Exercises for Stochastic Processes

Tutorial exercises:

T1. Show that (with the notations from the lecture)

$$\sum_{x\in V} \sup_{\eta} |f(\eta_x) - f(\eta)| < \infty$$

does not imply the continuity of a function $f : \{0,1\}^V \to \mathbb{R}$.

T2. Fix $K > 0$. For $x \in V$, fix $\beta_x, \delta_x \in (0, K]$ and define

$$c(x,\eta) := \beta_x \mathbb{1}_{\{\eta(x)=0\}} + \delta_x \mathbb{1}_{\{\eta(x)=1\}} .$$

(a) Show that there exists a spin system $(\eta_t)_{t\geq 0}$ with rate function c.

(b) Prove that it is ergodic and find its invariant distribution.!

T3. Let η_t and ζ_t be two spin systems satisifying the assumptions of Lemma 5.5. Write down the generator for the coupled process (η_t, ζ_t) for which $\eta_t \leq \zeta_t$ almost surely.

T4. Consider the one-dimensional stochastic Ising model with $\beta < 0$ and rates given by

$$c_\beta(x,\eta) := \exp\left(-\beta \sum_{y:y\sim x} (2\eta(x)-1)(2\eta(y)-1)\right) .$$

Prove that this spin system is ergodic and find its invariant measure.

(Hint: consider the bijection $\phi : S \to S$ that flips the spin of all odd vertices and calculate $c_\beta(x, \phi(\eta))$.)

→ Page 2

Homework exercises:

H1. Let p be a stochastic matrix on V with $p(x,x) = 0$ for all $x \in S$. Fix $\beta, \delta \geq 0$ and set

$$c(x,\eta) := \sum_{y:\eta(y)\neq\eta(x)} p(x,y) + \beta 1_{\eta(x)=0} + \delta 1_{\eta(x)=1}\,.$$

(a) Show that there exists a spin system $(\eta_t)_{t\geq 0}$ with rate function c.

(b) Give necessary and sufficient conditions for its ergodicity.

H2. Let (V,E) be a graph with bounded degree. For $x, y \in V$ we write $x \sim y$ if $(x,y) \in E$. Set

$$c(x,\eta) := |\{y \in V \mid y \sim x, \eta(y) = \eta(x)\}|\,.$$

(a) Show that there exists a spin system $(\eta_t)_{t\geq 0}$ with rate function c.

(b) Check its ergodicity for the cases

- $V := \mathbb{Z}^d$, $E := \{(x,y) \mid |x-y| = 1\}$ with $d \in \mathbb{N}$,
- $V := \mathbb{Z}/m\mathbb{Z}$, $E := \{(x,y) \mid y = x+1 \text{ or } y = x+(m-1)\}$ for m even,
- $V := \mathbb{Z}/m\mathbb{Z}$, $E := \{(x,y) \mid y = x+1 \text{ or } y = x+(m-1)\}$ for m odd.

H3. Consider the voter model with $V := \mathbb{Z}$ and $q(x,y) = \frac{1}{2}$ for $|x-y| = 1$, i.e.

$$c(x,\eta) := \frac{1}{2}\left(1_{\eta(x+1)\neq\eta(x)} + 1_{\eta(x-1)\neq\eta(x)}\right)\,.$$

The initial configuration $\eta_0 \in \{0,1\}^S$ shall be given by

$$\eta_0(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \geq 1\,. \end{cases}$$

Show that $\delta_{\eta_0} T_t$ converges weakly as $t \to \infty$ and determine the limit.

Deadline: Monday, 6.02.17