Solutions to tutorial exercises for stochastic processes

T1. Let $Z \sim N(\mu, \sigma^2)$. Let $N \in \mathbb{N}$ and let $(Z_n)_{1\leq n\leq N}$ be i.i.d. random variables with $Z_1 \sim N(\frac{\mu}{N}, \frac{\sigma^2}{N})$. Then

$$\sum_{n=1}^{N} Z_n \sim N(\mu, \sigma^2),$$

so that Z is infinitely divisible.

Let $\lambda > 0$ and $X \sim \mathrm{POI}(\lambda)$. Let $N \in \mathbb{N}$ and let $(X_n)_{1\leq n\leq N}$ be i.i.d. random variables with $X_1 \sim \mathrm{POI}(\frac{\lambda}{N})$. Then

$$\sum_{n=1}^{N} X_n \sim \mathrm{POI}(\lambda),$$

so that X is infinitely divisible.

T2. Let X be a random variable with finite support and $N \in \mathbb{N}$. Suppose $\mathrm{Var}(X) = 0$, so that X is constant. Then

$$\sum_{i=1}^{N} \frac{X}{N} \overset{d}{=} X,$$

so that X is infinity divisible. Now suppose $\mathrm{Var}(X) > 0$. Without loss of generality we can assume $\mathbb{E}[X] = 0$ and $\mathrm{Var}(X) = 1$. Suppose there exists i.i.d. random variables $(X_i^N)_{1\leq i\leq N}$ with $\sum_i X_i^N \overset{d}{=} X$. Then $\mathbb{E}[X_i^N] = 0$ and $\mathrm{Var}(X_i^N) = \frac{1}{N}$ for all i. We know that X has finite support, so that there exists $m \in \mathbb{R}$ such that

$$m = \inf\big\{A \in \mathbb{R} \,:\, \mathbb{P}\big(X \in [-A, A]\big) = 1\big\},$$

so that $\mathbb{P}(X > m) = 0$. It follows that

$$0 = \mathbb{P}(X > m) = \mathbb{P}\left(\sum_{i=1}^{N} X_i^N > m\right) \geq \mathbb{P}\left(X_1^N > \frac{m}{N}\right)^N,$$

so that $\mathbb{P}\left(X_1^N > \frac{m}{N}\right) = 0$. Similarly we can prove that $\mathbb{P}\left(X_1^N < -\frac{m}{N}\right) = 0$. Let $\varepsilon > 0$. It now follows that the triangular array $(X_i^N)_{N\in\mathbb{N}, 1\leq i\leq N}$ satisfies the Lindeberg condition:

$$\sum_{i=1}^{N} \mathbb{E}\left[\left(X_i^N\right)^2 \mathbb{1}_{\{|X_i^N|>\varepsilon\}}\right] = N\mathbb{E}\left[\left(X_1^N\right)^2 \mathbb{1}_{\{|X_1^N|>\varepsilon\}}\right] \leq n\frac{m^2}{n^2} = \frac{m^2}{N} \to 0 \quad \text{as} \quad N \to \infty.$$

The Lindeberg-Feller theorem now states that $\sum_i X_i^N$ converges weakly to a random variable with standard normal distribution as $N \to \infty$. However, $X \stackrel{d}{=} \sum_i X_i^N$ has finite support, which is a contradiction. It follows that the only infinitely divisible random variables with finite support are constants.

T3. We first show that X has stationary increments. Let $0 \leq s \leq t$. We have

$$X_t - X_s = \sum_{n=N_s}^{N_t} Y_n \stackrel{d}{=} \sum_{n=1}^{N_t - N_s} Y_n.$$

Since N is a Poisson process it has stationary increments. So the distribution of $N_t - N_s$ only depends on $t-s$. So the distribution of $X_t - X_s$ only depends on $t-s$.

Let $0 \leq s < t$. We prove that $X_t - X_s$ and X_s are independent. Denote by $\phi_Y(\cdot)$ the characteristic function of Y_1. Let $a, b \in \mathbb{R}$. We have

$$\begin{aligned}
\mathbb{E}\big[\exp(iaX_s + ib(X_t - X_s))\big] &= \mathbb{E}\left[\exp\left(ia\sum_{j=1}^{N_s} Y_j\right)\exp\left(ib\sum_{j=N_s+1}^{N_t} Y_j\right)\right] \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \mathbb{E}\left[\exp\left(ia\sum_{j=1}^{k} Y_j\right)\exp\left(ib\sum_{j=k+1}^{k+l} Y_j\right)\mathbb{1}_{\{N_s=k,\, N_t=k+l\}}\right] \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \phi_Y(a)^k \phi_Y(b)^l e^{-\lambda s}\frac{(\lambda s)^k}{k!} e^{-\lambda(t-s)}\frac{(\lambda(t-s))^l}{l!},
\end{aligned}$$

since Y_j are independent of each other and of N, and since N has stationary and independent increments. It follows that

$$\begin{aligned}
\mathbb{E}\big[\exp(iaX_s + ib(X_t - X_s))\big] &= \sum_{k=0}^{\infty} \phi_Y(a)^k e^{-\lambda s}\frac{(\lambda s)^k}{k!} \sum_{l=0}^{\infty} \phi_Y(b)^l e^{-\lambda(t-s)}\frac{(\lambda(t-s))^l}{l!} \\
&= \phi_{X_s}(a)\phi_{X_t - X_s}(b),
\end{aligned}$$

so that X_s and $X_t - X_s$ are independent. It can be proven similarly that n increments are independent.

T4. Let $Z \sim \text{Exp}(1)$. Then Z has characteristic function

$$\mathbb{E}[e^{i\theta Z}] = \frac{1}{1 - i\theta} = e^{-\psi(\theta)},$$

where

$$\psi(\theta) := \log(1 - i\theta).$$

If we show that $\psi(\theta)$ satisfies the Lévy-Khinchin formula, then there exists a Lévy process with $\mathbb{E}[\exp(i\theta X_1)] = \exp(-\psi(\theta))$, so that $X_1 \sim \text{Exp}(1)$. The derivative of ψ satisfies

$$\psi'(\theta) = -i\frac{1}{1 - i\theta} = -\int_0^\infty ie^{i\theta x}e^{-x}\mathrm{d}x.$$

We now have

$$\psi(\theta) = \psi(0) + \int_0^\theta \psi'(s)\mathrm{d}s = -\int_0^\theta \int_0^\infty ie^{isx}e^{-x}\mathrm{d}x\mathrm{d}s = -\int_0^\infty e^{-x}\int_0^\theta ie^{isx}\mathrm{d}s\mathrm{d}x,$$

where we used Fubini's theorem to switch the integrals. It follows that

$$\psi(\theta) = -\int_0^\infty \frac{e^{-x}}{x}(e^{i\theta x} - 1)\mathrm{d}x = \int_0^\infty \left(1 - e^{i\theta x} - i\theta x \mathbb{1}_{\{x<1\}}\right)\pi(\mathrm{d}x) + i\theta \int_0^1 x\frac{e^{-x}}{x}\mathrm{d}x,$$

where $\pi(\mathrm{d}x) = \mathbb{1}_{\{x>0\}}\frac{e^{-x}}{x}\mathrm{d}x$. Finally we have

$$\psi(\theta) = i\theta\left(1 - \frac{1}{e}\right) + \int_0^\infty \left(1 - e^{i\theta x} - i\theta x \mathbb{1}_{\{x<1\}}\right)\pi(\mathrm{d}x).$$

Moreover, π is a Lévy-measure:

$$\int_{\mathbb{R}\setminus\{0\}} (x^2 \wedge 1)\pi(\mathrm{d}x) = \int_0^1 xe^{-x}\mathrm{d}x + \int_1^\infty \frac{e^{-x}}{x}\mathrm{d}x < 2 < \infty.$$

So $\psi(\theta)$ satisfies the Lévy-Khinchin formula with triplet $(1 - 1/e, 0, \pi)$.