STRICT $\mathbb{A}^1$-INVARIANCE OF PRESHEAVES WITH FRAMED TRANSFERS

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Abstract. These are notes for a topics course at LMU Munich in winter 2021/2.

Contents

1. Introduction

1.1. **Statement of the theorem.** In [EHK+21] there was defined a category $\mathrm{hCor}_k^{\mathrm{fr}}$ (there spelled with two "r") with objects the smooth k-schemes and morphisms certain kinds of correspondences. There is a functor $\mathrm{Sm}_k \to \mathrm{hCor}_k^{\mathrm{fr}}$ which is the identity on objects. The aim of these notes is to prove the following result.

Theorem 1.1. *Let k be a perfect field. Let $F : (\mathrm{hCor}_k^{\mathrm{fr}})^{\mathrm{op}} \to \mathrm{Ab}$ be a functor (i.e. presheaf) such that $F(X \amalg Y) \simeq F(X) \times F(Y)$ and F is* $\mathbb{A}^1$-invariant*, i.e., for every $X \in \mathrm{Sm}_k$ the canonical map $F(X) \to F(\mathbb{A}^1 \times X)$ is an isomorphism.*

Then F is strictly $\mathbb{A}^1$-invariant*, that is, for every $X \in \mathrm{Sm}_k$ and $i \geq 0$ the canonical map $H^i_{\mathrm{Nis}}(X, F) \to H^i_{\mathrm{Nis}}(\mathbb{A}^1 \times X, F)$ is an isomorphism.*

Here $H^i_{\mathrm{Nis}}(X, F)$ denotes cohomology in the Nisnevich topology on the small étale site of X, of the presheaf thereon obtained from F by restriction along $\mathrm{Et}_X \to \mathrm{Sm}_k \to \mathrm{hCor}_k^{\mathrm{fr}}$.

1.2. **Situation in the literature.** The first theorem along these lines was proved by Voevodsky in [Voe00a, Theorem 5.6]. The statement is essentially the same; the only difference is that the category $\mathrm{hCor}_k^{\mathrm{fr}}$ is replaced by a related one. This result is one of the cornerstones of Voevodsky's celebrated theory of a derived category of motives [Voe00b].

More generally, from any category of correspondences one can attempt to build a theory of generalized motives. Proving an "$\mathbb{A}^1$-invariant implies strictly $\mathbb{A}^1$-invariant" theorem will be an important step in establishing that the theory of generalized motives is well-behaved. This program has been carried out several times [GP20, Dru17, Kol17, DK20]. For a textbook account in the case of Voevodsky correspondences, see [MVW06, Lecture 24].

In some sense, the most general "reasonable" category of correspondences is $\mathrm{hCor}_k^{\mathrm{fr}}$, and Theorem 1.1 is thus in some sense the strongest strict $\mathbb{A}^1$-invariance result. While it can be deduced from results in the literature (see the proof of [EHK+21, Theorem 3.4.11]), the arguments are more indirect than is necessary. We aim to remedy this in these notes.

Date: February 11, 2022.

1.3. **Preliminaries.** We assume basic knowledge of scheme theory, including but not limited to differentials, conormal sheaves, flat, smooth and étale morphisms, and elementary category theory. Beyond this, we require the following.

- Some theory of local complete intersection (lci) and syntomic (i.e. flat lci) morphisms; see e.g. [Liu02, §6.3].
- Some theory of cohomology of sheaves on sites; see e.g. [Mil80, §II and §III]. In particular we need basic homological algebra, including projective modules, derived functors etc.

1.4. **Outline.** We begin by given an elementary definition of the category $\mathrm{hCor}_k^{\mathrm{fr}}$. This requires us to first treat the algebraic K-theory groupoid and the stable normal bundle. After that we prove strict $\mathbb{A}^1$-invariance, following [DK20]. While this reference treats a different category of correspondences, their arguments apply to $\mathrm{hCor}_k^{\mathrm{fr}}$ with only superficial changes.

1.5. **Notation and terminology.** All schemes are assumed noetherian.

Most notation is introduced throughout the text.

Here are some comments for the experts. By a category we always mean an ordinary 1-category. We use notation consistent with common use, but sometimes with non-standard definitions. For example we write $K(X)_{\leq 1}$ for the 1-truncation of the K-theory space of X; this is an ordinary groupoid (well-defined up to equivalence) which we define directly. Given an lci morphism $f : X \to Y$, we define the "stable normal bundle" $L_f \in K(X)_{\leq 1}$; this is the opposite of the class of the cotangent complex. We make these definitions only in the case of affine schemes; this is enough for our arguments and simplifies everything considerably.

2. Lower algebraic K-theory

We define the algebraic K-theory groupoid of a symmetric monoidal category. In particular, we define K_0 and K_1 of a ring. Our presentation follows lectures 5 and 6 of [Hoy20].

For a more thorough introduction to algebraic K-theory see [Wei13], or [Mil71] for a historic perspective. The first two subsections recall standard definitions which can be found in any book on category theory.

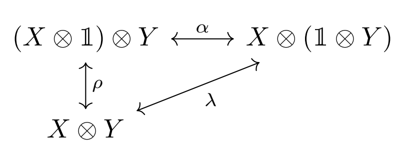
2.1. Symmetric monoidal categories and functors.

Definition 2.1. A *monoidal category* is a category $\mathcal{C}$ with a functor $\otimes : \times\mathcal{C} \to \mathcal{C}$ and an object $\mathbb{1} \in \mathcal{C}$, together with natural isomorphisms (for $X, Y, Z \in \mathcal{C}$)

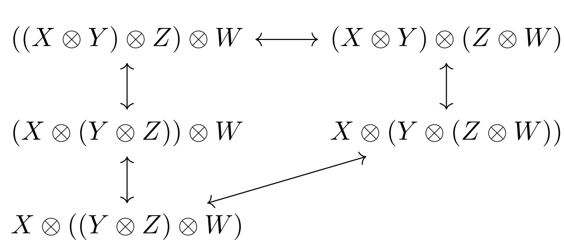
$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z),$$
$$\lambda_X : \mathbb{1} \otimes X \simeq X,$$
$$\rho_X : X \otimes \mathbb{1} \simeq X,$$

such that the following diagrams commute:

- (unit axiom)



- (pentagon axiom)



(Here all isomorphisms are instances of α tensored with identities.)

Remark 2.2. The pentagon axiom implies that $X_1 \otimes \cdots \otimes X_n$ is well-defined up to canonical isomorphism, independent of the choice of parenthesization. This is known as *MacLane's coherence theorem.*

Definition 2.3. A *symmetric monoidal category* is a monoidal category $\mathcal{C}$ together with a natural isomorphism

$$\gamma_{X,Y} : X \otimes Y \simeq Y \otimes X$$

such that $\gamma^2 = \mathrm{id}$ and the following diagram commutes:

- (hexagon axiom)

$$\begin{array}{ccc} (X \otimes Y) \otimes Z & \xleftrightarrow{\alpha} & X \otimes (Y \otimes Z) \\ \updownarrow{\scriptstyle\gamma} & & \updownarrow{\scriptstyle\gamma} \\ (Y \otimes X) \otimes Z & & (Y \otimes Z) \otimes X \\ \updownarrow{\scriptstyle\alpha} & & \updownarrow{\scriptstyle\alpha} \\ Y \otimes (X \otimes Z) & \xleftrightarrow{\gamma} & Y \otimes (Z \otimes X) \end{array}$$

Remark 2.4. The hexagon axiom implies that $X_1 \otimes \cdots \otimes X_n$ is independent of order, up to canonical isomorphism.

Example 2.5. If R is a commutative ring, then the category R-$\mathcal{M}$od of R-modules is a symmetric monoidal category, with $\otimes$ the usual tensor product.

Definition 2.6. A *monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ is a functor together with an isomorphism $F(\mathbb{1}_{\mathcal{C}}) \overset{\epsilon}{\simeq} \mathbb{1}_{\mathcal{D}}$ and a natural isomorphism

$$\mu_{X,Y} : F(X \otimes Y) \simeq F(X) \otimes F(Y),$$

such that certain evident diagrams commute.

A *symmetric monoidal functor* is a monoidal functor such that $F(\gamma^{\mathcal{C}}_{X,Y}) \simeq \gamma^{\mathcal{D}}_{FX,FY}$ via μ.

A *monoidal natural transformation* $\varphi : F \to G$ is a natural transformation such that the following diagrams commute:

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$$\begin{array}{ccc} F(\mathbb{1}_{\mathcal{C}}) & \xrightarrow{\varphi} & F(\mathbb{1}_{\mathcal{D}}) \\ \updownarrow{\scriptstyle\epsilon^F} & \nearrow\!\!\swarrow{\scriptstyle\epsilon^G} & \\ \mathbb{1}_{\mathcal{D}} & & \end{array}$$

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$$\begin{array}{ccc} F(X \otimes Y) & \xrightarrow{\varphi} & G(X \otimes Y) \\ \updownarrow{\scriptstyle\mu^F} & & \updownarrow{\scriptstyle\mu^G} \\ F(X) \otimes F(Y) & \xrightarrow{\varphi} & G(X) \otimes G(Y). \end{array}$$

Definition 2.7. Let $\mathcal{C}, \mathcal{D}$ be symmetric monoidal categories. We can form the category $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ with objects the symmetric monoidal functors and morphisms the monoidal natural transformations.

2.2. **Picard groupoids.**

Definition 2.8. A *groupoid* is a category in which all morphisms are isomorphisms.

Example 2.9. Let $\mathcal{C}$ be any category. We denote by $\mathcal{C}^{\simeq} \subset \mathcal{C}$ the maximal sub-groupoid, i.e., the objects of $\mathcal{C}^{\simeq}$ are the same as the objects of $\mathcal{C}$, and the morphisms of $\mathcal{C}^{\simeq}$ are the isomorphisms of $\mathcal{C}$. This is a groupoid.

Example 2.10. Let S be a set. We can view S as a category with set of objects S, and only identity morphisms. This is a groupoid.

Example 2.11. Let G be a group. We can form the category BG with one object $*$, and $\mathrm{Hom}(*, *) = G$. This is a groupoid.

Definition 2.12. A *(symmetric) monoidal groupoid* is an essentially small (symmetric) monoidal category which is also a groupoid.

Scholium 2.13. A category $\mathcal{C}$ is called *essentially small* if the class of isomorphism classes of objects is in fact a set. We shall largely ignore size issues in the sequel.

Definition 2.14. For an essentially small groupoid $\mathcal{C}$, we denote by $\pi_0(\mathcal{C})$ the set of isomorphism classes of objects. For a monoidal groupoid $\mathcal{C}$, we put $\pi_1(\mathcal{C}) = \mathrm{Hom}(\mathbb{1}, \mathbb{1})$.

If $\mathcal{C}$ is a (symmetric) monoidal groupoid, then $\pi_0(\mathcal{C})$ is a (commutative) monoid.

Definition 2.15. A *Picard groupoid* is a symmetric monoidal groupoid $\mathcal{C}$ such that the commutative monoid $\pi_0(\mathcal{C})$ is a group.

In other words, for every $X \in \mathcal{C}$ there must exist $Y \in \mathcal{C}$ such that $X \otimes Y \overset{\eta}{\simeq} \mathbb{1}$. We call the pair (Y, η) an *inverse of* X. One verifies immediately that if (Y', η') is another inverse of X, then there is a unique isomorphism $Y \simeq Y'$ compatible in the evident sense with (η, η').

Remark 2.16. We will often call just Y an inverse of X, suppressing η. But note that without a given choice of η, inverses are (in general) unique only up to non-unique isomorphism.

Remark 2.17. Let $\mathcal{C}$ be a Picard groupoid and $X \in \mathcal{C}$. Then

$$\pi_1(\mathcal{C}) \xrightarrow{\otimes X} \mathrm{Hom}(X, X)$$

is an isomorphism. Indeed if Y is an inverse of X, then $\otimes Y$ induces an inverse of the above map.

2.3. Group completion.

Definition 2.18. Let $\mathcal{C}$ be a symmetric monoidal groupoid. A *group completion* is a symmetric monoidal functor $\eta : \mathcal{C} \to \mathcal{C}^{\mathrm{gp}}$, where $\mathcal{C}^{\mathrm{gp}}$ is a Picard groupoid, such that for every Picard groupoid $\mathcal{D}$ the canonical functor

$$\mathrm{Fun}^{\otimes}(\mathcal{C}^{\mathrm{gp}}, \mathcal{D}) \xrightarrow{\circ \eta} \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$$

is an equivalence of categories.

Remark 2.19. One may prove (e.g. using the 2-categorical adjoint functor theorem) that $\mathcal{C}^{\mathrm{gp}}$ always exists.

Example 2.20. Let $\mathcal{D}$ be a Picard groupoid and $F : \mathcal{C} \to \mathcal{D}$ a symmetric monoidal functor. Under the equivalence $\mathrm{Fun}^{\times}(\mathcal{C}^{\mathrm{gp}}, \mathcal{D}) \simeq \mathrm{Fun}^{\times}(\mathcal{C}, \mathcal{D})$, F corresponds to some symmetric monoidal functor $\tilde{F} : \mathcal{C}^{\mathrm{gp}}, \mathcal{D}$ such that $\tilde{F} \circ \eta \simeq F$. In fact, given any two symmetric monoidal functors $\tilde{F}, \tilde{F}' : \mathcal{C}^{\mathrm{gp}} \to \mathcal{D}$ with this property, there exists a unique natural isomorphism $\kappa : \tilde{F} \simeq \tilde{F}'$ compatible with the other data.

Remark 2.21. Following this line of argument, we easily prove that any two group completions are canonically equivalent. Moreover $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{gp}}$ is a functor in an appropriate sense: given $F : \mathcal{C}_1 \to \mathcal{C}_2$ there is an induced symmetric monoidal functor $F^{\mathrm{gp}} : \mathcal{C}_1^{\mathrm{gp}} \to \mathcal{C}_2^{\mathrm{gp}}$, well-defined up to unique natural isomorphism, and furthermore $(G \circ F)^{\mathrm{gp}}$ coincides up to unique natural isomorphism with $G^{\mathrm{gp}} \circ F^{\mathrm{gp}}$.

Scholium 2.22. There is an analog of this construction "one category level down": the fully faithful embedding

$$\mathrm{Ab} \hookrightarrow \mathrm{AbMon}$$

has a left adjoint $M \mapsto M^{\mathrm{gp}}$. This is also known as the *Grothendieck construction.* For example $\mathbb{N}^{\mathrm{gp}} \simeq \mathbb{Z}$.

The two kinds of group completion are related.

Lemma 2.23. *Let $\mathcal{C}$ be a symmetric monoidal groupoid. Then $\pi_0(\mathcal{C}^{\mathrm{gp}}) \simeq \pi_0(\mathcal{C})^{\mathrm{gp}}$.*

(Here on the left hand side $(-)^{\mathrm{gp}}$ refers to Definition 2.18, whereas on the right hand side it refers to Scholium 2.22.)

Proof. For a (commutative) monoid M, we can also view M as a discrete (symmetric) monoidal groupoid, as in Example 2.10. Then for any symmetric monoidal groupoid $\mathcal{C}$ we find that

$$\pi_0 \mathrm{Fun}^{\otimes}(\mathcal{C}, M) \simeq \mathrm{Hom}_{\mathrm{AbMon}}(\pi_0 \mathcal{C}, M).$$

(In fact $\mathrm{Fun}^{\otimes}(\mathcal{C}, M)$ is discrete, but we do not need this.) Hence for any abelian group A we find that

$$\mathrm{Hom}_{\mathrm{AbMon}}(\pi_0 \mathcal{C}, A) \simeq \pi_0 \mathrm{Fun}^{\otimes}(\mathcal{C}, A) \simeq \pi_0 \mathrm{Fun}^{\otimes}(\mathcal{C}^{\mathrm{gp}}, A) \simeq \mathrm{Hom}_{\mathrm{AbMon}}(\pi_0(\mathcal{C}^{\mathrm{gp}}), A).$$

It follows that the abelian group $\pi_0(\mathcal{C}^{\mathrm{gp}})$ satisfies the same universal property as $\pi_0(\mathcal{C})^{\mathrm{gp}}$, whence the two are canonically isomorphic. □

Here is a non-obvious property of symmetric monoidal groupoids.

Lemma 2.24. *Let $\mathcal{C}$ be a symmetric monoidal groupoid. Then the group $\pi_1(\mathcal{C})$ is abelian.*

Proof. We have the two operations

$$\circ, \otimes : \pi_1(\mathcal{C}) \times \pi_1(\mathcal{C}) \to \pi_1(\mathcal{C}).$$

The second one is commutative, $\mathcal{C}$ being symmetric. The first one is the one is the one defining the group operation in $\pi_1(\mathcal{C})$. We shall show that the two coincide. In fact given $f, g : \mathbb{1} \to \mathbb{1}$ we have

$$f \otimes g = (f \circ \mathrm{id}) \otimes (\mathrm{id} \circ g) = (f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g) = f \circ g,$$

since $\otimes$ is a functor. This concludes the proof. □

Note that if $X \in \mathcal{C}$, then $\mathrm{Hom}(X, X)$ is a group, but it need not be abelian (unless $X = \mathbb{1}$). Of course if $\mathcal{C}$ is a Picard groupoid, then $\mathrm{Hom}(X, X) \simeq \pi_1(\mathcal{C})$ *is* abelian. We can use this to get a good handle on $\mathcal{C}^{\mathrm{gp}}$, in some cases.

Theorem 2.25. *Let $\mathcal{C}$ be a symmetric monoidal groupoid and suppose given an object $X \in \mathcal{C}$ such that for every $Y \in \mathcal{C}$ there exists $Z \in \mathcal{C}$ and $n \geq 0$ with $Y \otimes Z \simeq X^{\otimes n}$. Then $\pi_1(\mathcal{C}^{\mathrm{gp}}) \simeq G_\infty^{ab}$, where*

$$G_\infty = \operatorname*{colim}_n \mathrm{Hom}(X^{\otimes n}, X^{\otimes n})$$

and $(-)^{ab}$ denotes the maximal abelian quotient of a group.

Proof sketch. We build a symmetric monoidal category $\mathcal{C}'$. Its objects are the pairs (c, n) with $c \in \mathcal{C}$ and $n \in \mathbb{Z}$, thought of as $c \otimes X^{\otimes n}$. Define

$$\mathrm{Hom}_{\mathcal{C}'}(c \otimes X^{\otimes n}, d \otimes X^{\otimes m}) = \left(\operatorname*{colim}_r \mathrm{Hom}_{\mathcal{C}'}(c \otimes X^{\otimes n+r}, d \otimes X^{\otimes m+r})\right)^{ab}.$$

The symmetric monoidal structure is the "obvious" one (e.g. the switch on $X^{\otimes n}$ is given by $\gamma_{X,X}^{\circ n}$, for $n \in \mathbb{Z}$). One checks that $\mathcal{C} \to \mathcal{C}'$ satisfies the axioms of a group completion. □

Exercise 2.26. *Supply the details of the above proof.*

2.4. K-theory.

Definition 2.27. For a ring R, we denote by $\mathrm{Proj}_R \subset R\text{-}\mathcal{M}\mathrm{od}$ the category of finitely generated, projective R-modules. View this as a symmetric monoidal category via *direct sum.* We set

$$K(R)_{\leq 1} = (\mathrm{Proj}_R^{\simeq})^{\mathrm{gp}}$$

and

$$K_i(R) = \pi_i K(R)_{\leq 1}, \text{ for } i = 0, 1.$$

Remark 2.28. We use the symmetric monoidal structure on Proj_R given by direct sum, not tensor product. In particular this definition makes sense even for non-commutative rings. We shall have no use for this fact.

Remark 2.29. We have

$$K_0(R) \simeq \pi_0(\mathrm{Proj}_R^{\simeq})^{\mathrm{gp}} \quad \text{and} \quad K_1(R) \simeq \mathrm{GL}_\infty(R)^{ab}.$$

Indeed the first isomorphism is Lemma 2.23. For the second, we put $\mathrm{GL}_\infty(R) = \mathrm{colim}_n \mathrm{GL}_n(R)$, where $\mathrm{GL}_n(R) = \mathrm{Aut}(R^n)$, and apply Theorem 2.25 (with $X = R$).

Example 2.30. Let k be a field (or PID). Then $K_0(k) = \mathbb{Z}$, since $\pi_0(\mathrm{Proj}_k^{\simeq}) = \mathbb{N}$ (i.e. any finitely generated torsion-free k-module is isomorphic to k^n for some n).

Example 2.31. The maps $\det : \mathrm{GL}_n(R) \to R^\times$ are compatible, and hence induce $\det : \mathrm{GL}_\infty(R) \to R^\times$. The target being abelian, we obtain

$$\det : K_1(R) \to R^\times.$$

One may prove that this is an isomorphism if R is a semilocal ring.

Scholium 2.32. One may also define $K_i(R)$ for $i > 1$. In fact there is a *space* (also known as ∞-groupoid) $K(R)$ such that $K_i(R) = \pi_i(K(R))$. This is the subject of *higher algebraic K-theory.*

2.4.1. *Functoriality.* Let $\alpha : R \to S$ be a ring homomorphism. Then extension of scalars defines a symmetric monoidal functor $\mathrm{Proj}(\alpha) : \mathrm{Proj}_R \to \mathrm{Proj}_S$, and hence via Example 2.20 we obtain an essentially unique functor

$$K(\alpha) : K(R)_{\le 1} \to K(S)_{\le 1}.$$

Since naturally isomorphic functors induce the same maps on π_i, we obtain from this unique homomorphisms

$$K_i(\alpha) : K_i(R) \to K_i(S).$$

Given $\beta : S \to T$ another ring homomorphism, there is a canonical natural isomorphism $\mathrm{Proj}(\beta \circ \alpha) \simeq \mathrm{Proj}(\beta) \circ \mathrm{Proj}(\alpha)$ and hence also

$$K(\beta \circ \alpha)_{\le 1} \simeq K(\beta)_{\le 1} \circ K(\alpha)_{\le 1}.$$

Again since isomorphic functors induce the same maps on π_i we find that

$$K_i(\beta \circ \alpha) = K_i(\beta) \circ K_i(\alpha).$$

Thus $K_i(-)$ is a functor.

Scholium 2.33. Since $K(-)_{\le 1}$ is only well-defined up to (canonical) natural isomorphism, it does not really make sense to ask if it is a functor, at least in the usual sense. The theory of 2-categories can be used to make precise the idea that $K(-)_{\le 1}$ is a functor in every relevant sense.

2.4.2. *K-theory and exact sequences.* We have defined K-theory in terms of group completion, that is, by converting direct sums of modules to sums in our Picard groupoid. However, it turns out that K-theory also interacts favorably with (non-split) short exact sequences. Thus let

$$0 \to P_1 \to P_2 \to P_3 \to 0$$

be an exact sequence in Proj_R. There exists a splitting $s : P_3 \to P_2$, inducing an isomorphism $\varphi : P_2 \simeq P_1 \oplus P_3$. In general φ depends on s, but this is no longer the case in K-theory.

Proposition 2.34. *In the above situation, the image of φ in $K(R)_{\le 1}$ is independent of s.*

Proof. Let $s' : P_3 \to P_2$ be another splitting. Identifying P_2 with $P_1 \oplus P_3$ via φ, s' takes the form $(\alpha, \mathrm{id}_{P_3})$, for some (indeed any) map $\alpha : P_3 \to P_1$. We consequently obtain a commutative diagram of isomorphisms

$$\begin{array}{ccc} P_2 & \xrightarrow{\varphi} & P_1 \oplus P_3 \\ \downarrow{\scriptstyle \varphi'} & \nearrow{\scriptstyle M} & \\ P_1 \oplus P_3, & & \end{array}$$

where

$$M = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}.$$

It is sufficient (and necessary) to show that the automorphism M maps to the identity in $K(R)_{\le 1}$. Writing P_1, P_3 as summands of free modules, we may as well assume that $P_1 = R^m$ and $P_3 = R^n$. Then $M \in \mathrm{GL}_{n+m}(R)$ and we must write this as a commutator. Since we can write M as a product of elementary matrices, it suffices to deal with this case. We may assume that $m + n > 2$. Denoting by $E_{ij}(r)$ the elementary matrix with entry r in off-diagonal spot (i, j), one verifies that $E_{ij}(r)$ is the commutator of $E_{il}(r)$ and $E_{lj}(1)$, for any $i \neq l \neq j$. □

2.4.3. *K-theory of affine schemes.*

Definition 2.35. If X is an affine scheme, we put $K(X)_{\le 1} = K(\mathcal{O}_X(X))_{\le 1}$ and similarly for $K_i(X)$. Given $f : X \to Y$ a morphism of affine schemes, we obtain $f^* : K(Y)_{\le 1} \to K(X)_{\le 1}$ and similarly for K_i.

Note that $\mathrm{Proj}_{\mathcal{O}_X(X)}$ can be equivalently described as the category of vector bundles (locally free sheaves) on X.

Remark 2.36. K-theory can be extended to non-affine schemes, but the definition is more complicated (it takes into account the fact that there are non-split exact sequences of vector bundles). We shall not need this.

3. The (homotopy) category of framed correspondences

In this section we define the category $\mathrm{hCor}^{\mathrm{fr}}_A$, which is the basis for the definition of presheaves with framed transfers. The definition can be extracted from [EHK+21, §3.2.2], but we give a self-contained account.

3.1. **The stable normal bundle.** Let $f : X \to Y$ be an lci morphism of affine schemes. Being locally of finite type and affine, we may factor f as $X \overset{i}{\hookrightarrow} \mathbb{A}^n_Y \to Y$, for some $n \geq 0$. Then i is automatically a regular immersion, and so the conormal sheaf C_i is locally free. We now put

$$L^i_f = C_i - \Omega^1_{\mathbb{A}^n_Y/Y}|_X \in K(X)_{\leq 1}.$$

Remark 3.1. Of course $\Omega^1_{\mathbb{A}^n_Y/Y}|_X \simeq \mathcal{O}^n_X$.

Scholium 3.2. What we really mean by this definition is that L^i_f is a pair (M, η), where $M \in K(X)_{\leq 1}$ and η is an isomorphism between $M + \Omega^1_{\mathbb{A}^n_Y/Y}|_X$ and C_i in $K(X)_{\leq 1}$. If (M', η') is another such pair, then there is a unique isomorphism $M \simeq M'$ compatible with the other data; see also Remark 2.16.

It turns out that L^i_f is essentially independent of i.

Theorem 3.3. *(1) Let $X \overset{j}{\hookrightarrow} \mathbb{A}^m_Y \to Y$ be another factorization of f. We construct an isomorphism*

$$\alpha_{ij} : L^i_f \simeq L^j_f \in K(X)_{\leq 1}.$$

(2) Let $X \overset{k}{\hookrightarrow} \mathbb{A}^p_Y \to Y$ be yet another factorization of f. Then the following diagram commutes in $K(X)_{\leq 1}$

$$\begin{array}{ccc} L^i_f & \overset{\alpha_{ik}}{\longleftrightarrow} & L^k_f \\ \updownarrow{\scriptstyle \alpha_{ij}} & \swarrow\!\!\nearrow{\scriptstyle \alpha_{jk}} & \\ L^j_f. & & \end{array}$$

Once this is proved, we find that for all relevant purposes, $L^?_f$ is independent of the chosen factorization. We write $L_f \in K(X)_{\leq 1}$ for this canonically defined object. If no confusion can arise, we also denote this as $L_{X/Y}$. It satisfies the following further "properties".

Theorem 3.4. *(3) Let $g : Y \to Z$ be lci, with Z affine. We construct*

$$\beta_{g,f} : L_{X/Z} \simeq L_{X/Y} + f^* L_{Y/Z} \in K_{\leq 1}(X).$$

(4) Given furthermore $h : Z \to W$ lci, with Z affine, the following diagram commutes

$$\begin{array}{ccc} L_{X/W} & \longleftrightarrow & L_{X/Z} + (gf)^* L_{Z/W} \\ \updownarrow & & \updownarrow \\ L_{X/Y} + f^* L_{Y/W} & \longleftrightarrow & L_{X/Y} + g^* L_{Y/Z} + (gf)^* L_{Z/W}. \end{array}$$

Here all isomorphisms are induced by instances of β.

(5) Suppose given a cartesian square

$$\begin{array}{ccc} X' & \overset{p'}{\longrightarrow} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \overset{p}{\longrightarrow} & Y. \end{array}$$

Assume Y' is affine and either f or p is flat. We construct

$$\gamma_{f,p} : L_{X'/Y'} \simeq p'^* L_{X/Y} \in K(X')_{\leq 1}.$$

(6) Given another morphism of affine schemes $q : Y'' \to Y'$, set $X'' = Y'' \times_{Y'} X'$, and so on. Assume that either f is flat or both p and q are. Then up to the natural identifications we have

$$\gamma_{f',q} \circ \gamma_{f,p} = \gamma_{f,qp} \in \mathrm{Hom}_{K(X'')_{\leq 1}}(L_{X''/Y''}, (q'p')^* L_{X/Y}).$$

(7) β and γ are compatible in the evident way.

(8) *Suppose that f is étale. We construct*

$$\delta_f : L_f \simeq 0 \in K(X)_{\leq 1}.$$

The isomorphism δ is compatible with composition and base change (i.e. β and γ) in the evident way.

Proof of Theorems 3.3 and 3.4. (1) Consider $X \hookrightarrow (i,j)\mathbb{A}^{n+m}$. This provides yet another factorization, and it suffices to compare L_f^i and $L_f^{(i,j)}$. The embedding $X \hookrightarrow j\mathbb{A}^m$ corresponds to regular functions $\bar{c}_1, \ldots, \bar{c}_m$ on X. Consider the embedding $\mathbb{A}^n \hookrightarrow h\mathbb{A}^{n+m}$ corresponding to the regular functions $(x_1, \ldots, x_n, c_1, \ldots, c_m)$ on $\mathbb{A}^n$, where x_i is the i-th coordinate and c_j lifts $\bar{c}_j$. Then the composite hi is (i,j). Using the short exact sequence for the conormal sheaf in a composite of regular immersions and the interaction of exact sequences with K-theory (Proposition 2.34), we find that

$$C_{X/\mathbb{A}^{m+n}} \simeq C_{X/\mathbb{A}^n} + C_{\mathbb{A}^n/\mathbb{A}^{m+n}}|_X \in K(X)_{\leq 1}.$$

Since the closed embedding h is cut out by $(x_{n+i} - c_i)_{1 \leq i \leq m}$, the second conormal sheaf is canonically isomorphic to $\mathcal{O}^m$. It follows that

$$L_f^{(i,j)} \simeq C_{X/\mathbb{A}^n} + C_{\mathbb{A}^n/\mathbb{A}^{m+n}}|_X - \mathcal{O}^{m+n} \simeq C_{X/\mathbb{A}^n} - \mathcal{O}^n \simeq L_f^i,$$

defining the desired isomorphism.

(3) Consider the diagram

$$\begin{array}{ccccccc} & & & \mathbb{A}^{n+m}_Z & & & \\ & & \nearrow & & \searrow & & \\ & \mathbb{A}^n_Y & & & & \mathbb{A}^m_Z & \\ \nearrow & & \searrow & & \nearrow & & \searrow \\ X & & & Y & & & Z \end{array}$$

in which all maps are the evident ones, and the square is cartesian. We find that

$$C_{X/\mathbb{A}^{n+m}_Z} \simeq C_{X/\mathbb{A}^n} + C_{\mathbb{A}^n_Y/\mathbb{A}^{n+m}_Z}|_X \simeq C_{X/\mathbb{A}^n} + C_{Y/\mathbb{A}^m_Z}|_X,$$

where we have also used the stability of conormal sheaves under flat base change. From this we produce the desired isomorphism, as before.

(5) Consider the diagram of cartesian squares

$$\begin{array}{ccc} X' & \xrightarrow{p'} & X \\ \downarrow & & \downarrow \\ \mathbb{A}^n_{Y'} & \longrightarrow & \mathbb{A}^n_Y \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{p} & Y. \end{array}$$

The assumptions imply that $X' \to \mathbb{A}^n_{Y'}$ is a regular immersion. It follows that $C_{X/\mathbb{A}^n_Y}|_{X'} \simeq C_{X'/\mathbb{A}^n_{Y'}}$, from which we obtain the desired isomorphism as before.

(8) Since $X \to Y$ is smooth, we have an exact sequence

$$0 \to C_{X/\mathbb{A}^n_Y} \xrightarrow{d} \Omega^1_{\mathbb{A}^n_Y/Y} \to \Omega^1_{X/Y} \to 0.$$

Since X/Y is étale, $\Omega^1_{X/Y} = 0$ and so d is an isomorphism. This yields

$$L_{X/Y} = C_{X/\mathbb{A}^n_Y} - \Omega^1_{\mathbb{A}^n_Y/Y} \simeq 0,$$

as needed.

It remains to show that the stated compatibilities hold. This is tedious but straightforward. □

Example 3.5. Let $X \to Y$ be a regular immersion. Then we have the "factorization" $X \to \mathbb{A}^0_Y \to Y$, showing that $L_{X/Y} \simeq N_{X/Y}$.

Example 3.6. Let $X \to Y$ be smooth, and $X \to \mathbb{A}^n_Y \to Y$ a factorization. Then one has the exact sequence

$$0 \to N_{X/\mathbb{A}^1_Y} \to \Omega^1_{\mathbb{A}^n_Y/Y} \to \Omega^1_{X/Y} \to 0.$$

It follows that

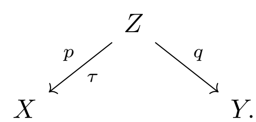
$$L_{X/Y} \simeq N_{X/\mathbb{A}^1_Y} - \Omega^1_{\mathbb{A}^n_Y/Y} \simeq -\Omega^1_{X/Y}.$$

3.2. **The groupoid of framed correspondences.**

Definition 3.7. Let A be an affine scheme, and $X, Y \in \mathrm{Aff}_A$. A *framed correspondence from X to Y over A* consists of

- a finite syntomic morphism (of A-schemes) $Z \to X$,
- any morphism (of A-schemes) $Z \to Y$, and
- an isomorphism
$$\tau : L_{Z/X} \simeq 0 \in K(Z)_{\le 1}.$$

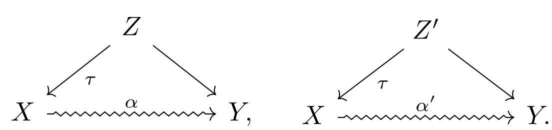
We think of framed correspondences as some kinds of generalized maps, and often write $\alpha : X \rightsquigarrow Y$ to mean that α is a framed correspondence from X to Y (over A). We often summarize the data of a framed correspondence in a diagram



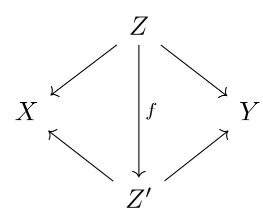
In this case the label p is the name of the morphism $Z \to X$, q is the name of $Z \to Y$, and τ is the trivialization of $L_{Z/X}$; any of these may be omitted for clarity.

Remark 3.8. Let $p : Z \to X$ be finite syntomic. Then it may or may not be the case that L_p is isomorphic to 0 in $K(Z)_{\le 1}$. If so, then the set of such isomorphisms is a torsor over $K_1(Z)$.

Definition 3.9. Let $X, Y \in \mathrm{Aff}_A$ and suppose given two framed correspondences



An isomorphism between α and α' consists of an isomorphism $f : Z \to Z'$ such that the following diagram commutes



and the composite

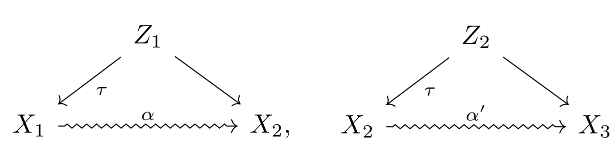
$$L_{Z/X} \simeq L_{Z'/X}|_Z + L_{Z/Z'} \overset{\tau'}{\simeq} 0 + 0 \simeq 0 \in K(Z)_{\le 1}$$

is τ. (Here we have also used Theorem 3.4(3,8).)

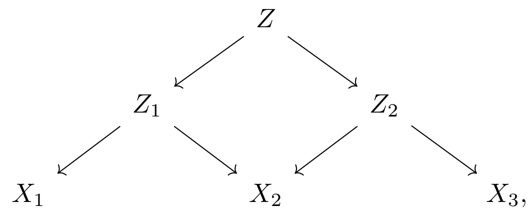
Definition 3.10. Isomorphism of framed correspondences can be composed in an evident way. We hence obtain the *groupoid of framed correspondences* $\mathrm{Cor}^{\mathrm{fr}}_A(X, Y)_{\le 1}$.

3.3. **The category** $\mathrm{hCor}^{\mathrm{fr}}_A$**.**

Definition 3.11. Let $X_1, X_2, X_3 \in \mathrm{Aff}_A$ and



be framed correspondences. Consider the diagram



in which the square is cartesian. We have an isomorphism

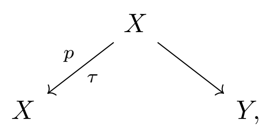
$$L_{Z/X_1} \simeq L_{Z/Z_1} + L_{Z_1/X}|_Z \simeq L_{Z_2/X_2}|_Z + L_{Z_1/X}|_Z \overset{\tau,\tau'}{\simeq} 0.$$

This data defines a framed correspondence $\alpha' \circ \alpha : X_1 \rightsquigarrow X_3$ called the *composition of* α *and* α'.

Theorem 3.12. *There is a category* $\mathrm{hCor}_A^{\mathrm{fr}}$ *as follows. The objects are the affine A-schemes, the set of morphisms from X to Y is* $\pi_0(\mathrm{Cor}_A^{\mathrm{fr}}(X, Y)_{\le 1})$ *(i.e. framed correspondences up to isomorphism), and* $[\alpha'] \circ [\alpha] = [\alpha' \circ \alpha]$.

Proof. Straightforward verification. □

Lemma 3.13. *There is a functor* $\mathrm{Aff}_A \to \mathrm{hCor}_A^{\mathrm{fr}}$ *which is the identity on objects and sends a morphism* $X \to Y$ *to the framed correspondence*



where $p = \mathrm{id}$ *and* τ *is the canonical isomorphism (e.g. obtained via Theorem 3.4(8)).*

Scholium 3.14. Just as there is a "higher version" $K(R)$ of $K(R)_{\le 1}$, which is an "∞-groupoid", there is a "higher version" $\mathrm{Cor}_A^{\mathrm{fr}}$ of $\mathrm{hCor}_A^{\mathrm{fr}}$ which is an "∞-category". Presheaves of sets on $\mathrm{Cor}_A^{\mathrm{fr}}$ are the same as presheaves on $\mathrm{hCor}_A^{\mathrm{fr}}$; this is what allows us to eschew the infinitudes. On the other hand, when considering presheaves of ∞-groupoids, the differences become pronounced.

Moreover, using more complicated definitions of K-theory valid for non-affine schemes, the category $\mathrm{(h)Cor}_A^{\mathrm{fr}}$ can be enlarged to also contain non-affine schemes.

Remark 3.15. There is a variant of the theory of framed correspondences, where τ (i.e. the framing) is omitted from the definitions. This yields a groupoid $\mathrm{Cor}_A^{\mathrm{fsyn}}(X, Y)_{\le 1}$ and a category $\mathrm{hCor}_A^{\mathrm{fsyn}}$. We shall not use this in the sequel.

3.4. **Examples of presheaves with framed transfer.** Suppose $F \in \mathcal{P}(\mathrm{SmAff}_k)$ is $\mathbb{A}^1$-invariant. We may wish to prove that F is strictly $\mathbb{A}^1$-invariant. By the main theorem, it will suffice to find $\tilde{F} \in \mathcal{P}(\mathrm{hCor}_k^{\mathrm{fr}})$ extending F. Suppose instead that we can find $\hat{F} \in \mathcal{P}(\mathrm{hCor}_k^{\mathrm{fsyn}})$ extending F. Then we can restrict $\hat{F}$ along the evident functor $\mathrm{hCor}_k^{\mathrm{fr}} \to \mathrm{hCor}_k^{\mathrm{fsyn}}$ to obtain $\tilde{F}$. In other words, providing F with transfers in the sense of $\mathrm{hCor}_k^{\mathrm{fsyn}}$ is *harder* than providing it with transfers in the sense of $\mathrm{hCor}_k^{\mathrm{fr}}$. We shall consider two examples to illustrate the difference.

Consider first the presheaf $F(X) = K_0(X)$, i.e. the group completion of the set of isomorphism classes of vector bundles. To lift this to $\mathrm{hCor}_k^{\mathrm{fsyn}}$, for every finite syntomic morphism $p : X \to Y$ we must find a transfer map $p_* : K_0(X) \to K_0(Y)$. Viewing a vector bundle V on X as a locally free sheaf, we have the sheaf p_*V on Y. Since p is finite flat, p_*V is locally free again. Clearly $p_*(V \oplus W) \simeq p_*(V) \oplus p_*(W)$. It follows that there is an induced map $p_* : K(X)_{\le 1} \to K(Y)_{\le 1}$, and taking π_0 we obtain the desired transfer. One may prove that this lifts $K_0(-)$ to a presheaf on $\mathrm{hCor}_k^{\mathrm{fsyn}}$.

Now we shall consider a variant of this construction which does not lift to $\mathrm{hCor}_k^{\mathrm{fsyn}}$, but which *does* lift to $\mathrm{hCor}_k^{\mathrm{fr}}$. By a *symmetric bilinear bundle* on X we mean a pair of a vector bundle V and a bilinear map $V \times V \to \mathcal{O}$ which is symmetric and such that the induced map $V \to V^*$ is an isomorphism. Symmetric bilinear bundles can be added in an evident way, and we write $GW_0(X)$ for the group completion of the set of isomorphism classes of symmetric bilinear bundles. This is clearly a functor on $\mathrm{Aff}_k^{\mathrm{op}}$. Now let $p : X \to Y$

by a finite syntomic morphism, and V a symmetric bilinear bundle on X. We would like to make $p_*(V)$ into a symmetric bilinear bundle on Y. The functor p_* is lax symmetric monoidal, so we obtain a symmetric map $p_*(V) \otimes p_*(V) \to p_*(\mathcal{O}_X)$. However, to turn this into a symmetric bilinear form in a natural way, surely we need a (reasonable) further map $p_*(\mathcal{O}_X) \to \mathcal{O}_Y$ to compose with. Duality theory in algebraic geometry shows that

$$\mathrm{Hom}(p_*(\mathcal{O}_X), \mathcal{O}_Y) \simeq \mathrm{Hom}(\mathcal{O}_X, \det L_p).$$

Here we have used a functor $\det : K(X)_{\le 1} \to Shv(X)$ induced by the determinant of a vector bundle. Consequently we can build a natural transfer whenever we are provided with a trivialization of $\det L_p$, for example, if we are provided with a trivialization of L_p itself. One may prove that this construction lifts $GW_0(-)$ to a presheaf on $\mathrm{hCor}_k^{\mathrm{fr}}$.

4. More about $\mathrm{hCor}_k^{\mathrm{fr}}$

4.1. Semiadditive categories.

Definition 4.1. A category $\mathcal{C}$ is called *pointed* there exists an object $0 \in \mathcal{C}$ which is both initial and final, i.e., for every $X \in \mathcal{C}$ we have $\mathrm{Hom}(X, 0) = * = \mathrm{Hom}(0, X)$.

Remark 4.2. Equivalently, $\mathcal{C}$ has an initial object $\emptyset$, a final object $*$, and the unique map $\emptyset \to *$ is an isomorphism.

Definition 4.3. A pointed category $\mathcal{C}$ is called *semiadditive* if for all $X, Y \in \mathcal{C}$ the sum $X \amalg Y$ and the product $X \times Y$ exist, and the canonical map

$$X \amalg Y \simeq (X \times 0) \amalg (0 \times Y) \to (X \times Y) \amalg (X \times Y) \to X \times Y$$

is an isomorphism. We write $X \oplus Y$ for the common object $X \amalg Y \simeq X \times Y$.

Lemma 4.4. *The category* $\mathrm{hCor}_A^{\mathrm{fr}}$ *is semiadditive. The* 0 *object corresponds to the empty scheme, and the operation* $\oplus$ *corresponds to the disjoint union of schemes.*

Proof. We have $\mathrm{Sch}_\emptyset \simeq \{\emptyset\}$ and $K(\emptyset)_{\le 0} \simeq *$. From this we deduce that

$$\mathrm{Cor}_A^{\mathrm{fr}}(\emptyset, Z) \simeq * \simeq \mathrm{Cor}_A^{\mathrm{fr}}(Z, \emptyset)$$

for any $Z \in \mathrm{Aff}_A$, i.e., $\mathrm{hCor}_A^{\mathrm{fr}}$ is pointed.

Now let $X, Y \in \mathrm{Aff}_A$. We get $\mathrm{Sch}_{X \amalg Y} \simeq \mathrm{Sch}_X \times \mathrm{Sch}_Y$, and also $K(X \amalg Y)_{\le 1} \simeq K(X)_{\le 1} \times K(Y)_{\le 1}$. From this one easily deduces that

$$\mathrm{Cor}_A^{\mathrm{fr}}(X \amalg Y, Z) \simeq \mathrm{Cor}_A^{\mathrm{fr}}(X, Z) \times \mathrm{Cor}_A^{\mathrm{fr}}(Y, Z) \quad \text{and} \quad \mathrm{Cor}_A^{\mathrm{fr}}(Z, X \amalg Y) \simeq \mathrm{Cor}_A^{\mathrm{fr}}(Z, X) \times \mathrm{Cor}_A^{\mathrm{fr}}(Z, Y),$$

naturally in $X, Y, Z \in \mathrm{Aff}_A$. This implies semiadditivity. □

4.2. Presheaves on a semiadditive category.

Definition 4.5. Let $\mathcal{C}$ be a category with finite coproducts. Denote by $\mathcal{P}_\Sigma(\mathcal{C})_{\le 0} \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ the subcategory of those presheaves F such that

$$F(X \amalg Y) \xrightarrow{(i_X^*, i_Y^*)} F(X) \times F(Y)$$

is an isomorphism for all $X, Y \in \mathcal{C}$, and $F(\emptyset) = *$. We sometimes call the objects in $\mathcal{P}_\Sigma(\mathcal{C})_{\le 0}$ Σ-*presheaves* or even Σ-*sheaves.*

Scholium 4.6. Let $F : \mathcal{A} \to \mathcal{B}$ be any functor, and $p : I \to \mathcal{A}$ a diagram. Assume that p has a limit L, and $F \circ p : I \to \mathcal{B}$ has a limit L'. There is a canonical map $F(L) \to L'$, induced by the maps $F(L \to p(x))$ for $x \in I$ and the universal property of L'. We say that F preserves the limit of p if $F(L) \to L'$ is an isomorphism. More generally, it makes sense to ask if F preserves all limits of a certain shape (or shapes), assuming all such limits exist in $\mathcal{A}$ and $\mathcal{B}$.

Now if $\mathcal{C}$ has finite coproducts, then $\mathcal{C}^{\mathrm{op}}$ has finite products, and the condition for $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ to be in $\mathcal{P}_\Sigma(\mathcal{C})_{\le 0}$ is precisely equivalent to asking that F preserves all finite products (including the empty one). For this reason we also write

$$\mathcal{P}_\Sigma(\mathcal{C})_{\le 0} := \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}).$$

Construction 4.7. Let $\mathcal{C}$ be semiadditive, $F \in \mathcal{P}_\Sigma(\mathcal{C})$ and $X \in \mathcal{C}$. Define an operation $+ : F(X) \times F(X) \to F(X)$ via the composite

$$F(X) \times F(X) \simeq F(X \oplus X) \xrightarrow{\Delta^*} F(X),$$

where $\Delta : X \to X \oplus X \simeq X \times X$ is the diagonal.

Lemma 4.8. *In the above situation, $(F(X), +)$ is a commutative monoid, natural in $X \in \mathcal{C}$.*

Proof. Commutativity of the addition operation corresponds to the fact that the composite

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{\tau} X \oplus X$$

coincides with Δ, where τ is the swap map. Similarly associativity follows from the fact that

$$X \xrightarrow{\Delta, \mathrm{id}} (X \oplus X) \oplus X \simeq X \oplus X \oplus X$$

is the triple diagonal, and similarly for (id, Δ) in place of (Δ, id). Naturality of the monoid structure along $f : X \to Y \in \mathcal{C}$ follows from the following commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \oplus X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \oplus f} \\ Y & \xrightarrow{\Delta} & Y \oplus Y. \end{array}$$

□

Lemma 4.9. *Let $\mathcal{C}$ be semiadditive, $X \in \mathcal{C}$ and $F \in \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathrm{AbMon})$. The map*

$$F(X) \times F(X) \simeq F(X \oplus X) \xrightarrow{\Delta^*} F(X)$$

is addition in the commutative monoid $F(X)$.

Proof. The composite

$$F(X) \times F(X) \xrightarrow{p_1^* + p_2^*} F(X \oplus X) \simeq F(X) \times F(X)$$

is the identity. Indeed composition with the first projection is $i_1^* \circ (p_1^* + p_2^*)$ and $p_1 i_1 = \mathrm{id}$ whereas $p_2 i_1 = 0$, and similarly for the second projection. It follows that the composite in question is given by $\Delta^* \circ (p_1^* + p_2^*) = \mathrm{id} + \mathrm{id}$, as needed. □

Proposition 4.10. *Let $\mathcal{C}$ be semiadditive. The forgetful functor $\mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathrm{AbMon}) \to \mathcal{P}_\Sigma(\mathcal{C})_{\le 0}$ is an equivalence of categories.*

Proof. Lemma 4.8 supplies a functor in the other direction, and Lemma 4.9 shows that the two functors are inverse *isomorphisms*.[1] □

Definition 4.11. Let $F \in \mathcal{P}_\Sigma(\mathcal{C})_{\le 0}$, where $\mathcal{C}$ is semiadditive. We call F *grouplike* if each of the commutative monoids $F(X)$ for $X \in \mathcal{C}$ is an abelian group. Denote by $\mathcal{P}_\Sigma(\mathcal{C})^{\mathrm{gp}}_{\le 0} \subset \mathcal{P}_\Sigma(\mathcal{C})_{\le 0}$ the full subcategory on grouplike Σ-presheaves.

Corollary 4.12. *The forgetful functor $\mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{P}_\Sigma(\mathcal{C})_{\le 0}$ is an equivalence onto $\mathcal{P}_\Sigma(\mathcal{C})^{\mathrm{gp}}_{\le 0}$.*

Proof. Indeed $\mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \subset \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathrm{AbMon})$ is the subcategory of sectionwise grouplike functors, so this is immediate from Proposition 4.10. □

Definition 4.13. We write $\mathcal{P}_{\mathrm{Ab}}(\mathcal{C})$ for the common category $\mathcal{P}_\Sigma(\mathcal{C})^{\mathrm{gp}}_{\le 0} \simeq \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.

[1] But the notion of isomorphism of categories is evil, so we do not state this stronger conclusion in the proposition.

4.3. **Local group completion.** Suppose that $\mathcal{C}$ is a category with finite coproducts. The Yoneda embedding $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ factors through $\mathcal{P}_\Sigma$. Consequently, if $\mathcal{C}$ is semiadditive, then for $X, Y \in \mathcal{C}$ the set $\mathrm{Hom}_\mathcal{C}(X, Y)$ acquires a commutative monoid structure (being identified with the sections over X of the presheaf represented by Y).

Remark 4.14. Chasing through the definitions, we find that given $f, g : X \to Y \in \mathcal{C}$, $f + g$ is given by the composite

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y.$$

We can group-complete the commutative monoids comprising the category $\mathcal{C}$.

Lemma 4.15. *Let $\mathcal{C}$ be semiadditive. There is a category $\mathcal{C}^{\mathrm{gp}}$ with the same objects as $\mathcal{C}$, and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{gp}}}(X, Y) = \mathrm{Hom}_\mathcal{C}(X, Y)^{\mathrm{gp}}$. The category $\mathcal{C}^{\mathrm{gp}}$ is semiadditive and the canonical functor $\mathcal{C} \to \mathcal{C}^{\mathrm{gp}}$ preserves finite coproducts.*

Proof. Straightforward checking. □

The category $\mathcal{C}^{\mathrm{gp}}$ satisfies a universal property.

Lemma 4.16. *The forgetful functor $\mathcal{P}_\Sigma(\mathcal{C}^{\mathrm{gp}})_{\le 0} \to \mathcal{P}_\Sigma(\mathcal{C})_{\le 0}$ is an equivalence onto the subcategory of grouplike objects.*

Proof. By Remark 4.14, the composite

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{\mathrm{id}, -\mathrm{id}} X \oplus X \xrightarrow{\nabla} X$$

is the zero map. This implies that if $F \in \mathcal{P}_\Sigma(\mathcal{C}^{\mathrm{gp}})_{\le 0}$ and $a \in F(X)$, then $a + (-\mathrm{id})^*(a) = 0$; hence $F(X)$ is grouplike. The forgetful functor thus factors through $\mathcal{P}_\Sigma(\mathcal{C})^{\mathrm{gp}}_{\le 0}$. Next we claim that if $F \in \mathcal{P}_\Sigma(\mathcal{C})^{\mathrm{gp}}$ then F admits a *unique* extension to a presheaf on $\mathcal{C}^{\mathrm{gp}}$. Granting this, we obtain a functor $\mathcal{P}_\Sigma(\mathcal{C})^{\mathrm{gp}} \to \mathcal{P}_\Sigma(\mathcal{C}^{\mathrm{gp}})$, and the two functors are seen to be inverse *isomorphisms*. It remains to prove the claim. If $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{gp}}}(X, Y)$ then $f = f_1 - f_2$, where $f_1, f_2 \in \mathrm{Hom}_\mathcal{C}(X, Y)$ (not uniquely determined). The only choice for $f^* : F(Y) \to F(X)$ is $f_1^* - f_2^*$. It is straightforward to verify that this is well-defined, concluding the proof. □

4.4. **Presheaves with framed transfers.**

Definition 4.17. Let $\mathcal{C} \subset \mathrm{Aff}_k$ be closed under finite coproducts. Write $\mathrm{hCor}^{\mathrm{fr}}_k(\mathcal{C}) \subset \mathrm{hCor}^{\mathrm{fr}}_k$ for the full subcategory on the objects of $\mathcal{C}$.

A presheaf with framed transfers (over k, relative to $\mathcal{C}$) means an object of $\mathcal{P}_{\mathrm{Ab}}(\mathrm{hCor}^{\mathrm{fr}}_k(\mathcal{C}))$.

The most common choice for us will be $\mathcal{C} = \mathrm{SmAff}_k$, the subcategory of smooth affine k-schemes. Note that for any $\mathcal{C}$ as in the definition, $\mathrm{hCor}^{\mathrm{fr}}_k(\mathcal{C})$ is semiadditive.

Remark 4.18. Lemma 3.13 yields a functor $\mathcal{C} \to \mathrm{hCor}^{\mathrm{fr}}_k(\mathcal{C})$. Restriction along this turns an object of $\mathcal{P}_{\mathrm{Ab}}(\mathrm{hCor}^{\mathrm{fr}}_k(\mathcal{C}))$ into an abelian presheaf on $\mathcal{C}$ itself, called the *underlying presheaf*.

4.5. **$\mathbb{A}^1$-invariance.** Let $\mathcal{C} \subset \mathrm{Aff}_k$ be closed under finite coproducts, and product with $\mathbb{A}^1$. We call $F \in \mathcal{P}_{\mathrm{Ab}}(\mathrm{hCor}^{\mathrm{fr}}_k(\mathcal{C}))$ $\mathbb{A}^1$-invariant if the underlying presheaf is, that is, if given $X \in \mathcal{C}$ the canonical map $F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism.

There are many equivalent characterizations of $\mathbb{A}^1$-invariance.

Lemma 4.19. *Let $F \in \mathcal{P}_{\mathrm{Ab}}(\mathcal{C})$. The following conditions are equivalent:*

1. *F is $\mathbb{A}^1$-invariant, i.e., for $X \in \mathcal{C}$ the canonical map $p^* : F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism.*
2. *For $X \in \mathcal{C}$ the canonical map $i_0^* : F(X \times \mathbb{A}^1) \to F(X)$ is an isomorphism.*
3. *The maps p^* are surjective (for every X).*
4. *The maps i_0^* are injective (for every X).*
5. *The maps i_0^* and i_1^* are equal (for every X).*

Here $i_s : X \to X \times \mathbb{A}^1$ is the inclusions at $s \in \mathbb{A}^1$.

Proof. Since $p \circ i_s = \mathrm{id}$, the map p^* is an isomorphism if and only if i_s^* is, in which case the two are inverse. Thus in particular $i_0^* = (p^*)^{-1} = i_1^*$. Hence (1) is equivalent to (2), and either implies (5). Moreover again since $i_0^* p^* = \mathrm{id}$, the map p^* is always injective and i_0^* is always surjective, so that (1) is equivalent to (3) and (2) is equivalent to (4). It remains to prove that (5) implies any of the other conditions. Consider the map $m : \mathbb{A}^1 \times \mathbb{A}^1 \times X \to \mathbb{A}^1 \times X$ induced by multiplication $(x, y, t) \mapsto (xy, t)$. Then mi_0 is the map $(x, t) \mapsto (0, x, t) \mapsto (0, t)$, i.e. the same as $i_0 p$. On the other hand mi_1 is the map $(x, t) \mapsto (1, x, t) \mapsto (x, t)$, i.e. the identity. It follows that

$$p^* i_0^* = m^* i_0^* \overset{(5)}{=} m^* i_1^* = \mathrm{id}^*,$$

and hence p^* is surjective as needed. □

Definition 4.20. We write $\mathcal{P}_{\mathrm{Ab}}^{\mathbb{A}^1}(\mathrm{hCor}_k^{\mathrm{fr}}(\mathcal{C})) \subset \mathcal{P}_{\mathrm{Ab}}(\mathrm{hCor}_k^{\mathrm{fr}}(\mathcal{C}))$ for the full subcategory on $\mathbb{A}^1$-invariant presheaves.

Definition 4.21. Let

$$\overline{\mathrm{hCor}_k^{\mathrm{fr}}}(X, Y) = \mathrm{hCor}_k^{\mathrm{fr}}(X, Y)^{\mathrm{gp}}/\mathrm{im}\alpha,$$

where $\alpha : \mathrm{hCor}_k^{\mathrm{fr}}(\mathbb{A}^1 \times X, Y)^{\mathrm{gp}} \to \mathrm{hCor}_k^{\mathrm{fr}}(X, Y)^{\mathrm{gp}}$ is given by $\alpha(h) = h \circ i_0 - h \circ i_1$.

Remark 4.22. Two morphisms of the form $h \circ i_0, h \circ i_1$ are called (directly) $\mathbb{A}^1$-homotopic.

Lemma 4.23. *We have a category* $\overline{\mathrm{hCor}_k^{\mathrm{fr}}}(\mathcal{C})$ *with the evident objects, mapping sets, and composition. It is semiadditive. The forgetful functor* $\mathcal{P}_{\mathrm{Ab}}(\overline{\mathrm{hCor}_k^{\mathrm{fr}}}(\mathcal{C})) \to \mathcal{P}_{\mathrm{Ab}}(\mathrm{hCor}_k^{\mathrm{fr}}(\mathcal{C}))$ *is an equivalence onto the subcategory of* $\mathbb{A}^1$*-invariant presheaves.*

Proof. The first assertion follows easily from the fact that $\mathbb{A}^1$-homotopic maps are stable under composition on either side. For the second assertion, since $(\mathrm{hCor}_k^{\mathrm{fr}})^{\mathrm{gp}}(\mathcal{C}) \to \overline{\mathrm{hCor}_k^{\mathrm{fr}}}(\mathcal{C})$ is surjective on mapping sets, it follows from Lemma 4.16 that the functor is fully faithful. It remains to prove that given $F \in \mathcal{P}_{\mathrm{Ab}}(\mathrm{hCor}_k^{\mathrm{fr}}(\mathcal{C}))$, F extends (necessarily uniquely) over $\overline{\mathrm{hCor}_k^{\mathrm{fr}}}(\mathcal{C})$ if and only if F is $\mathbb{A}^1$-invariant. If F is $\mathbb{A}^1$-invariant, $a \in F(X)$ is a section and $h \in \mathrm{hCor}_k^{\mathrm{fr}}(Y \times \mathbb{A}^1, X)$, then

$$\alpha(h)^*(a) = i_0^* h^*(a) - i_1^* h^*(a) = 0$$

by Lemma 4.19(5). Conversely, in the category $\overline{\mathrm{hCor}_k^{\mathrm{fr}}}(\mathcal{C})$ the two maps $i_0, i_1 : X \to \mathbb{A}^1 \times X$ are equal by construction, and so $i_0^* = i_1^*$ whenever F extends. Thus we conclude by Lemma 4.19(5) again. □

Scholium 4.24. In the category $\overline{\mathrm{hCor}_k^{\mathrm{fr}}}(\mathcal{C})$, the map $p : \mathbb{A}^1 \times X \to X$ is an isomorphism with inverse i_0 $(= i_1)$. This follows from the previous result via the Yoneda lemma, or by directly noting that the composite $i_0 p - \mathrm{id}$ equals $\alpha(m)$ and so is 0. In the sequel, many arguments about $\mathbb{A}^1$-invariant presheaves with transfers will be expressed as arguments in the category $\overline{\mathrm{hCor}_k^{\mathrm{fr}}}(\mathcal{C})$.

5. A prototypical result

Up to certain formal arguments explained in the next section, proving strict $\mathbb{A}^1$-invariance largely involves studying the category $\overline{\mathrm{hCor}_k^{\mathrm{fr}}}$. Before delving into the formalism, we want to illustrate what we mean by the latter in the simplest case.

5.1. Construction of framed correspondences.

Lemma 5.1. *Let* $X \to Y$ *be smooth of relative dimension* d*. Let* $f_1, \ldots, f_n \in \mathcal{O}(X)$ *and put* $Z = Z(f_1, \ldots, f_n)$*. If all non-empty fibers of* $Z \to Y$ *have relative dimension* $d - n$*, then* $X \to Y$ *is syntomic.*

Proof. Since the problem is local on X and Y, we may assume that X, Y are affine and the morphism factors as $X \to \mathbb{A}_Y^d \to Y$, with $X \to \mathbb{A}_Y^d$ étale. Now apply [EHK+21, Lemma 2.1.15] (and note that relative "global complete intersections" are syntomic [Sta18, Tag 00SW]). □

Example 5.2. The most common example of a framed correspondence is as follows. Let $f_1, \ldots, f_n \in \mathcal{O}(\mathbb{A}_X^n)$ and assume that $Z = Z(f_1, \ldots, f_n)$ is finite over X. Then by the above, $Z \to X$ is syntomic. Moreover

$$L_{Z/X} = C_{Z/\mathbb{A}_X^n} - \Omega^1_{\mathbb{A}_X^n/X}$$

is trivialized, since both terms are canonically isomorphic to free modules of rank n (with bases $[f_1], \ldots, [f_n]$ and $dT_1, \ldots, dT_n$, respectively). This presentation is sometimes called an *equational framing.*

We shall employ only the case of relative dimension 1.

Construction 5.3. Suppose given $X, Y \in \mathrm{Aff}_k$, $p : U \to X$ smooth of relative dimension 1, $g : U \to Y$, $f \in \mathcal{O}(U)$, a decomposition $Z(f) = Z \amalg Z'$ with $Z \to X$ finite, and an isomorphism $\mu : \Omega^1_{U/X} \simeq \mathcal{O}_U$. We denote by

$$\mathrm{div}(f)^{\mu,g}_Z \in \mathrm{Cor}^{\mathrm{fr}}_k(X, Y)$$

the correspondence

$$\begin{array}{ccccc} & & Z & & \\ & {\scriptstyle p}\swarrow_{\tau} & & \searrow{\scriptstyle g} & \\ X & & & & Y, \end{array}$$

where $Z \to X, Z \to Y$ are obtained from the inclusion $Z \to U$ by composition with the maps $U \to X, U \to Y$. The map p is finite syntomic by Lemma 5.1. The trivialization τ is given by

$$L_{Z/X} \simeq L_{Z/U} + L_{U/X}|_Z \simeq N_{Z/U} - \Omega^1_{U/X}|_Z \overset{(f,\mu)}{\simeq} \mathcal{O}_Z - \mathcal{O}_Z \simeq 0.$$

Warning 5.4. The notation $\mathrm{div}(f)^{\mu,g}_Z$ can be confusing if X, Y or p are not clear from the context.

This construction has the following properties, all of which are straightforward from the definition.

- If $\lambda \in \mathcal{O}^\times(U)$ then $\mathrm{div}(\lambda f)^{\lambda\mu,g}_Z = \mathrm{div}(f)^{\mu,g}_Z$.
- If $Z = Z_1 \coprod Z_2$, then $\mathrm{div}(f)^{\mu,g}_Z = \mathrm{div}(f)^{\mu,g}_{Z_1} + \mathrm{div}(f)^{\mu,g}_{Z_2}$.
- If $Z \to X$ and $Z \to Y$ are both isomorphisms of schemes, then $\mathrm{div}(f)^{\mu,g}_Z$ is invertible.
- Given $h : Y \to W$ we have $h \circ \mathrm{div}(f)^{\mu,g}_Z = \mathrm{div}(f)^{\mu,hg}_Z$.
- Given $h : W \to X$ we have $\mathrm{div}(f)^{\mu,g}_Z \circ h = \mathrm{div}(f')^{h^*\mu,gh}_{h^{-1}Z}$. Here we are using $f' : U' := U \times_X W \to W$ and the isomorphism $\Omega^1_{U'/W} \simeq h^*\Omega^1_{U/X}$.

5.2. **Injectivity for the relative affine line.** Using the above method for constructing correspondences, we can prove our first non-trivial result about presheaves with framed transfers.

Theorem 5.5. *Let $U \in \mathrm{SmAff}_k$, $V_1 \subset V_2 \subset \mathbb{A}^1_U$ affine and open. Assume that $\mathbb{A}^1_U \setminus V_2$ and $V_2 \setminus V_1$ are finite over U. Then for $F \in \mathcal{P}^{\mathbb{A}^1}_{\mathrm{Ab}}(\mathrm{hCor}^{\mathrm{fr}}_k(\mathrm{SmAff}_k))$, the restriction map $F(V_2) \to F(V_1)$ is injective.*

Example 5.6. If $U = \mathrm{Spec}(k)$, then any open subschemes $\emptyset \neq V_1 \subset V_2 \subset \mathbb{A}^1_k$ satisfy the hypotheses.

In order to establish Theorem 5.5, it will be enough to show the following.

Theorem 5.7. *Let U, V_1, V_2 as above. Then there exists $\Phi : V_2 \rightsquigarrow V_1$ such that*

$$[i \circ \Phi] = [\mathrm{id}_{V_2}] \in \overline{\mathrm{hCor}^{\mathrm{fr}}_k}.$$

Here $i : V_1 \to V_2$ denotes the inclusion.

Indeed given F as in Theorem 5.5, it extends to a presheaf on $\overline{\mathrm{hCor}^{\mathrm{fr}}_k}(\mathrm{SmAff}_k)$ (by Lemma 4.23). Then $\Phi^* i^* = \mathrm{id}$, whence i^* must be injective.

Proof of Theorem 5.7. We begin by constructing certain functions $f, g \in k[\mathbb{A}^1_U \times_U V_2]$. We shall denote the first coordinate by y and the second by x. We will arrange that f, g are respectively monic in y of degrees n and $n-1$ (for some n sufficiently large). Moreover, we shall ensure that

$$f|_{(\mathbb{A}^1_U \setminus V_1) \times_U V_2} = 1$$
$$g|_{(\mathbb{A}^1_U \setminus V_2) \times_U V_2} = (y - x)^{-1}$$
$$g|_{(V_2 \setminus V_1) \times_U V_2} = 1 \quad g|_{Z(y-x)} = 1.$$

To do this, first note that each of the subschemes we are restricting to is finite over V_2, and hence proper closed in $\mathbb{A}^1_{V_2}$. Now apply Lemma 5.8 below.

Let $h \in k[\mathbb{A}^1 \times V_2 \times \mathbb{A}^1]$ be given by

$$h = (1 - \lambda) f + \lambda (y - x) g;$$

here λ denotes the third coordinate. Note that h is monic in y. Define

$$\Phi' = \mathrm{div}(f|_{V_1 \times_U V_2})^{dy,pr_1}_{Z(f)} : V_2 \rightsquigarrow V_1$$
$$\Theta = \mathrm{div}(h|_{V_2 \times_U V_2 \times \mathbb{A}^1})^{dy,pr_1}_{Z(h)} : V_2 \times \mathbb{A}^1 \rightsquigarrow V_2.$$

Since f, h are monic in y, their vanishing loci are finite (over V_2, respectively $V_2 \times \mathbb{A}^1$). By construction, they have no zeros if $y \notin V_1$ or $y \notin V_2$, respectively. Hence the vanishing loci relevant for the $\mathrm{div}(\dots)$ are also finite.

Since $h|_{\lambda=0} = f$ we find that

$$\Theta' \circ i_0 = i \circ \Phi'.$$

Since $h|_{\lambda=1} = (y-x)g$ has vanishing locus splitting into two disjoint pieces, we find that

$$\Theta' \circ i_1 = \mathrm{div}((y-x)g|_{V_2 \times_U V_2})^{dy,pr_1}_{Z(y-x)} + \mathrm{div}((y-x)g|_{V_2 \times_U V_2})^{dy,pr_1}_{Z(g)}.$$

Define

$$\Phi^- = \mathrm{div}((y-x)g|_{V_1 \times_U V_2})^{dy,pr_1}_{Z(g)} : V_2 \rightsquigarrow V_1.$$

As before the vanishing locus is finite. Finally put $\Phi = \Phi' - \Phi^-$. We find that

$$[i] \circ [\Phi] = [i \circ \Phi'] - [i \circ \Phi'] = [\Theta \circ i_1] - [i \circ \Phi'] = \mathrm{div}((y-x)g|_{V_2 \times_U V_2})^{dy,pr_1}_{Z(y-x)},$$

which is invertible as needed. □

Lemma 5.8. *Let U be an affine scheme and $Z \subset \mathbb{A}^1_U$ a closed subscheme which is finite over U. Let $\bar{f} \in \mathcal{O}(Z)$. Then for n sufficiently large there exists a monic $f \in \mathcal{O}(\mathbb{A}^1_U)$ with $f|_Z = \bar{f}$.*

Proof. Let $U = \mathrm{Spec}(A)$, $Z = \mathrm{Spec}(A[T]/I)$. Since Z is finite, there exist $g_1, \dots, g_r \in A[T]$ whose images generate $A[T]/I$ as an A-module. Let n be larger than the maximum of the degrees of the g_i. We claim that f as desired can be found for such n. Indeed note that *any* $\bar{h} \in A[T]/I$ admits a lift $h \in A[T]$ of degree $< n$; in fact we can choose the lift to be an A-linear combination of the g_i. Now let f_1 be an arbitrary lift of $\bar{f} - T^n$ of degree $< n$, and put $f = T^n + f_1$. □

6. The formal argument

Fix a perfect field k.

6.1. Axiomatics.

Definition 6.1. $X \in \mathrm{Aff}_k$ is called *essentially smooth* if it can be written as a cofiltered limit of objects in SmAff_k with affine transition maps. Denote by $\mathrm{EssSmAff}_k \subset \mathrm{Aff}_k$ the category of essentially smooth schemes.

Example 6.2. Let $X \in \mathrm{Sm}_k$ and $x \in X$. Then $X_x, X_x^h, X_x^{sh} \in \mathrm{EssSmAff}_k$.

Example 6.3. Let K/k be a finitely generated field extension. Then $\mathrm{Spec}(K) \in \mathrm{EssSmAff}_k$. Indeed $\mathrm{Spec}(K)$ is the generic point of a smooth k-variety, by generic smoothness (using that k is perfect).

Example 6.4. $\mathbb{A}^\infty = \lim_i \mathbb{A}^i$ is *not* essentially smooth.

Given $F \in \mathcal{P}(\mathrm{SmAff}_k)$, we can left Kan extend F to $\mathcal{P}(\mathrm{EssSmAff})$. Thus for $X = \lim_i X_i$ with $X_i \in \mathrm{SmAff}_k$ we put

$$F(X) = \operatorname*{colim}_i F(X_i).$$

We shall use this freely in the sequel.

Definition 6.5 (IA)**.** We say that F satisfies *injectivity on the affine line* (short *IA*) if the following holds. For any finitely generated field extension K/k (automatically essentially smooth) and open subschemes $\emptyset \neq V_1 \subset V_2 \subset \mathbb{A}^1_K$ (automatically affine), the restriction $F(V_2) \to F(V_1)$ is injective.

Definition 6.6 (EA)**.** We say that F satisfies *excision on the relative affine line* (short *EA*) if the following holds. For any essentially smooth affine scheme U and affine open subscheme $V \subset \mathbb{A}^1_U$ containing 0_U, restriction induces an isomorphism

$$F(\mathbb{A}^1_U \setminus 0_U)/F(\mathbb{A}^1_U) \simeq F(V \setminus 0_U)/F(V).$$

(Note that $\mathbb{A}^1_U \setminus 0_U$ and $V \setminus 0_U$ are indeed affine.)

Furthermore we require that if K/k is a finitely generated field extension, $z \in \mathbb{A}^1_K$ a closed point, $V \subset \mathbb{A}^1_K$ an open neighborhood of z, then

$$F(\mathbb{A}^1_K \setminus z)/F(\mathbb{A}^1_K) \simeq F(V \setminus z)/F(V).$$

Definition 6.7 (IL)**.** We say that F satisfies *injectivity for henselian local schemes* (short *IL*) if the following holds. For any essentially smooth, henselian local scheme U with generic point η, the restriction $F(U) \to F(\eta)$ is injective.

Definition 6.8 (EE)**.** We say that F satisfies *étale excision* (short *EE*) if the following holds. Let $\pi : X' \to X$ be an étale morphism of essentially smooth, local k-schemes. Let $Z \subset X$ be a principal closed subscheme such that $\pi^{-1}(Z) \to Z$ is an isomorphism. Then the canonical map

$$F(X \setminus Z)/F(X) \to F(X' \setminus \pi^{-1}(Z))/F(X')$$

is an isomorphism.

We shall also use the notion of contraction.

Definition 6.9. Let F be a presheaf. We denote by F_{-1} the presheaf $X \mapsto F(X \times \mathbb{G}_m)/F(X)$, and by F_{-n} the n-fold iterate of this construction.

The main result of this section is as follows.

Theorem 6.10. *Let k be a perfect field. Let $\mathcal{C}$ be a collection of abelian presheaves on SmAff_k which is closed under $F \mapsto F_{-1}$ and $F \mapsto H^i(-, F)$. Assume that whenever $F \in \mathcal{C}$ is $\mathbb{A}^1$-invariant then it satisfies IA, EA, IL and EE.*

Let $F \in \mathcal{C}$ be $\mathbb{A}^1$-invariant. Then for every essentially smooth (not necessarily affine) k-scheme X we have

$$H^i(X \times \mathbb{A}^1, F) \simeq H^i(X, F).$$

Remark 6.11. We shall eventually prove that $\mathcal{C} = \{\text{presheaves admitting framed transfers}\}$ satisfies the assumptions of Theorem 6.10. This will prove Theorem 1.1, i.e., achieve the goal of these notes.

6.2. $\mathbb{A}^1$-invariance of the associated sheaf.

Lemma 6.12. *Let K/k be a finitely generated field extension and F a presheaf satisfying IA, EA, IL and EE. Let $U \subset \mathbb{A}^1_K$ be open. Then $F(U) \simeq (a_{\mathrm{Nis}}F)(U)$ and $H^i(U, F) = 0$ for $i > 0$.*

Proof. Let $X = \mathbb{A}^1_K$.

We first establish the following claim: $(*)$ if $U \subset \mathbb{A}^1_K$ is open, $z_1, \ldots, z_n \in U$ are distinct closed points, then

$$F(U \setminus \{z_1, \ldots, z_n\})/F(U) \simeq \bigoplus_{i=1}^{n} F(U^h_{z_i} \setminus z_i)/F(U^h_{z_i}).$$

If $n = 1$, this follows by combining EA and EE. Now let $n > 1$, and assume the claim proved for $n - 1$. Combining IA and the case $n = 1$, we have a short exact sequence

$$0 \to F(U \setminus \{z_1, \ldots, z_{n-1}\})/F(U) \to F(U \setminus \{z_1, \ldots, z_n\})/F(U) \to F(U^h_{z_n} \setminus z_n)/F(U^h_{z_n}) \to 0.$$

By induction the source is isomorphic to $\bigoplus_{i=1}^{n-1} F(U^h_{z_i} \setminus z_n)/F(U^h_{z_i})$, and we can thus split the sequence. This proves the claim.

Consider the following sequence of presheaves on X_{Nis}

$$0 \to F \to \bigoplus_{\eta \in U^{(0)}} F(\eta) \to \bigoplus_{z \in U^{(1)}} F(U^h_z \setminus z)/F(U^h_z) \to 0;$$

here $U \to X$ is some étale scheme (automatically affine). Observe that the second two terms are skyscraper sheaves. In particular they are sheaves, and even acyclic. We argue that this sequence is exact after sheafification. For this we need only consider the case $U = \eta$ (a generic point of some étale X-scheme), and the case where U is henselian local of dimension 1, so in particular has only two points. Both sequences exact; the only non-trivial point is injectivity of $F(U) \to F(\eta)$ which is IL.

It follows that we may compute $H^i(U, F)$ using the above resolution; in particular $H^i = 0$ for $i > 1$. Let $U \subset X$. We first compute $H^0(U, F)$: it consists of those elements $a \in F(\eta)$ (where η is the generic point of U) such that for every closed point $z \in U$, a is in the image of $F(X_z^h) \to F(X_z^h \setminus z)$. Let a be such an element. Then there exists $\emptyset \neq V \subset U$ and $a' \in F(V)$ such that $a = a'|_\eta$. Let $z \in U \setminus V$ and put $V' = V \cup \{z\}$. Note that $V' \subset U$ is open. By $(*)$ with $n = 1$ we have $F(V)/F(V') \simeq F((V_z')^h \setminus z)/F((V_z')^h)$. The image of a' in the right hand group vanishes by assumption, hence it vanishes in the left hand group. In other words there exists $a'' \in F(V')$ extending a'. Repeating this argument finitely many times we conclude that $F(U) \to H^0(U, F)$ is surjective. The map is injective by IA, and hence an isomorphism.

It remains to prove that $H^1(U, F) = 0$. In other words, given distinct closed points $z_1, \dots, z_n \in U$ we must prove that $F(\eta) \to \bigoplus_{i=1}^n F(U_{z_i}^h \setminus z_i)/F(U_{z_i}^h)$ is surjective. This follows from $(*)$, since it identifies the right hand side with a quotient of $F(U \setminus \{z_1, \dots, z_n\})$. □

Remark 6.13. Suppose that in addition F is $\mathbb{A}^1$-invariant. Then we get

$$H^0(\mathbb{A}^1_K, F) \simeq F(\mathbb{A}^1_K) \simeq F(K) \simeq H^0(\mathrm{Spec}(K), F)$$

and also

$$H^i(\mathbb{A}^1_K, F) = 0 = H^i(\mathrm{Spec}(K), F), i > 0,$$

since the Nisnevich cohomological dimension of fields is zero. Of course these conclusions would be expected if F was strictly $\mathbb{A}^1$-invariant.

In fact $\mathbb{A}^1$-invariance of $H^0(-, F) = a_{\mathrm{Nis}} F$ follows easily from the restricted case considered above.

Corollary 6.14. *Suppose that F is $\mathbb{A}^1$-invariant and satisfies IA, EA, IL and EE. Then $a_{\mathrm{Nis}} F$ is $\mathbb{A}^1$-invariant.*

Proof. Let $X \in \mathrm{SmAff}_k$. We must prove that $H^0(X \times \mathbb{A}^1, F) \to H^0(X, F)$ is injective (see Lemma 4.19). Consider the diagram

$$\begin{array}{ccc}
H^0(X, F) & \longrightarrow & \prod_{\eta \in X^{(0)}} H^0(\eta, F) \\
\uparrow & & \uparrow \\
H^0(\mathbb{A}^1_X, F) & \longrightarrow & \prod_{\eta \in X^{(0)}} H^0(\mathbb{A}^1_\eta, F) \\
\downarrow & & \downarrow \\
\prod_{x \in \mathbb{A}^1_X} H^0((\mathbb{A}^1_X)^h_x, F) & \longrightarrow & \prod_{x \in \mathbb{A}^1_X} H^0(\eta_x, F),
\end{array}$$

where η_x is the generic point of $(\mathbb{A}^1_X)^h_x$. This lies over a point of $\mathbb{A}^1_\eta$, so the bottom right hand map is defined and the diagram commutes. The bottom left hand map is injective and the bottom map is injective by IL; hence the middle map is injective. Consequently the top left hand map is injective as soon as the top right hand map is. This reduces the claim to the case $X = \eta$, which holds by Lemma 6.12. □

6.3. **$\mathbb{A}^1$-invariance of cohomology.** We have to work harder for the higher cohomology groups.

To be added late.

7. Transfers on cohomology

Let us first study colimits in abelian presheaves.

Lemma 7.1. *Let $\mathcal{C}$ be a category with finite coproducts. Then colimits in $\mathcal{P}_{\mathrm{Ab}}(\mathcal{C})$ are computed sectionwise.*

That is, given a diagram $F : I \to \mathcal{P}_{\mathrm{Ab}}(\mathcal{C})$ and $c \in \mathcal{C}$, we have

$$(\operatorname{colim}_i F_i)(c) \simeq \operatorname{colim}_i F_i(c).$$

Note that the colimit on the left hand side is computed in in $\mathcal{P}_{\mathrm{Ab}}(\mathcal{C})$ and the colimit on the right hand side is computed in Ab.

Proof. We know that colimits in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ are computed sectionwise. It thus suffices to prove that if $F_i \in \mathcal{P}_{\mathrm{Ab}}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ for every i, then $\mathrm{colim}_i F_i$ computed in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ lies in $\mathcal{P}_{\mathrm{Ab}}(\mathcal{C})$. In other words, the colimiting presheaf G must still satisfy $G(c \coprod d) \simeq G(c) \times G(d)$. For this it is enough to show: given $A, B : I \to \mathrm{Ab}$, we have

$$\operatorname*{colim}_i [A_i \times B_i] \simeq \left[\operatorname*{colim}_i A_i\right] \times \left[\operatorname*{colim}_i B_i\right].$$

It suffices to prove this for I being a filtered category, or the category corresponding to a finite coproduct, or a reflexive coequalizer [?]. Since finite coproducts are also finite products in Ab, they commute with finite products as needed. The case of filtered colimits and reflexive coequalizers is easily verified by hand. □

Corollary 7.2. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between categories with finite coproducts, preserving finite coproducts. Then in the adjunction*

$$f : \mathcal{P}_{\mathrm{Ab}}(\mathcal{C}) \rightleftarrows \mathcal{P}_{\mathrm{Ab}}(\mathcal{D}) : f^*$$

(where the left adjoint is left Kan extension), f^* preserves all limits and colimits.

Proof. Preservation of limits is clear, being a right adjoint. Preservation of colimits follows from Lemma 7.1 since colimits are computed sectionwise in both categories. □

In particular, we can compute colimits in presheaves with transfers just as we would without transfers; more precisely the forgetful functor commutes with colimits.

We shall prove that cohomology presheaves admits transfers by employing a trick: we shall find a specific resolution (the Godement resolution) which actually is a resolution by presheaves with framed transfers.

Construction 7.3. Let $F \in \mathcal{P}_{\mathrm{Ab}}(\mathrm{SmAff}_k)$. Define $E^0F \in \mathcal{P}_{\mathrm{Ab}}(\mathrm{SmAff}_k)$ by

$$(E^0F)(X) = \prod_{x \in X} F(X^h_x),$$

with the evident structure maps. Denote by $F \to E^0F$ the map induced by restriction.

To define the "evident" structure maps, suppose that $X \to Y$ is a map in SmAff_k, taking $x \in X$ to $y \in Y$. Then since $(X \times_Y Y^h_y)^h_x \simeq X^h_x$, there is a canonical induced map $X^h_x \to Y^h_y$, giving a component of $(E^0F)(Y) \to (E^0F)(X)$.

Lemma 7.4. *(1) $F \to E^0F$ is a well-defined map of presheaves.*
(2) E^0F is a sheaf.
(3) The induced map $a_{\mathrm{Nis}}F \to E^0F$ is injective.

Proof. (1,2) Tedious checks.

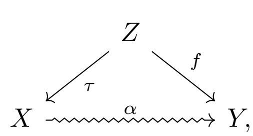
(3) Let $X \in \mathrm{SmAff}_k$, $x \in X$. The composite $F(X) \to (E^0F)(X) \to F(X^h_x)$, where the second map is projection, is just restriction. Taking the colimit over étale neighborhoods of x, we find that the induced map $F(X^h_x) \to (E^0F)(X^h_x)$ admits a retraction, and so is injective. The map is thus injective on stalks, as needed. □

Now we show that the Godement construction is compatible with transfers.

Lemma 7.5. *Let $F \in \mathcal{P}_{\mathrm{Ab}}(\mathrm{hCor}^{\mathrm{fr}}_k(\mathrm{SmAff}_k))$. Then $F \to E^0F$ lifts canonically to $\mathcal{P}_{\mathrm{Ab}}(\mathrm{hCor}^{\mathrm{fr}}_k(\mathrm{SmAff}_k))$.*

We shall only sketch the proof of this result. For a very careful treatment of a closely related result, see [Ivo07].

Proof sketch. We explain how to define transfers on E^0F and leave the rest to the reader. Consider a framed correspondence



with $X, Y \in \mathrm{SmAff}_k$. For $x \in X, y \in Y$ we shall define $\alpha^*_{xy} : F(Y^h_y) \to F(X^h_x)$; these will be the components of $\alpha^* : (E^0F)(Y) \to (E^0F)(X)$. Let $x_1, \ldots, x_n \in Z$ be the fibers above x. We set $\alpha^*_{xy} = \alpha^*_1 + \cdots + \alpha^*_n$. If

$f(x_i) \neq y$ we let $\alpha_i^* = 0$. Otherwise by [Sta18, Tag 08HR] there is a unique map $Z^h_{x_i} \to Y^h_y$ compatible with $x_i \to y$. Moreover since (use [Sta18, Tag 04GH(1)])

$$X^h_x \simeq \coprod_{i=1}^n Z^h_{x_i},$$

we obtain a framed correspondence

$$\begin{array}{ccc} & Z^h_{x_i} & \\ \swarrow^{\tau} & & \searrow \\ X^h_x & \overset{\alpha_i}{\rightsquigarrow} & Y^h_y. \end{array}$$

This concludes the construction. □

Remark 7.6. The proof illustrates one reason why it is crucial to use the Nisnevich instead of Zariski topology: $X_x \times_X Z$ need not be a finite disjoint union of local rings, and the argument will break down.

Corollary 7.7. *If $F \in \mathcal{P}_{\mathrm{Ab}}(\mathrm{hCor}^{\mathrm{fr}}_k(\mathrm{SmAff}_k))$, then $H^i(-, F)$ also has canonical framed transfers.*

Proof. Consider the sequence

$$F \to E^0 F \to E^1 F \to \ldots,$$

where

$$E^{i+1}F := E^0(E^i F/\mathrm{im} E^{i-1}F)$$

(and $E^{-1}F := F$), with $E^i F \to E^{i+1}F$ the evident composite. By construction, this is a complex, and since $G \to E^0 G$ is Nisnevich locally injective and each $E^i F$ is a Nisnevich sheaf (Lemma 7.4), $a_{\mathrm{Nis}} F \to E^\bullet F$ is a resolution. Since moreover each $E^i F$ is flasque, we can use the resolution to compute cohomology presheaves. Thus in order to prove the result, it suffices to prove that $E^\bullet F$ is a complex of presheaves with framed transfers. This follows from Lemma 7.5 (and the fact that colimits of presheaves can be computed with or without transfers, by Lemma 7.2). □

8. Injectivity for semilocal schemes

In this section we shall prove the following result.

Theorem 8.1. *Let $X \in \mathrm{EssSmAff}_k$ have finitely many closed points $x_1, \ldots, x_n$. Let $Z \subset X$ be a principal closed subscheme containing all the closed points. Let $F \in \mathcal{P}^{\mathbb{A}^1}_{\mathrm{Ab}}(\mathrm{hCor}^{\mathrm{fr}}_k)$. Then $F(X) \to F(X \setminus Z)$ is injective.*

Before going any further, let us see how to deduce axiom IA from this.

Example 8.2. Let X be henselian local with generic point η. We have

$$\eta \simeq \lim_{\emptyset \neq U \subset X} U = \lim_{Z \subset X} X \setminus Z.$$

Here the limit is over all non-empty open subschemes of X, which we may (for cofinality reasons) assume to be principal. We can index this equivalently on the complements, which are the proper principal closed subschemes Z. Now applying Theorem 8.1 to each such Z and using that a filtered colimit of injections is injective, we deduce that

$$F(X) \to \operatorname*{colim}_Z F(X \setminus Z) \simeq F(\eta)$$

is injective. This is IL.

To prove the theorem, we shall use the following lemma.

Lemma 8.3. *Assume k is infinite.*

Let $V \in \mathrm{SmAff}_k$, $x_1, \ldots, x_n \in V$, $Z \subset V$ principal closed containing all the x_i, X the semilocalization of V in the x_i. Then there is a framed correspondence $\Phi : X \rightsquigarrow V \setminus Z$ such that

$$\begin{array}{ccc} & & V \setminus Z \\ & \overset{\Phi}{\nearrow} & \downarrow \\ X & \hookrightarrow & V \end{array}$$

commutes up to $\mathbb{A}^1$-homotopy.

Proof of Theorem 8.1. We assume that k is infinite. In the case of finite fields a "coprime extensions" trick can be used to reduce to the infinite case.

Write $X = \lim_i X_i$, where $X_i \in \mathrm{SmAff}_k$. Since Z is a principal closed subscheme, without loss of generality we may assume that $Z = \lim_i Z_i$, where $Z_i \subset X_i$ is principal closed. Let $y_1^i, \dots, y_n^i \in X_i$ denote the images of the x_i. We may assume that $y_j^i \in Z_i$ for all j. Let Y_i be the semilocalization of X_i in the y_j^i. Noting that $\lim_i Y_i \simeq X$, it suffices to prove that $F(Y_i) \to F(Y_i \setminus Z_i)$ is injective. Let $V \subset X_i$ be an open affine neighborhood of the $\{y_j^i\}_j$. Lemma 8.3 shows that

$$\ker(F(V) \to F(V \setminus Z) \subset \ker(F(V) \to F(Y_i)).$$

Taking the (filtered, whence exact) colimit over all such open neighborhoods we deduce that

$$\ker(F(Y_i) \to F(Y_i \setminus Z_i)) \subset \ker(F(Y_i) \to F(Y_i)) = 0.$$

This concludes the proof. □

Remark 8.4. To prove Lemma 8.3, we may assume that $x_1, \dots, x_n \in V$ are closed. Indeed if not, pick closed specializations $y_1, \dots, y_n$. Note that $y_i \in Z$ (Z being closed) and $x_i \in V_{y_1,\dots,y_n}$ ($V_{y_1,\dots,y_n}$ being an intersection of open subsets containing y_i). Applying the Lemma with the y_i in place of the x_i yields the right hand half of the following diagram

$$\begin{array}{ccccc} & & & & V \setminus Z \\ & & & \overset{\Phi'}{\nearrow} & \downarrow \\ V_{x_1,\dots,x_n} & \xrightarrow{j} & V_{y_1,\dots,y_n} & \hookrightarrow & V. \end{array}$$

Setting $\Phi = \Phi' \circ j$ proves what we want.

To prove Lemma 8.3, we shall use the following geometric result. It is established in the next section.

Proposition 8.5. *Assume k infinite.*

Let $V \in \mathrm{SmAff}_k$, $Z \subset V$ a closed subscheme (everywhere) of positive codimension, $x_1, \dots, x_n \in Z$ closed points, $U = V_{x_1,\dots,x_n}$. Then we have a diagram in $\mathrm{EssSmAff}_k$

$$V \xleftarrow{v} C \xrightarrow{j} \overline{C} \xrightarrow{p} U$$

with the following properties:

1. *$p : \overline{C} \to U$ is a relative projective curve, j is an open immersion, and $p \circ j : C \to U$ is smooth.*
2. *$p \circ j$ admits a section Δ such that $U \xrightarrow{\Delta} C \xrightarrow{v} V$ is the canonical map.*
3. *$v^{-1}(Z)$ is finite over U.*
4. *$\overline{C} \setminus C$ is finite over U.*
5. *$\Omega^1_{C/U}$ is trivial.*

Proof of Lemma 8.3. We shall use several times that any line bundle on a semilocal scheme is trivial. We shall also use that if X is projective over an affine scheme A, $F \to G$ a surjection of coherent sheaves on X, then $H^0(X, F(n)) \to H^0(X, G(n))$ is surjective for n sufficiently large.

Choose data as in Proposition 8.5. We use U instead of X now. Put $Z' := v^{-1}(Z)$, $D := \overline{C} \setminus C$. Since D is finite over U, if T is a positive dimensional component of the fiber of $\overline{C} \to U$ over a closed point, then T is not contained in D. For every such component, pick a closed point $x_T \in T$ with $x_T \notin D$, and write S for the finite set of points obtained in this way. Since $\mathcal{O}_{\overline{C}} \to \mathcal{O}_{D \amalg (Z' \cup \Delta \cup S)}$ is surjective, for r sufficiently large $H^0(\overline{C}, \mathcal{O}(r)) \to H^0(D \amalg (Z' \cup \Delta \cup S), \mathcal{O}(r))$ is surjective. Since $\mathcal{O}_{Z' \cup \Delta \cup S}(n)$ is trivial, it admits a non-vanishing section 1. We can thus find $d \in H^0(\overline{C}, \mathcal{O}(r))$ mapping to

$$(0, 1) \in H^0(D \amalg (Z' \cup \Delta \cup S), \mathcal{O}(r)) \simeq H^0(D, \mathcal{O}(r)) \times H^0(Z' \cup \Delta \cup S, \mathcal{O}(r)).$$

Note that $D \subset Z(d)$. Replace D by $Z(d)$, C by $\overline{C} \setminus C$, $\mathcal{O}(1)$ by $\mathcal{O}(r)$. By construction $Z' \cup \Delta \subset C$. Also $D \to U$ is finite. Indeed it is proper, so it suffices to prove quasi-finiteness, and so by semicontinuity of fiber dimension [Sta18, Tag 0D4I] (and using that $\overline{C} \to U$ is a relative curve) it suffices to prove that if T is a positive dimensional component of a fiber of $\overline{C} \to U$ over a closed point, then $T \not\subset D$. But $x_T \notin D$ by construction. Hence all hypotheses still hold.

Note that $\Delta : U \to C$ is a regular immersion of codimension 1, and hence $\Delta \subset \overline{C}$ is a divisor. In particular $\mathcal{O}(-\Delta)$ (the ideal sheaf defining Δ) is a line bundle on $\overline{C}$, with inverse $\mathcal{O}(\Delta)$. We claim that for n sufficiently large, we can find sections satisfying the following hypotheses:

$$s \in H^0(\overline{C}, \mathcal{O}(n)), \quad \tilde{s} \in H^0(\overline{C} \times \mathbb{A}^1, \mathcal{O}(n)), \quad s' \in H^0(\overline{C}, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta)), \quad \delta \in H^0(\overline{C}, \mathcal{O}(\Delta))$$
$$Z(s) \cap (Z' \amalg D) = \emptyset, \quad Z(s') \cap (Z' \cup D \cup \Delta) = \emptyset, \quad Z(\delta) = \Delta$$
$$Z(\tilde{s}) \text{ finite over } \mathbb{A}^1_U$$
$$\tilde{s}|_{\overline{C} \times 0} = s, \quad \tilde{s}|_{\overline{C} \times 1} = s' \otimes \delta, \quad Z(\tilde{s}) \cap D \times \mathbb{A}^1 = \emptyset.$$

The last two conditions imply that also $Z(s), Z(s' \otimes \delta)$ are finite over U, and so is $Z(s')$ (being a closed subscheme of $Z(s' \otimes \delta)$). Pick an isomorphism $\mu : \Omega^1_{C/U} \simeq \mathcal{O}_C$, and recall the map $v : C \to V$. We consider the following framed correspondences, built using Construction 5.3 from the relative smooth curves $C \setminus Z' \to U$ and $C \times \mathbb{A}^1 \to U \times \mathbb{A}^1$ (plus extra data)

$$\Phi' = \operatorname{div}(s/d^n)^{\mu,v}_{Z(s)} - \operatorname{div}(s' \otimes \delta/d^n)^{\mu,v}_{Z(s')} \in \operatorname{Cor}^{\mathrm{fr}}_k(U, V \setminus Z)^{\mathrm{gp}}$$

$$\Theta' = \operatorname{div}(\tilde{s}/d^n)^{\mu,v}_{Z(\tilde{s})} - \operatorname{div}(s' \otimes \delta/d^n)^{\mu,v}_{Z(s')} \circ pr_U^{\mathbb{A}^1 \times U} \in \operatorname{Cor}^{\mathrm{fr}}_k(\mathbb{A}^1_U, V)^{\mathrm{gp}}.$$

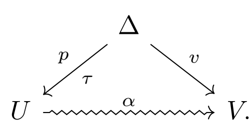
We compute

$$\Theta' \circ i_0 = i \circ \Phi'$$

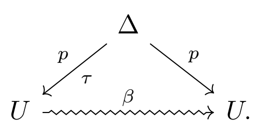
and

$$\Theta' \circ i_1 = \operatorname{div}(s' \otimes \delta/d^n)^{\mu,v}_{Z(\delta)}.$$

The latter is a framed correspondence



Consider also the framed correspondence



Then since $U \simeq \Delta \xrightarrow{v} V$ is $j : U \to V$ we find that $\alpha = j \circ \beta$. On the other hand β is invertible. Put $\Phi' = \Phi \circ \beta^{-1}$ and $\Theta = \Theta' \circ \beta^{-1}$. One checks that $\Theta \circ i_0 = i \circ \Phi$ and $\Theta \circ i_1 = \alpha \circ \beta^{-1} = j$, as needed.

It remains to construct s, s', δ and $\tilde{s}$. For δ, we just take the tautological section of $\mathcal{O}(\Delta)$ (dual to the inclusion $\mathcal{O}(-\Delta) \subset \mathcal{O}$). For n large enough, both

$$H^0(\overline{C}, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta)) \to H^0(Z' \cup D \cup \Delta, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta))$$

and

$$H^0(\overline{C}, \mathcal{O}(n)) \to H^0(Z' \amalg D, \mathcal{O}(n))$$

are surjective. Pick s' such that $s'|_{Z' \cup D \cup \Delta}$ has no zeros (this is possible because $Z' \cup D \cup \Delta$ is semilocal, so any line bundle on it is trivial). Pick s in such a way that $s|_D = s' \otimes \delta|_D$ and $s|_{Z'}$ is non-vanishing. Finally put $\tilde{s} = (1-t)s + ts' \otimes \delta$, where t is the coordinate on $\mathbb{A}^1$. We are done once we prove that $Z(\tilde{s}) \to \mathbb{A}^1 \times U$ is finite (note that $Z|_{D \times \mathbb{A}^1}$ is just given by $s|_D = s' \otimes \delta|_D$, which is non-vanishing). The map being proper and $\overline{C} \times \mathbb{A}^1 \to U \times \mathbb{A}^1$ being a relative curve, it is enough to show that if T is a positive dimensional component of a fiber, then $T \not\subset Z(\tilde{s})$. But $\mathcal{O}(1)|_T$ is ample whence non-trivial, so $D = Z(d)$ meets T. We conclude since $Z(\tilde{s})$ and D are disjoint. □

9. A moving lemma

In order to prove Proposition 8.5, we shall use the method of general projections. The following result is enough for what we need.

Theorem 9.1. *Let k be a field, $X \subset \mathbb{A}^N_k$ a closed subscheme of dimension d, $Z \subset \mathbb{A}^N$ of dimension $\le d-1$, $S \subset \mathbb{A}^N$ a finite set of closed point (i.e. a subscheme of dimension 0). Then for a general linear projection $\pi : \mathbb{A}^N \to \mathbb{A}^d$, the following hold:*

(1) $\pi|_X : X \to \mathbb{A}^d$ is finite.
(2) If X is smooth, then $\pi|_X$ is étale at all points of $S \cap X$.
(3) $\pi^{-1}(\pi(S)) \cap Z \subset S$.

Scholium 9.2. Linear maps $\mathbb{A}^N \to \mathbb{A}^d$ are parametrized by the variety $P = \mathbb{A}^{N\times d}$, in the sense that given $x \in P$ we obtain a linear map $\pi_x : \mathbb{A}^N_{k(x)} \to \mathbb{A}^d_{k(x)}$. The meaning of the theorem is that there exists a dense open subset $U \subset P$ such that for all $x \in U$, the map π_x satisfies conditions (1), (2) and (3) (slightly modified to take into account the base change to $k(x)$).

Remark 9.3. Note that a dense open subset of affine space over an *infinite* field contains a rational point. It follows that in the case of an infinite field, there is an actual linear projection $\pi : \mathbb{A}^N_k \to \mathbb{A}^d_k$ satisfying all the properties.

Scholium 9.4. To prove Theorem 9.1, we can use the following simplifications. Firstly, in any topological space, finite intersections of dense open sets are dense open. It follows that in order to prove that a general map satisfies a finite conjunction of properties, it suffices to prove separately for each condition that it is satisfied by a general map. Furthermore, affine space being irreducible, an open set being dense is equivalent to it being nonempty. Thus it will suffice to prove that the set of maps satisfying some condition is open, and then exhibit a single map satisfying the condition.

Proof of Theorem 9.1. Write $\pi = (\pi_1, \ldots, \pi_d)$, where $\pi_i : \mathbb{A}^N \to \mathbb{A}^d$ is given by $\pi_i = \sum_{j=1}^N a^i_j x_j$ (with $a^i_j \in k$). We treat each of the conditions in turn.

Condition (1). Let $\bar\pi_i = a^i_1 X_1 + \cdots + a^i_N X_N$ and $\bar\pi_0 = X_0$ and put $V = Z(\bar\pi_0, \bar\pi_1, \ldots, \bar\pi_d) \subset \mathbb{P}^N$. We obtain a map

$$\bar\pi : \mathbb{P}^N \setminus V \to \mathbb{P}^d, (X_0 : \cdots : X_N) \mapsto (\bar\pi_0 : \cdots : \bar\pi_d).$$

Write $\mathbb{P}^{N-1}_\infty \subset \mathbb{P}^N, \mathbb{P}^{d-1}_\infty \subset \mathbb{P}^d$ for the hyperplanes at infinity. Note that $\bar\pi^{-1}(\mathbb{P}^{d-1}_\infty) \subset \mathbb{P}^{N-1}_\infty$, $\bar\pi^{-1}(\mathbb{A}^d) \subset \mathbb{A}^N$ and $\bar\pi|_{\mathbb{A}^N} = \pi$. Let $\bar X \subset \mathbb{P}^N$ be the closure of X. If $\bar X \cap V = \emptyset$ then we obtain a proper map $\bar\pi|_{\bar X} : \bar X \to \mathbb{P}^d$. Note that $\bar X \cap \mathbb{A}^N = X$ whence $\bar\pi|_{\bar X}^{-1}(\mathbb{A}^d) = X$, and so $\pi|_X : X \to \mathbb{A}^d$ is both proper and affine, whence finite. It thus remains to prove that for general π, $\bar X \cap V = \emptyset$.

By assumption, $\dim X = d$ and hence $\dim X \cap \mathbb{P}^{N-1}_\infty = d-1$. This is a well-known fact, a proof of which we now sketch. Let G be the Grassmannian of $(N-1-d)$-dimensional linear subspaces of $\mathbb{P}^{N-1}_\infty$. Let $U \subset P$ be the open subset of full rank linear maps $\mathbb{A}^N \to \mathbb{A}^d$. Sending π to V defines a morphism of schemes $U \to G$. We shall prove that there is an open subset of G such that the corresponding V misses $\bar X \cap \mathbb{P}^{N-1}_\infty =: Y$. By assumption, $\dim X \le d$ and hence $\dim Y < d$. Let $T \subset Y \times G$ be the set of pairs (y, V) with $y \in V$. The fibers of $T \to Y$ are given by Grassmannians of subspaces spaces containing a specific point, which is the same is $(N-d-2)$-dimensional linear subspaces of $\mathbb{P}^{N-2}$. Using that $\dim G = d(N-d)$ we find that

$$\dim T \le \dim(\text{fibers}) + \dim Y = d(N-d-1) + d - 1 = d(N-d) - 1 < \dim G.$$

It follows that $T \to G$ is not surjective, and hence its image is a proper closed subset. This was to be shown.

Condition (2). Since X is smooth, étaleness is equivalent to proving that $\Omega^1_{\mathbb{A}^d/k}|_{S\cap X} \to \Omega^1_{X/k}|_{S\cap X}$ is an isomorphism. This is a linear map between finite dimensional vector spaces over fields of the same dimension, and so being an isomorphism is determined by the non-vanishing of a determinant. This is visibly an open condition. To prove non-emptiness, we may assume that k is algebraically closed and $S = \{x\} \subset X$. The map $\Omega^1_{\mathbb{A}^d/k}|_x \to \Omega^1_{X/k}|_x$ has image spanned by $d\pi_1, \ldots, d\pi_d$. The map $\Omega^1_{\mathbb{A}^N/k}|_x \to \Omega^1_{X/k}|_x$ is a surjection, and $d\pi_1, \ldots, d\pi_d$ define d arbitrary elements in the source. It remains to prove: given a surjection of finite-dimensional k-vector spaces $V \to W$ with $\dim W = d$, then the images of d general elements of V in W are linearly independent. We have already proved that this condition is open, so it suffices to prove that it is non-empty, which is clear (lift a basis of W).

Condition (3). The argument is similar to (1). We may assume that k is algebraically closed, $S = \{s\}$. $\pi^{-1}(\pi(s))$ is a general affine-linear subspace of $\mathbb{A}^N$ containing s of dimension $N-d$. Translating to the origin, these are in bijection with linear subspaces, and so parametrized by a Grassmannian G. Let $T \subset Z \times G$ be the subset of pairs (z, V) where $z \in V$ and $z \neq s$. As before, considering the fibers, we find that $\dim T < \dim G$. The closure of the image of T in G is thus a proper closed subset, and hence its open complement is nonempty, as needed. □

Proof of Proposition 8.5. If V is not pure, argue separately for each connected component. Hence we may assume that V is pure of dimension d. Pick an embedding $V \hookrightarrow \mathbb{A}^N$. Applying Theorem 9.1, we find a linear map $\pi : \mathbb{A}^N \to \mathbb{A}^d$ such that $\pi|_V : V \to \mathbb{A}^d$ is finite, $\pi|_V$ is étale at $S = \{z_1, \ldots, z_n\}$, and $\pi^{-1}(\pi(S)) \cap Z \subset S$. Let $V' \subset V$ be a open neighbourhood of S such that $\pi|_{V'}$ is étale. Note that in proving the theorem, we may replace V by an open neighborhood of S at will; hence redesignate V' as V, V as $\overline{V}$, $Z \cap V'$ as Z and Z as $\overline{Z}$. Applying Theorem 9.1 again, we find a linear map $\varpi : \mathbb{A}^d \to \mathbb{A}^{d-1}$ such that $\varpi|_{\pi(\overline{Z}) \cup \pi(\overline{V} \setminus V)}$ is finite, and $\varpi^{-1}(\varpi(\pi(S))) \cap \pi(\overline{Z} \setminus Z) \subset \pi(S)$. Now we form the following pullbacks

$$\begin{array}{ccccccc} V & \longrightarrow & \overline{V} & \xrightarrow{\pi} & \mathbb{A}^d & \xrightarrow{\varpi} & \mathbb{A}^{d-1} \\ \uparrow & & \uparrow{\scriptstyle v} & & \uparrow & & \uparrow{\scriptstyle \varpi\pi} \\ C & \longrightarrow & \overline{C}_0 & \longrightarrow & \mathbb{A}^1_U & \longrightarrow & U. \end{array}$$

The canonical map $U \to V$ induces a section $\Delta : U \to C$. By construction, $v^{-1}(\overline{Z} \setminus Z) \to U$ is finite, but also misses the closed points of U (since $\overline{Z} \setminus Z \to \mathbb{A}^{d-1}$ misses the image of S). This implies that $v^{-1}(\overline{Z} \setminus Z)$ is empty, i.e., $v^{-1}(Z) = v^{-1}(\overline{Z})$. Thus $v^{-1}(Z) \to U$ is finite. By Zariski's main theorem, the composite $\overline{C}_0 \to \mathbb{A}^1_U \to \mathbb{P}^1_U$ factors as $\overline{C}_0 \xrightarrow{g} \overline{C} \xrightarrow{f} \mathbb{P}^1_U$, where f is finite and g is a dense open immersion. Now $i : \overline{C}_0 \to f^{-1}(\mathbb{A}^1_U)$ is a dense open immersion, but also $\overline{C}_0 \to \mathbb{A}^1_U$ is finite, so i is also a closed immersion, whence an isomorphism. We claim that $\overline{C} \setminus C_0 \to U$ is finite. Being proper, we just need to check quasi-finiteness. By what we just said,

$$\overline{C} \setminus \overline{C}_0 = f^{-1}(\mathbb{P}^1_U \setminus \mathbb{A}^1_U) \to \mathbb{P}^1_U \setminus \mathbb{A}^1_U \simeq U$$

is finite, hence quasi-finite. On the other hand $\overline{C}_0 \setminus C \to U$ is finite by construction, so also quasi-finite; this proves the claim. It follows that this construction of $C \subset \overline{C}$ satisfies all the required properties (e.g. $C \to \mathbb{A}^1_U$ is étale, so C is smooth over U and $\Omega^1_{C/U}$ is trivial). □

10. Vista

The axioms EE and EA are proved by similar but more elaborate arguments. I will add them in a future revision.

This concludes the proof of strict $\mathbb{A}^1$-invariance of presheaves with framed transfers (over infinite perfect fields).

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