

Exercises for Stochastic Processes

Tutorial exercises:

T1. Show that an irreducible Markov chain on a countable state space with one recurrent state is recurrent (i.e. all its states are).

T2. A stochastic process X is called **strictly stationary** if for all $n \in \mathbb{N}$ and all $t_1 < t_2 < \dots < t_n$ the distribution of $(X_{t_1+s}, \dots, X_{t_n+s})$ does not depend on s .

Let X be a Markov chain on S with transition probability $p_t(\cdot, \cdot)$ and starting distribution π . Show that X is strictly stationary if and only if π is invariant.

T3. Let X be a Markov chain on a finite set S with transition probability $p_t(\cdot, \cdot)$ and Q-matrix $q(\cdot, \cdot)$. Let π be a strictly positive measure on S . Show that π is reversible if and only if

$$\pi(x)q(x, y) = \pi(y)q(y, x) \quad \forall x, y \in S.$$

(N.B. this result also holds for countable S .)

T4. Let $(a_n)_{n \in \mathbb{N}}$ be a nonnegative subadditive sequence, i.e.,

$$0 \leq a_{n+m} \leq a_n + a_m \quad \forall n, m \in \mathbb{N}.$$

Show that

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} = \inf \left\{ \frac{a_n}{n}, n \in \mathbb{N} \right\}.$$

Homework exercises:

- H1. Show that an irreducible Markov chain on a countable state space with one positive recurrent state is positive recurrent (i.e. all its states are).
- H2. (a) Show that recurrent Markov chains on a countable state space have a stationary (not necessarily normalizable) measure.
(b) Show that an irreducible Markov chain on a countable state space has a stationary distribution if and only if it is positive recurrent.
- H3. Compute the transition function for the “linear birth chain”, i.e. the (unique) continuous time Markov chain on \mathbb{N}_0 with Q-matrix given by

$$q(m, n) := \begin{cases} \rho m & \text{if } n = m + 1, \\ -\rho m & \text{if } n = m, \end{cases}$$

(where $\rho > 0$ is an intensity parameter.)

- H4. Let $0 < p < 1$. Find all invariant measures for the asymmetric random walk on \mathbb{Z} defined by the Q-matrix

$$q(x, x - 1) = 1 - p, \quad q(x, x) = -1, \quad q(x, x + 1) = p \quad \forall x \in \mathbb{Z}.$$

Deadline: Monday, 12.12.16