Solutions to tutorial exercises for stochastic processes

T1. Consider the function $f : \{0,1\}^V \to \{0,1\}$ given by

$$f(\eta) = \mathbb{1}\big\{|x \in V \ : \ \eta(x) = 1| = \infty\big\}$$

. Then $\sup_\eta |f(\eta_x) - f(\eta)| = 0$ for all $x \in V$, so that

$$\sum_{x \in V} \sup_\eta |f(\eta_x) - f(\eta)| = 0.$$

However we can show that f is not continuous. Let $(x_k)_{k\in\mathbb{N}}$ be an enumeration of V and let η be such that $f(\eta) = 1$. Define the sequence

$$\eta_n(x) = \begin{cases} \eta(x) & \text{if } x = x_k \text{ for some } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\eta_n(x) \to \eta(x)$ pointwise as $n \to \infty$, so also $\eta_n \to \eta$ by T4 of Sheet 11. However $f(\eta_n) = 0$ for all $n \in \mathbb{N}$.

T2. (a) Let $x \neq u$, then $c(x,\eta) = c(x,\eta_u)$ for all $\eta \in S$. So we have $\gamma(x,u) = \sup_\eta |c(x,\eta) - \eta(x,\eta_u)| = 0$ and also $M = \sup_x \sum_{u \neq x} \gamma(x,u) = 0 < \infty$. Define the generator

$$\mathcal{L}f(\eta) = \sum_{x\in V} c(x,\eta)\big(f(\eta_x) - f(\eta)\big)$$

on the domain

$$\mathcal{D}(\mathcal{L}) = \left\{ f \in C(S) \ : \ \sum_{x\in V} \sup_\eta |f(\eta_x) - f(\eta)| < \infty \right\}.$$

Then it follows from Theorem 5.2 that $\bar{\mathcal{L}}$ is a probability generator. So there exists a spin system with rate function $c(x,\eta)$.

(b) Let $(x_i)_{i\in\mathbb{N}}$ be an enumeration of V and consider $V_n = \{x_1, \ldots, x_n\}$. Similar to (a) there exists a spin system $\eta_t^n \in \{0,1\}^{S_n}$ with rate function $c(x,\eta)$, so that $M = 0$ and

$$\varepsilon_n = \inf_{1\leq i \leq n,\, \eta} \big(c(x_i,\eta) - c(x_i,\eta_{x_i})\big) = \min_{1 \leq i \leq n} \beta_{x_i} + \delta_{x_i} > 0.$$

It follows that the spin system η_t^n is ergodic. Furthermore the unique stationary measure is given by μ_n under which the state of all x_i, $1 \leq i \leq n$, is Bernoulli distributed with parameter $\frac{\beta_{x_i}}{\beta_{x_i}+\delta_{x_i}}$, independently of each other. This follows from the following calculation:

$$\begin{aligned}\int \mathcal{L}_n f(\eta)\mu_n(\mathrm{d}\eta) &= \sum_{i=1}^{n} \int \big(\beta_{x_i}\mathbb{1}_{\{\eta(x_i)=0\}} + \delta_{x_i}\mathbb{1}_{\{\eta(x_i)=1\}}\big)\big(f(\eta_{x_i}) - f(\eta)\big)\mu_n(\mathrm{d}\eta) \\ &= \sum_{i=1}^{n} \frac{\delta_{x_i}\beta_{x_i}}{\beta_{x_i}+\delta_{x_i}} \left(\sum_{\eta:\eta(x_i)=0} \big(f(\eta_{x_i}) - f(\eta)\big) + \sum_{\eta:\eta(x_i)=1} \big(f(\eta_{x_i}) - f(\eta)\big) \right) \\ &= 0.\end{aligned}$$

Let μ be the distribution under which the state of all $x \in V$ is Bernoulli distributed with parameter $\frac{\beta_x}{\beta_x+\delta_x}$, independently of each other, so that $\mu|_{S_n} = \mu_n$. Consider the spin system $(\eta_t)_{t\geq 0}$ with initial distribution ν. Let $\eta_t^n = \eta_t|_{S_n}$, then (η_t^n) is an ergodic spin system and

$$\mathbb{P}^{\nu}_{\eta_t^n} \to \mu_n \quad \text{weakly as } t \to \infty.$$

Since $\bigcup_n S_n = S$ and S is compact it follows from the convergence from the finite-dimensional distributions $\mathbb{P}^{\nu}_{\eta_t^n}$ that the measures on the entire S also converge, i.e.,

$$\mathbb{P}^{\nu}_{\eta_t} \to \mu \quad \text{weakly as } t \to \infty.$$

T3. Let $\mathcal{L}$ be the generator of the coupled spin system with domain

$$\mathcal{D}(\mathcal{L}) = \left\{ f \in C\left(S^2\right) \,:\, \sum_{x\in V} \sup_{(\eta,\zeta)} |f(\eta_x,\zeta) - f(\eta,\zeta)| + |f(\eta,\zeta_x) - f(\eta,\zeta)| + |f(\eta_x,\zeta_x) - f(\eta,\zeta)| < \infty \right\}.$$

For $f \in \mathcal{D}(\mathcal{L})$ we can write

$$\begin{aligned}\mathcal{L}f(\eta,\zeta) = \sum_{x\in V} &\tilde{c}_1(x,\eta,\zeta)\big(f(\eta_x,\zeta) - f(\eta,\zeta)\big) \\ &+ \tilde{c}_2(x,\eta,\zeta)\big(f(\eta,\zeta_x) - f(\eta,\zeta)\big) + \tilde{c}_3(x,\eta,\zeta)\big(f(\eta_x,\zeta_x) - f(\eta,\zeta)\big),\end{aligned}$$

where $\tilde{c}_1, \tilde{c}_2$ and $\tilde{c}_3$ are the rates at which (η,ζ) changes to $(\eta_x,\zeta), (\eta,\zeta_x)$ and (η_x,ζ_x) respectively. These are given by

$$\begin{aligned}\tilde{c}_1(x,\eta,\zeta) &= \mathbb{1}_{\{\eta(x)=0,\zeta(x)=1\}}c_1(x,\eta) + \mathbb{1}_{\{\eta(x)=1\}}\big(c_1(x,\eta) - c_2(x,\zeta)\big), \\ \tilde{c}_2(x,\eta,\zeta) &= \mathbb{1}_{\{\zeta(x)=0\}}\big(c_2(x,\zeta) - c_1(x,\eta)\big) + \mathbb{1}_{\{\eta(x)=0,\zeta(x)=1\}}\big(c_1(x,\eta) - c_2(x,\zeta)\big), \\ \tilde{c}_3(x,\eta,\zeta) &= \mathbb{1}_{\{\zeta(x)=0\}}c_1(x,\eta) + \mathbb{1}_{\{\eta(x)=1\}}c_2(x,\zeta).\end{aligned}$$

T4. Consider the bijection $\phi : S \to S$ given by

$$\phi(\eta)(x) = \begin{cases} \eta(x) & \text{if } x \text{ even}, \\ \eta_x(x) & \text{if } x \text{ odd}. \end{cases}$$

Define the generator $\mathcal{L} = \mathcal{L}_\beta$ and its domain $\mathcal{D}(\mathcal{L})$ in the usual way. Then $f \in \mathcal{D}(\mathcal{L})$ if and only if $f \circ \phi \in \mathcal{D}(\mathcal{L})$ since $\phi(\eta_x) = \phi(\eta)_x$ and therefore

$$\sup_\eta |f(\phi(\eta_x)) - f(\phi(\eta))| = \sup_\xi |f(\xi_x) - f(\xi)|,$$

since ϕ is a bijection. Moreover

$$\begin{aligned} c_\beta(x, \phi(\eta)) &= \exp\left(-\beta \sum_{y:y\sim x} (2\phi(\eta)(x) - 1)(2\phi(\eta)(y) - 1)\right) \\ &= \exp\left(\beta \sum_{y:y\sim x} (2\eta(x) - 1)(2\eta(y) - 1)\right) = c_{-\beta}(x, \eta). \end{aligned}$$

Therefore $\mathcal{L}_\beta(f \circ \phi) = \mathcal{L}_{-\beta} f$. We know that the stochastic Ising model with parameter $-\beta > 0$ is ergodic. Therefore there exists a unique stationary measure μ such that

$$0 = \int \mathcal{L}_{-\beta} f \mathrm{d}\mu = \int \mathcal{L}_\beta (f \circ \phi) \mathrm{d}\mu,$$

for all $f \in \mathcal{D}(\mathcal{L})$. Let $g \in \mathcal{D}(\mathcal{L})$ and take $f = g \circ \phi^{-1}$, so that $f \circ \phi = g$, since ϕ is bijective. We conclude that for all $g \in \mathcal{D}(L)$

$$\int \mathcal{L}_\beta g \mathrm{d}\mu = 0,$$

so that the spin system is ergodic with unique stationary measure μ.