proof of Theorem 2.50 in *Brownian Motion* by Mörters and Peres it follows that

$$B_{\tau\wedge t} \leq \sup_{0\leq s\leq 4^\alpha} B_s =: M^+,$$

with $\alpha = \lceil \log_4 \tau \rceil$ and $M^+ \in L^1$. Similarly

$$-B_{\tau\wedge t} \leq -\inf_{0\leq s\leq 4^\alpha} B_s =: M^-.$$

Since $B \overset{d}{=} -B$, we have $M^- \overset{d}{=} M^+$, so that $M^- \in L^1$. We now have $|B_{\tau\wedge t}| \leq M^+ \vee M^- \in L^1$. It follows that $B_{\tau\wedge t}$ is uniformly integrable. We claim that from uniform integrability it follows that $\mathbb{E}[B_\tau|\mathfrak{F}_t] = B_{\tau\wedge t}$ almost surely. We now use Jensen's inequality to obtain

$$\mathbb{E}[B_{\tau\wedge t}^2] = \mathbb{E}\big[\mathbb{E}[B_\tau \mid \mathfrak{F}_t]^2\big] \leq \mathbb{E}\big[\mathbb{E}[B_\tau^2 \mid \mathfrak{F}_t]\big] = \mathbb{E}[B_\tau^2].$$

It remains to prove the claim. By the continuity of paths we have that $B_{\tau\wedge t} \to B_\tau$ almost surely. Combining this with the uniform integrability it follows that $B_{\tau\wedge t} \to B_\tau$ in L^1 as well. Therefore

$$\mathbb{E}[B_\tau \mid \mathfrak{F}_t] = \lim_{s\to\infty} \mathbb{E}[B_{\tau\wedge s} \mid \mathfrak{F}_t].$$

Now let $s > t$. We have

$$\begin{aligned}
\mathbb{E}[B_{\tau\wedge s} \mid \mathfrak{F}_t] =& \mathbb{E}[B_{(\tau\wedge s)\vee t}\mathbb{1}_{\{\tau\wedge s>t\}} \mid \mathfrak{F}_t] + \mathbb{E}[B_{\tau\wedge t}\mathbb{1}_{\{\tau\wedge s\leq t\}} \mid \mathfrak{F}_t] \\
=& \mathbb{1}_{\{\tau\wedge s>t\}}\mathbb{E}[B_{(\tau\wedge s)\vee t} \mid \mathfrak{F}_t] + B_{\tau\wedge t}\mathbb{1}_{\{\tau\wedge s\leq t\}} \\
=& \mathbb{1}_{\{\tau\wedge s>t\}}B_t + B_{\tau\wedge t}\mathbb{1}_{\{\tau\wedge s\leq t\}} \\
\xrightarrow{s\to\infty}& \mathbb{1}_{\{\tau>t\}}B_t + B_{\tau\wedge t}\mathbb{1}_{\{\tau\leq t\}} \\
=& \mathbb{1}_{\{\tau>t\}}B_{\tau\wedge t} + B_{\tau\wedge t}\mathbb{1}_{\{\tau\leq t\}} \\
=& B_{\tau\wedge t}.
\end{aligned}$$

T3. We use the Markov property to find

$$\begin{aligned}
\mathbb{E}^0[X_t \mid \mathfrak{F}_s] &= -\frac{1}{2}\int_0^s f''(B_u)\mathrm{d}u + \mathbb{E}^0\left[f(B_t) - \frac{1}{2}\int_s^t f''(B_u)\mathrm{d}u \,\middle|\, \mathfrak{F}_s\right] \\
&= -\frac{1}{2}\int_0^s f''(B_u)\mathrm{d}u + \mathbb{E}^0\left[\left(f(B_{t-s}) - \frac{1}{2}\int_0^{t-s} f''(B_u)\mathrm{d}u\right)\circ\theta_s \,\middle|\, \mathfrak{F}_s\right] \\
&= -\frac{1}{2}\int_0^s f''(B_u)\mathrm{d}u + \mathbb{E}^{B_s}\left[f(B_{t-s}) - \frac{1}{2}\int_0^{t-s} f''(B_u)\mathrm{d}u\right].
\end{aligned}$$

Since $f'' \in L^1$ we can apply Fubini's theorem to find

$$\mathbb{E}^0[X_t \mid \mathfrak{F}_s] = -\frac{1}{2}\int_0^s f''(B_u)\mathrm{d}u + \mathbb{E}^{B_s}[f(B_{t-s})] - \frac{1}{2}\int_0^{t-s}\mathbb{E}^{B_s}[f''(B_u)]\mathrm{d}u. \tag{1}$$

We now focus on the integrand of the last term. Let $p(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\big(-(x-y)^2/2t\big)$ be the normal density, then by using integration by parts twice we find

$$\begin{aligned}
\mathbb{E}^{B_s}[f''(B_u)] &= \int_{\mathbb{R}} p(u,B_s,y)f''(y)\mathrm{d}y \\
&= \int_{\mathbb{R}} \frac{\partial^2 p(u,B_s,y)}{\partial y^2}f(y)\mathrm{d}y.
\end{aligned}$$

The normal density $p(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\big(-(x-y)^2/2t\big)$ satisfies the differential equation $\frac{\partial}{\partial t}p = \frac{1}{2}\frac{\partial^2}{\partial y^2}p$, so that

$$\mathbb{E}^{B_s}[f''(B_u)] = 2\int_{\mathbb{R}} \frac{\partial p(u,B_s,y)}{\partial u}f(y)\mathrm{d}y.$$

We again use Fubini's theorem:

$$\begin{aligned}
\frac{1}{2}\int_0^{t-s}\mathbb{E}^{B_s}[f''(B_u)]\mathrm{d}u &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\varepsilon}^{t-s}\frac{\partial p(u,B_s,y)}{\partial u}\mathrm{d}u f(y)\mathrm{d}y \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\big(p(t-s,B_s,y) - p(\varepsilon,B_s,y)\big)f(y)\mathrm{d}y \\
&= \mathbb{E}^{B_s}[f(B_{t-s})] - \lim_{\varepsilon\to 0}\mathbb{E}^{B_s}[f(B_\varepsilon)] \\
&= \mathbb{E}^{B_s}[f(B_{t-s})] - f(B_s),
\end{aligned}$$

where we use the dominated convergence theorem in the last step. Combining this with (1) gives

$$\mathbb{E}^0[X_t \mid \mathfrak{F}_s] = f(B_s) - \frac{1}{2}\int_0^s f''(B_u)\mathrm{d}u.$$