

## Exercises for Stochastic Processes

### Tutorial exercises:

- T1. (a) Show that  $\mathcal{L}f := \frac{1}{2}f''$  defined on  $\mathcal{D}(\mathcal{L}) := \{f \in C_0(\mathbb{R}) \mid f', f'' \in C_0(\mathbb{R})\}$  is a probability generator.  
(b) Show that it generates Brownian motion.  
(c) Show that the resolvent associated to one-dimensional Brownian motion is given by

$$U_\alpha f(x) = \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{\infty} f(y) e^{-\sqrt{2\alpha}|x-y|} dy.$$

- T2. Show that the operator on  $\{f \in C_0(\mathbb{R}) \mid f', f'' \in C_0(\mathbb{R})\}$  given by

$$\mathcal{L}f := af' + \frac{b}{2}f'',$$

with  $a \in \mathbb{R}$ ,  $b \geq 0$ , is a probability generator. What is the corresponding semigroup and process?

- T3. Let  $\mathcal{L}$  be the generator of  $n$ -dimensional Brownian motion, i.e., a process given by  $B_t := (B_t^1, \dots, B_t^n)$ , where  $B^1, \dots, B^n$  are independent one-dimensional Brownian motions, and  $f \in C_0^2(\mathbb{R}^n)$ .

- (a) Show that  $\mathcal{L}f = \frac{1}{2}\Delta f$ .  
(b) Show that  $u(t, x) := \mathbb{E}^x(f(B_t))$  satisfies the heat equation

$$\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \Delta u(t, x)$$

with initial condition

$$u(0, x) = f(x).$$

- T4. A probability measure  $\mu$  is called stationary for a Feller process on  $S$  with semigroup  $(T_t)$  if  $\int T_t f d\mu = \int f d\mu$  for all  $f \in C_0(S)$  and  $t \geq 0$ .

- (a) How is this definition related to the usual notion of stationarity?  
(b) Show that  $\mu$  is stationary for a process with generator  $\mathcal{L}$  if and only if  $\int \mathcal{L}f d\mu = 0$  for all  $f \in \mathcal{D}(\mathcal{L})$ .

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### Homework exercises:

- H1. Show that there cannot be two probability generators  $\mathcal{L}_1, \mathcal{L}_2$  on the same state space with  $\mathcal{D}(\mathcal{L}_1) \subsetneq \mathcal{D}(\mathcal{L}_2)$  and  $\mathcal{L}_1 = \mathcal{L}_2$  on  $\mathcal{D}(\mathcal{L}_1)$ .
- H2. Let  $(X_t)$  be the Feller process with generator  $\mathcal{L}f := \frac{1}{2}f'' - f'$  defined on  $\mathcal{D}(\mathcal{L}) := \{f \in C_0(\mathbb{R}) \mid f', f'' \in C_0(\mathbb{R})\}$  and start in  $x \in \mathbb{R}$ . Compute the mean value of its first hitting time of the origin.
- H3. Let  $\mu T_t$  denote the distribution at time  $t$  of the Feller process with semigroup  $(T_t)$  and initial distribution  $\mu$ . Show that, if  $\mu T_t \Rightarrow \nu$  weakly as  $t \rightarrow \infty$ , then  $\nu$  is stationary.
- H4. Let  $\varepsilon > 0$  small and let  $\mathcal{L}$  be a probability generator. Define for all  $f \in C_0(S)$

$$\mathcal{L}_\varepsilon = \mathcal{L}(I - \varepsilon\mathcal{L})^{-1}.$$

- (a) Show that  $\mathcal{L}_\varepsilon$  is a probability generator.
- (b) Show that  $\{T_{\varepsilon,t} := \exp(t\mathcal{L}_\varepsilon) : t \geq 0\}$  is a probability semigroup and

$$\mathcal{L}_\varepsilon = \lim_{t \downarrow 0} \frac{T_{\varepsilon,t}f - f}{t}.$$

**Deadline:** Monday, 16.01.17