Lecture 8: Relations

Last time: If G is a pro-p-group then any minimal generating system has rank $\dim H^1(G)$ $\left(= H^1(G, \mathbb{F}_p)\right)$

If $H^2(G) = 0$ then G is free.

Shall see: $H^2(G) \leftrightarrow$ # of relations.

Preliminary: inflation & transgression

Let G be a profinite group, $H \subset G$ normal subgroup, $A \in \mathcal{C}_G$. Recall inflation

$$\text{inf}: H^*(G/H, A^H) \to H^*(G, A^H) \to H^*(G, A)$$

For $g \in G$ build autom $\tilde{g}$ of $H^*(H, A)$ via

$$[H, A] \longrightarrow [H, A]$$

$$h \overset{\varphi}{\longmapsto} ghg^{-1}$$

$$a \overset{\psi}{\longmapsto} ga$$

$$\left[\ \varphi(ha) = g.ha \overset{\shortparallel}{=} \varphi(h)\psi(a) = ghg^{-1}.ga\ \right]$$

Observe that $\tilde{g}_2 \tilde{g}_1 = \widetilde{g_1 g_2}$, i.e. obtain action of G on $H^*(H, A)$.

Note that if $g \in H$ or $ga = a$ $\forall a \in A$ then $\tilde{g} = \text{id}$ on H^0 & hence on $H^*(H, A)$ by dimension shifting.

$\rightsquigarrow$ $H^*(H, A)$ is a discrete G/H-module.

Thm $H^n(G,A) \to H^n(H,A)$ has G/H-invariant image.

Proof

$$\begin{array}{ccc} H^n(G,A) & \longrightarrow & H^n(H,A) \\ \text{id} \downarrow \tilde{g} & G & \downarrow \tilde{g} \\ H^n(G,A) & \longrightarrow & H^n(H,A) \end{array} \quad \square$$

Shall construct the <u>transgression</u>

$$\text{tra} : H^1(H,A)^{G/H} \longrightarrow H^2(G/H, A^H).$$

Let $\bar{a} \in H^1(H,A)^{G/H}$ & $a \in ZC^1(H,A)$ a repres. cocycle (i.e. a crossed hom^n).

First define $b : G \to A$ extending s.t.

(I) b is cts

(II) $g\, b(g^{-1}hg) - b(h) = h\, b(g) - b(g)$

(III) $b(hg) = b(h) + h b(g)$. $\forall g \in G, h \in H$

Let $s : G/H \to G$ be a ct. section.

Since $\bar{a}$ is G-invariant, for $\gamma \in G/H$ find $b(\gamma)$ s.t.

$$h \mapsto (s\gamma)\, a\big((s\gamma)^{-1} h\, (s\gamma)\big) - a(h) = \text{a coboundary}$$
$$= h \cdot b(\gamma) - b(\gamma)$$

For $g = hs\gamma$ put $b(g) = a(h) + h b(s\gamma)$.

Check that this satisfies (I), (II) & (III).

Put $f(g_1, g_2) = s(g_1) + g_1 s(g_2) - s(g_1 g_2)$

(II) $\Rightarrow hf = f$

$\Rightarrow f(h_1 \cdot g_1, h_2 \cdot g_2) = f(g_1, g_2)$

Now define $\varphi \in C^2(G/H, A^H)$ by

$$\varphi(\gamma_1, \gamma_2) = f(s\gamma_1, s\gamma_2).$$

Check that: - φ is a cocycle

- $\bar{\varphi} \in H^2(G/H, A^H)$ is indep. of choices

<u>Def</u> $tr(\bar{a}) = \bar{\varphi}$.

<u>Thm</u> Suppose $H \subset G$ is a normal subgp. of a profinite group. Then tr is a hom &

$$0 \to H^1(G/H, A^H) \to H^1(G, A) \to H^1(H, A)^{G/H} \to H^2(G/H, A^H) \to H^2(G, A).$$

<u>Pf</u> Tedious exercise.

Ex Suppose that all elts of A have order prime to G/H. Then $H^*(G/H, A^H) = 0$ (annihilated by $[G:H]$) & hence $H^1(G, A) \cong H^1(H, A)^{G/H}$.

Systems of relations

Def^{n} G a pro-p-group.

Call an exact sequence $1 \to R \to F \xrightarrow{\varphi} G \to 1$ with F free a __presentation__ of G. If F is free on $\{t_i \mid i \in I\}$ & $\{\varphi(t_i) \mid i \in I\} \subset G$ is a minimal generating set, then call the presentation minimal.

Call $E \subset R$ a (generating) set of relations if the only closed normal subgroup of F ctg. E is R & for every open normal subgroup $U \subset R$, $E \setminus U$ is finite.

Call E __minimal__ if no subset of E generates.

Now let $\{G_i \mid i \in I\}$ be pro-p-groups & $\varphi_i : G_i \to G$.

For $i \in I$ let $T_i \subset G_i$ be a normal subgroup.
Call $\{\varphi_i\}, \{T_i\}$ <u>admissible</u> if G_i/T_i is free
& if $U \subset G$ is open & normal, then
$\varphi(T_i) \subset U$ for almost all i.

<u>Lemma</u> Let $\{\varphi_i\}, \{T_i\}$ be admissible. Suppose given for every i a presentation $1 \to R_i \to F_i \xrightarrow{\psi_i} G_i \to 1$
& also a presentation $1 \to R \to F \to G \to 1$.
Then there exist commutative diagrams

$$\begin{array}{ccccc} R_i & \to & F_i & \to & G_i \\ \downarrow \bar{\chi}_i & & \downarrow \chi_i & & \downarrow \varphi_i \\ R & \to & F & \to & G \end{array}$$

s.t. $\{\chi_i\}, \{R_i\}$ is admissible.

[i.e. if $N \subset R$ open & normal, then $R_i \subset N$ for almost all i]

Call all of this data an admissible presentation of $\{\varphi_i\}, \{T_i\}$.

<u>Pf</u> Pick section $\sigma: G \to F$ with $\sigma(1) = 1$.

Pick generating systems $\{t_u \mid u \in I_i\}$ for the free groups F_i s.t. $I_i = I_i^1 \sqcup I_i^2$ where the images of $\{t_u \mid u \in I_i^1\}$ in G_i/T_i form gen. sys. & all $t_u \in I_i^2$ are mapped to 1.

Define $\chi_i(t_u) = \sigma \varphi_i \psi_i(t_u)$.

Exercise: this has the desired property. □

<u>Lemma</u> In the above situation:

1) for $n \geq 2$, $\alpha \in H^n(G)$: $\underset{\substack{\Uparrow \\ H^n(G_i)}}{\varphi_i^* \alpha} = 0$ for almost all i

2) for $\alpha \in H^1(R)$: $\underset{\substack{\Uparrow \\ H^1(R_i)}}{\chi_i^*(\alpha)} = 0$ for almost all i.

<u>Pf</u> 1) Pick $U \subset G$ open normal s.t.

$$\begin{array}{ccccc} H^n(G/U) & \longrightarrow & H^n(G) & & \text{hits } \alpha \\ \downarrow & & \downarrow & & \\ 0 = H^n(G_i/T_i) & \longrightarrow & H^n(G_i) & & \text{Suppose } \varphi_i(T_i) \subset U \end{array}$$

b/c G_i/T_i is free. //

2)
$$\begin{array}{ccccc} R_i & \longrightarrow & R & & \\ \downarrow & \nearrow & & \searrow & \\ U & \longrightarrow & \{e\} & \longrightarrow & R/U \end{array}$$

& apply H^1.

$\square$

Hence we obtain $\chi^*: H^1(R) \longrightarrow \bigoplus_{i\in I} H^1(R_i)$

$\varphi^*: H^2(G) \longrightarrow \bigoplus_{i\in I} H^2(G_i)$.

<u>Thm</u> $\{\chi_i(R_i) \mid i \in I\}$ generate $R \subset F$ as a normal subgroup if & only if

$$\chi^*: H^1(R)^G \longrightarrow \bigoplus_{i\in I} H^1(R_i)$$

is injective.

<u>Pf</u> Let $(f: R \to \mathbb{F}_p) \in H^1(R)^G$, $\chi^*(f)=0$.
If R is gen. by the $\chi(R_i)$ as normal subgp, then f vanishes on $h^{-1}\chi_i(R_i)h$ $\forall h \in F$ & hence on all of R.

$\therefore f=0$, i.e. χ^* is injective.

Conversely, suppose that χ^* is injective, $R' \subset R$ be the normal subgp. gen. by $\chi_i(R_i)$.

$$H^1(R) \xrightarrow{\alpha} H^1(R') \longrightarrow \bigoplus_{i \in I} H^1(R_i)$$

$$H^1(R)^G \hookrightarrow H^1(R), \qquad H^1(R)^G \to H^1(R') \ \ \therefore \text{inj.}, \qquad H^1(R)^G \xrightarrow{\text{inj.}} \bigoplus_{i \in I} H^1(R_i)$$

$\ker(\alpha)^G = 0.$

$\therefore \ker(\alpha) = 0$ [pro-p-groups cannot act w/o fixed pts on $\mathbb{F}_p$-v.s.]

$\therefore R' \to R$ is surjective □

Main result

Def^n $E \subset R$ is called <u>complementary set</u> if

(I) $E \cup \bigcup_i \chi_i(R_i)$ generates R as a normal subgroup of F

(II) $U \subset R$ open normal $\Rightarrow$ $E \setminus U$ finite.

Call E <u>minimal</u> if no proper subset is complementary.

Thm Given admissible minimal presentations

$$\begin{array}{ccccccccc} 1 \to & R_i & \to & F_i & \to & G_i & \to & 1 \\ & \downarrow \tilde{\chi}_i & & \downarrow \chi_i & & \downarrow \varphi_i & & \\ 1 \to & R & \to & F & \to & G & \to & 1 \end{array}$$

& $E \subset R$ a minimal complement.

Then $|E| = \dim \ker \varphi^*$.

Pf For $j \in E$, let R_j be the subgp. of R generated by j, $\chi_j : R_j \hookrightarrow R$. Note that $\{\chi_j \mid j \in I \cup E\}$ $\{R_j\}$ is admissible & $\bigcup_{j \in I \cup E} R_j$ generate R as a normal subgp.

$$\therefore H^1(R)^G \hookrightarrow \bigoplus_{j \in I \cup E} H^1(R_j)^{G_j}$$

Let $\varphi^* : H^1(R)^G \longrightarrow \bigoplus_{i \in I} H^1(R_i)^G$.

Then $\ker \varphi^* \hookrightarrow \bigoplus_{j \in E} H^1(R_j)$

— in fact iso by prev. thm & minimality of E.

$$\therefore \ \dim \ker \varphi^* = \dim \bigoplus_{i \in E} \underbrace{H^1(R_i)}_{\dim 1} = |E|.$$

Infl.-tra.-diagram:

$$\begin{array}{ccccccc}
 & & H^1(G) & \longrightarrow & \bigoplus_i H^1(G_i) \\
 & & \downarrow \cong \text{ by min. gen. set} & & \downarrow \cong \\
 & & H^1(F) & \longrightarrow & \bigoplus_i H^1(F_i) \\
 & & \downarrow \therefore 0 & & \downarrow \therefore 0 \\
0 \to \ker \varphi^* & \longrightarrow & H^1(R)^G & \xrightarrow{\varphi^*} & \bigoplus_i H^1(R_i)^{G_i} \\
\downarrow \therefore \cong & & \downarrow \text{tra} \ \therefore \cong & & \downarrow \therefore \cong \\
0 \to \ker \varphi^! & \longrightarrow & H^2(G) & \xrightarrow{\varphi^!} & \bigoplus_i H^2(G_i) \\
 & & \downarrow & & \downarrow \\
 & & H^2(F) & \longrightarrow & \bigoplus_i H^2(F_i) \\
 & & \| & & \| \\
 & & 0 & & 0
\end{array}$$

$\square$

<u>Cor</u> $\dim H^2(G) =$ card. of a minimal sys. of rel$^{\text{ns}}$.

$G =$ any pro-p-group

<u>Pf</u> Take $I = \emptyset$.

Cor $\mathcal{R}$ generated by the $\lambda_i(R_i) \iff \varphi^\sigma$ injective.

Pf $E = \emptyset \iff |E| = 0 \iff \varphi^\sigma$ inj. $\square$