Solutions to tutorial exercises for stochastic processes

T1. X_t is Gaussian since it is a linear combination of B_{s+t} and B_s and B is Gaussian. Furthermore

$$\mathbb{E}[X_t] = \mathbb{E}[B_{s+t}] - \mathbb{E}[B_s] = 0,$$

and for some $t, r > 0$ we have

$$\begin{aligned}\mathrm{Cov}(X_t, X, r) &= \mathrm{Cov}(B_{t+s} - B_s, B_{r+s}, B_s)\\ &= -\mathrm{Cov}(B_{t+s}, B_s) + \mathrm{Cov}(B_{t+s}, B_{r+s}) - \mathrm{Cov}(B_s, B_{r+s}) + \mathrm{Cov}(B_s, B_s)\\ &= -s + s + t \wedge r - s + s = t \wedge r.\end{aligned}$$

Lastly, X_t is continuous, since B_t is continuous. So X_t is Brownian motion.

Y_t is Gaussian, since it is a rescaling of B_t and B is Gaussian. Furthermore $\mathbb{E}[Y_t] = \frac{1}{\sqrt{c}}\mathbb{E}[B_{ct}] = 0$, and for some $t, r > 0$

$$\mathrm{Cov}(Y_t, Y_r) = \mathrm{Cov}\left(\frac{B_{ct}}{\sqrt{c}}, \frac{B_{cr}}{\sqrt{c}}\right) = \frac{1}{c}(ct \wedge cr) = t \wedge r.$$

Finally, Y_t is continuous since B_t is continuous. So Y_t is also Brownian motion.

T2. We will use the monotone class theorem to prove the result. Define the space of functions $\mathcal{H}$ as follows:

$$\mathcal{H} = \{f : \mathbb{R}^2 \to \mathbb{R} \ : \ \mathbb{E}[f(X,Y)|\mathfrak{G}] = g(X)\}.$$

By the linearity of the expectation, $\mathcal{H}$ is a linear space. Furthermore for $A, B \in \mathcal{B}(\mathbb{R})$, $\mathcal{H}$ contains the indicator $\mathbb{1}_{A\times B}$:

$$\mathbb{E}[\mathbb{1}_{A\times B}(X,Y)|\mathfrak{G}] = \mathbb{E}[\mathbb{1}_A(X)\mathbb{1}_B(Y)|\mathfrak{G}] = \mathbb{1}_A(X)\mathbb{E}[\mathbb{1}_B(Y)|\mathfrak{G}] = \mathbb{1}_A(X)\mathbb{P}(Y \in B) = g(X).$$

To apply the monotone class theorem it remains to show that for a sequence $f_n \in \mathcal{H}$ with $f_n \uparrow f$ and f bounded it holds that $f \in \mathcal{H}$. By the monotone convergence theorem we have

$$\mathbb{E}[f(X,Y)|\mathfrak{G}] = \lim_{n\to\infty} \mathbb{E}[f_n(X,Y)|\mathfrak{G}] = \lim_{n\to\infty} g_n(X).$$

Furthermore, again using the monotone convergence theorem, it holds that

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \mathbb{E}[f_n(x,Y)] = \mathbb{E}[f(x,Y)] = g(x),$$

so that $f \in \mathcal{H}$. The monotone class theorem now states that $\mathcal{H}$ contains all bounded and measurable functions f, which is the desired result.

T3. Let Z_t be defined as follows.

$$Z_t = \begin{cases} 0 & \text{if } t = 0 \\ tB_{1/t} & \text{if } t > 0, \end{cases}$$

where B is standard Brownian motion. Then Z is Brownian motion as well and $\lim_{t\to 0} Z_t = 0$ almost surely. By applying the change of variables $s := 1/t$ we find

$$0 = \lim_{t\to 0} Z_t = \lim_{s\to\infty} \frac{B_s}{s}. \quad a.s.$$

An alternative is to prove that $B_t/t \to 0$ in L^2 and subsequently use the martingale convergence theorem.

T4. Since B is almost surely continuous we can write

$$\int_0^t B_s \mathrm{d}s = \lim_{k\to\infty} \frac{t}{k} \sum_{i=1}^{k} B_{i\frac{t}{k}}.$$

Similarly for points $t_1, \ldots, t_n > 0$ and constants $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ we can write for the linear combination

$$\sum_{j=1}^{n} \alpha_j \int_0^{t_j} B_s \mathrm{d}s = \lim_{k\to\infty} \sum_{j=1}^{n} \frac{\alpha_j t_j}{k} \sum_{i=1}^{k} B_{i\frac{t_j}{k}} =: \lim_{k\to\infty} Z_k.$$

Since B is a Gaussian process, Z_k has a normal distribution for every $k \in \mathbb{N}$. The above limit is in the almost sure sense, therefore we conclude that $\lim_{k\to\infty} Z_k$ has a normal distribution as well, so that X is Gaussian.

To calculate $\mathbb{E}[X_t]$ we need to apply Fubini's theorem. Therefore we first need to check that $X \in L^1$:

$$\mathbb{E}\left|\int_0^t B_s \mathrm{d}s\right| \le \mathbb{E}\int_0^t |B_s| \mathrm{d}s \le \int_0^t \mathbb{E}|B_s| \mathrm{d}s \le t^2 < \infty,$$

where we used Fubini's theorem in the second inequality. This is allowed since $B \in L^1$. Now we can calculate $\mathbb{E}[X_t]$:

$$\mathbb{E}[X_t] = \mathbb{E}\left[\int_0^t B_s \mathrm{d}s\right] = \int_0^t \mathbb{E}[B_s] \mathrm{d}s = 0.$$

Now let $0 \le s \le t$. We again use Fubini's Theorem to calculate the covariance:

$$\begin{aligned} \mathrm{Cov}(X_s, X_t) &= \int_0^s \int_0^t \mathbb{E}[B_u B_v] \mathrm{d}u \mathrm{d}v = \int_0^s \int_0^t u \wedge v \, \mathrm{d}u \mathrm{d}v \\ &= \int_0^s \int_0^v u \, \mathrm{d}u \mathrm{d}v + \int_0^s \int_v^t v \, \mathrm{d}u \mathrm{d}v \\ &= \frac{1}{2} t s^2 - \frac{1}{6} s^3. \end{aligned}$$