Exercises for Stochastic Processes

Let B denote standard Brownian motion.

Tutorial exercises:

T1. Find stopping times σ and τ with $\mathbb{E}[\sigma] < \infty$, $\sigma \le \tau$ almost surely and

$$\mathbb{E}[B_\sigma^2] > \mathbb{E}[B_\tau^2].$$

T2. (a) Show that there exists a stopping time τ with $\mathbb{E}[\tau] = \infty$ and $\mathbb{E}[B_\tau^2] < \infty$.

(b) Show that for every stopping time τ with $\mathbb{E}[\tau] = \infty$ and $\mathbb{E}[\sqrt{\tau}] < \infty$, we have

$$\mathbb{E}[B_\tau^2] = \infty$$

(Hint: from $\mathbb{E}[\sqrt{\tau}] < \infty$ it follows that $B_{\tau\wedge t} \le \sup_{s\le 4^\tau} B_s$, furthermore $\sup_{s\le 4^\tau} B_s \in L^1$. See Theorem 2.50 in *Brownian Motion* by Mörters and Peres.)

T3. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and twice continuously differentiable with bounded first derivative and suppose that for all $t > 0$ and all $x \in \mathbb{R}$ we have $\mathbb{E}^x|f(B_t)| < \infty$ and $\mathbb{E}^x[\int_0^t |f''(B_s)|\mathrm{d}s] < \infty$. Show that the process defined by

$$X_t := f(B_t) - \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s$$

is a martingale.

(Hint: The normal density $p(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\big(-(x-y)^2/2t\big)$ satisfies the differential equation $\frac{\partial}{\partial t}p = \frac{1}{2}\frac{\partial^2}{\partial y^2}p$.)

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Homework exercises:

H1. Let μ be a probability distribution with three values $a < 0 < b < c$ and mean zero, consider

$$\tau_s := \min\left(\inf\{t \geq 0 \mid B_t = a\}, \inf\{t \geq s \mid B_t = b\}, \inf\{t \geq 0 \mid B_t = c\}\right) .$$

Show that the distribution of B_{τ_s} varies continuously from the one on $\{a, b\}$ with mean zero to the one on $\{a, c\}$ if s is varied from 0 to ∞ and conclude that, for some $s \geq 0$, B_{τ_s} has distribution μ!

H2. Let $(X_t)_{0 \leq t \leq 1}$ be a Brownian bridge:

$$X_t := B_t - tB_1.$$

Show that for all $x > 0$:

$$\mathbb{P}\left(\sup_{t \in [0,1]} X_t > x\right) = \exp\big(-2x^2\big).$$

(Hint: Recall H3(b) of sheet 2.)

H3. Let $(X_n)_{n \in \mathbb{N}}$ be i.i.d. with mean 0 and variance 1, and let $S_k = \sum_{i=0}^{k} X_i$. Show that

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{1}{n} \max\{k \leq n \mid S_k S_{k+1} \leq 0\} \leq t\right) = \frac{2}{\pi} \arcsin \sqrt{t}$$

for $0 \leq t \leq 1$!
(Hint: Recall that the same arcsine distribution solved problem T3.b) on sheet 3.)

Deadline: Monday, 28.11.16