Solutions to tutorial exercises for stochastic processes

T1. $\Rightarrow$: Suppose $\mathbb{X}$ is $\mathfrak{F}-\mathfrak{S}^T$ measurable. For any $t \in T$ we have by the definition of $\mathfrak{S}^T$ that $\mathbb{X}^{-1}(\Pi_t^{-1}(S)) \in \mathfrak{F}$ for any $S \in \mathfrak{S}$, where Π_t denotes the projection on the tth coordinate. Finally $\mathbb{X}^{-1}(\Pi_t^{-1}(S)) = \{\omega : X_t(\omega) \in S\} = X_t^{-1}(S) \in \mathfrak{F}$.

$\Leftarrow$: Suppose all projections X_t are $\mathfrak{F}-\mathfrak{S}$-measurable. Let $S \in \mathfrak{S}$ and $t \in T$. Then $\mathbb{X}^{-1}(\Pi_t^{-1}(S)) = X_t^{-1}(S) \in \mathfrak{F}$. So $\mathbb{X}$ is measurable on the set $\{\Pi_t^{-1}(S) : t \in T, S \in \mathfrak{S}\}$. This set generates $\mathfrak{S}^T$, so $\mathbb{X}$ is $\mathfrak{F}-\mathfrak{S}^T$ measurable.

T2. Since X_t is continuous and since $\mathbb{Q}$ is dense in $\mathbb{R}$ we have that

$$\sup_{t\in\mathbb{R}} X_t = \sup_{t\in\mathbb{Q}} X_t.$$

Let $a \in \mathbb{R}$. Then

$$\left\{\sup_{t\in\mathbb{R}} X_t \le a\right\} = \left\{\sup_{t\in\mathbb{Q}} X_t \le a\right\} = \bigcap_{t\in\mathbb{Q}} \{X_t \le a\} \in \mathfrak{F}.$$

So $\sup_{t\in\mathbb{R}} X_t$ is measurable on the set $\{(-\infty, a] : a \in \mathbb{R}\}$, which generates $\mathfrak{B}$. So $\sup_{t\in\mathbb{R}} X_t$ is $\mathfrak{F}-\mathfrak{B}$-measurable. Furthermore

$$\left\{\sup_{t\in\mathbb{R}} X_t = \infty\right\} = \bigcap_{n=1}^{\infty} \bigcup_{t\in\mathbb{Q}} \{X_t \ge n\} \in \mathfrak{F},$$

and similarly

$$\left\{\sup_{t\in\mathbb{R}} X_t = -\infty\right\} = \bigcap_{n=1}^{\infty} \bigcup_{t\in\mathbb{Q}} \{X_t \le n\} \in \mathfrak{F},$$

so that the events $\{\sup_{t\in\mathbb{R}} X_t = \infty\}$ and $\{\sup_{t\in\mathbb{R}} X_t = -\infty\}$ are measurable as well.

T3. We first show by induction that for some $s > 0$, N_s is Poisson distributed with parameter λs. Firstly $\mathbb{P}(N_s = 0) = e^{-\lambda s}$. Now suppose that $\mathbb{P}(N_s = k) = e^{-\lambda s}\frac{(\lambda s)^k}{k!}$ for all $s > 0$. Then by conditioning on τ_1 we find

$$\mathbb{P}(N_s = k+1) = \int_0^s \lambda e^{-\lambda x} \mathbb{P}(N_{s-x} = k) \mathrm{d}x = \int_0^s \lambda e^{-\lambda x} e^{-\lambda(s-x)} \frac{(\lambda(s-x))^k}{k!} \mathrm{d}x$$
$$= e^{-\lambda s} \frac{(\lambda s)^{k+1}}{(k+1)!}.$$

So N_s is indeed Poisson distributed with parameter λs. Let $T_k := \sum_{i=1}^k \tau_i$ be the sequence of arrivals. We can write

$$N_s = \sum_{k=1}^{\infty} \mathbb{1}_{\{T_k \le s\}},$$

and similarly

$$N_t - N_s = \sum_{k=N_s+1}^{\infty} \mathbb{1}_{\{T_k \le t\}} = \sum_{k=1}^{\infty} \mathbb{1}_{\{T_{N_s+k} \le t\}}.$$

We know that $T_{N_s+1} > s$. In fact by the memorylessness of the exponential distribution we have $T_{N_s+1} \sim s + \mathrm{EXP}(\lambda)$. Similarly $T_{N_s+k} \sim s + T'_k$, where T'_k is an i.i.d. copy of T_k. Furthermore T_{N_s+k} is independent of N_s, since N_s is independent of $\tau_{N_s+1}, \tau_{N_s+2}, \ldots$. Finally we have

$$\begin{aligned}
\mathbb{P}(N_s = x, N_t - N_s = y) &= \mathbb{P}(N_s = x)\mathbb{P}(N_t - N_s = y \mid N_s = x) \\
&= \mathbb{P}(N_s = x)\mathbb{P}\left(\sum_{k=1}^{\infty} \mathbb{1}_{\{T_{N_s+k} \le t\}} = y \,\middle|\, N_s = x\right) \\
&= \mathbb{P}(N_s = x)\mathbb{P}\left(\sum_{k=1}^{\infty} \mathbb{1}_{\{s+T'_k \le t\}} = y\right) \\
&= e^{-\lambda s}\frac{(\lambda s)^x}{x!} e^{-\lambda(t-s)}\frac{(\lambda(t-s))^y}{y!}.
\end{aligned}$$

T4. The process $(M+N)_t$ is increasing and right-continuous, since M_t and N_t are increasing and right-continuous. Furthermore

$$(M+N)_t - (M+N)_s = M_t - M_s + N_t - N_s \sim \mathrm{POI}\big((\lambda+\mu)(t-s)\big),$$

since $M_t - M_s$ and $N_t - N_s$ are independent and Poisson distributed with parameter $\lambda(t-s)$ and $\mu(t-s)$ respectively. It remains to show that $(M+N)_t$ has steps of size 1 almost surely. Construct the process M'_t by placing $X_i \sim \mathrm{POI}(\lambda)$ points, $x_1^i, \ldots, x_k^i$, uniformly at random in the interval $[i, i+1)$, so that $M'_t \overset{d}{=} M_t$. Similarly construct $N'_t \overset{d}{=} N_t$ by placing the points $y_1^i, \ldots, y_l^i$ in the interval $[i, i+1)$. Then $(M'+N')_t \overset{d}{=} (M+N)_t$. Suppose $(M'+N')_t$ has a jump of size 2. Then there exists an interval $[i, i+1)$ such that $x_v^i = y_w^i$ for some $v, w \in \mathbb{N}$. Now,

$$\begin{aligned}
\mathbb{P}\big((M'+N')_t \text{ has jump of size 2}\big) &\le \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{v=1}^{k}\sum_{w=1}^{l} \mathbb{P}(X_i = k, Y_i = l, x_v^i = y_w^i) \\
&= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{v=1}^{k}\sum_{w=1}^{l} \mathbb{P}(X_i = k)\mathbb{P}(Y_i = l)\mathbb{P}(x_v^i = y_w^i) \\
&= 0,
\end{aligned}$$

since $\mathbb{P}(x_v^i = y_w^i) = 0$. So $(M'+N')_t$ has steps of size 1 almost surely and thus $(M+N)_t$ as well.