ALGEBRAIC K-THEORY FROM THE VIEWPOINT OF MOTIVIC HOMOTOPY THEORY

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Abstract. These are notes for an online mini course "at" Peking University that I gave in spring 2021. After an informal introduction to ∞-categories, we define algebraic K-theory of a ring and the motivic unstable and stable ∞-categories of a scheme. We construct the spectrum KGL and show that it represents algebraic K-theory. We define the slice filtration and hence construct the motivic spectrum $H\mathbb{Z}$ and the motivic filtration on algebraic K-theory. After stating without any indication of proofs the fundamental theorems of motivic cohomology, we illustrate the use of the slice filtration by computing the p-adic algebraic K-theory of fields containing enough roots of unity.

Contents

1. ∞-categories and algebraic K-theory

1.1. **The world of ∞-categories.** We attempt to describe the concept of an ∞-category, without giving an actual definition. While this seems unsatisfying, we feel forced to do this due to time constraints. For a much more thorough introduction, see [Lur09].

Let $\mathcal{C}$ be an ∞-category and $X, Y \in \mathcal{C}$ be objects. Then there is a *space of maps* $\mathrm{Map}_{\mathcal{C}}(X, Y)$.[1] Given a third object Z, there is a *homotopy class of composition maps* $\mathrm{Map}_{\mathcal{C}}(X, Y) \times \mathrm{Map}_{\mathcal{C}}(Y, Z) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$. The *homotopy category* $\mathrm{h}\mathcal{C}$ is a 1-category, by which we mean a category in the usual sense. Its objects are the same as those of $\mathcal{C}$, and $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)$. We also write $[X, Y]_{\mathcal{C}} := \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$. The composition in $\mathrm{h}\mathcal{C}$ is induced by the composition maps of $\mathcal{C}$; in particular composition in $\mathcal{C}$ is associative up to homotopy and so on.

Given an ∞-category $\mathcal{C}$ and objects $X, Y \in \mathcal{C}$, by a *morphism* $X \to Y$ we mean a point of $\mathrm{Map}_{\mathcal{C}}(X, Y)$. A morphisms is called an *equivalence* if it determines an isomorphism between X and Y in $\mathrm{h}\mathcal{C}$.

Here are some examples of ∞-categories:[2]

Example 1.1. If $\mathcal{C}$ is a 1-category, then we may view $\mathcal{C}$ as an ∞-category with $\mathrm{Map}_{\mathcal{C}}(X, Y)$ the discrete space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. The homotopy category $\mathrm{h}\mathcal{C}$ is just $\mathcal{C}$ viewed as a 1-category again.

Example 1.2. The ∞-category $\mathcal{S}\mathrm{pc}$ of spaces has as objects the CW complexes (or Kan complexes, if we wish) and as morphism spaces the function complexes (i.e. in the case of CW complexes the sets of continuous maps with the compact-open topology, perhaps replaced by a weakly equivalent CW complex, or in case of Kan complexes just mapping simplicial sets). Then $\mathrm{h}\mathcal{S}\mathrm{pc}$ is the *homotopy category of spaces*, i.e. CW complexes (respectively Kan complexes) with (simplicial) homotopy classes of maps as Hom sets.

Example 1.3. The ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories.

Date: March 15, 2021.

[1] By a space we may mean a CW complex or a Kan complex, depending on one's taste.

[2] Since we have not given an actual definition of ∞-category, we also cannot rigorously construct the examples.

Example 1.4. If $\mathcal{C}$ is an ∞-category and $\mathcal{C}_0$ is a collection of objects, we can form the *full subcategory* $\mathcal{C}_0$, with $\mathrm{Map}_{\mathcal{C}_0}(X,Y) = \mathrm{Map}_{\mathcal{C}}(X,Y)$.

1.1.1. *Functor categories.* Given ∞-categories $\mathcal{C},\mathcal{D}$, there exists an ∞-category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ of (∞-)functors from $\mathcal{C}$ to $\mathcal{D}$. Again we do not define what an ∞-functor means, but try to describe them by some of their properties. Firstly, if $\mathcal{C},\mathcal{D}$ are 1-categories viewed as ∞-categories, then $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ is itself a 1-category viewed as an ∞-category, namely the usual category of functors (with morphisms the natural transformations). Secondly, there is a map $\mathrm{hFun}(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathrm{h}\mathcal{C},\mathrm{h}\mathcal{D})$.[3] Thus an ∞-functor consists of a functor between the homotopy categories in the usual sense, but with some additional data.

Warning 1.5. The amount of "additional data" is, in general, infinite, and it is, in general, *very* hard to write down ∞-functors directly.

Nonetheless some properties of functor categories generalize from the case of 1-categories to ∞-categories:

Example 1.6. Given a morphism $\alpha : F \to G$ in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$, α is an equivalence if and only if the image of α in $\mathrm{Fun}(\mathrm{h}\mathcal{C},\mathrm{h}\mathcal{D})$ is. In other words, this happens if and only if for every object $X \in \mathcal{C}$, the morphism $\alpha_X : F(X) \to G(X)$ is an equivalence in $\mathcal{D}$.

Example 1.7. We have $\mathrm{Fun}(*,\mathcal{C}) \simeq \mathcal{C}$, where $*$ denotes the terminal 1-category, viewed as an ∞-category (which is in fact the terminal ∞-category).

1.1.2. *Limits and colimits.* The language of ∞-categories is essentially the same as that of 1-categories, just interpreted in a different class of objects.[4] Using the fragment of the language of ∞-categories introduced above, we can formulate most of the usual concepts from category theory.

Example 1.8 (initial objects). An object $X \in \mathcal{C}$ is called *initial* if for all $Y \in \mathcal{C}$ we have $\mathrm{Map}_{\mathcal{C}}(X,Y) \simeq *$. In particular, X is initial as an object of $\mathrm{h}\mathcal{C}$.

For the next example, we need the following preparation. Given ∞-categories $\mathcal{C},\mathcal{D}$, there is a functor $\mathcal{D} \to \mathrm{Fun}(\mathcal{C},\mathcal{D})$ which sends an object $d \in \mathcal{D}$ to the functor $d^* : \mathcal{C} \to * \xrightarrow{d} \mathcal{D}$.[5]

Example 1.9 (colimits). Given a functor $F : \mathcal{C} \to \mathcal{D}$, by a colimit of F we mean an object $d \in \mathcal{D}$ together with a morphism $\alpha : F \to d^*$ in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ such that for every object $X \in \mathcal{D}$, the composite

$$\mathrm{Map}_{\mathcal{D}}(d,X) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(d^*,X^*) \xrightarrow{\alpha\circ} \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F,X^*)$$

is an equivalence of spaces.

Exercise 1.1. *Show that a colimit of F, if it exists, is determined up to an equivalence which is unique up to homotopy.*[6]

Warning 1.10. In general, a colimit in $\mathcal{C}$ is *very different* from a colimit in $\mathrm{h}\mathcal{C}$.

Remark 1.11. If $\mathcal{C},\mathcal{D}$ are 1-categories, then the above notion of colimit recovers the usual one.

Example 1.12 (limits). Limits can be defined completely analogously, reversing some appropriate arrows.

Example 1.13 (products). If $\mathcal{C}$ is the 1-category with two objects and no non-identity morphisms (viewed as an ∞-category), then limits over $\mathcal{C}$ are called *(binary) products*. One may show that $\mathrm{Fun}(\mathcal{C},\mathcal{D}) \simeq \mathcal{D}\times\mathcal{D}$ (via evaluation at the two objects). Thus a product of two objects $X,Y \in \mathcal{D}$ consists explicitly of an object $P \in \mathcal{D}$ together with morphisms $P \to X, P \to Y$ such that for any $Z \in \mathcal{D}$ we have $\mathrm{Map}_{\mathcal{D}}(Z,P) \simeq \mathrm{Map}_{\mathcal{D}}(Z,X) \times \mathrm{Map}_{\mathcal{D}}(Z,Y)$ (via the canonical map).

The following example is instructive to illuminate Warning 1.10.

Example 1.14 (pushouts). Let $\mathcal{C}$ be the 1-category $\bullet \leftarrow \bullet \to \bullet$. Colimits over $\mathcal{C}$ are called *pushouts*. Unravelling the definitions, one finds the following: given objects $X,Y,Z \in \mathcal{D}$ and morphisms $Y \xleftarrow{\alpha} X \xrightarrow{\beta} Z$, in order to specify a pushout of this span we have to produce an object $P \in \mathcal{D}$, morphisms $f : Y \to P, g :$

[3] Which is usually not an equivalence unless $\mathcal{D}$ is a 1-category.

[4] See [Cis19] for (many) more details about this idea.

[5] To rigorously construct this, we may use that $\mathrm{h}\mathcal{C}\mathrm{at}_\infty$ is cartesian closed, with Fun right adjoint to the cartesian product.

[6] In fact much more is true: the space of possible colimits is contractible.

$Z \to P$, as well as a homotopy $f\alpha \overset{H}{\simeq} g\beta \in \mathrm{Map}_{\mathcal{C}}(X,P)$. In other words, H witnesses commutativity of the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & P. \end{array}$$

This *data* has to satisfy the following *condition*: given another object P', the canonical map

$$\mathrm{Map}_{\mathcal{C}}(P',P) \to \mathrm{Map}_{\mathcal{C}}(Y,P') \times_{\mathrm{Map}_{\mathcal{C}}(X,P')} \mathrm{Map}_{\mathcal{C}}(Z,P')$$

is an equivalence, where the right hand side means the homotopy pullback of spaces. This homotopy pullback can be made explicit as follows: it is the space of triples $\{f',g',H'\}$ of maps $f' : Y \to P', g' : Z \to P'$ and homotopies $f'\alpha \overset{H'}{\simeq} g'\beta$.

1.2. **Commutative monoids.** Let $\mathcal{C}$ be an ∞-category with finite products.

Definition 1.15. We denote by $\mathrm{CMon}(\mathcal{C}) \subset \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C})$ the full subcategory on those functors $M : \mathrm{Fin}_* \to \mathcal{C}$ such that for all $n \geq 0$, the map $M(\underline{n}) \to M(\underline{1})^n$, induced by the n pointed maps

$$\underline{n} = \{*, 1, \dots, n\} \to \underline{1}$$

with fiber over $*$ of size n, is an equivalence. We call $M(\underline{1}) \in \mathcal{C}$ the *underlying object*, which we often also denote by just M.

Example 1.16. We get $M(\underline{0}) \simeq M(\underline{1})^0 \simeq *$, the terminal object of $\mathcal{C}$.

Example 1.17. Given $M \in \mathrm{CMon}(\mathcal{C})$, we get a canonical map

$$M(\underline{1}) \times M(\underline{1}) \simeq M(\underline{2}) \to M(\underline{1})$$

called addition, induced by the map $\{*,1,2\} \to \{*,1\}$ sending $1,2$ to 1 and $*$ to $*$.

Exercise 1.2. *If $\mathcal{C}$ is a 1-category, then $\mathrm{CMon}(\mathcal{C})$ is a 1-category, equivalent to the usual category of commutative monoids in $\mathcal{C}$.*

Let $M \in \mathrm{CMon}(\mathcal{S}\mathrm{pc})$. Composing with the functor $\pi_0 : \mathcal{S}\mathrm{pc} \to \mathit{Set}$ we obtain a commutative monoid $\pi_0 M$ in the usual sense.

Definition 1.18. $M \in \mathrm{CMon}(\mathcal{S}\mathrm{pc})$ is called *grouplike* if $\pi_0 M$ is a grouplike commutative monoid in the usual sense. Denote by $\mathrm{CMon}(\mathcal{S}\mathrm{pc})^{\mathrm{gp}} \subset \mathrm{CMon}(\mathcal{S}\mathrm{pc})$ the full subcategory on grouplike monoids. By a *group-completion* of M we mean $M^{\mathrm{gp}} \in \mathrm{CMon}(\mathcal{S}\mathrm{pc})^{\mathrm{gp}}$ together with a map $X \to X^{\mathrm{gp}}$ such that for any $Z \in \mathrm{CMon}(\mathcal{S}\mathrm{pc})^{\mathrm{gp}}$ we have $\mathrm{Map}_{\mathrm{CMon}(\mathcal{S}\mathrm{pc})}(M,Z) \simeq \mathrm{Map}_{\mathrm{CMon}(\mathcal{S}\mathrm{pc})}(M^{\mathrm{gp}},Z)$.

As usual, M^{gp} is unique in an appropriate sense.

Exercise 1.3. *Show that $\pi_0(M^{\mathrm{gp}}) \simeq \pi_0(M)^{\mathrm{gp}}$. You may use the following fact: if $\mathcal{C}' \subset \mathcal{C}$ is a full subcategory, then $\mathrm{Fun}(\mathcal{D},\mathcal{C}') \to \mathrm{Fun}(\mathcal{D},\mathcal{C})$ is fully faithful (i.e. induces equivalences on mapping spaces).*

Theorem 1.19 (McDuff–Segal)**.** *Let $M \in \mathrm{CMon}(\mathcal{S}\mathrm{pc})$. Then M^{gp} exists and*

$$H_*(M^{\mathrm{gp}}) \simeq H_*(M)[\pi_0(M)^{-1}],$$

where by the right hand side we mean the localization with respect to the Pontryagin ring structure.

Warning 1.20. We usually do *not* have $\pi_*(M^{\mathrm{gp}}) \simeq \pi_*(M)[\pi_0(M)^{-1}]$.

To use the group completion theorem from above, the following fact is very helpful.

Proposition 1.21. *If $\alpha : X \to Y$ is a morphism of commutative, grouplike H-spaces (i.e. a morphism in $\mathrm{CMon}(\mathrm{h}\mathcal{S}\mathrm{pc})^{\mathrm{gp}}$) then α is an equivalence if and only if $H_*(\alpha)$ is an isomorphism.*

1.3. **Algebraic K-theory of rings.** Let $\mathcal{C}$ be a symmetric monoidal 1-category. For $X \in \mathrm{Fin}_*$ put $X' = X \setminus \{*\}$. Given $\alpha : X \to Y \in \mathrm{Fin}_*$, we would like to define

$$\alpha_\otimes : \mathcal{C}^{X'} \to \mathcal{C}^{Y'}$$

via

$$\alpha_\otimes(\{c_x\}_{x \in X'}) = \{\otimes_{x \in f^{-1}(y)} c_x\}_{y \in Y'}.$$

To make sense of this, we have to choose a way of ordering and associating the tensor product. The axioms of a symmetric monoidal category ensure that the resulting object is well-defined up to canonical isomorphism, but unfortunately it is not in general well-defined "on the nose". Thus in order to define $\alpha_\otimes$ we would have to make arbitrary choices, which would mean that $\alpha_\otimes \beta_\otimes \neq (\alpha\beta)_\otimes$.[7] We can avoid this issue as follows.

Construction 1.22. We build a functor $\mathcal{C}^\otimes : \mathrm{Fin}_* \to \mathcal{C}\mathrm{at}_1$ as follows. An object of $\mathcal{C}^\otimes(X)$ consists of an object $\{c_x\} \in \mathcal{C}^{X'}$ together with the following extra data: for every $f : X \to Y \in \mathrm{Fin}_*$ a choice $f^*_\otimes(\{c_x\}) \in \mathcal{C}^{Y'}$. The morphisms are just the morphisms in $\mathcal{C}^{X'}$. Given $\alpha : X \to Y \in \mathrm{Fin}_*$, define $\alpha_\otimes : \mathcal{C}^\otimes(X) \to \mathcal{C}^\otimes(Y)$ by sending an object as above to $\alpha^*_\otimes(\{c_x\})$ with the evident extra data.

One may check that the axioms of a symmetric monoidal category imply that this is a well-defined functor, and in fact an element of $\mathrm{CMon}(\mathcal{C}\mathrm{at}_1)$. In particular $\alpha_\otimes \beta_\otimes = (\alpha\beta)_\otimes$ now holds "on the nose".

The "classifying space" or "nerve" construction yields an ∞-functor

$$B : Gpd_1 \to \mathcal{S}\mathrm{pc},$$

where $Gpd_1 \subset \mathcal{C}\mathrm{at}_1$ denotes the full subcategory of groupoids (categories in which every morphism is invertible). Given a symmetric monoidal category $\mathcal{C}$, we have $\mathcal{C}^\otimes \in \mathrm{CMon}(\mathcal{C}\mathrm{at}_1) \subset \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C}\mathrm{at}_1)$. We can compose with the functor

$$\mathcal{C}\mathrm{at}_1 \to Gpd_1, \mathcal{D} \mapsto \mathcal{D}^\simeq$$

discarding the non-invertible morphisms and then with the nerve B to obtain a functor

$$B\mathcal{C}^{\otimes,\simeq} : \mathrm{Fin}_* \to \mathcal{S}\mathrm{pc}.$$

This is still a commutative monoid.

Definition 1.23. Let R be a (possibly non-commutative) ring. We denote by $Proj(R)$ the category of finitely generated projective (left) R-modules, viewed as a symmetric monoidal category for the direct sum operation $\oplus$. The *K-theory space of R* is

$$K(R) = (BProj(R)^{\otimes,\simeq})^{\mathrm{gp}} \in \mathrm{CMon}(\mathcal{S}\mathrm{pc}).$$

The *algebraic K-theory groups of R* are

$$K_i(R) = \pi_i K(R).$$

1.4. **Further exercises.**

Exercise 1.4. *Let k be a field (or more generally local ring). Show that $K_0(k) = \mathbb{Z}$.*

Exercise 1.5. *Let $M \in \mathrm{CMon}(\mathcal{S}\mathrm{pc})$ and $m \in M$. Construct a map*

$$tel(m) := \mathrm{colim}(M \xrightarrow{+m} M \xrightarrow{+m} \dots) \to M^{\mathrm{gp}}.$$

Exercise 1.6. *There exists a commutative monoid*

$$B = \coprod_{n \geq 0} BU_n \in \mathrm{CMon}(\mathcal{S}\mathrm{pc}),$$

where BU_n denotes the classifying space of the topological *group U_n of $n \times n$ unitary matrices. Pick $m \in \mathrm{BU}_1$.*

(a) Show that $tel(m) \simeq \mathbb{Z} \times BU$, where $U = \mathrm{colim}_n U_n$ is the infinite unitary group.
(b) Show that $\mathbb{Z} \times BU$ is a commutative H-space.
(c) Deduce that $B^{\mathrm{gp}} \simeq \mathbb{Z} \times BU$.[8]

[7] This holds up to canonical isomorphism again.

[8] The space B^{gp} is known as the *complex K-theory space*. It should not be confused with $K(\mathbb{C})$.

2. Unstable motivic homotopy theory

2.1. Presheaves and adjoint functors.

Warning 2.1 (size issues). A plethora of problems arise out of the well-known conundrum that there is no "set of all sets". These problems are "technical", in the sense that they are not usually relevant unless we are asking unreasonable questions. For this reason we shall mostly *ignore* them in these notes. Nonetheless: look twice before crossing the street, and ensure all your mathematics is determined by a small amount of data!

2.1.1. *Presheaves.*

Theorem 2.2. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be ∞-categories. Assume that $\mathcal{E}$ has colimits (respectively limits) of shape $\mathcal{D}$. Then so does $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$, and for any $X \in \mathcal{C}$, the functor $ev_X : \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \to \mathcal{E}$ preserves them.*

Definition 2.3. If $\mathcal{C}$ is an ∞-category, we denote by $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{pc})$ the category of (space-valued) *presheaves on $\mathcal{C}$*.[9]

One may show that $\mathcal{S}\mathrm{pc}$ has all limits and colimits, so by Theorem 2.2 the same is true for presheaf categories.

Remark 2.4. It is possible to show that the Yoneda lemma holds in the usual way: there is an embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ sending $c \in \mathcal{C}$ to the representable presheaf $R_c = \mathrm{Map}_{\mathcal{C}}(-, c)$, and for $F \in \mathcal{P}(\mathcal{C})$ we have $\mathrm{Map}_{\mathcal{P}(\mathcal{C})}(R_c, F) \simeq \mathcal{F}(c)$.

2.1.2. *Adjoint functors.* Let $R : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories. Consider the ∞-category

$$\mathcal{E} = \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{D},\mathcal{D})} \mathrm{Fun}([1], \mathrm{Fun}(\mathcal{D}, \mathcal{D})) \times_{\mathrm{Fun}(\mathcal{D},\mathcal{D})} \{\mathrm{id}\},$$

where the limit is formed in $\mathcal{C}\mathrm{at}_\infty$, the map $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ is composition with R and the two maps $\mathrm{Fun}([1], \mathrm{Fun}(\mathcal{D}, \mathcal{D})) \to \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ are evaluation at 0 and 1, respectively. In other words this is the category of pairs $(L : \mathcal{C} \to \mathcal{D},\ \alpha : LR \to \mathrm{id}_{\mathcal{D}})$. Denote by $\mathcal{E}_0 \subset \mathcal{E}$ the full subcategory on those pairs (L, α) such that for every object $X \in \mathcal{C}$, $Y \in \mathcal{D}$ the composite

$$\mathrm{Map}_{\mathcal{C}}(X, RY) \xrightarrow{L} \mathrm{Map}_{\mathcal{C}}(LX, LRY) \xrightarrow{\alpha_Y \circ} \mathrm{Map}(LX, Y)$$

is an equivalence.

Definition 2.5 (adjoints)**.** By a *left adjoint of* R we mean an object of $\mathcal{E}_0$.

This seems highly non-unique, but it is not:

Proposition 2.6. *The ∞-category $\mathcal{E}_0$ is either empty or equivalent to the terminal ∞-category $*$. In particular, any two left adjoints are equivalent, in an essentially unique way.*

There is a similar definition of right adjoints.

2.1.3. *Aside: presentability and the adjoint functor theorem.* It may be a good idea to skip this section on first reading. In order to construct ∞-functors, a very helpful tool is an existence criterion for adjoints known as the *adjoint functor theorem.* We sketch the idea here.

Certain particularly nice ∞-categories are called *presentable.* We will not define this notion, but collect the following properties.

Theorem 2.7.
1. *Presentable categories have limits and colimits for all diagrams.*
2. *Let $\mathcal{D}$ be a diagram of ∞-categories. Suppose that all the categories are presentable, and either all the functors preserve colimits or all the functors preserve limits. Then the category $\lim \mathcal{D}$ is presentable.*
3. *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{P}(\mathcal{C})$ is presentable.*

Example 2.8. Let $\mathcal{C}$ be a presentable ∞-category and $\mathcal{C}_i \subset \mathcal{C}$ be a family of full presentable subcategories. Then $\bigcap_{i \in I} \mathcal{C}_i \subset \mathcal{C}$ is a full presentable subcategory.

[9] We have not talked about it so far, but the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories has an involution $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ "reversing the arrows".

Example 2.9. Let $\mathcal{E} \subset \mathrm{Fun}([1], \mathcal{S}\mathrm{pc}) =: \mathcal{D}$ denote the full subcategory on those functors corresponding to equivalences in $\mathcal{S}\mathrm{pc}$. One may show that $\mathcal{E} \simeq \mathcal{S}\mathrm{pc}$, and so is presentable. Since limits in categories of presheaves are computed sectionwise (Theorem 2.2), $\mathcal{E} \to \mathcal{D}$ is a functor of presentable categories preserving limits. Now let $\mathcal{C}$ be a presentable ∞-category and $f : X \to Y$ a morphism in $\mathcal{C}$. Evaluation at f defines a functor $\mathcal{C} \to \mathrm{Fun}([1], \mathcal{S}\mathrm{pc})$ which preserves limits. Consequently in the pullback diagram of ∞-categories

$$\begin{tikzcd} \mathcal{C}_f \arrow[r, "\alpha"'] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathcal{E} \arrow[r] & \mathrm{Fun}([1], \mathcal{S}\mathrm{pc}), \end{tikzcd}$$

$\mathcal{C}_f$ is presentable. Unravelling the definitions, $\mathcal{C}_f \subset \mathcal{C}$ is the full subcategory on those objects T such that $f^* : \mathrm{Map}_{\mathcal{C}}(Y, T) \to \mathrm{Map}_{\mathcal{C}}(X, T)$ is an equivalence.

For once, we need to talk about size. If κ is a regular cardinal, then there is a class of "κ-small" ∞-categories, roughly characterized by saying that their set of objects is κ-small, and all homotopy groups of all mapping spaces also are.

Definition 2.10. Let κ be a regular cardinal and $\mathcal{C}$ an ∞-category. We call $\mathcal{C}$ *κ-filtered* if for every κ-small ∞-category $\mathcal{D}$ and every functor $F : \mathcal{D} \to \mathcal{C}$, there exists an object $c \in \mathcal{C}$ and a morphism $F \to c^* \in \mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

Theorem 2.11 (adjoint functor theorem). *Let $R : \mathcal{D} \to \mathcal{C}$ be a functor of presentable categories preserving limits and κ-filtered colimits for some κ. Then R admits a left adjoint L.*

Example 2.12. The category $\mathrm{CMon}(\mathcal{S}\mathrm{pc})$ is presentable, being an intersection of subcategories of a presheaf category determined by certain maps being an equivalence (see Examples 2.8 and 2.9). Its subcategory $\mathrm{CMon}(\mathcal{S}\mathrm{pc})^{\mathrm{gp}}$ is also presentable: being grouplike is the same as the shearing map

$$M \times M \xrightarrow{(m_1, m_2) \mapsto (m_1, m_1 + m_2)} M \times M$$

being an equivalence. This also implies that $\mathrm{CMon}(\mathcal{S}\mathrm{pc})^{\mathrm{gp}} \to \mathrm{CMon}(\mathcal{S}\mathrm{pc})$ preserves limits. One may also show using Theorem 2.13 below that the inclusion preserves κ-filtered colimits for any κ. We deduce that that group-completion is a functor $(-)^{\mathrm{gp}} : \mathrm{CMon}(\mathcal{S}\mathrm{pc}) \to \mathrm{CMon}(\mathcal{S}\mathrm{pc})^{\mathrm{gp}}$.

The following fact is very useful.

Theorem 2.13. *In the category $\mathcal{S}\mathrm{pc}$, finite limits commute with κ-filtered colimits for any κ.*

Warning 2.14. An ∞-category being finite is a very strong condition; it in particular requires that all mapping spaces are finite complexes. Binary products are finite limits, equalizers are not!

2.2. **Motivic spaces.** Let S be a scheme. We write Sm_S for the category of smooth schemes of finite type over S.

Definition 2.15. We call $F \in \mathcal{P}(\mathrm{Sm}_S)$ *$\mathbb{A}^1$-invariant* if for every $X \in \mathrm{Sm}_S$ the canonical map $F(X) \to F(X \times \mathbb{A}^1)$ is an equivalence. We denote the subcategory of $\mathbb{A}^1$-invariant presheaves by $L_{\mathbb{A}^1}\mathcal{P}(\mathrm{Sm}_S)$.

It follows from the discussion in the previous section that the inclusion $L_{\mathbb{A}^1}\mathcal{P}(\mathrm{Sm}_S) \subset \mathcal{P}(\mathrm{Sm}_S)$ admits a left adjoint, which we denote by $L_{\mathbb{A}^1}$. The subcategory of $\mathbb{A}^1$-invariant presheaves is closed under all limits and colimits, those being computed sectionwise.

Definition 2.16. Let X be a scheme. A *Nisnevich square* is a cartesian diagram of schemes

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"'] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

where i is an open immersion, p is étale, and $V \times_X (U \setminus X) \simeq V \setminus W$.

Definition 2.17. We call $F \in \mathcal{P}(\mathrm{Sm}_S)$ Nisnevich local if for all $X, Y \in \mathrm{Sm}_S$ we have $F(X \amalg Y) \simeq F(X) \times F(Y)$, $F(\emptyset) \simeq *$ and for every Nisnevich square as above, the following square of spaces is cartesian

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ F(V) \arrow[r] & F(W). \end{tikzcd}$$

We write $L_{\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_S) \subset \mathcal{P}(\mathrm{Sm}_S)$ for the full subcategory of Nisnevich local presheaves.

As before (this time using Theorem 2.13, i.e. the commutativity of filtered colimits and pullbacks in spaces), the inclusion admits a left adjoint L_{Nis}.

Definition 2.18. We call $\mathcal{S}\mathrm{pc}(S) := L_{\mathrm{mot}}\mathcal{P}(\mathrm{Sm}_S) := L_{\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_S) \cap L_{\mathbb{A}^1}\mathcal{P}(\mathrm{Sm}_S)$ the category of *motivic spaces.*

Warning 2.19. If F is $\mathbb{A}^1$-local, $L_{\mathrm{Nis}}F$ need not be, and similarly if F is Nisnevich local $L_{\mathbb{A}^1}F$ need not be. Thus $L_{\mathrm{mot}} \neq L_{\mathrm{Nis}}L_{\mathbb{A}^1}$ and $L_{\mathrm{mot}} \neq L_{\mathbb{A}^1}L_{\mathrm{Nis}}$ (in general). However, it follows from general principles that L_{mot} can be reached by repeating L_{Nis} and $L_{\mathbb{A}^1}$ alternatingly sufficiently (infinitely) many times.

2.2.1. *$\mathbb{A}^1$-localization.* A map $f : F \to G \in \mathcal{P}(\mathrm{Sm}_S)$ is called an $\mathbb{A}^1$-equivalence if $L_{\mathbb{A}^1}f$ is an equivalence. Essentially by construction, for every $X \in \mathrm{Sm}_S$ the projection $X \times \mathbb{A}^1 \to X$ is an $\mathbb{A}^1$-equivalence.

Two maps $f_0, f_1 : F \to G$ are called $\mathbb{A}^1$-homotopic if there exists $H : F \times \mathbb{A}^1 \to G$ such that $f_i \simeq H \circ i_s$, where $i_s : * \to \mathbb{A}^1$ is the inclusion at the point s.

Exercise 2.1. *Show that if $F \in \mathcal{P}(\mathrm{Sm}_S)$, then $F \times \mathbb{A}^1 \to F$ is an equivalence. Deduce that $\mathbb{A}^1$-homotopy equivalences (i.e. maps admitting an inverse up to $\mathbb{A}^1$-homotopy) are $\mathbb{A}^1$-equivalences. (You may use that any object in a presheaf category is a colimit of representables.)*

The $\mathbb{A}^1$-localization can be made very explicit. Write Δ for the simplex category, i.e. the category of finite non-empty totally ordered sets. A skeleton is given by the objects $[n] = \{0, 1, 2, \ldots, n\}$.

Definition 2.20. The *standard cosimplicial scheme* is

$$\mathbb{\Delta}^\bullet : \Delta \to \mathrm{Sch}_S, [n] \mapsto \mathbb{\Delta}^n_S \subset \mathbb{A}^{n+1}_S,$$

where $\mathbb{\Delta}^n$ is determined by the equation $T_0 + \cdots + T_n = 1$, and the structure maps are induced by partial projections and face inclusions. (For details, see e.g. [Wei13, Definition 11.3].)

Theorem 2.21. *Let $F \in \mathcal{P}(\mathrm{Sm}_S)$ and $X \in \mathrm{Sm}_S$. Then*

$$(L_{\mathbb{A}^1}F)(X) \simeq \operatorname*{colim}_{[n] \in \Delta^{\mathrm{op}}} F(X \times \mathbb{\Delta}^n).$$

2.2.2. *Nisnevich localization.* $L_{\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_S)$ is a category of sheaves. We can explain this as follows.

Definition 2.22. Let X be a scheme and $x \in X$ a point. We denote by X^h_x the (cofiltered) limit of all pointed schemes (Y, y) with an étale map $Y \to X$ sending y to x and inducing $k(y) \simeq k(x)$. (See e.g. [Sta18, Tag 04GV] for details.) We call X^h_x the *henselization* of X in x.

Theorem 2.23. *Suppose that $\dim S < \infty$ and $f : F \to G \in \mathcal{P}(\mathrm{Sm}_S)$. Then $L_{\mathrm{Nis}}f$ is an equivalence if and only if for every $X \in \mathrm{Sm}_S$ and $x \in X$ the induced map*

$$f(X^h_x) : \operatorname*{colim}_{(Y,y) \to (X,x)} F(Y) \to \operatorname*{colim}_{(Y,y) \to (X,x)} G(Y) \in \mathcal{S}\mathrm{pc}$$

is an equivalence.

Remark 2.24. We do not have time to discuss this, but the assumption that $\dim S < \infty$ is important. In the world of ∞-categories, there is more than one notion of sheaf, and only one of them (the so-called hypersheaves) interact with stalks in the way one is used to from the world of 1-categories.

2.3. K-theory.

Theorem 2.25. *Let S be a regular noetherian scheme of finite dimension. Then there exists $K \in \mathcal{S}\mathrm{pc}(S)$ such that for every affine scheme $X = Spec(A) \in \mathrm{Sm}_S$ we have $K(X) \simeq K(A)$, the algebraic K-theory space of the ring A.*

Remark 2.26. The space $K(X)$ for $X \in \mathrm{Sm}_S$ not affine also has a name: it is called the *Thomason–Trobaugh* K-theory of X.

Remark 2.27. The Theorem says in particular that the presheaf $Spec(A) \mapsto K(A)$ is $\mathbb{A}^1$-invariant (i.e. $K(A) \simeq K(A[t])$) and satisfies Nisnevich descent. Neither of these properties is obvious from the definition of K-theory that we gave. This also shows that the assumption that S be regular is important: otherwise K-theory is not $\mathbb{A}^1$-invariant (in general)!

Proof. Using Example 2.12, we construct a functor $F : \mathrm{Sm}_S^{\mathrm{op}} \to \mathcal{S}\mathrm{pc}$ with $F(X) = K(\mathcal{O}_X(X))$. It will suffice to show that the canonical map $F \to L_{\mathrm{mot}}F =: K$ induces an equivalence on sections over affine schemes. Some formal manipulations show that this is equivalent to showing that F is $\mathbb{A}^1$-local and Nisnevich local "on affine schemes", which translates into properties of K-theory. Namely we need to know that $K(A[t]) \simeq K(A)$, and a similar condition for Nisnevich squares. These are well-known (but highly non-trivial) properties of the K-theory of regular rings [Wei13, Theorem V.6.3, Examples V.10.10]. □

We have presheaves of groups GL_n on Sm_S, sending X to $\mathrm{GL}_n(\mathcal{O}_X(X))$. Taking the colimit, we obtain the presheaf of groups GL. Taking classifying spaces sectionwise, one obtains $B\mathrm{GL} \in \mathcal{P}(\mathrm{Sm}_S)$

Theorem 2.28. *Assumptions as in the previous theorem. We have $K \simeq L_{\mathrm{mot}}(\mathbb{Z} \times B\mathrm{GL}) \in \mathcal{S}\mathrm{pc}(S)$.*

Proof. Consider the presheaf of commutative monoids

$$B : X \mapsto \coprod_{n \geq 0} B\mathrm{GL}_n(\mathcal{O}_X(X)) \in \mathrm{Fun}(\mathrm{Sm}_S^{\mathrm{op}}, \mathrm{CMon}(\mathcal{S}\mathrm{pc})).$$

Using that $B(X) \simeq Proj(X)^{\simeq}$ if X is local, we see that $K \simeq L_{\mathrm{mot}}B^{\mathrm{gp}}$. Let $m \in B\mathrm{GL}_1$ correspond to the trivial line bundle. As in Exercise 1.5 we can construct a map $tel_m(B) \to K$, where

$$tel_m(B) : X \mapsto tel_m(B(X))$$

is formed sectionwise. By direct computation we have $tel_m(B) \simeq \mathbb{Z} \times B\mathrm{GL}$. We have thus constructed a canonical map $\mathbb{Z} \times B\mathrm{GL} \to K$ which we need to prove is a motivic equivalence. We shall prove that it becomes an equivalence upon applying $L_{\mathrm{Nis}}L_{\mathbb{A}^1}$. By Theorems 2.23 and 2.21, it is enough to prove the following: if R is a (henselian) local ring, then

$$\operatorname*{colim}_{n \in \Delta^{\mathrm{op}}} [\mathbb{Z} \times B\mathrm{GL}(R[t_0, \ldots, t_n])] \simeq K(R).$$

One may show (see Exercise 2.4) that the left hand side is a grouplike H-space (and of course so is the right hand side), and hence by Proposition 1.21 it suffices to prove that the map induces an isomorphism in homology. One may also show that if $X_\bullet$ is a Δ^{op}-indexed diagram of spaces such that the induced maps in homology $H_*(X_n) \to H_*(X_m)$ are all isomorphisms, then $H_* \operatorname{colim}_{\Delta^{\mathrm{op}}} X_\bullet \simeq H_*X_0$. The group completion theorem 1.19 implies that $H_*(\mathbb{Z} \times B\mathrm{GL}(R)) \simeq H_*K(R)$ for any ring R such that $K_0(R) = \mathbb{Z}$ (see Exercise 2.2). We thus conclude by $\mathbb{A}^1$-invariance of $K(-)$ and Exercise 1.4. □

Theorem 2.29. *Denote by $Gr_n(\mathbb{A}^m)$ the Grassmannian scheme of n-planes in $\mathbb{A}^m$, and $Gr_n := \operatorname{colim}_m Gr_n(\mathbb{A}^m)$. We have $L_{\mathrm{mot}}B\mathrm{GL}_n \simeq L_{\mathrm{mot}}Gr_n$, and so*

$$K \simeq L_{\mathrm{mot}}(\mathbb{Z} \times Gr_\infty).$$

Proof. See Exercise 2.5. □

2.4. Exercises.

Exercise 2.2. *Let R be a ring. Show that $K(R) \simeq (\amalg_{n \geq 0} B\mathrm{GL}_n(R))^{\mathrm{gp}}$ if and only if $K_0(R) \simeq \mathbb{Z}$.*

Exercise 2.3. *Show that the commutative diagram of schemes*

$$\begin{array}{ccc} \mathbb{A}^1_S \setminus 0 & \longrightarrow & S \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathbb{P}^1_S \end{array}$$

becomes a pushout in $\mathcal{S}\mathrm{pc}(S)$.

Exercise 2.4. *Construct a natural isomorphism* $R^\infty \oplus R^\infty \simeq R^\infty$. *Deduce the existence of a map* $a : B\mathrm{GL} \times B\mathrm{GL} \to B\mathrm{GL} \in \mathcal{P}(\mathrm{Sm}_S)$. *Show that a is associative, unital and commutative up to (non-direct)* $\mathbb{A}^1$*-homotopy and deduce that* $L_{\mathrm{mot}} B\mathrm{GL}$ *is a presheaf of grouplike commutative H-spaces.*

Exercise 2.5. *Show that* $B\mathrm{GL}_n$ *is motivically equivalent to* Gr_n*, as follows.*

(1) Show that

$$Gr_n(\mathbb{A}^m) \simeq L_{\mathrm{Zar}}[GL_{n+m}/GL_n \times GL_m],$$

where the right hand side is the quotient of presheaves of groups.

(2) Show that $\mathrm{colim}_n\, GL_{n+m}/GL_n \simeq *$.

(3) Deduce the desired result. (You may use that "the colimit of a free action is a homotopy colimit".)

3. Stable motivic homotopy theory

3.1. Pointed categories.

Definition 3.1. Let $\mathcal{C}$ be an ∞-category. We say $X \in \mathcal{C}$ is a 0-object if X is both initial and final. We say that $\mathcal{C}$ is pointed if it admits a 0-object.

We denote by Pr the ∞-category of presentable ∞-categories (see §2.1.3) and colimit preserving functors.

Proposition 3.2. *Let $\mathcal{C}$ be presentable ∞-category with a final object $*$. Then*

$$\mathcal{C}_* := \{*\} \times_{\mathcal{C}} \mathrm{Fun}([1], \mathcal{C})$$

is the initial pointed presentable category under $\mathcal{C}$ in Pr*, where the functor* $\mathcal{C} \to \mathcal{C}_*$ *is given by* $c \mapsto (c \to c \coprod *)$.[10]

In other words, an object of $\mathcal{C}_*$ consists of an object $c \in \mathcal{C}$ and a morphism $* \to \mathcal{C}$.

Definition 3.3. We denote by $\mathcal{S}\mathrm{pc}(S)_*$ the category of *pointed motivic spaces.*

Exercise 3.1. *Show that for $X \in \mathcal{C}_*$ there is a pushout square in $\mathcal{C}_*$*

$$\begin{array}{ccc} *_+ & \longrightarrow & X_+ \\ \downarrow & & \downarrow \\ * & \longrightarrow & X. \end{array}$$

Remark 3.4. Pushout squares of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ * & \longrightarrow & C \end{array}$$

are also called *cofiber sequences* and denoted $A \to B \to C \simeq B/A$.

[10] More formally, the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ admits a left adjoint with the claimed action on objects.

3.2. Symmetric monoidal categories.

Definition 3.5. A symmetric monoidal ∞-category is an object of $\mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty)$. A presentably symmetric monoidal ∞-category is a symmetric monoidal ∞-category $\mathcal{C}$ such that $\mathcal{C}$ is presentable and the tensor product preserves colimits in each variable separately. We write $\mathrm{CAlg}(\mathrm{Pr})$ for the category of presentably symmetric monoidal categories and symmetric monoidal, cocontinuous functors.

We denote the unit object of a symmetric monoidal category generically by $\mathbb{1}$.

Since symmetric monoidal ∞-categories are examples of commutative monoids, one expects them to be difficult to write down directly. Here is one way.

Theorem 3.6. *Let $\mathcal{C}$ be an ∞-category with finite products. Then there exists a canonical symmetric monoidal category $\mathcal{C} = \mathcal{C}^\times \in \mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty)$ with underlying object $\mathcal{C}$ and tensor product given by the cartesian product.*

In particular, $\mathcal{S}\mathrm{pc}(S)$ is a symmetric monoidal ∞-category under cartesian product. It is easy to see that $\mathcal{S}\mathrm{pc}(S)$ is cartesian closed (i.e. the cartesian product has a right adjoint), and hence presentably symmetric monoidal.

Here is another way of constructing symmetric monoidal structures.

Theorem 3.7. *Let $\mathcal{C}$ be a presentable ∞-category. Write $\mathcal{C}_*^\wedge$ for the initial object of $\mathrm{CAlg}(\mathrm{Pr})$ under $\mathcal{C}^\times$ such that the underlying category is pointed. Then the underlying category of $\mathcal{C}_*^\wedge$ is $\mathcal{C}_*$, the symmetric monoidal operation is given by*

$$X \otimes Y := X \wedge Y = X \times Y / X \vee Y.$$

(Here $X \vee Y = X \amalg_* Y$ is the coproduct in $\mathcal{C}_*$.)

In particular $\mathcal{S}\mathrm{pc}(S)_*$ is presentably symmetric monoidal.

Exercise 3.2. *Let $S^1 \in \mathcal{S}\mathrm{pc}(S)_*$ denote the image of the constant presheaf with values the circle S^1, and $\mathbb{G}_m$ the image of the representable sheaf $\mathbb{A}^1 \setminus 0$ pointed at 1. Show that $S^1 \wedge \mathbb{G}_m \simeq \mathbb{P}^1$.*

3.3. Stabilization.

Definition 3.8. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr})$ and $X \in \mathcal{C}$. Given $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr})_{\mathcal{C}/}$ we say that X acts invertibly on $\mathcal{D}$ if the functor $\mathcal{D} \to \mathcal{D}$ informally described as $d \mapsto X \otimes d$ is an equivalence. We denote by $\mathcal{C}[X^{-1}]$ the initial object of $\mathrm{CAlg}(\mathrm{Pr})$ under $\mathcal{C}$ on which $\mathcal{X}$ acts invertibly, and call it the *stabilization of $\mathcal{C}$ with respect to X.*

Example 3.9. Let $\mathcal{C} = \mathcal{S}\mathrm{pc}_*$ (the usual ∞-category of pointed spaces) and $X = S^1$. Then the category $\mathcal{C}[(S^1)^{-1}]$ is called the stable ∞-category. We denote it by $\mathcal{SH}$. $\mathrm{h}\mathcal{SH}$ is the *stable homotopy category*, as constructed by Boardman. The canonical functor $\mathcal{S}\mathrm{pc}_* \to \mathcal{SH}$ is customarily denoted Σ^∞. The traditional notation for the tensor product in $\mathcal{SH}$ is $\wedge$.

Definition 3.10. Let $\mathbb{P}^1 \in \mathcal{S}\mathrm{pc}(S)_*$ denote $\mathbb{P}^1$ pointed at 1. The category $\mathcal{SH}(S) := \mathcal{S}\mathrm{pc}(S)_*[(\mathbb{P}^1)^{-1}]$ is called the *stable motivic ∞-category.* We denote the canonical functor $\mathcal{S}\mathrm{pc}(S)_* \to \mathcal{SH}(S)$ by Σ^∞, and the tensor product in $\mathcal{SH}(S)$ by $\wedge$.

In order to get a handle on the very abstractly defined stabilizations, the following result is very helpful.

Theorem 3.11. *Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr})$ and $X \in \mathcal{C}$. Suppose that for some $n \geq 2$, the cyclic permutation on $X^{\otimes n}$ is homotopic to the identity. Then the presentable category underlying $\mathcal{C}[X^{-1}]$ is equivalent to the category*

$$Sp^{\mathbb{N}}(\mathcal{C}, X) = eq(\mathrm{Fun}(\mathbb{N}, \mathcal{C}) \rightrightarrows \mathrm{Fun}(\mathbb{N}, \mathcal{C})),$$

where the two endomorphisms are given by the identity $(X_n)_n \mapsto (X_n)_n$ and $\Omega : (X_n)_n \mapsto (\Omega_X X_{n+1})_n$. Here Ω_X denotes the right adjoint of $(-) \otimes X : \mathcal{C} \to \mathcal{C}$.

In other words, objects of $Sp^{\mathbb{N}}(\mathcal{C}, X)$ are collections $(c_1, c_2, \dots)$ with $c_i \in \mathcal{C}$, together with *bonding maps* $c_i \simeq \Omega_X c_{i+1}$, and morphisms are the evident commutative diagrams.

3.4. **The motivic stable homotopy category.** It follows from Exercise 3.2 that the operations $\Sigma^{1,1} := \Sigma_{\mathbb{G}_m} := (-) \wedge \Sigma^\infty \mathbb{G}_m : \mathcal{SH}(S) \to \mathcal{SH}(S)$ and $\Sigma^{1,0} := \Sigma_{S^1} := (-) \wedge \Sigma^\infty S^1 : \mathcal{SH}(S) \to \mathcal{SH}(S)$ are invertible. For $p, q \in \mathbb{Z}$ we put $\Sigma^{p,q} = (\Sigma^{1,1})^{\circ p}(\Sigma^{1,0})^{\circ q-p}$.

Definition 3.12. For $E \in \mathcal{SH}(S)$, the groups[11]

$$\pi_{i,j}(E) := [\Sigma^{i,j}\mathbb{1}, E]$$

are called the *bigraded homotopy groups of E*. We also use the notation

$$\pi_i(E)_j = [\Sigma^i \mathbb{1}, \Sigma^{j,j} E] = \pi_{i-j,-j}(E).$$

Remark 3.13. Suspension $\Sigma^{1,0}$ being invertible, $\mathcal{SH}(S)$ is a *stable* ∞-category. It follows in particular that $\mathrm{h}\mathcal{SH}(S)$ is a triangulated category. So for example if $A \to B \to C \in \mathcal{SH}(S)$ is a cofiber sequence, then for any $q \in \mathbb{Z}$ we obtain a long exact sequence

$$\cdots \to \pi_{*,q}A \to \pi_{*,q}B \to \pi_{*,q}C \to \pi_{*-1,q}A \to \dots .$$

Definition 3.14 (Milnor–Witt K-theory)**.** Let k be a field. The graded ring $K_*^{MW}(k)$ called *Milnor–Witt K-theory of k* is defined to be the quotient of the free non-commutative algebra on generators $[a]$ in degree 1 for $a \in k^\times$ and a generator η in degree -1, subject to the following relations

(1) $\eta[a] = [a]\eta$,
(2) $[a][1-a] = 0$ for $a \in k \setminus \{0, 1\}$,
(3) $[ab] = [a] + [b] + \eta[a][b]$, and
(4) $\eta(2 + \eta[-1]) = 0$.

The graded ring $K_*^M(k)$ called *Milnor K-theory of k* is defined as the quotient of $K_*^{MW}(k)$ by the central element η.

For $a \in k^\times$, there is a corresponding map $[a] : * \to \mathbb{G}_m \in \mathcal{S}\mathrm{pc}(k)$ yielding also a stable map $[a] : \mathbb{1} \to \Sigma^{1,1}\mathbb{1} \in \mathcal{SH}(k)$. There is also the map $H : \mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$. By Exercises 3.2 and 3.6, $\Sigma^{-1,-1}\Sigma^\infty H$ defines a map $\eta : \Sigma^{1,1}\mathbb{1} \to \mathbb{1} \in \mathcal{SH}(k)$.

Theorem 3.15 (Morel)**.** *Let k be a field. The graded ring $\pi_0(\mathbb{1}_k)_*$ is via the elements constructed above isomorphic to $K_*^{MW}(k)$.*

3.5. **The algebraic K-theory spectrum.** For a scheme X, write $\mathrm{Vect}(X)$ for the category of vector bundles on X, viewed as a symmetric monoidal category for direct sum. This yields $\mathrm{Vect}(X) \in \mathrm{CMon}(\mathcal{C}\mathrm{at}_1)$. Discarding non-invertible morphisms, taking classifying spaces and group-completing we obtain a space

$$K\mathrm{Vect}(X) = (B\mathrm{Vect}(X)^{\otimes,\simeq})^{\mathrm{gp}}$$

also called *direct sum K-theory* of X. If $X = Spec(A)$ is affine then $K\mathrm{Vect}(X) \simeq K(A)$. A finitely generated projective $\mathcal{O}_X(X)$-module (i.e. a summand of $\mathcal{O}_X(X)^n$ for some n) induced a vector bundle on X; this is a symmetric monoidal functor $Proj(\mathcal{O}_X(X)) \to \mathrm{Vect}(X)$. This way we obtain a morphism

$$K(\mathcal{O}(-)) \to K\mathrm{Vect}(-) \in \mathcal{P}(\mathrm{Sm}_S)$$

which induces an equivalence on affine schemes, and hence is a Zariski equivalence. Hence the motivic space $K \in \mathcal{S}\mathrm{pc}(S)$ can also be written as $K \simeq L_{\mathrm{mot}} K\mathrm{Vect}(-)$.

The element $0 \in K(S)$ yields a canonical map $* \to K \in \mathcal{S}\mathrm{pc}(S)$, i.e. a lift $K \in \mathcal{S}\mathrm{pc}(S)_*$. We have the tautological line bundle $\gamma \in \mathrm{Vect}(\mathbb{P}^1_S)$. For $X \in \mathrm{Sm}_S$, we obtain an additive functor $\otimes\gamma : \mathrm{Vect}(X) \to \mathrm{Vect}(X \times \mathbb{P}^1)$, naturally in X. This induces a morphism of commutative monoids $\gamma : K\mathrm{Vect}(X) \to K\mathrm{Vect}(X \times \mathbb{P}^1)$. We also have the pullback morphism $1 : K\mathrm{Vect}(X) \to K\mathrm{Vect}(X \times \mathbb{P}^1)$, and since $K\mathrm{Vect}(X)$ is grouplike, we can form the difference of morphisms $\gamma - 1$.

Theorem 3.16. *The above morphisms assemble into a morphism of presheaves*

$$\gamma - 1 : K\mathrm{Vect}(-) \to \Omega_{\mathbb{P}^1} K\mathrm{Vect}(-) \in \mathcal{P}(\mathrm{Sm}_S)_*.$$

If S is noetherian, regular and finite-dimensional[12]*, the induced map*

$$\gamma - 1 : K \simeq L_{\mathrm{mot}}(K\mathrm{Vect}(-)) \to \Omega_{\mathbb{P}^1} K \in \mathcal{S}\mathrm{pc}(S)_*$$

is an equivalence.

[11] Recall that $[A, B] := \pi_0\mathrm{Map}(A, B)$.
[12] These assumptions are actually unnecessary.

Proof. We put $K\mathrm{Vect}(-) =: K'$ for brevity. By naturality we obtain a morphism $K' \to \Omega_{\mathbb{P}^1_+} K' \in \mathcal{P}(\mathrm{Sm}_S)_*$, which by adjunction is the same as a morphism $\mathbb{P}^1_+ \wedge K' \to K'$. Exercise 3.1 gives us a cofiber sequence $*_+ \to \mathbb{P}^1_+ \to \mathbb{P}^1$, and hence to produce $\mathbb{P}^1 \wedge K' \to K'$ we need to show that the composite $K' \simeq *_+ \wedge K' \to \mathbb{P}^1_+ \wedge K' \to K'$ is null homotopic. Unwinding the definitions, it is induced by $\gamma|_S - 1$, which is null since $\gamma|_S$ is the trivial bundle. Passing over by adjoints again, we have constructed the desired map $K' \to \Omega_{\mathbb{P}^1} K'$. This induces $L_{\mathrm{mot}} K' \to L_{\mathrm{mot}} \Omega_{\mathbb{P}^1} K' \to \Omega^1_{\mathbb{P}} L_{\mathrm{mot}} K'$. In order to show that this is an equivalence, by Remark 2.26 we must show that for a noetherian regular finite-dimensional scheme X we have $K(X) \simeq K(X_+ \wedge \mathbb{P}^1)$, where $K(-)$ is Thomason–Trobaugh K-theory. This follows from the so-called *projective bundle formula* [Wei13, Theorem V.1.5]. □

Definition 3.17. We obtain an object

$$\mathrm{KGL} = \mathrm{KGL}_S := (K, K, K, \dots) \in Sp^{\mathbb{N}}(\mathcal{S}\mathrm{pc}(S)_*, \mathbb{P}^1) \simeq \mathcal{SH}(S),$$

with all bonding maps given by $\gamma - 1$.

Exercise 3.3. *Show that* $\Sigma^{2n,n}\mathrm{KGL} \simeq \mathrm{KGL}$.

Proposition 3.18. *Let S be regular, noetherian and finite-dimensional. Then $\pi_{p,q}\mathrm{KGL}_S \simeq K_{p-2q}(S)$ for $p \geq 2q$ and $= 0$ else.*

Proof. See Exercise 3.7. □

3.6. **The slice filtration.** Denote by $\mathcal{SH}(k)^{\mathrm{eff}} \subset \mathcal{SH}(k)$ the full subcategory generated under colimits by objects of the form $\Sigma^{n,0}\Sigma^\infty_+ X$ for $X \in \mathrm{Sm}_k, n \in \mathbb{Z}$. The adjoint functor theorem implies that the inclusion admits a right adjoint r, and we denote the composite

$$\mathcal{SH}(k) \xrightarrow{r} \mathcal{SH}(k)^{\mathrm{eff}} \to \mathcal{SH}(k)$$

by $f_0 : \mathcal{SH}(k) \to \mathcal{SH}(k)$. For $n \in \mathbb{Z}$ we put

$$f_n = \Sigma^{n,n} f_0 \Sigma^{-n,-n} : \mathcal{SH}(k) \to \mathcal{SH}(k).$$

Exercise 3.4. *Show that $f_n(\mathcal{SH}(k)) \subset \mathcal{SH}(k)^{\mathit{eff}}(n) := \Sigma^{n,n}\mathcal{SH}(k)^{\mathit{eff}}$ and that the restricted functor $f_n : \mathcal{SH}(k) \to \mathcal{SH}(k)^{\mathit{eff}}(n)$ is right adjoint to the inclusion $\mathcal{SH}(k)^{\mathit{eff}}(n) \to \mathcal{SH}(k)$.*

It follows that for $E \in \mathcal{SH}(k)$ we obtain a canonical bi-infinite tower

$$\cdots \to f_1 E \to f_0 E \to f_{-1} E \to \cdots \to E \in \mathcal{SH}(k),$$

called the *slice tower of* E. The successive cofibers

$$s_n(E) := \mathrm{cof}(f_{n+1}E \to f_n E)$$

are called the *slices of* E.

Definition 3.19. The motivic spectrum $\mathrm{kgl} := f_0\mathrm{KGL}$ is called the *effective algebraic K-theory spectrum.* The motivic spectrum $H\mathbb{Z} := s_0\mathrm{KGL}$ is called the *motivic cohomology spectrum.*

Exercise 3.5. *Show that $f_n\mathrm{KGL} \simeq \Sigma^{2n,n}\mathrm{kgl}$ and $s_n\mathrm{KGL} \simeq \Sigma^{2n,n}H\mathbb{Z}$.*

Definition 3.20. The decreasing filtration $\mathrm{im}(\pi_{*,0}f_n\mathrm{KGL}_k \to \pi_{*,0}\mathrm{KGL}_k)_n \subset K_*(k)$ is called the *motivic filtration.*

3.7. **Further exercises.**

Exercise 3.6. *Show that* $\mathbb{A}^n \setminus 0 \simeq \mathbb{G}_m^{\wedge n} \wedge S^{n-1}$.

Exercise 3.7. *Let S be regular, noetherian and finite-dimensional and $X \in \mathrm{Sm}_S$. Prove that $[\Sigma^{p,q}\Sigma^\infty_+ X, \mathrm{KGL}_S] = K_{p-2q}(X)$ for $p - 2q \geq 0$, and $= 0$ else, as follows.*

1. *Reduce to $q = 0$.*
2. *Show the result for $p \geq 0$ (and $q = 0$).*
3. *Show that the result for $p < 0$ will follow from the following statement: if $X \in \mathrm{Sm}_S$, then $K_0(X) \simeq K_0(X \times \mathbb{A}^1 \setminus 0)$.*

This last property is known as (part of) Bass' fundamental theorem, and holds in the regular noetherian situation. [Hints: *Use that* $\mathrm{KGL} \simeq \mathbb{P}^1 \wedge \mathrm{KGL}$ *and* $S^{-1} \wedge \mathbb{P}^1 \simeq \mathbb{G}_m$.]

4. Motivic cohomology

Throughout we work in $\mathcal{SH}(k)$, where k is a field.[13]

4.1. Homotopy ring spectra.

Definition 4.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. By a *homotopy ring object* of $\mathcal{C}$ we mean an object of $\mathrm{CAlg}(\mathrm{h}\mathcal{C})$, i.e. a commutative algebra object in the symmetric monoidal 1-category $\mathrm{h}\mathcal{C}$.[14]

In other words a homotopy ring object is $E \in \mathcal{C}$ together with maps $u : \mathbb{1} \to E$ and $m : E \otimes E \to E$ satisfying the axioms of a commutative algebra up to homotopy; e.g. the composite

$$E \simeq \mathbb{1} \otimes E \xrightarrow{u \otimes \mathrm{id}_E} E \otimes E \xrightarrow{m} E$$

should be homotopic to id_E.

Example 4.2. One may show that the tensor product of vector bunndles induces a homotopy ring structure on $\mathrm{KGL}_k \in \mathcal{SH}(k)$.

One checks immediately that lax symmetric monoidal functors preserve homotopy ring objects.

Exercise 4.1. *Show that* $\mathrm{kgl}, H\mathbb{Z} \in \mathcal{SH}(k)$ *are homotopy ring objects.*

4.2. Motivic cohomology groups.

Given $E \in \mathcal{SH}(k)$ and $X \in \mathrm{Sm}_k$, we put

$$E^{p,q}(X) = [\Sigma^\infty X_+, \Sigma^{p,q} E].$$

This is called the *bigraded cohomology theory represented by* E.

Example 4.3. $\pi_{p,q} E \simeq E^{-p,-q}(k)$

Example 4.4. If E is a homotopy ring spectrum, then $E^{**}(X)$ is a (not necessarily commutative) ring.

Definition 4.5. For an integer n and $E \in \mathcal{SH}(k)$, we define E/n as the pushout in the following diagram

$$\begin{array}{ccc} E & \xrightarrow{n} & E \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & E/n. \end{array}$$

Here the map $n : E \to E$ is the sum of n copies of the identity map, using additivity of $\mathcal{SH}(k)$.

Example 4.6. We have a long exact sequence

$$\cdots \to \pi_{p,q} E \xrightarrow{n} \pi_{p,q} E \to \pi_{p,q}(E/n) \to \pi_{p-1,q} E \xrightarrow{n} \ldots.$$

Remark 4.7. It is possible to show that $H\mathbb{Z}/n$ is a homotopy ring spectrum, in an essentially unique way.

Definition 4.8. The bigraded cohomology theory represented by $H\mathbb{Z}$ (respectively $H\mathbb{Z}/n$) is called *motivic cohomology* (respectively motivic cohomology with $\mathbb{Z}/n$ coefficients) and denoted by

$$H^{p,q}(X) = H^{p,q}(X, \mathbb{Z}) = H^p(X, \mathbb{Z}(q)) := H\mathbb{Z}^{p,q}(X)$$

(respectively $H^{p,q}(X, \mathbb{Z}/n) = H^p(X, \mathbb{Z}(q)/n) := (H\mathbb{Z}/n)^{p,q}(X)$).

4.3. The Bloch cycle complex.

Let $X \in \mathrm{Sm}_k$ and $d \geq 0$. We denote by $Z^d(X)$ the group of codimension d cycles on X, in other words the free abelian group on those points $x \in X$ such that the closure $\overline{\{x\}} \subset X$ has codimension d. The latter condition means hat $\dim \mathcal{O}_{X,x} = d$, or equivalently if X is equidimensional, $\dim \overline{\{x\}} + d = \dim X$. Now let $i : Y \to X$ be a closed immersion, with $Y \in \mathrm{Sm}_k$. If $c = \sum_n a_n x_n \in Z^d(X)$, we say that c *is in good position with respect to* i if every component of $Y \cap \overline{\{x_n\}}$ has codimension $\geq d$ on Y, for every n. Denote by $Z^d(X)_i \subset Z^d(X)$ the subgroup of cycles in good position with respect to i. Then there exists a pullback map $i^* : Z^d(X)_i \to Z^d(Y)$; see e.g. [MVW06, §17A.1]. If $c = x \in Z^d(X)_i$ then $i^*(c) = \sum e_n y_n$, where y_n runs through the generic points of $\overline{\{x\}} \cap Y$ of codimension d and $e_n \in \mathbb{Z}$ is called the *intersection multiplicity*.

[13] Some of the definitions make sense more generally, but we concentrate on the fields case for simplicity.

[14] There is a notion of commutative algebra in $\mathcal{C}$ itself, but the definition is more involved.

Definition 4.9 (Bloch cycle complex)**.** Let $X \in \mathrm{Sm}_k$. Denote by $z^d(X,n) \subset Z^d(X \times \mathbb{\Delta}^n)$ the subgroup of those cycles in good position with respect to all (iterated) faces $X \times \mathbb{\Delta}^i \subset X \times \mathbb{\Delta}^n$. Define

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^* : z^d(X,n) \to z^d(X,n-1),$$

where $d_i : \mathbb{\Delta}^{n-1} \to \mathbb{\Delta}^n$ is the inclusion of the i-th face. One may show that $(z^d(X,\bullet),\partial)$ is a chain complex. Its homology groups are denote by

$$\mathrm{CH}^d(X,n) = H_n(z^d(X,\bullet),\partial)$$

and called *higher Chow groups.*

Remark 4.10. Replacing $z^d(X,\bullet)$ by $z^d(X,\bullet)/n$, one obtains higher Chow groups with $\mathbb{Z}/n$ coffecients.

Example 4.11. $z^d(X,0) = Z^d(X)$ (sine $\mathbb{\Delta}^0$ has no faces), and so $\mathrm{CH}^d(X,0)$ is a quotient of $Z^d(X)$. One may show that it coincides with the ordinary *Chow group* $\mathrm{CH}^d(X)$.

Exercise 4.2. *Let X be connected. Show that $\mathrm{CH}^0(X,0) = \mathbb{Z}$ whereas $\mathrm{CH}^0(X,n) = 0$ for $n \neq 0$.*

Remark 4.12. It is not obvious, but $\bigoplus_{d,n} \mathrm{CH}^d(X,n)$ can be given a ring structure related to intersection of cycles.

Remark 4.13. Let $f : Y \to X$ be a flat morphism. Since flat morphisms preserve codimension of subschemes, there is an induced morphism $f^* : z^d(X,n) \to z^d(Y,n)$ which one may prove induces a morphism of complexes $z^d(X) \to z^d(Y)$. Hence we obtain *flat pullbacks* $\mathrm{CH}^d(X,n) \to \mathrm{CH}^d(Y,n)$.

4.4. Theorems.

Theorem 4.14 (Bloch [Blo86], bottom p. 269)**.** *Let X be the localization of a smooth scheme. Then $\mathrm{CH}^1(X,1) = \mathcal{O}_X(X)^\times$ and $\mathrm{CH}^1(X,n) = 0$ else.*

Theorem 4.15 (Nesterenko–Suslin [NS90], Totaro [Tot92])**.** *Let k be a field. We have $\mathrm{CH}^d(k,n) = 0$ for $n < d$ and $\mathrm{CH}^d(k,d) \simeq K_d^M(k)$.*

The above two theorems are, in principle, elementary. This is very far from true for the remaining theorems we are going to state.

Theorem 4.16 (Levine [Lev08])**.** *For $X \in \mathrm{Sm}_k$, $p,q \in \mathbb{Z}$ we have natural isomorphisms*

$$H^{p,q}(X) \simeq \mathrm{CH}^q(X, 2q-p)$$

and similarly for $\mathbb{Z}/n$ coefficients.

Apart from the fact that there is no obvious relationship a priori between motivic cohomology and higher Chow groups (as we defined them), they also have very different a priori properties. Let us unpack these properties to illuminate just how strong the above theorem is:

- By Remark 4.13, $\mathrm{CH}^d(-,n)$ is contravariantly functorial in *flat* morphisms. But since any cohomology theory represented by a motivic spectrum is contravariant in all morphisms of schemes, we deduce that the same must be true for $\mathrm{CH}^d(-,n)$.
- By construction, any cohomology theory represented by a motivic spectrum is $\mathbb{A}^1$-invariant. We deduce that $\mathrm{CH}^d(X,n) \simeq \mathrm{CH}^d(X \times \mathbb{A}^1, n)$.
- Similarly, any cohomology theory represented by a motivic spectrum has a Mayer–Vietoris sequences for distinguished Nisnevich squares, hence the same must be true for $\mathrm{CH}^d(-,\bullet)$.
- Combining this with Theorem 4.14, we deduce that

$$H^p(X,\mathbb{Z}(1)) \simeq H^{p-1}_{\mathrm{Nis}}(X,\mathbb{G}_m) \simeq \begin{cases} Pic(X) & p=2 \\ \mathcal{O}_X(X)^\times & p=1 \\ 0 & \text{else} \end{cases}.$$

- By construction we have $\mathrm{CH}^d(X,n) = 0$ for $n < 0$. Translated into motivic cohomology this says that $H^{p,q}(X) \simeq \mathrm{CH}^q(X,2q-p) = 0$ for $p > 2q$.

Exercise 4.3. *Without using the above theorem, show that $\mathrm{CH}^q(X,2q-p) = 0 = H^{p,q}(X)$ for $q < 0$.*

Remark 4.17. It is an open question if $H^{p,q}(X) = 0$ for $p < 0$. This is known as the *Beilinson–Soulé vanishing conjecture.*

Exercise 4.4. *Show that* $H^{p,p}(k, \mathbb{Z}/n) = K_p^M(k)/n$ *and* $H^{p,q}(k, \mathbb{Z}/n) = 0$ *for* $p > q$.

The next result is known as the norm residue isomorphism theorem, formerly the *Bloch–Kato conjecture.* See [HW19] for a textbook account.

Theorem 4.18 (Voevodsky). *Let n be coprime to the characteristic of k, $q \geq 0$ and $X \in \mathrm{Sm}_k$.*

1. *The complex $L_{\acute{e}t} z^d(X)/n$ (on the site of étale X-schemes) is quasi-isomorphic to $L_{\acute{e}t}\mu_n^{\otimes d}$.*
2. *The induced map $H^p(X, \mathbb{Z}/n(q)) \to H^p_{\acute{e}t}(X, \mu_n^{\otimes q})$ is an isomorphism for $p \leq q$.*

The really difficult part of the theorem (2), (1) is more elementary, going back to Bloch. Again let us expound on this result by illustrating some consequences.

- If $p < 0$, then $H^p_{\acute{e}t}(X, \mu_n^{\otimes q}) = 0$. Thus the Beilinson–Soulé vanishing conjecture holds with $\mathbb{Z}/n$ coefficients.
- If k is a field, then $H^p(k, \mathbb{Z}/n(q)) = 0$ for $p > q$ by Theorem 4.15. Hence we obtain

$$H^{p,q}(k, \mathbb{Z}/n) \simeq \begin{cases} H^p_{\acute{e}t}(k, \mu_n^{\otimes q}) & p \leq q, q \geq 0 \\ 0 & \text{else} \end{cases}.$$

- Using the other part of Theorem 4.15 we obtain a canonical isomorphism (of rings)

$$\bigoplus_{p \geq 0} H^p_{\acute{e}t}(k, \mu_n^{\otimes p}) \simeq K_*^M(k)/n.$$

Exercise 4.5. *Let k be a field containing a primitive n-th root of unity (so in particular n is coprime to the characteristic of k). Show that*

$$H^{**}(k, \mathbb{Z}/n) \simeq K_*^M(k)/n[\tau],$$

where τ is a certain element in $H^{0,1}(k, \mathbb{Z}/n)$.

Finally we have the following analog "at the characteristic".

Theorem 4.19 (Geisser–Levine [GL00]). *Let k have characteristic p. Then $H^{m,n}(k, \mathbb{Z}/p^r) = 0$ unless $m = n$.*

In other words, combining again with Theorem 4.15, we find that

$$H^{**}(k, \mathbb{Z}/p^r) \simeq K_*^M(k)/p^r,$$

where the elements of $K_q^M(k)/p^r$ are placed in bidegree (q, q).

5. K-theory of fields

Throughout we work in $\mathcal{SH}(k)$, where k is a field.

5.1. p-adic completion.

Proposition 5.1 (Milnor exact sequence). *Let $\cdots \to E_2 \to E_1 \in \mathcal{SH}(k)$ be an inverse system. Then there is a short exact sequence*

$$0 \to \mathrm{lim}^1_n \pi_{i+1,j} E_n \to \pi_{i,j}(\lim_n E_n) \to \lim_n \pi_{i,j}(E) \to 0.$$

Definition 5.2. Let $E \in \mathcal{SH}(k)$. We put

$$E_p^\wedge = \lim \left(\cdots \to E/p^3 \to E/p^2 \to E/p \right)$$

and call it the p-adic completion of E.

Example 5.3. The groups $\pi_{n,0}\mathrm{KGL}_p^\wedge$ are also denoted $K_n(k, \mathbb{Z}_p)$ and called the *p-adic K-theory groups of k.*

Proposition 5.4.

1. *The assignment $E \mapsto E_p^\wedge$ is a lax symmetric monoidal functor preserving limits and finite colimits.*

(2) *There are natural exact sequences*
$$0 \to L_p\pi_{i,j}(E) \to \pi_{i,j}(E^\wedge_p) \to \lim_n \pi_{i-1,j}(E)[p^n] \to 0$$
and
$$0 \to \mathrm{lim}^1_n \pi_{i,j}(E)[p^n] \to L_p\pi_{i,j}(E) \to \pi_{i,j}(E)^\wedge_p \to 0.$$
Here for an abelian group A, $A[p^n]$ means the p^n-torsion subgroup, and $A^\wedge_p = \lim_n A/p^n$ is the p-completion.

(3) *There is a cofiber sequence*
$$E'_p := \lim\Big(\dots \xrightarrow{p} E \xrightarrow{p} E \xrightarrow{p} E\Big) \to E \to E^\wedge_p.$$

(4) *We have $E^\wedge_p/p \simeq E/p$.*

Proof. (1) is formal nonsense, and (2) can be deduced from Proposition 5.1. (3) follows from the cofiber sequence of diagrams
$$\begin{tikzcd}
\dots \arrow[r, "p"] & E \arrow[r, "p"] \arrow[d, "p^3"'] & E \arrow[r, "p"] \arrow[d, "p^2"'] & E \arrow[d, "p"'] \\
\dots \arrow[r, "\mathrm{id}"] & E \arrow[r, "\mathrm{id}"] \arrow[d] & E \arrow[r, "\mathrm{id}"] \arrow[d] & E \arrow[d] \\
\dots \arrow[r] & E/p^3 \arrow[r] & E/p^2 \arrow[r] & E/p
\end{tikzcd}$$
by taking limits, noting that limits preserve cofiber sequences by stability. (4) can be deduced from (3) by showing that multiplication by p is an isomorphism of E'_p. $\square$

Exercise 5.1. *Suppose that $\pi_{**}E$ is bounded p-torsion. Show that $\pi_{**}(E^\wedge_p) \simeq \pi_{**}(E^\wedge_p)$.*

5.2. Weakly orientable spectra and Bott elements.

Definition 5.5. Let $E \in \mathcal{SH}(k)$ be a homotopy ring spectrum. We call E *weakly orientable* if the image of $\eta \in \pi_{-1,-1}\mathbb{1} \simeq K^{MW}_{-1}(k)$ under the unit map $\mathbb{1} \to E$ vanishes.

Exercise 5.2. *Let E be weakly orientable. Show that f_0E, s_0E are also weakly orientable.*

Example 5.6. KGL is weakly orientable, since $\pi_{-1,-1}(\mathrm{KGL}) = 0$ by Exercise 3.7. It follows from Exercise 5.2 that kgl, $H\mathbb{Z}$ are also weakly orientable.

Example 5.7. Let E be weakly orientable. Then the canonical map
$$K^{MW}_*(k) \simeq \pi_0(\mathbb{1})_* \to \pi_0(E)_*$$
annihilates η, and so we obtain
$$K^M_*(k) \to \pi_0(E)_*.$$

Remark 5.8. One may show that if E is weakly orientable, then the ring $E^{**}(X)$ is graded commutative (with the signs depending only on the first index).

Definition 5.9. We shall write $\mu_{p^\infty} \subset k$ to mean that $char(k) \neq p$ and also k contains all p^n-th roots of unity, for all n.

Construction 5.10. Let $\mu_{p^\infty} \subset k$ and $E \in \mathcal{SH}(k)$ a weakly orientable ring spectrum. Pick a sequence $(\zeta_n)_n \in k^\times$, where ζ_n is a primitive p^n-th root of unity and $\zeta_{n+1}^p = \zeta_n$. The elements $[\zeta_n] \in \pi_0(E)_1$ from Example 5.6 induce via the Milnor exact sequence (Proposition 5.1) an element (possibly non-unique) $\zeta \in \pi_0(E'_p)$ (notation from Proposition 5.4). By construction the image of ζ in $\pi_0(E)_1$ is $p[\zeta_1] = [\zeta_1^p] = [1] = 0$ (using Example 5.6 again for the first equality), and hence ζ lifts to an element (possibly non-unique) $\tau \in \pi_1(E^\wedge_p)_1 = \pi_{0,-1}(E^\wedge_p)$ called *a Bott element.*

Exercise 5.3. *Show that under the composite $E^\wedge_p \to E/p \xrightarrow{\partial} \Sigma E/p$, where ∂ is the Bockstein map (i.e. the composite of the boundary map $E/p \to \Sigma E$ and the reduction $\Sigma E \to \Sigma E/p$), the Bott element τ is sent to the image of $[\zeta_1]$.*

5.3. K-theory computations.

Lemma 5.11. *Suppose that $\mu_{p^\infty} \subset k$ and $\tau \in \pi_{0,-1} H\mathbb{Z}_p^\wedge$ is a Bott element. Then*

$$H^{**}(k, \mathbb{Z}_p) := \pi_{-*,-*} H\mathbb{Z}_p^\wedge \simeq K_*^M(k)_p^\wedge[\tau].$$

Alternatively if $char(k) = p$ we have $H^{**}(k, \mathbb{Z}_p) \simeq K_*^M(k)_p^\wedge$.

Proof. Let us prove the second statement as a warm-up. We have $H\mathbb{Z}_p^\wedge = \lim_n H\mathbb{Z}/p^n$, and $\pi_{**} H\mathbb{Z}/p^n \simeq K_*^M(k)/p^n$ by Theorem 4.19. By the Milnor exact sequence (Proposition 5.1), we need to prove that $\lim^1$ of this sequence vanishes. But this is true since all the transition maps are surjective. For the first statement we can argue similarly using the resolution of the Bloch–Kato conjecture; via Exercise 4.5 we are reduced to proving that the image τ_n of τ in $\pi_{0,-1}(H\mathbb{Z}/p^n)$ generates this group, which we know is isomorphic to $\mathbb{Z}/p^n$. Since $\mathbb{Z}/p^n$ is a local ring, this just means that the image in $\mathbb{Z}/p$ is non-zero; i.e. we need to prove that τ_1 is non-zero. But $\partial\tau_1 \neq 0$ by Exercise 5.3, concluding the proof. □

Theorem 5.12. *Let $\mu_{p^\infty} \subset k$ or $char(k) = p$. Write $\tau \in \pi_{0,-1}\mathrm{kgl}_p^\wedge$ for a choice of Bott element and $\beta = \gamma - 1 \in \pi_{2,1}\mathrm{kgl}$ for the periodicity generator. Let $R = \mathbb{Z}$ if $char(k) = p$ and $R = \mathbb{Z}[\tau]$ else. Then we have*

$$\pi_{**}\mathrm{kgl}_p^\wedge \simeq K_*^M(k)_p^\wedge[\beta] \otimes R \quad \text{and} \quad \pi_{**}\mathrm{KGL}_p^\wedge \simeq K_*^M(k)_p^\wedge[\beta, \beta^{-1}] \otimes R.$$

Proof. We have

$$\mathrm{KGL} = \mathrm{kgl}[\beta^{-1}] = \operatorname{colim}(\mathrm{kgl} \xrightarrow{\beta} \Sigma^{-2,-1}\mathrm{kgl} \xrightarrow{\beta} \dots),$$

whence the statement for kgl implies the one for KGL. We have seen in Exercise 3.7 that $\pi_n(\mathrm{kgl})_0 = \pi_n(\mathrm{KGL})_0 = 0$ for $n < 0$. One may use this to deduce [Bac17, §3] that $\pi_n(\mathrm{kgl})_* = 0$ if $n < 0$, and hence by Proposition 5.4(2) we get

$$\pi_n(\mathrm{kgl}_p^\wedge)_* = 0 \text{ if } n < 0. \tag{5.1}$$

Exercise 3.5 supplies us with a cofiber sequence $\Sigma^{2,1}\mathrm{kgl} \xrightarrow{\beta} \mathrm{kgl} \to H\mathbb{Z}$, and hence by p-completing we get

$$\Sigma^{2,1}\mathrm{kgl}_p^\wedge \xrightarrow{\beta} \mathrm{kgl}_p^\wedge \to H\mathbb{Z}_p^\wedge. \tag{5.2}$$

Now we determine $\pi_{**}\mathrm{kgl}_p^\wedge$. This is a ring by Example 4.4. Example 5.6 leads to a ring map in from $K_*^M(k)_p^\wedge$, and we also have elements β, τ (the latter if $char(k) \neq p$). Hence we obtain a ring morphism

$$\alpha : S := K_*^M(k)_p^\wedge[\beta] \otimes R \to \pi_{**}\mathrm{kgl}_p^\wedge,$$

which we shall show is an isomorphism. Lemma 5.11 shows that the right hand map in (5.2) is surjective on π_{**}, and hence we get a short exact sequence

$$0 \to \pi_{*-2,*-1}\mathrm{kgl}_p^\wedge \xrightarrow{\beta} \pi_{**}\mathrm{kgl}_p^\wedge \xrightarrow{r} K_*^M(k)_p^\wedge \otimes R \to 0.$$

Proof that α is surjective: Let $x_1 \in \pi_{**}\mathrm{kgl}_p^\wedge$. If $r(x_1) \neq 0$, then we find $y \in S$ with $r\alpha(y) = r(x_1)$. Replacing x_1 by $x_1 - \alpha(y)$, we may assume that $r(x_1) = 0$. Then $x_1 = \beta x_2$, for some $x_2 \in \pi_{*-2,*-1}\mathrm{kgl}_p^\wedge$, and it suffices to prove that x_2 is in the image of α. Repeating this argument, we eventually get to $x_n = 0$, by (5.1). Hence α is surjective as descried.

Proof that α is injective: Let $0 \neq y \in S$. We can write $y = \beta^n y'$, where y' is not divisible by β. Then $\alpha(y) = 0$ if and only if $\alpha(y') = 0$, because multiplication by β is injective on $\pi_{**}\mathrm{kgl}_p^\wedge$ (by our exact sequence). The image of y' in $S/\beta = K_*^M(k)_p^\wedge \otimes R$ is non-zero and coincides with $r\alpha(y')$, whence also $\alpha(y') \neq 0$. Thus α is injective. □

Corollary 5.13. *Assumptions as in the theorem. We have*

$$K_*(k, \mathbb{Z}_p) \simeq K_*^M(k)_p^\wedge \otimes R',$$

where $R' = \mathbb{Z}$ if $char(k) = p$ and otherwise $R' = \mathbb{Z}[t]$ (with $t = \tau\beta$ and so $|t| = 2$).

Exercise 5.4. *Show that $K_*(\mathbb{C}, \mathbb{Z}_p) \simeq \pi_*\mathrm{ku}_p^\wedge$, where* ku *is the topological spectrum known as connective complex K-theory. Can you produce an equivalence $K(\mathbb{C})_p^\wedge \simeq \mathrm{ku}_p^\wedge$?*

5.4. **Further results.** We can rewrite Theorem 5.12 as

$$\pi_{**}\mathrm{KGL}_p^\wedge \simeq \pi_{**}(H\mathbb{Z}_p^\wedge)[\beta, \beta^{-1}].$$

In this form the result holds for many more fields (i.e. without assuming that $\mu_{p^\infty} \subset k$ [Kah02]. It also holds rationally:

$$\pi_{**}(\mathrm{KGL}) \otimes \mathbb{Q} \simeq \pi_{**}(H\mathbb{Z})[\beta, \beta^{-1}] \otimes \mathbb{Q}$$

for any field k (in fact any smooth scheme over a field) [Rio10, Theorem 5.3.10]. For many more results about the K-theory of fields, see [Wei13, §VI].

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