Solutions to tutorial exercises for stochastic processes

T1. Suppose we have two Lévy-Khinchin triples (a, σ^2, π) and $(\tilde{a}, \tilde{\sigma}^2, \tilde{\pi})$ with $\psi(\theta) = \tilde{\psi}(\theta)$. If we can show that $\lim_{\theta\to\infty} \mathrm{Re}\left(\frac{\psi(\theta)}{\theta^2}\right) = -\frac{\sigma^2}{2}$ and $\lim_{\theta\to\infty} \mathrm{Re}\left(\frac{\tilde{\psi}(\theta)}{\theta^2}\right) = -\frac{\tilde{\sigma}^2}{2}$ then it follows that $\sigma^2 = \tilde{\sigma}^2$. We have

$$\mathrm{Re}\frac{\psi(\theta)}{\theta^2} = -\frac{\sigma^2}{2} + \int_{\mathbb{R}} \frac{\cos(\theta x) - 1}{\theta^2} \pi(\mathrm{d}x).$$

Since $1 - \cos(x) \leq x^2$ we can bound

$$\left| \frac{\cos(\theta x) - 1}{\theta^2} \right| \leq x^2 \mathbb{1}_{\{|x|\leq 1\}} + \frac{2}{\theta^2} \mathbb{1}_{\{|x|>1\}}.$$

Suppose $\theta > \sqrt{2}$, then

$$\left| \frac{\cos(\theta x) - 1}{\theta^2} \right| \leq x^2 \wedge 1,$$

which is integrable with respect to π. So we can apply the dominated convergence theorem to find

$$\lim_{\theta\to\infty} \mathrm{Re}\left(\frac{\psi(\theta)}{\theta^2}\right) = -\frac{\sigma^2}{2}.$$

Similarly we can compute the limit for $\tilde{\psi}(\theta)$.

T2. Denote by T the product topology on S. We will first show that the projection $\pi_v : S \to X$ given by $\pi_v(\eta) = \eta(v)$ is continuous. Let T_X denote the topology on X and let $A \in T_X$. For any $v \in V$ we have

$$\pi_v^{-1}(A) = \{\eta \in S \;:\; \eta(v) \in A\} = \prod_{w\neq v} X \times A \in T,$$

by the definition of the product topology. So π_v is continuous for all $v \in V$.

Now let T denote a topology on S such that π_v is continuous for all $v \in V$. So for all $A \in T_X$ we have

$$\pi_v^{-1}(A) = \prod_{w\neq v} X \times A \in T.$$

Suppose $B \subseteq S$ can be written as

$$B = \prod_{v\in V_1} B_v \times \prod_{v\in V_2} X,$$

where V_1 is finite, $V_1 \cup V_2 = S$ and $B_v \in T_X$. Then we can write

$$B = \bigcap_{v \in V_1} \left(\prod_{w \neq v} X \times B_v \right) \in T,$$

since V_1 is finite. It follows that T contains all sets included in the product topology.

T3. Let $A \in T_\rho$ and let $\eta \in A$ and $r \in \mathbb{R}$ such that $\rho(\eta, \xi)$ implies $\xi \in A$. Since α is summable we can write

$$\sum_{v \in \mathbb{Z}^d} \alpha(v) = \sum_{v \in V_1} \alpha(v) + \sum_{v \in V_2} \alpha(v),$$

with V_1 finite and

$$\sum_{v \in V_2} \alpha(v) < r.$$

Consider the set

$$B_\eta = \prod_{v \in V_1} \{\eta(v)\} \times \prod_{v \in V_2} \{0, 1\}.$$

Then for all $\xi \in B_\eta$ it holds that $\rho(\eta, \xi) < r$, so that $B_\eta \subset A$. Furthermore we have $A = \bigcup_{\eta \in A} B_\eta$, so that A is in the product topology. So T_ρ is a subset of the product topology.

Now let B be in the base of the product topology:

$$B = \prod_{v \in V_1} B_v \times \prod_{v \in V_2} \{0, 1\},$$

with V_1 finite. Now let $\eta \in B$ and take $r = \min_{v \in V_1} \alpha(v)$. Then if $\rho(\eta, \xi) < r$ it follows that $\eta(v) = \xi(v)$ for all $v \in V_1$, so that $\xi \in B$. It follows that B is in T_ρ. Since T_ρ is a topology and the base of the product topology is contained in T_ρ it follows that the product topology is a subset of T_ρ.

T4. $\Rightarrow$: Suppose $\eta_n \to \eta$ with respect to the product topology, i.e., for every $V \subset \mathbb{Z}^d$ finite there exists $N \in \mathbb{N}$ such that for all $n > N$ we have

$$\eta_n \in \prod_{v \in V} \{\eta(v)\} \times \prod_{\mathbb{Z}^d \setminus V} \{0, 1\}.$$

Let $x \in \mathbb{Z}^d$ and take $V = \{x\}$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$

$$\eta_n \in \{\eta(x)\} \times \prod_{\mathbb{Z}^d \setminus \{x\}} \{0, 1\},$$

so that $\eta_n(x) = \eta(x)$ for all n large enough. So η_n converges pointwise.

$\Leftarrow$: Suppose for every $x \in \mathbb{Z}^d$ there exists $N_x \in \mathbb{N}$ such that for all $n > N_x$ we have $\eta_n(x) = \eta(x)$. Let $V \subset \mathbb{Z}^d$ finite be given. Take $N = \max_{x \in V} N_x$. Then for all $n > N$ we have

$$\eta_n \in \prod_{x \in V} \{\eta(x)\} \times \prod_{\mathbb{Z}^d \setminus V} \{0, 1\},$$

so that η_n converges with respect to the product topology.